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TRANSCENDENTAL ANALYSIS OF MATHEMATICS: THE TRANSCENDENTAL CONSTRUCTIVISM (PRAGMATISM) AS THE PROGRAM OF FOUNDATION OF MATHEMATICS [1]

Abstract. Kant's transcendental philosophy (transcendentalism) is associated with the study and substantiation of objective validity both "a human mode of cognition" as whole, and specific kinds of our cognition (resp. knowledge) [KrV, B 25]. This article is devoted to Kant's theory of the construction of mathematical concepts and his understanding (substantiation) of mathematics as cognition "through construction of concepts in intuition" [KrV, B 752] (see also: "to construct a concept means to exhibit a priori the intuition corresponding to it"; [KrV, B 741]). Unlike the natural sciences the mathematics is an abstract – formal cognition (knowledge), its thoroughness "is grounded on definitions, axioms, and demonstrations" [KrV, B 754]. The article consequently analyzes each of these components.

Mathematical objects, unlike the specific 'physical' objects, have an abstract character (a–objects vs. the–objects) and they are determined by *Hume's principle* (Hume – Frege principle of abstraction). Transcendentalism considers the question of genesis and ontological status of mathematical concepts. To solve them Kant suggests the doctrine of *schematism* (Kant's schemata are "acts of pure thought" [KrV, B 81]), which is compared with the contemporary theories of mathematics. We develop the dating back to Kant original concept of the transcendental constructivism (pragmatism) as the as the program of foundation of mathematics.

"Constructive" understanding of mathematical acts is a significant innovation of Kant. Thus mathematical activity is considered as a two-level system, which supposes a "descent" from the level of rational under-standing to the level of sensual contemplation and a return "rise". In his theory Kant highlights ostensive (geometric) and symbolic (algebraic) constructing. The article analyses each of them and shows that it is applicable to modern mathematics, in activity of which both types of Kant's constructing are intertwined.

Keywords: Transcendental philosophy (transcendentalism) of Kant, transcendental constructivism (pragmatism), Kant's theory of the construction of mathematical concepts (mathematical cognition as construction of concepts in intuition).

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Introduction

According to Kant, one of the main questions of transcendental research is the question of "how the [pure] math is possible?" [2] (Kant, 1998; [B 21][3]) that suggests the substantiation of "objective general validity" [B 122] of this kind of cognition (resp. knowledge)[4]. Moreover, exactly with such — *semantic* — perspective of Kant (as the substantiation of the objective importance of our mental

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[2] In *Prologemena* Kant calls it "*a main transcendental question*" (Kant, 2004).

[3] All references to the *Critique of Pure Reason* (abbreviated as *Critique*; KrV*)* are to the standard A/B pagination of the 1st and 2nd edns and cite the translation of Paul Guyer and Allen Wood, *The Cambridge Edition of the Works of Immanuel Kant* (Cambridge: Cambridge University Press, 1998; see (Kant, 1998)).

[4] The problem of objective validity of a priori principles of reason, which underlay the natural science, is solved by *deduction*, the problem of which are discussed by Kant in §13 «On the principles of a transcendental deduction in general» *Critique* [B 117-125 and beyond]. However, in relation to the problem of the objective validity of the principles of mathematical cognition, which have not only rational, but also sensual nature, Kant does not hold systematic studies.

conceptions) the second (after the Neo-Kantianism of the end of XIX century) "discovery" of Kant in the 80s of the XX century in the Anglo-Saxon [analytical] tradition is associated, which resulted also in a transition from theory (interpretation) of "two objects/worlds" to the theory of "two aspects", in which the Kantian *thing-in-itself* and *thing-for-us* are interpreted not as two different ontological essences, but as "two sides" [B XIX footnotes], or as a "dual method" [B XXVII] — sensual and rational — considerations of the same real existing object[5]. Cognitive-semantic understanding of transcendentalism became a further development of this approach in the XXI century[6]. The basis for such interpretation of Kant (resp. cognitive-semantic reading of the *Critique*) is Kant's letter to M. Herz (21.02.1772), where conception of his *Critique* is determined as finding "the key to the whole mystery of metaphysics", which is associated with the solution of the [semantic] problem (question): "What is the ground of the relation of that in us which we call "representation" to the object?" (Kant, 2004)[7].

Thus transcendentalism acts as a program of semantic substantiation of our knowledge that is the very essence of transcendental philosophy (*TrPh*), which Kant defines as the study "that is occupied not so much with objects but rather with our mode of cognition of objects insofar as this is to be possible a priori" [B 25]. In this case, the definition of *TrPh* sets both *general* problem of transcendentalism which is associated with the study of human cognitive faculty as whole[8] and *applied* problem associated with the analysis [of specificity] and substantiation of certain ways of cognition, one of which is a mathematical activity in which we are interested in[9].

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[5] The following works: 1) Strawson, P. *The Bounds of Sense: An Essay on Kant's Critique of Pure Reason*, 1966; 2) Sellars, W. *Science and Metaphysics: Variations on Kantian Themes*, 1968, — put the beginning of this analytical "discovery" of Kant. The work of Prauss (see: Prauss, G. *Kant und das Problem der Dinge an sich*, 1974) became the next milestone and the forerunner of the new reading (interpretation) of Kant. The development of such 'revolutionary' interpretation of Kantian transcendentalism is associated primarily with the works: 1) Allison H., *Kant's Transcendental Idealism: An Interpretation and Defense,* 2004; 2) Bird G. *The Revolutionary Kant: A Commentary on the Critique of Pure Reason,* 2006 and others. See (Rohlf, 2010)

[6] For example see: (Hanna, 2001; 2007).

[7] In the preface to the 2nd edition Kant implicitly reproduce this defining intention when he speaks about the specifics of his decision, which he connects with the 'altered method of thinking' and his 'Copernican revolution (turn)': "…else I assume that the objects, or what is the same thing, the experience in which alone they can be cognized (as given objects [or *thing-for-us*. – S.K.]) conforms to those [a priori] concepts,… we assume as the altered method of our way of thinking, namely that we can cognize of things *a priori* only what we ourselves have put into them" [B XVIII].

[8] Here we won't discuss these problems in detail. See (Katrechko, 2012a; 2012b; 2013a; 2014a; 2014b; 2014c).

[9] See (Shabel, 2013).The most significant studies dedicated to the Kant's conception of mathematics presented in the last 30 years see: Posy, C., *Kant's Philosophy of Mathematics: Modern Essays* (1992), which contains articles by such well-known philosophers as J. Hintikka, Ch. Parsons, G. Brittan, M. Friedman, and others (*forthcoming*: Posy, C., Rechter, O. (eds.), *Kant's Philosophy of Mathematics*, 2 vol., Cambridge: Cambridge University Press.). Review of the Kant's theory of mathematics can be found in: 1) Brittan, G. *Kant's Theory of Science*, 1978 [see also: Brittan, G., *Kant's Philosophy of Mathematics*, in G. Bird (ed.), A Companion to Kant, 2009, pp. 222-235; 2) Friedman, M. *Kant and the Exact Sciences*, 1992; 3) Shabel, L. *Kant's Philosophy of Mathematics*, in P. Guyer (ed.), The Cambridge Companion to Kant, 2006, pp. 94-128. See contemporary interpretations of Kant's philosophy of mathematics in: Shabel, L. (ed), *Mathematics in Kant's Critical Philosophy*, Canadian Journal of Philosophy, Vol. 44, Issue 5-6, 2014 (http://www.tandfonline.com/toc/rcjp20/44/5-6) [*forthcoming*, 2015: Carson, E., Shabel, L. (eds), Kant: Studies on Mathematics in the Critical Philosophy]. See also our paper (Katrechko, 2003; 2007; 2008b; 2009; 2014d; 2014e).

In *Section 1* we cite Plato's, Aristotle's and Kant's approach to the definition of the specificity of mathematics. In *Section 2* we implement the transcendental analysis of mathematical activity as cognition "through construction of concepts in intuition" [CPR, B 752] (see also: "to construct a concept means to exhibit a priori the intuition corresponding to it" [CPR, B 741]). In *Section 3* on the basis of Kant's theory of the construction of mathematical concepts (mathematics) we are developing the doctrine of transcendental constructivism (pragmatism) as the foundation program of the mathematics (as the program of foundation of mathematics).

1. The mathematics as special abstract - formal mode of human cognition

Let's start with the question of the differentiation of different types of cognition in order to identify the specificity of the mathematical method of cognition. One of the first versions of this classification belongs to Plato[10], but the classification of Aristotle, who in his treatise *On the Soul* (Latin: De Anima) distinguishes *physical, mathematical, philosophical* ways of cognition (Aristotle, 2011). is more relevant for the purposes of our analysis. According to Aristotle, a *physicist (reasoning about nature)* examines "the state of certain body and certain matter," for example, "that the house is made up of stones, bricks and logs", i.e. actually existing *concrete* objects (by their *matter* (chemistry) and/or *motion* (physics)), while a *mathematician* studies "properties that although inseparable from the body, but as they are not in the state of a specific body and are taken abstract from the body" or *shape* of the body (in their "diversion" from matter/motion), for example *geometric forms*, i.e. *abstract objects*, and a *dialectician* (metaphysics) studies *things in existence as such*, "separated from all corporeal". It worth paying attention to the separation of *two* types of *objects/methods of cognition*: *concrete* objects of "physics" (natural science) and *abstract* objects of mathematics, constituting the relevant types of cognition, of which we are interested in the latter.

Transcendentalism [of Kant] as the study, that 'is occupied… with our mode of cognition [including mathematical kind. — S.K.]' [B 25] generally accepts this Plato-Aristotle's distinction, while at the same time, on the one hand, refines it, and, on the other hand, based on its analysis of the human mode of cognition, — proves it.

First of all, transcendentalism distinguishes *cogitation* (thinking) and *cognition*, as to *cogitate* (think) and *cognize* an object is not the same: 'to think of an object and to cognize an object are thus not the same' [B 146][11]. Thus Kant captures the *objective* character of our cognition, its sensual (in the broadest sense of the this word) character, because it is sensuality (as susceptibility) which

[10] See, for example, the Plato's concept of "Divided Line". Previously, we have shown the similarities between Plato's and Kant's theory of mathematics (Katrechko, 2013b).
[11] Kant speaks about this more detailed in his answering the questions of I. G.K. Kiesewetter (Kant, 1862).

“delivers” [through sensuous intuition] an object to our cogitation[12]. In this respect, *philosophy*, although Kant calls it sometimes *cognition[13]* is still not a complete *cognition*, but only [pre-cognitive] cogitation, because it is not contemplative, it does not have an *objective* nature.

Thesis about the existence in human mode of cognition of two main “stems of human cognition,… namely sensibility and understanding” [B 29], a certain combination of which predetermines the specifics of a particular type [of objective] cognition is fundamental for Kant. If *experimental natural science* begins with *sensuous intuition* of empirical object, which subsequently is conceptualized (recognized) by understanding through concepts (respectively, the scheme of natural science is: “sensuality (contemplated object) + understanding (the concept of it)"), then mathematics Kant defines as cognition ‘through construction of concepts in intuition” [B 752][14], which means joint activity of *understanding* and *imagination*, but in order reverse to the natural science: firstly understanding creates, i.e. construct “a pure [sensual] concept”, which then must be presented – with the help of imagination and determining power of judgment (resp. Kant’s schematism) – as a intuitive structure: for example, the concept of the triangle should be drawn as a figure. Thus Kant, following Aristotle, distinguishes two types of *objective* cognition, namely *physics* (natural science) and *mathematics* as different types of *objective* cognition, the first of which is *empirical*, and the second is *formal* (abstract) *cognition[15]*. Accordingly, *concrete* empirical objects are the subject of study of the first, and created by our mind *formal abstract objects* are the subject of the second, thus, they have different ontological status.

Pay your attention to one important difficulty in justifying the “objective validity” of mathematical cognition/knowledge. Objective character of natural science is provided by “external intuition”: the existence of *empirical* objects is certified, possibly, through a number of theoretical concepts with the help of their perception by our senses or instruments, as any our *empirical* cognition, without a doubt, begins with *experience* (B 1 *Critique’s* paraphrase). To suppose such a *natural* status of mathematical objects is absurd even for ordinary mind, because of “nature there is no *circles, squares*...” (Galileo Galilei): mathematical objects “are not lying on the road.”

In this regard, we turn to the important Ch. «*On the ground of the distinction of all objects in general into phenomena and noumena*» *Critique* [B 298–300 and onwards], which in concentrated

[12] Moreover, we need *external intuitions* for the substantiation of the *validity* of categories.
[13] See: “Philosophical cognition is rational cognition from concepts…” [B 741].
[14] See also: “mathematical cognition that from the construction of concepts, but to construct a concept means to exhibit a priori the intuition corresponding to it” [B 741].
[15] That does not exclude that the theoretical branches of modern physics are constituted by the type of mathematics, i.e. start with postulating of some *a la* mathematical abstract objects. This indicates that modern physics is increasingly becoming not empirical but mathematical, because as was noted by Kant, “in any particular doctrine about nature the sciences in their *own* sense can be found [i.e., of theoretical system of knowledge rather, not an empirical data. – S.K.] just as much, as there is mathematics in it” (Kant, 1786: 5).

form contains both the substance of transcendentalism as a whole with its semantic issues, and the Kantian approach to semantic and ontological validity of the mathematical way of cognition. There Kant emphasizes that without an *object* (resp. "empirical intuition, i.e., to data for possible experience" [B 298]) the *concepts*, including the concepts of mathematics, "have no sense [and objective validity at all]" [B 298], because they are "rather a mere play, whether it be with representations of the imagination or of the understanding" [B 298]. And that's why we "**make** an abstract [for example, mathematical. – S.K.] concept **sensible**[16], i.e., display the object, that corresponds to it in intuition, since without this the concept would remain... without **sense**, i.e., without significance"[17]. And further Kant when emphasizing the specificity of mathematics in its difference from physics, continues: "mathematics fulfills this requirement by means of the construction of the figure, which is an appearance present to the senses (even though brought about a priori)" [B 299; the emphasis is mine][18].

The objective character of mathematics is also associated with sensual intuition, because as per Kant there is no other intuitions, however, in contrast to the "physics" the nature of mathematics' intuitions are not empirical but *a prior*: respectively, Kant calls mathematical concepts "*pure* sensible concepts" [B 181]. And here we must not be misled by the empirical character, for example, of geometric drawings, because, as noted by Plato, drawings as 'visible forms' are a likeness of what geometers can see by them mind's eye[19].

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[16] Here's another Kant's fragment on the subject, although here Kant is less categorical. "Further, we are now also able to determine our concepts of an **object** in general more correctly. All representations, as representations, have their object (or *denotation*. – S.K.), and can themselves be objects [= *denotation*] of other representations in turn. Appearances are the only objects that can be given to us immediately, and that in them which is immediately related to the object is called intuition. However, these appearances are not things in themselves, but themselves only representations, which in turn have their object, which therefore cannot be further intuited by us, and that may therefore be called the non-empirical, i.e., transcendental object = *X*" [A 108–9]. Here Kant admits the possibility of a hierarchy of views, i.e. allows for the possibility of sending a single abstract view (concept) to another less abstract view, but at the bottom of this hierarchy of abstractions, in the end, a sensual [empirical] intuition, through which a real object is given, should lay. And this applies to all theoretical [scientific] abstract concepts, but in the first place to mathematical abstractions (i.e. mathematical *objects*).

[17] Comp. with the contemporary (modern) fundamental Frege's semantic concept (Frege, 1892).

[18] Comp. with [B 741, B 752], in which Kant expresses his [constructive] understanding of mathematics, to the analysis of which we turn later in this paper [B 740-766]. There, in particular, Kant writes: "all of our cognition is in the end related to possible intuitions: for through these alone is an object given" [B 747].

[19] See: "[510 c] Students of geometry and reckoning and such subjects first postulate the odd and the even and the various figures and three kinds of angles and other things akin to these in each branch of science, regard them as known, and, treating them as absolute assumptions, do not deign to render any further account of them to themselves or others, taking it for granted that they are obvious to everybody. They take their start [510 d] from these, and pursuing the inquiry from this point on consistently, conclude with that for the investigation of which they set out…And do you not also know that they further make use of the visible forms and talk about them, though they are not thinking of them but of those things of which they are a likeness, pursuing their inquiry for the sake of the square as such and the diagonal as such, and not for the sake of the image of it which they draw? [510e] And so in all cases. The very things which they mould and draw, which have shadows and images of themselves in water, these things they treat in their turn as only images, but what they really seek is to get sight of those realities which can be seen" (Plato, Rep. 6.510 c-e)

Thus mathematics is a special type of knowledge, which has abstract formal rather than substantive character that distinguishes it not only from natural sciences, but also from humanities[20].

2. Transcendental analysis of mathematics activity: Kant's theory of mathematics as cognition "through construction of concepts in intuition"

We now turn to a more detailed analysis of mathematical activity. Speaking about the Kantian understanding (resp. substantiation) of mathematics, they usually regard section "Transcendental Aesthetic" of the *Critique* as the defining, where the *a priori* forms of sensibility are the conceptual foundations of mathematics: for geometry — a priori form of space, and for arithmetic — a priori form of time. I would like to draw the attention to the last section "Transcendental Doctrine of Method" (Ch. I. "The Discipline of Pure Reason") of the *Critique*, in which Kant gives a detailed analysis of mathematical activity, the thoroughness of which "grounded on definitions, axioms, and demonstrations" [B 754].

The *mathematical definitions* that "make the [mathematical] concept itself" [B 758], and thus "to exhibit originally the exhaustive concept of a thing within its boundaries[21] " [B 756], but not only *explain* it as it occurs in the natural science and philosophy are the determinant — primary and constitutive for next two components of mathematical knowledge — in this triad. This ensures *mathematical concepts* that they fully comply with the [mathematical] *objects* (resp. *intuitions*), while the *empirical concepts* [of natural science] and *a priori concepts* [of metaphysics] in general do not have such compliance: in the case of natural science things are usually "richer" than their concepts (e.g., *table* and the *concept of the table* do not match, and the concept of the table can not convey all information about the real table in all the nuances of its existence), while metaphysical concepts (categories) are generally "richer" than their empirical usage, because they can be applied not only to the object of our sensible intuition, i.e. *things-for-us* (as "objects of a possible experience" [B298]), but also to "things in general" [B 298][22].

The identity of mathematical *objects* and *concepts*, as the first are created by the second (resp. by means of definitions), is the basis for such full compliance. However, it clarifies Kant's 'Copernican revolution', the essence of which is that "we can cognize of things a priori only what

[20] However, the Math is similar to other *formal sciences*, such as the Logic or the Grammatica.

[21] "*Exhaustiveness* signifies the clarity and sufficiency of marks; *boundaries*, the precision, that is, that there are no more of these than are required for the exhaustive concept; *original*, however, that this boundary-determination is not derived from anywhere else and thus in need of a proof, which would make the supposed definition' incapable of standing at the head of all judgments about an object". [B 756 footnote].

[22] Application of categories that go beyond experience Kant calls transcendent (or transcendental): "the transcendental use of a concept in any sort of principle consists in its being related to things in general and in themselves its empirical use, however, in its being related merely to appearances, i.e., objects of a possible experience"[B 298].

we ourselves have put into them" [B XVIII]: if in relation to natural (physical) objects [of perception/sensation] Kant's thesis seems too radical, then with respect to the mathematical abstract objects it is trivial. Our knowledge of mathematical objects is like the "knowledge" of master, who makes this or that thing. However, any mathematical definition has a *constructive* nature, it contains a way of generating of its object, or, as Kant says, "containing an arbitrary synthesis which can be constructed a priori" [B 757]. In Ch. "On the schematism of the pure concepts of the understanding" Kant notes that they (schemata) "signifies a rule of the synthesis of the imagination [of mathematical objects such as figures [for example, schema of the triangle] with regard to pure shapes in space" [B 180/A 141][23] Thus, certain *mental actions* of our consciousness are at the heart of mathematical activity[24], and mathematical concepts are *schematized concepts* set by a constructive way. For example, the *circle* is defined by Kant through *constructive* [genetic] *definition*: "Thus the common explanation of the circle, that it is a curved line every point of which is the same distance from a single one (the center-point)" [B 759–60].

In fact, Kant by its introduction of mathematical objects through definitions specifies them as *abstract* ones, as opposed to *specific* (physical) objects. In modern philosophy of mathematics in this context is said about the Hume – Frege principle of abstraction (*Hume's principle*[25]*): for any* $(\alpha)(\beta)$ $[(\sum(\alpha) = \sum(\beta)) \leftrightarrow (\alpha \approx \beta)]$, where $\sum(\alpha)/\sum(\beta)$ means the newly introduced abstract object using meta-language symbol Σ[26]. Classic (paradigmatic) example is the introduction of a new abstraction, such as Frege's "introduction" of the new concept of "*direction* (straight line)", denoted by D(α)/D(β), which "made" of [already familiar] conceptual design of lower level of "*parallelism* of a and в straight lines": D(α) = D(β) ↔ straight line α is parallel to the straight line β. The principle of abstraction captures the fact that new — "secondary" — abstract object is obtained from the "primary" abstract object by implicit definition. The fact that we can write in the form of (*quasi*–)definition indicates its similarity to [Kant's] definition. Moreover, we can write the principle of abstraction in the form of a standard definition $\sum(\alpha/\beta) =_{df} (\alpha\approx\beta)$, though with the loss of part of information about how the object is constructed. This indicates the similarity of principles of abstraction and definition, since abstract entities are set by both. More precisely, Hume's principle

[23] See: "The schema of sensible concepts (such as figures in space) is a product and... a monogram of pure a priori imagination" [B 181].
[24] Comp. with already quoted above remark of Kant about "acts of pure thinking/thought" [B 81].
[25] Hume's principle appears in Frege's Foundations of Arithmetic, which quotes from Part III of Book I of David Hume's A Treatise of Human Nature: "Algebra and arithmetic [are] the only sciences, in which we can carry on a chain of reasoning to any degree of intricacy, and yet preserve a perfect exactness and certainty. We are possessed of a precise standard, by which we can judge of the equality and proportion of numbers; and according as they correspond or not to that standard, we determine their relations, without any possibility of error. *When two numbers are so combined, as that the one has always a unit answering to every unit of the other, we pronounce them equal*; and it is for want of such a standard of equality in [spatial] extension, that geometry can scarce be esteemed a perfect and infallible science." See: http://en.wikipedia.org/wiki/Hume's_principle or http://plato.stanford.edu/entries/frege-logic/.
[26] By means of brackets (α) / (β) in the formula designated universal quantifier.

of abstraction is a type of definition, in which the *method* of constructing of an abstract object is fixed, information about what is crucial in the implementation of [Kant's] constructing. So if again to turn to Kant's definition of the *circle*, it is a meta-object - a *line* made up of lower-level objects – *points* equidistant from the center, where the *sign* of "equidistance from the center" is the *basis* or definiens (in the formula: $\alpha \approx \beta$) for the generation of this [new] abstraction (resp. a new definiendum (*Dfd*) of the circle; in the formula: $\Sigma(\alpha)$).

However, in a formalized principle of Frege's abstraction two important points are not clarified, namely: the mechanism of formation of an introduced new abstraction, i.e. the question of what our actions are hidden behind the expression «$\alpha \approx \beta$» (comp. with Husserl's theory of abstraction, which also solves this problem), and the question of specificity of thus obtained mathematical concepts. Transcendentalism, aimed at the study of our way of cognition, gives the answers to these questions. We can say that in contrast to the currently prevailing *logic-formal* approach to the analysis of mathematics, Kant develops a *pragmatic* approach aimed at identifying the specificity of mathematics as a human activity, the study of "mathematics with a human face."

Mathematical objects introduced by definition represent a special type of abstraction, different from both standard abstractions [of natural sciences], obtained by abstracting (abstraction) from any given characteristics of specific objects (Aristotle, Locke) and *eidetic intuition* (Husserl), based on a variation procedure[27]. In general, any concept, according to Kant, is a *synthesis*, the union in its composition of many similar objects (for one or another of its characteristic) and *generalization* of this similarity exactly in this *concept*. The specific character of mathematics (as per Kant) is that its "pure sensible concepts" [B 181], which the mathematical abstractions are, are the *generalizations upon similarity of action* (resp. relation). In general, the relation of equity type ("equal", "identity", "isomorphic", "congruent", etc.) based on the *comparison operation* is the primary mathematical action, denoted in formal record by the symbol '≈'. Accordingly, in the case of Frege's "*direction*" the act of checking (or detection) of *parallelism* of straight lines is such, as in case with Kant's *circle* - the *action* of checking (detection) of *equidistance* of points from the center. Kant defines such [mathematical] concepts as *schemata*. Here's what he writes about an algorithm of construction: "Thus, if I place five points in a row..., this is an image of the number five. On the contrary, if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself, which in this case I could survey and compare with the

[27] Since phenomenological description yields ideal species, it involves what Husserl was later (notably in *Ideas–1*) to call "eidetic reduction", i.e., an unfolding of abstract features shared by appropriate sets of fictitious or real-life examples, by way, e.g., of free imaginative variation on an arbitrarily chosen initial example (for the method of "free variation", see: Husserl, E. *Experience and Judgement*, sec. 87). Kant also speaks of variation: abstract objects of mathematics are formed by "an arbitrary [or *free* — S.K.] synthesis which can be constructed a priori" [B 757].

concept only with difficulty. Now this representation of a general procedure [= algorithm of construction] of the imagination for providing a concept with its image is what I call the schema for this concept” [B 180].

In this case in the resulting scheme this action sort of “declines”, moves from the surface (*Dfd*) level on a depth (*Dfn*) level[28], but for the person who practices math activities behind this symbolism the forming it mathematical act is guessable. For example, in [natural] number it is the *sum* of its units or the *product* of its factors, etc.

Transcendental distinction as part of our cognitive faculties of the two main «stems of human cognition, sensibility and understanding” in principle not reducible to each other, is the basis for this revolutionary understanding of abstraction. In this case, it means that for any result of cognition, what is the concept of the understanding, we must look for some (mental) “action”, possibly already relating to the sensuality (imagination) as its transcendental condition or foundation. And because of this the abstraction is not the operation to divert some signs of initial concept to get a more abstract concept, similar in its effect to the operation of the logical generalization (the Hume – Frege principle), and not eidetic intuition of Husserl[29], but some pre-rational [mental] *action* associated with the construction of a intuitive analogue of pre-concept – of a Kantian *schema*. So, some action for equidistant location of points from the center of the circle lays in the basis (of construction) of mathematical abstraction of a *circle* (which “do not occur in nature” [Galileo]).

Thus Kant instead of *logical approach* to the analysis of mathematics (including, to the formation of mathematical abstractions), characteristic to the modern — *logical* — programmes of foundations of mathematics (i.e. *logicism, formalism, constructivism* and *structuralism*), offers (as part of his transcendentalism) a *transcendental-pragmatic* approach, the essence of which is expressed by the maxima: look for relating, i.e. lying at its base [mental] action behind every rational concept. Accordingly, the validity of introduced mathematical abstracts is justified by Kant not through *axiomatic method* (though Kant says about it too), thereby properties of an abstract object are implicitly specified, but by searching for lying in the basis of a particular abstraction of “[mental] actions” on its construction: characteristics and scope of a one or another abstraction are caused by *possible actions* with one or another abstraction and/or prohibition of those [mathematical] actions, which are not possible, i.e. inconsistent with the definition of abstraction.

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[28] Comp. with distinction “Surface Information vs. Depth Information” (Hintikka, 1978). Below we'll talk about the two-levelness of mathematical knowledge: its declarative and procedural levels.

[29] Note that Kant rejects any kind of intellectual intuition except sensual intuition, although Husserl’s *free variation* fits the Kantian transcendental constructivism (pragmatism).

We also pay attention to the fact that in its pragmatic and constructive direction transcendentalism is similar to so-called Erlangen School of [German] constructivism as one of the program of foundation of mathematics (Lorenzen, 1974), but its fundamental difference is in the status of *actions*: for transcendentalism this is not some kind of physical action ("structure") justifying a certain mathematical concept such as correlation of "straight line" with a beam of light, but a certain "[mental] action of cogitation", which Kant calls the *scheme*. In this case we can show that Kantian transcendentalism underlies the well-known programs of foundations of mathematics in the XX century: *formalism* (formal setting of objects), *intuitionism* (reliance on the sensual contemplations) and *constructivism* — although conceptually is the closest to *mathematical intuitionism* (Katrechko, 2007b).

Concluding the theme of mathematical objects as abstractions, briefly touch a few more points. Firstly, the above-discussed principle of abstraction can be applied iteratively, creating abstractions of increasingly higher levels. On the other hand, there is the problem of "descent" and identifying of *primary* mathematical objects and actions. Contemporary mathematics solves this problem: finding of some universal primary elements and related actions, which constitute the foundation of the rest of mathematical abstractions — by highlighting some fundamental mathematical proto–theory (or even "language of mathematics"), at the end of the XIX century *set theory* acted as such, and from the middle of the XX century — the *category theory* claims to be it. Secondly, the abstract nature of mathematical objects gives them impersonal nature, in contrast to the concrete-natural objects: we can not, for example, distinguish one point from another, or one two from the another two[30], although because of this impersonality of mathematical reasoning are *apodictic*: we prove the theorem for the impersonal mathematical object, and thus for any object of this type, such as a triangle in general[31].

Thirdly, it seems absolutely fair to the mathematical abstractions, the following going back to E. Mally (Mally, 1912) "splitting" of standard predication into two types: *exemplification* and *coding* (Linsky B., Zalta E., 1995). In standard way the predication «*x is F*» expresses exemplification of predicate (property) of F in [physical] object x: [object] x has the property F, i.e. *exemplifies* it. Accordingly, this type of predication can be written as *F(x)*. The case is not the same

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[30] In the literature the uncertainty of mathematical objects was called Frege's '*Caesar problem*': "the referent of a certain number [as an abstract object] can be Julius Caesar" (Frege, 1844: § 56; see also: http://rgheck.frege.org/pdf/published/JuliusCaesarObjection.pdf). Comp. also with the well-known aphorism of D. Hilbert: "*Correctness of axioms and theorems won't waver if we replace the usual terms 'point, straight line, plane', by as conditional others: 'chair, table, mug of beer'!*".

[31] Comp. with Kant's characteristics of schemes as "non-empirical intuitions of universal validity… for all possible intuitions that belong under the same concept" ([B 741; see also [B 125]). Thus, any triangle: rectangular, obtuse or acute-angled is covered by the scheme of triangle, which can be expressed by the rule [algorithm] of construction a 'figure formed by a double bend of straight line'.

for abstract objects. On the one hand these objects are *incomplete*, because they do not have the full set of features characterizing specific objects[32]. On the other hand, the expression "x is F» represents a *coding* of property F by introduced object x. Thus, the phrase "two is a prime number" is to be understood as the introduction of the object "2 (two)", which *encodes* the property of "being a prime number" what can be recorded by *(x)F*. If the "two" is introduced by definition as *simple even number*, "two" does not have other characteristics than defined in the definition (these are properties of simplicity and parity): the content of abstract entities poorer than concrete physical objects, but all of their "coded" content is contained *completely* in their definition.

In view of this *impersonality* and *uncertainty* mathematical objects are not objects in the exact (physical, empirical) sense of the word. Thus, along with an understanding of mathematical abstractions as full ontological entities, although belonging perhaps to another world of *Forms* (*Ideas*), i.e. full-blooded *mathematical Platonism*, whose representatives were, for example, such famous mathematicians as Bernays and Godel, one can identify other three possible interpretations of abstract objects that are weaker from the ontological point of view[33].

Firstly, this understanding of abstract objects as un(in)certain specific *objects*, i.e. their interpretation in the mode of *possibility*, not reality (R. Ingarden, G. Hellman and others; (Hellman, 1989; 1996). This interpretation tends to nominalism, and in its radical versions — to *fictionalism* (Field, 1989). Secondly, this understanding of abstract objects as substantivized set of properties was developed in the works of neologicists (E.Zalta, B. Linsky and others). This is quite influential, along with the *objective* ['full-blooded'] Platonism of Gödel – Bernays, version of mathematical Platonism, which, however, leaves open the ontological question of properties of *what* the mathematical abstractions are, what seems to implicitly be supposed as specific (physical) objects[34].

Thirdly, this understanding of mathematical "objects" within the framework of actively developing in the second half of the XX century *mathematical structuralism* (P. Benacerraf, M. Resnik, S. Shapiro and others; see (Shapiro, 1997)), which puts forward very radical thesis of

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[32] The indication of the incompleteness of mathematical abstractions is interesting from the point of view of the approach to solving the medieval *problem of universals* in the light of discernment of *universals* and *abstractions*. Abstract objects are not common, but *a*–objects ('*a'* is an indefinite article). Thus, Frege (Frege, 1844) correlates such logical-mathematical "objects" as the numbers with uncertain objects, i.e. *a*–objects. In mathematics, abstract objects are modeled by variables, which must be replaced (by substitution) by *the*–object, i.e. individual (specific object). The Husserl's distinction between *generalization* and *formalization* from § 13 Ideas–1 is interesting in this regard. Husserl conceives formalization as a special type of abstraction, meant to explicate the structural characteristics of one or another mathematical action (comp. with Kantian schemata), although it is possible that all mathematical abstractions predominantly have such a formal nature.

[33] Comp. with Beth's thesis: "The philosophy of mathematics ... is the ontology of mathematical objects" (Beth, 1965).

[34] Representatives of contemporary *neologicism* (Neo-Fregean) are R. Hale, G. Boolos, K. Fine, R. Hale, R. Heck, C. Wright and others. In Stanford the *Metaphysics Research Lab* (https://mally.stanford.edu/) was founded by Zalta, which mission is a study of meta-physical [i.e. mathematical] objects. This shows that the current Platonism understands abstraction in a wide conceptual range from objects to properties.

no-object nature of mathematical knowledge: mathematics deals not with *objects* but *structures*, which determine the relative place/position of mathematical (*quasi-*)objects inside the structures, but there are no mathematical objects as full entities. For example, *three* is not an independent mathematical object (resp. number), but only something that takes "place" between two and four[35]. Moreover, such a weak ontological understanding of mathematics is enough to solve the main problems of mathematical activity, namely: performing mathematical operations and answering questions such as "Is three bigger than two?", "Is three smaller than four?" In its radical versions structuralism advances the thesis that mathematics can do even without structures (H. Field, G. Hellman, J. Burgess and others[36]), what align it with the *extreme nominalism* and/or *instrumentalism*. We can say that in structuralism a third of the possible interpretations of the mathematical abstracts, as not as *objects* or *properties*, but as a *relation,* is represented[37]. Moreover, in all its variations structuralism tends to *anti-realism*, in which either a *nominalist* understanding of mathematical structures as our language constructs, or a *conceptual* understanding of mathematical activity as our mental constructs is possible (Kant, Husserl, mathematical intuitionism).

3. Transcendental constructivism as the program of foundation of mathematics

The abstract nature of mathematical objects determines the abstract nature of the two others, marked by Kant, necessary components of mathematical knowledge: *axioms* and *demonstrations* (constructions and calculations).

First of all, the axioms of mathematics, by which the relation between the introduced by definition mathematical objects (such as "A straight line is the shortest distance between two points") or between different properties of the same object is fixed, are the basic a priori-synthetic (basic)provisions, thanks to which abstract mathematical knowledge is informative, meaningful in nature and, therefore, can be applied to physical reality. Accordingly, the *axioms* of mathematics are the counterparts of physical *laws*. Axioms are the next (second) level of synthetic character of mathematical knowledge (cognition): if mathematical *concepts* serve as a synthesis of similar objects, then axioms serve as a synthesis of different concepts into a single structure, from which it the excretion (through demonstrations) of consequences — mathematical theorems — is possible. At the same time, according to Kant, the axioms have not *discursive* (rational) but *intuitive* (or rather, discourse-intuitive) character associated with the fact that "mathematics… is capable of axioms, e.g., that three points always lie in a plane, because by means of the construction of

[35] The detailed interpretation was proposed by Benacerraf, the author of the article: *Benacerraf P.* What Numbers Could not Be? (Benacerraf, 1965), — which became a manifesto of the structuralism in the 70s of the XX century.
[36] See: Field H. *Science without Numbers: a Defense of Nominalism,* 1980; Hellman G. *Structuralism without Structures, 1996;* Burgess J., Rosen G. *Subject with No Object*, 1999.
[37] See more about this kind of ontology in my article: (Katrechko, 2008).

concepts in the intuition of the object it can connect the predicates of the latter a priori and immediately." [B 760–1].

Moreover, in his postulation of mathematical axioms Kant anticipated the development of mathematics in the XX century, which is now (thanks to the efforts of Hilbert's school) is unthinkable without the *axiomatic method*. However, the epistemological status of *axioms* in modern mathematics is not fully clarified. More precisely, different statements both *declarative* and *procedural* in nature appear under this name. In this regard we can recall the first scientific treatise on mathematics, Euclid's "Elements", where "postulates" of three types are marked as the "elements" of mathematical knowledge: *definitions*, *axioms* [as general provisions] and *postulates* [as constructions]. Note that this distinction is taken into account in logic, where in addition to *axioms* the *rules of conclusion* are marked too. This demonstrates the need for further development of the *axiomatic method*, and transcendental analysis of different types of cognition/knowledge is to play a crucial role here.

However, the most significant innovation of Kant, serving a continuation and consequence of his constructive-pragmatic approach to the treatment of mathematics as a "work" with abstract objects (resp. an abstract type of cognition), is his understanding of mathematical activity as "For the construction of a concept, therefore, a *non-empirical* intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept" [B 741]. In its most general form, this concept serves as a solution to his methodological principle of the need "to *make* an abstract concept *sensible*, i.e., display the object that corresponds to it in intuition" [B 299/A 240].

In the text of *Critique* there are many examples of such "structures" (constructions), but following fragment, through which Kant illustrates the introduced by him concept of transcendental object[38] acts as one of the paradigmatic one: "Thus we think of a triangle as an object by being conscious of the composition of three straight lines in accordance with a rule according to which such an intuition can always be exhibited. Now this unity of rule determines every manifold, and limits it to conditions that make the unity of apperception possible, and the concept of this unity is the representation of the object = *X*, which I think through those predicates of a triangle" [A 105; emphasis is mine. — S.K.][39]. The following fragment of *Critique*, on which we implicitly relied

[38] *Objective* (as *object)* character of our cognition is associated with us having the concept of the *transcendental object* (see: [A 108–9]), thanks to which we constitute perceived sensual diversity as objects: for example, a system of three lines as a triangle.

[39] See: 1) "We cannot think of a line without drawing it in thought, we cannot think of a circle without describing it, we cannot represent the three dimensions of space at all without placing three lines perpendicular to each other at the same

above, offering my own interpretation of Kant's *transcendental constructivism*[40], is the key one in this respect:

> Philosophical cognition is rational cognition from concepts, mathematical cognition that from the construction of concepts. But to construct a concept means to exhibit a priori the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely a priori, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. [B 741–2/A 714]

To clarify his thesis Kant as an example cites the proof of the theorem on the equality of the sum of the angles of a triangle [B 744]. In this case, the original (declarative) concept of the triangle as an object (= figure consisting of three angles) "decomposes" into their components: segments of straight lines and angles — what allows to perform the additional construction (holding the line through one of its vertices and the continuation of the other two sides) and bring the information to prove the equality of corresponding angles. Due to the introduction of such new objects and activities with them[41] we are able to synthesize a new knowledge about the triangle: to prove the required theorem that the sum of the angles of any triangle is 180° (in the Euclidean plane).

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point, and we cannot even represent time without, in drawing a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold through which we successively determine the inner sense, and thereby attending to the succession of this determination in inner sense" [B 154]; 2) "But in order to cognize something in space, e.g., a line, I must draw it, and thus synthetically bring about a determinate combination of the given manifold, so that the unity of this action is at the same time the unity of consciousness (in the concept of a line), and thereby is an object first cognized' ([B 138]; emphasis and insert are mine. — S.K.; comp. with Kant's "acts of pure thought" [B 81]; [B 741–766], [B 155]). See also: [B 103, 112, 124–125, 423–430].

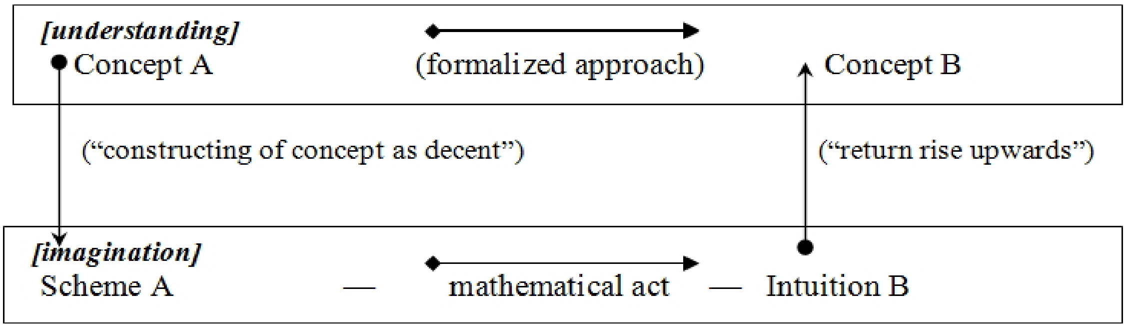
[40] See: (Katrechko, 2007; 2008b; 2014d; 2014e).

[41] In his analysis Kant says that when proving geometer is guided only by "individual drawn figure (or empirical intuition)" as representations, what seems to be not absolutely correct. More precisely we would say that "drawn figure" is one of the necessary conditions for the implementation of mathematical operations: geometric constructions (acts) and subsequent discretion [equities]. Not only "drawn figure", but the process of drawing (constructing) it: "The geometer… begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question" ([B 745/A 717]; emphasis and is mine. — S.K.).

Kantian idea of *constructing concepts* stands as fundamental and key one for his conception (understanding) of mathematics. Although Kant's constructing is present, as we explained above, at the level of definitions and axioms, but only in mathematical activity *constructing* finds its essential meaning. In this case mathematics is conceived by Kant as a *complex* two-level (two-component) way of cognition. It begins with the creation by using definitions "pure sensible concept" [B 180]. Their specificity is that they are formed by "arbitrary synthesis" [B 757], i.e., contain some *mathematical* [mental/mind] *action*. Further, when we *constructing concepts* we implement *descent* to the level of (quasi)sensuality (imagination) and relating of the concept to universally valid contemplation — *scheme*. Here, as if upon the reverse reading (from left to right) of the *Hume's principle*, happens the *decoding* (or construction) of the concept, i.e., the transition to a deep information level: from rational (declarative) level of the concept on contemplative (procedural pragmatic) level of schemes. This can be represented as an expansion of the original abstract concept to lower-level objects that are in some [space-time] *environment* and with which (therefore) we can perform certain mathematical operations. Actually exactly here the *creative* mathematical activity of the corresponding type is performed: geometric constructions, algebraic calculations or logical-mathematical proofs, each of which, in turn, represents a certain set of permissible in this environment local action – operations (like drawing the line, division of numbers, etc.). We can say that in this "descent" through sensory *intuition* the egress beyond the original concepts and the [synthetic] increment of knowledge occurs, as any [dynamic] *action* (as opposed to static concepts) is a *synthesis* of at least two views[42]. The result of this synthesis by reverse return (*rise*) on the cerebral (conceptual) level is fixed as a formal result of the construction, calculation or the proved theorem.

Schematically, mathematical acts can be represented as follows:

[42] In structural terms, any *action* can be represented as the synthesis of *pair* of representations "initial state – final state" (as the result of an action)". Therefore, any action is synthetic. This, in particular, clarifies the Kantian thesis that, for example, the expression "5 + 7 = 12", symbolizing the operation of *addition* of two numbers, has a *synthetic* character (see: "§ 1. Mathematical judgments are all synthetic"; [B 14–17]). Its synthetic nature is connected with the *action* of addition, which is synthesized into a coherent whole (amount) the members of adding: "action" [addition] and gives the synthetic nature to the amount. Therefore, it is impossible, as Frege does it, while criticizing Kant in his *Foundations of Arithmetic* (Frege, 1884) to interpreted the expression "5 + 7 = 12" only as a *formal* equality [of understanding] because behind it the real [mental] action of constructing of this concept implicitly presents, actually *adding* which occurs at the level of [sensual] contemplation as *combining* units (or points) contained in 5 and 7. Accordingly, the mathematical sign "=" means not equity of left and right sides of the formula, but the justification of the *transition* from the left side of the mathematical expression to its right side.



Here, following by I. Lakatos (Lakatos, 1976), we should distinguish the actual mathematical activity as a certain sequence of mathematical actions in contemplative environments (space and time) in the lower part of the scheme and its *logical design*, which can be represented in the upper block of the scheme as a formal-logical transition ("conclusion") from the concept (formula) *A* to the concept (formula) *B*. And the first can not be completely reduced to the second, as the task of the *formal proof* is not in the modeling of real mathematical activity (for example, as a process of mathematical constructions in the proof of the theorem on the sum of the angles of triangle), but in ensuring the logical rightness (correctness) of its implementation. Therefore, the *structure* of the real mathematical process differs from its logical design in some formal meta-language.

It is also clear that the situation in modern mathematics is much more difficult, since the above principle of abstraction can be applied iteratively, generating abstractions of increasingly higher levels. Accordingly, the specifying these abstractions "descents" will also be multi-level ones, and intermediate "descents" will likely be not "descents" to the level of sensuality (imagination), but on a preceding more specific rational level. However, the Kantian thesis about the need to "to **make** an abstract concept **sensible**" [B 299] should be the principal one here, it suggests a final "descent" to the level of sensible intuition, for example, geometric drawing.

In his analysis of mathematical activity, Kant distinguishes two types of constructing: *geometric* and *algebraic*. Along with the *ostensive* (from Lat. Ostentus — showing) constructing, based on the spatial intuition, or the intuition in space, Kant also distinguishes founded by time intuition — *symbolic construction*, underlying the algebraic operations:

> But mathematics does not merely construct magnitudes (quanta), as in geometry, but also mere magnitude (quantitatem), as in algebra, where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitude. In this case it chooses a certain notation for all construction of magnitudes in general (numbers), as well as addition, subtraction, extraction of roots, etc. and, after it has also designated the general concept of quantities in accordance with their different relations, it then exhibits all the procedures through which magnitude is generated and altered in accordance with certain rules in intuition; where one magnitude is to be

divided by another, it places their symbols together in accordance with the form of notation for division, and thereby achieves by a symbolic construction equally well what geometry does by an ostensive or geometrical construction (of the objects themselves), which discursive cognition could never achieve by means of mere concepts. [B 745/A 717] (emphasis is mine. – S.K.)

Turning to the analysis of more abstract *algebra* as a kind of mathematical activity we would like to draw attention to two things. Firstly, the use of "language of X-s and Y-s" or transition compared to the arithmetic meta-language of *variables*, which allows us to "work" not only with certain values as in arithmetic (e.g., distinguish even and odd numbers), but also with *abstract values* whose validity is spontaneous, i.e. with *variables*, is one of the major constituents of algebra[43]. Secondly, the language of algebra allows expressing not only abstract symbols, but also [arithmetic] *operations* ("actions"), which can be done with these symbols, is no less important factor, although it almost falls out of the scope of the analysis. Thus the algebraic language is, unlike *declarative* language that is used, for example, in metaphysics ("philosophy vs. mathematics"), a specific *procedural* language in which the possible ways to work with mathematical objects is fixed, i.e. *how* we need to make certain mathematical operations. Moreover, if earlier such procedural language was the prerogative of algebra only, now it applies to all branches of the modern, which has become super-abstract, mathematics: no logical-mathematical language without special characters for expressing operations on mathematical objects is possible. It is important to note that the *symbolic* constructing is much more transparent than ostensive, since in the latter actions are is not expressly "affect" but only "shown" (Wittgenstein) through their real implementation in geometric constructions, although they may be explicated by describing the methods of constructing, in meta-languages[44]. Such codification of possible mathematical actions makes mathematics more rigorous, although it restricts its heuristic potential, as it becomes impossible to introduce new mathematical structures.

However, the *symbolic* constructing, as indicated by its title, is an *abstract* one in another respect too. In essence, by mathematical symbols of [algebraic] operations the latter are only coded, i.e. presented in abbreviated [symbolic] form, what suggests their real existence already beyond formulaic expressions. So, formulaic (symbolic) record of multiplying of two numbers "$a \times b$" supposes the real action of multiplying of *a* and *b*, for example, by *multiplying in column*, which serves *a la* geometric construction.

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[43] This is the fundamental difference between algebra and arithmetic. Solving [of specific] problem '**2 + 3 = 5**' is an arithmetic problem and the solution of the equation '**3 + x = 5**' is already an algebraic one. The appearance of actual algebra can be associated with the treatise of Diophantus "Arithmetic", in which algebraic symbols, i.e., language of *X*-s and *Y*-s began to be used.

[44] As it, for example, was done in the Euclid's "Elements", which limited set of geometric operations by compass-and-straightedge constructions, and a list of possible actions — in the *postulates*.

Conclusion. Thus mathematical activity represents, despite the predominance of symbolic formulaic structures, a "mixture" of both types of constructing: *in modern abstract mathematics both types of Kant's constructing, which are closely intertwined within the same mathematical structure, are valid*[45]. Kantian *constructing* of *concepts* in general is an explication of their [procedural] sense through the contemplations of general value, i.e. by placing them in a spatial or temporal *environment* where some (valid) mathematical *actions* that make up the "deep" constructive – pragmatic basis of this concept can perform with them.

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[45] So *ostensive structures* are widely used in the logical-mathematical theories. Here you can specify the use of a syllogism of Euler *pie diagrams* that graphically represent the relation between the concepts, or the theory of graphs, as well as *sequential trees* and *subordinative conclusions natural calculations* that are contemplative structures of conclusions in logical calculations of different types.

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