Interactive optimization as a tool for finding the complex periodic solutions in nonlinear dynamics

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I n t r o d u c t i o n. We consider essentially nonlinear dynamical systems with the ability to implement a chaotic behavior and deterministic solutions of various kinds. Among the deterministic solutions, we will highlight a variety of periodic solutions of different periods. Problems of control of dynamic regimes in such systems discussed in [1]. This work is devoted to numerical algorithms for constructing and analyzing the stability of periodic solutions of strongly nonlinear dynamical systems.

P r o b l e m S t a t e m e n t. We will consider the strongly nonlinear system, which places no restrictions on the value of the individual components. In the framework of this approach we can analyze the linear and quasi-linear system, but the focus will be given to an essentially nonlinear systems of general form. The only requirement that we make to a dynamic system is the ability to construct numerical solutions of the Cauchy problem with the required precision.

We will use a general approach to the problem of constructing periodic solutions of nonlinear systems of ordinary differential equations. This approach H. Poincare [2] formulated as follows:
Suppose that

$$dx_i(t)/dt=X_i \qquad (i=1,2,\ldots,n) \tag{1}$$

is the system of differential equations, where X_i - data, unambiguous function of the variables x_1, x_2, ..., x_n, and maybe, time t.
Suppose now that

$$x_1=\varphi_1(t),\ x_2=\varphi_2(t),\ \ldots,\ x_n=\varphi_n(t) \tag{2}$$

is particular solution of this system.
Imagine that at time T n variables x_i take their initial values, so that $\varphi_i(0)= \varphi_i(T)$. It is clear that at time T, we will be in the same conditions as at time 0, and hence for any t $\varphi_i(t) = \varphi_i(t + T)$. In other words, the functions $\varphi_i(t)$ are periodic functions of t.

Variant 1. The system (1) is autonomous, that is, the right parts X_i are not depend on time t. In this case, the period T of the solution is unknown.

Variant 2. The system (1) is non-autonomous, that is, the right parts X_i are depend on time t:

$$X_i=X_i(t,x_1,x_2,\ldots,x_n) \tag{3}$$

In this case, the period T of the system (1) is known:

$$X_i(t,x_1,x_2,\ldots,x_n)=X_i(t+T,x_1,x_2,\ldots,x_n) \tag{4}$$

A periodic solution can have a multiple of the period kT, k=1,2,…:

$$\varphi_i(t)=\varphi_i(t+kT),\ \ i=1,2,\ldots,n. \tag{5}$$

H. Poincare for finding periodic solutions of the system implies the existence of a small parameter. We use his approach on the initial conditions of the periodic solution, but let's not assume the existence of a small parameter in the system (1).

Problem Statement of constructing periodic solutions of strongly nonlinear autonomous dynamical system (1) (variant 1):

Find the initial conditions $\varphi_i(0)(i=1,\ldots,n)$, corresponding to the periodic solution and the period T of this solution: $\varphi_i(0)= \varphi_i(T)$, $(i=1,\ldots,n)$, and therefore $\varphi_i(t)= \varphi_i(t+T)$, $(i=1,\ldots,n)$.

Problem Statement of constructing periodic solutions of strongly nonlinear autonomous dynamical system (1) (variant 2):

Find the initial conditions $\varphi_i(0)(i=1,\ldots,n)$, corresponding to the kT-periodic solution: $\varphi_i(0)= \varphi_i(kT)$, $(i=1,\ldots,n,\ k=1,2,\ldots)$, and therefore $\varphi_i(t)= \varphi_i(t+kT)$, $(i=1,\ldots,n)$.

Note that the dimension of variant 1 is n + 1, the dimension of variant 2 is n. After finding the initial conditions of the periodic solution it is built using the numerical integration in one period.

Consider the algorithm for determining the initial conditions of the periodic solution of nonautonomous nonlinear dynamics problems (variant 2). The period T of the system is known. The period of solutions given by integer parameter k at the start. Note that the increase in the integer parameter k does not significantly restrict the form of the solution. For example, for k = 12 in the search box, the solution with k = 1,2,3,4,6,12 are included. (Fig.1).

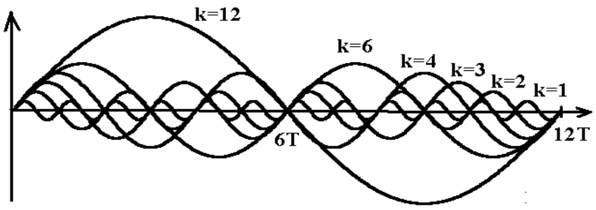

Fig. 1.

Denote the unknown initial conditions, corresponding to kT-periodic solution $Y_i= x_i(0)$. Obviously, for periodic solutions to satisfy the equality

$$x_i(kT)=Y_i,\ i=1, 2, ..., n \tag{6}$$

This is done only when the initial conditions corresponding to a periodic solution.

Consider the function (Fig.2)

$$F(Y_1, Y_2, ..., Y_n)=\sqrt{\sum_{i=1}^{n}[Y_i - x_i(kT\ )]^2} \tag{7}$$

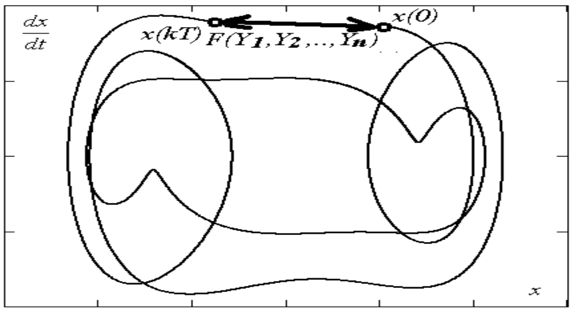

Fig. 2.

This function defines the discrepancy in fulfilling the conditions of periodicity. Obviously, for a periodic solution

$$F(Y_1, Y_2, ..., Y_n)=0 \tag{8}$$

Therefore, to determine the initial conditions corresponding to a periodic solution, it is possible to use optimization algorithms with the objective function

$$F(Y_1, Y_2, ..., Y_n) \rightarrow \min \tag{9}$$

Comparative numerical experiments have shown that the more effective is another way to determine the initial conditions Y_i, i=1, 2, ..., n corresponding to a periodic solution.

To find the initial conditions Y_1, Y_2, ..., Y_n corresponding a periodic solution we use a system of nonlinear algebraic equations

$$Y_i - x_i(kT) = 0, \; i = 1, 2, ..., n \tag{10}$$

This system is not divided into separate equations, since the quantities $x_i(kT)$ are determined from the original nonlinear system of ordinary differential equations (1) by numerical calculation of the Cauchy problem on the interval [0,kT]. To solve this system we have used Newton's method. In the computer implementation of this algorithm includes the possibility of interactive control of calculations. This allows us to specifically control the calculations to find the most interesting of periodic solutions.

For an autonomous system of ordinary nonlinear differential equations (1) (variant 1) the period T of the solution is also unknown. In this case, since the initial time is arbitrary, we assume that Yn = 0. Then to find the initial conditions of Y_1, Y_2, ..., Y_{n-1} and period T of solutions we have the system of nonlinear algebraic equations

$$\begin{aligned} &Y_i - x_i(T) = 0, i = 1,2,\ldots,n-1 \\ &x_n(T) = 0 \end{aligned} \tag{11}$$

This system is solved using Newton's method. After finding the initial conditions Y_1, Y_2, ..., Y_{n-1} and period T of solution we numerically calculate periodic solution itself.

Note that the dimensions of systems of nonlinear algebraic equations for variant 1 (11) and variant 2 (10) are the same and equal to n.

To determine the stability of periodic solutions found, we construct the variational system and calculate the multipliers.

Note that the presented algorithm for finding periodic solutions of strongly nonlinear dynamical systems is iterative, and stability analysis algorithm is finite.

Interactive algorithm. The complexity of the dynamic system behavior is described by periodic solutions of strongly nonlinear systems of ordinary differential equations determines the branching structure of the algorithm for constructing periodic solutions. Here we investigate the evolution of these solutions when changing parameters of the dynamical system.

Note that the convergence of Newton's method depend on starting initial conditions. These conditions can be defined either randomly or on the results of the previous step on the parameter of the dynamical system. The dynamical system may not have periodic solutions of a certain period. To overcome these and other computational problems, our algorithm for finding periodic solutions of nonlinear systems of ordinary differential equations is realized in an interactive mode (Fig. 3). Interim results of finding periodic solutions are shown to the user in real time. The user can intervene in the computation and change the parameters of dynamic systems, finding solution numerical methods, tactics of searching.

This approach allows us to find most complex periodic solutions of nonlinear dynamical systems with several degrees of freedom even in the field of dynamical chaos, where all the periodic solutions are unstable. Also, this approach allows us to investigate bifurcation of periodic solutions.

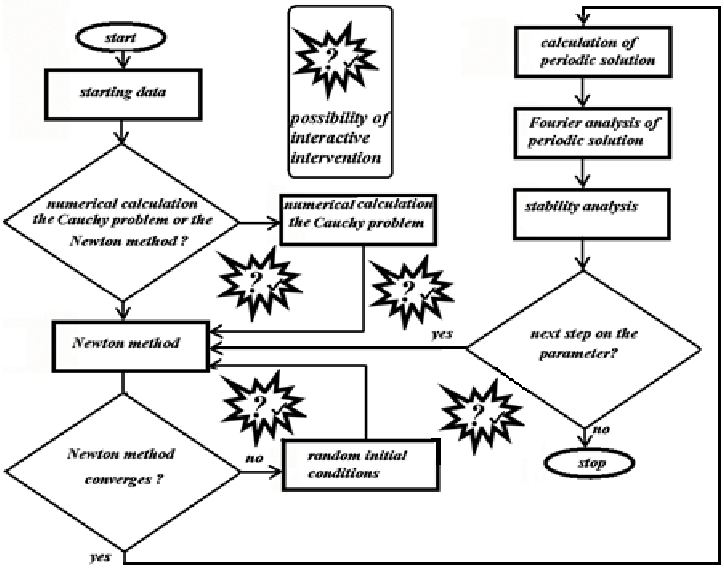

Fig. 3.

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