

The Hausdorff Dimension of the Support of the Erdős Measure and Symbolic Dynamics

Z. I. Bezhaeva & V. I. Oseledets

Journal of Dynamical and Control Systems

ISSN 1079-2724

Volume 19

Number 4

J Dyn Control Syst (2013) 19:569-573

DOI 10.1007/s10883-013-9195-2

Volume 19, Number 4

October 2013

19(4) 471–624 (2013)

ISSN 1079-2724

***Journal of
Dynamical
and
Control
Systems***

 Springer

 Springer

Your article is protected by copyright and all rights are held exclusively by Springer Science +Business Media New York. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

The Hausdorff Dimension of the Support of the Erdős Measure and Symbolic Dynamics

Z. I. Bezhaeva · V. I. Oseledets

Received: 12 October 2012 / Published online: 28 September 2013
© Springer Science+Business Media New York 2013

Abstract Algorithm is given for computation of the Hausdorff dimension of the support of the Erdős measure for a Pisot number.

Keywords Hausdorff dimension · Pisot number · Erdős measure · Sofic set · Deterministic automaton · Entropy

Mathematics Subject Classifications (2010) 28D05 · 28D20 · 37B10 · 11K55

1 Introduction

Let D be a finite set of nonnegative integers ($0 \in D$, $|D| > 1$) and let

$$\zeta = \zeta_1 \rho + \zeta_1 \rho^2 + \dots,$$

where $0 < \rho < 1$, variables ζ_1, ζ_2, \dots are i.i.d. random variables, and $P(\zeta_i = d) = 1/|D|$.

The distribution of the random variable ζ is called the Erdős measure on the real line. Denote by Λ the support of the Erdős measure on the real line.

The problem of computing the Hausdorff dimension of the set Λ is considered in [3–5].

Z. I. Bezhaeva (✉)
MIEM NRU HSE, Moscow, Russia
e-mail: zbejaeva@hse.ru

V. I. Oseledets
Moscow State University, Moscow, Russia

V. I. Oseledets
Financial University, Moscow, Russia
e-mail: oseled@gmail.com

Let $\beta = \frac{1}{\rho}$ be a Pisot number. Recall that the a Pisot number is an algebraic integer whose algebraic conjugates lie inside the unit disk. The Hausdorff dimension of Λ equals the Hausdorff dimension of some sofic compact space (see [1, 2]). This last dimension equals the ratio of topological entropy h to $\log \beta$.

The topological entropy of the sofic compact space equals the entropy of some regular language. The entropy of a regular language equals $\log \lambda$, where λ is the spectral radius of the adjacency matrix of the deterministic automaton recognizing this language.

For the “difficult” case considered in [5] ($D = \{0, 1, 3\}$, $\beta = 1 + \sqrt{3}$), we give an estimate of the Hausdorff dimension of the set Λ more exact than that in [5].

For the “very difficult” case in [4] $D = \{0, 1, 3\}$, $\beta^3 = 2\beta^2 + 2\beta + 2$ (where the Lalley formula is impractical), we determine the Hausdorff dimension of the set Λ up to 10^{-15} .

2 Formula for Computing the Hausdorff Dimension

For a Pisot number β , there exists a finite partition ξ of the unit $[0, 1)$ whose atoms are the intervals Δ_i , $i = 1, 2, \dots, N$, such that:

1. $\xi \geq \eta$, where η is the partition with the atoms $\{x : [\beta x] = k\}$, $k = 0, 1, \dots$;
2. For $V : x \mapsto \{\beta x\} = \beta x - [\beta x]$,

$$V\Delta_i = \cup_j q_{ij}\Delta_j, \quad q_{ij} \in \{0, 1\}, \quad 0\Delta_j = \emptyset, \quad 1 \cdot \Delta_j = \Delta_j.$$

The partition ξ is the V -Markovian partition. Let Q be the matrix $(q_{ij}, i, j = 1, \dots, N)$.

The Markov compact set X_Q with alphabet $\{1, 2, \dots, N\}$ is the set of all infinite words $x = x_1x_2\dots x_k\dots$, $x_i \in A$, with $q_{x_i x_{i+1}} = 1$, $i = 1, 2, \dots$

The metric is given by the formula $d(x, y) = \rho^{n(x, y)}$, where $n(x, y)$ is the length of the largest common prefix of words x and y .

Let $\psi : [0, 1] \rightarrow X_Q$ be $\psi(x) = x_1\dots x_k\dots$, where $x \in \Delta_{x_1}$, $Vx \in \Delta_{x_2}$, \dots , $V^{n-1}x \in \Delta_{x_n}$, \dots

The shift $\sigma : x_1x_2\dots \mapsto x_2x_3\dots$ on X_Q corresponds to the map V on the unit interval, $\psi(Vx) = \sigma\psi(x)$. The invariant Erdős measure on the unit interval gives the invariant Erdős measure on the space X_Q (see [1, 2]). Let Λ_Q be the support of this measure. It is easy to prove (follow [1, 2]) that

$$\dim_H(\Lambda_Q) = \dim_H(\Lambda).$$

Let Δ be the interval $(-1, \frac{\max D}{\beta-1})$. Define Γ_k by the formula

$$\Gamma_k x = \{x' : \exists j \in D, x' = \beta x - j + k\} \cap \Delta, \quad k = 0, 1, \dots, \tilde{N} - 1,$$

where

$$\tilde{N} = \begin{cases} [\beta], & \beta \text{ is not an integer,} \\ \beta - 1, & \beta \text{ is an integer.} \end{cases}$$

For a Pisot number β , there exists a minimal finite set $S \subset \Delta$ such that

$$0 \in S, \Gamma_k S \subset S, \quad k = 0, 1, \dots, \tilde{N} - 1.$$

Let $P(S) = 2^S$ and $\Gamma x = \cup_k \Gamma_k x$. We define the set S' as the minimal Γ -invariant subset of the set $P(S)$ such that $\{\{s\} : s \in S\} \subset S'$.

Let G_k be the graph (S', E_k) with vertex set S' and edge set

$$E_k = \{(s', s'') : s'' = \Gamma_k s'\}, \quad k = 0, 1, \dots, \tilde{N} - 1.$$

Denote by m_k the adjacency matrix of this graph.

Define the matrix M_j as the matrix m_k if $\Delta_j \subset \{x : [\beta x] = k\}$. Introduce the block matrix $M = (q_{ij} m_j)$ following [1, 2].

Let λ be the spectral radius of the matrix M . Then,

$$\dim_H(\Lambda_Q) = \frac{\log(\lambda)}{\log \beta}.$$

Indeed, the matrix M is the adjacency matrix of deterministic automaton which recognizes the language of the sofic set Λ_Q .

3 Simple Example [4]

Let $\beta = 3$, $D = \{0, 1, 3\}$. Here, $\xi = \eta = \{[0, 1/3), [1/3, 2/3), [2/3, 1)\}$. All entries of the matrix Q are equal to 1, $S' = \{\{0\}, \{1\}, \{0, 1\}\}$, and

$$m_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad m_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$M(1) = m_0, \quad M(2) = m_1, \quad M(3) = m_2.$$

The spectral radius of the matrix M equals the spectral radius λ of the matrix $m_0 + m_1 + m_2$. The spectral radius λ is the largest root of the polynomial $1 - 3x + x^2$,

$$\dim_H(\lambda) = \frac{\log\left(\frac{3+\sqrt{5}}{2}\right)}{\log 3}.$$

See [3].

4 Difficult Example [3, 5]

Let $\beta = 1 + \sqrt{3}$, $D = \{0, 1, 3\}$. Here, $q_{ij} = 1$, $(i, j) \neq (3, 3)$, $q_{33} = 0$ and $N = 3$, $\tilde{N} = 2$.

The set S is the set $\{-2\rho, -1 + 2\rho, 0, 1 - 2\rho, 1 - 4\rho, 2\rho, 1, 2 - 2\rho\}$. The cardinality of S' equals 45.

The spectral radius λ of the matrix M is the root (with the largest absolute value) of the polynomial

$$\begin{aligned} p(x) &= \text{Det} [Id_{45}x^2 - (m_0 + m_1)x - m_2m_0 - m_2m_1] \\ &= (-1 + x)^4 x^{67} (1 + x)^3 (1 + x + x^2) q(x), \end{aligned}$$

where

$$\begin{aligned} q(x) &= -14 - 8x^2 + 5x^3 - 8x^4 - 10x^5 + 9x^6 - 9x^7 \\ &\quad - 4x^8 + 2x^9 + 9x^{10} + 8x^{11} - 6x^{12} - 2x^{13} + x^{14}. \end{aligned}$$

We have $q(\lambda) = 0$ and $\lambda = 2.65574542447572\dots$. The Hausdorff dimension is

$$\dim_H(\Lambda) = \frac{\ln(\lambda)}{\ln(\beta)} = 0.9718152436329895\dots$$

In [5], Lalley proved that

$$2.63855 \leq \lambda \leq 2.65584,$$

$$0.963855 \leq \dim_H(\lambda) \leq 0.971847.$$

5 Very Difficult Example [5]

Let

$$\beta^3 = 2\beta^2 + 2\beta + 2, \quad \beta > 1, \quad D = \{0, 1, 3\}.$$

Then,

$$\begin{aligned} S = \{ & -2\rho - 2\rho^2, -2 + 2\rho + 4\rho^2, 1 - 4\rho - 4\rho^2, -1 + 2\rho^2, -2\rho, -2 + 4\rho, \\ & 1 - 4\rho - 2\rho^2, -1 + 2\rho - 2\rho^2, -1 + 4\rho^2, -4\rho^2, -2\rho + 2\rho^2, \\ & -2 + 4\rho + 2\rho^2, 1 - 4\rho, -1 + 2\rho, -2\rho^2, -2 + 4\rho + 4\rho^2, \\ & 1 - 2\rho - 4\rho^2, -1 + 2\rho + 2\rho^2, 0, 1 - 2\rho - 2\rho^2, -1 + 2\rho + 4\rho^2, \\ & 2 - 4\rho - 4\rho^2, 2\rho^2, 1 - 2\rho, -1 + 4\rho, 2 - 4\rho - 2\rho^2, \\ & 2\rho - 2\rho^2, 4\rho^2, 1 - 4\rho^2, 1 - 2\rho + 2\rho^2, -1 + 4\rho + 2\rho^2, 2 - 4\rho, \\ & 2\rho, 1 - 2\rho^2, -1 + 4\rho + 4\rho^2, 2 - 2\rho - 4\rho^2, 2\rho + 2\rho^2, 1, \\ & 2 - 2\rho - 2\rho^2, 2\rho + 4\rho^2, 1 + 2\rho^2, -1 + 6\rho + 2\rho^2, 2 - 2\rho, 4\rho, \\ & 1 + 2\rho - 2\rho^2, 1 + 4\rho^2, -1 + 6\rho + 4\rho^2, 2 - 4\rho^2, 2 - 2\rho + 2\rho^2\}. \end{aligned}$$

The cardinality of S equals 49. The cardinality of S' equals 482. The matrix M is

$$M = \begin{pmatrix} m_0 & m_1 & m_2 & m_2 \\ m_0 & m_1 & m_2 & m_2 \\ m_0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \end{pmatrix}.$$

It is easy to prove that $p(\lambda) = 0$, where

$$p(x) = \text{Det} [-Id_{482}x^3 + (m_0 + m_1)x^2 + (m_2m_0 + m_2m_1)x + m_2m_0 + m_2m_1],$$

$$p(x) = (-1 + x)^3 x^{1380} (1 + x + x^2)^2 (-2 + x^3)^2 q(x),$$

$$\begin{aligned} q(x) = & 12 - 4x - 16x^2 + 16x^3 + 36x^4 + 60x^5 + 75x^6 + 99x^7 - 52x^8 - 69x^9 - 184x^{10} \\ & + 15x^{11} - 109x^{12} + 147x^{13} + 131x^{14} + 137x^{15} + 54x^{16} - 124x^{17} + 34x^{18} \\ & - 220x^{19} - 25x^{20} + 36x^{21} + 164x^{22} + 101x^{23} - 7x^{24} - 208x^{25} - 164x^{26} \\ & - 481x^{27} + 358x^{28} - 182x^{29} + 801x^{30} - 145x^{31} + 659x^{32} - 440x^{33} - 251x^{34} \\ & - 420x^{35} - 194x^{36} + 267x^{37} - 77x^{38} + 410x^{39} - 99x^{40} + 223x^{41} - 253x^{42} \\ & + 14x^{43} - 170x^{44} + 71x^{45} + 76x^{47} - 7x^{48} - 16x^{50} + x^{53}. \end{aligned}$$

We have $q(\lambda) = 0$ and

$$\lambda = 2.1965202182188763....$$

The Hausdorff dimension is

$$\dim_H(\Lambda) = \frac{\log(\lambda)}{\log(\beta)} = 0.7343944361851578....$$

Acknowledgements The work was supported by RFBR grant 11-01-00982. The study was carried out within "The National High School of Economics" Academic fund program in 2013–2014 (grant no. 12-01-005C).

References

1. Bezhaeva ZI, Oseledets VI. Erdős measures, sofic measures, and Markov chains. *Zap Nauch Sem POMI*. 2005;326:28–47 (in Russian).
2. Bezhaeva ZI, Oseledets VI. Erdős measures, sofic measures, and Markov chains. *J Math Sci*. 2007;120:28–47.
3. Pollicott M, Simon K. Hausdorff dimension λ -expansions with deleted digits. *Trans Am Math Soc*. 1995;347:967–83.
4. Kean M, Smorodinsky M, Solomyak B. On the morphology of γ -expansions with deleted digits. *Trans Am Math Soc*. 1995;347:955–66.
5. Lalley Steven P. β -expansions with deleted digits for Pisot numbers β . *Trans Am Math Soc*. 1997;347:4355–65.