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Critical Elements in Combinatorially Closed Families of Graph Classes

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Abstract—The notions of boundary and minimal hard classes of graphs, united by the term "critical classes," are useful tools for analysis of computational complexity of graph problems in the family of hereditary graph classes. In this family, boundary classes are known for several graph problems. In the paper, we consider critical graph classes in the families of strongly hereditary and minor closed graph classes. Prior to our study, there was the only one example of a graph problem for which boundary classes were completely described in the family of strongly hereditary classes. Moreover, no boundary classes were known for any graph problem in the family of minor closed classes. In this article, we present several complete descriptions of boundary classes for these two families and some classical graph problems. For the problem of 2-additive approximation of graph bandwidth, we find a boundary class in the family of minor closed classes. Critical classes are not known for this problem in the other two families of graph classes.

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INTRODUCTION

A *graph class* is a set of simple graphs closed under isomorphism. A graph class is *hereditary* if it is closed under vertex deletion. Any hereditary class (and only a hereditary class) of graphs $\mathcal{X}$ can be defined by a set $\mathcal{S}$ of its forbidden induced subgraphs. This is denoted by $\mathcal{X} = \mathrm{Free}(\mathcal{S})$. If a hereditary graph class can be defined by a finite set of forbidden induced subgraphs then it is called *finitely defined*.

In this article, alongside the family $\mathbb{H}$ of all hereditary graph classes, we consider its two subfamilies. The first is the family $\mathbb{SH}$ of all *strongly hereditary graph classes*, i.e., the classes closed under vertex and edge deletion. The second is the family $\mathbb{M}$ of all *minor closed graph classes*, i.e., of strongly hereditary classes closed also under edge contraction. Each strongly hereditary graph class $\mathcal{X}$ can be defined by a set $\mathcal{S}$ of its forbidden subgraphs, and we write $\mathcal{X} = \mathrm{Free}_s(\mathcal{S})$. Obviously, $\mathbb{SH}$ can be defined by a finite set of forbidden subgraphs if and only if it can be defined by a finite set of forbidden induced subgraphs. Therefore, the term *finitely defined class* has the same meaning in $\mathbb{SH}$ as in $\mathbb{H}$. By the known Robertson–Seymour theorem [20], each minor closed class $\mathcal{X}$ can be defined by a finite set $\mathcal{S}$ of its forbidden minors; we write this as $\mathcal{X} = \mathrm{Free}_m(\mathcal{S})$.

Let Π be a graph problem in the class NP. The term "graph problem" is understood intuitively: this is a question that must be answered for given input data one of which is a graph.

Definition 1. A hereditary graph class is called Π-*easy* if the problem Π is polynomially solvable in this class. A hereditary graph class is called Π-*hard* if it is not Π-easy.

Throughout the article, we assume that P$\neq$NP, and this assumption is not explicitly included in the statements of the results. For example: If Π is NP-hard in some hereditary class then the class is Π-hard. Another example is given by the assertion that, for a problem Π, every hereditary class is either only Π-easy or only Π-hard.

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The notion of boundary graph class serves as a useful tool for the analysis of computational complexity of a graph problem in the family of hereditary graph classes, and especially on the family of finitely defined classes. This notion was introduced in [5] for the independent set problem and generalized in [6] to the case of an arbitrary graph problem in the class NP. Prior to the present article, the notion of a boundary graph class was considered only within the family $\mathbb{H}$, and here we extend this notion to the case of all three families $\mathbb{H}$, $\mathbb{SH}$, and $\mathbb{M}$.

Definition 2. Let $\mathbb{F} \in \{\mathbb{H}, \mathbb{SH}, \mathbb{M}\}$. A graph class $\mathcal{X}$ is called $(\Pi, \mathbb{F})$-*limit* if there exists an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \dots$ consisting of Π-hard graph classes each of which belongs to $\mathbb{F}$ and such that $\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i$.

Definition 3. An inclusion-wise minimal $(\Pi, \mathbb{F})$-limit class is called $(\Pi, \mathbb{F})$-*boundary*.

The meaning of this notion is revealed by Theorem 1, which can be proved in the same way as the corresponding assertions in [5, 6]:

Theorem 1. *Let $\mathbb{F} \in \{\mathbb{H}, \mathbb{SH}\}$. A finitely defined class in $\mathbb{F}$ is Π-hard if and only if it includes a $(\Pi, \mathbb{F})$-boundary class. A minor closed class is Π-hard if and only if it includes a $(\Pi, \mathbb{M})$-boundary class.*

Thus, by Theorem 1, knowledge of all the boundary graphs leads to a complete classification of all finitely defined classes (or all minor closed) by the complexity of a graph problem. Moreover, Theorem 1 implies that boundary classes exist for any NP-complete graph problem since the class of all graphs can be defined by an empty set of forbidden (induced) graphs or by the empty set of forbidden minors.

It was proved in [2] that, for the edge 3-coloring problem and the family $\mathbb{H}$, the set of boundary classes has the continuum cardinality, which indirectly confirms the fundamental impossibility of obtaining a complete description of the set of boundary classes for this problem. By analogy with arguments in [2], one can prove the validity of this result also for the family $\mathbb{SH}$.

Definition 4. Let $\mathbb{F} \in \{\mathbb{H}, \mathbb{SH}, \mathbb{M}\}$. A class in $\mathbb{F}$ is called $\mathbb{F}$-*minimal* Π-*hard* if every its proper subclass in $\mathbb{F}$ is already Π-easy.

Henceforth we unite the notions of a $(\Pi, \mathbb{F})$-boundary graph class and an $\mathbb{F}$-minimal Π-hard graph class by the term "$\mathbb{F}$-*critical class*."

It was proved in [3] that, for some classical problems on graphs, the family $\mathbb{H}$ contains no minimal classes. It is not hard to prove (by analogy with the corresponding arguments in [3]) that, for some problems on graphs, minimal hard classes are either absent in $\mathbb{SH}$.

The notions of $(\Pi, \mathbb{M})$-boundary class and $\mathbb{M}$-minimal Π-hard class are identical for every problem Π; moreover, the set of $(\Pi, \mathbb{M})$-boundary graph classes is at most countable for every problem Π. This easily follows from the Robertson–Seymour theorem.

Boundary graph classes are known for some graph problems in the family $\mathbb{H}$ [2, 4–7]. However, prior to the results of the present article, there was the only one example of a graph problem (the independent set problem) for which boundary classes were revealed in $\mathbb{SH}$ [5]. Moreover, no boundary class was known for any graph problem in $\mathbb{M}$.

The goal of this article is to establish the boundariness of some graph classes in $\mathbb{SH}$ and $\mathbb{M}$ for some graph problems. Namely, we give a complete description of the family of boundary classes for $\mathbb{SH}$ and $\mathbb{M}$ and a number of classical graph problems. Note that at present there is only one example of a problem admitting a complete description of boundary classes in $\mathbb{H}$ [4]. It is interesting that the results obtained for $\mathbb{SH}$ and $\mathbb{M}$ differ for the same graph problems. Another main result is a proof of the $\mathbb{M}$-criticality of some graph class for the problem of 2-additive approximation of graph bandwidth. Note that neither $\mathbb{H}$-critical classes nor $\mathbb{SH}$-critical classes are known for this problem and the problem of the graph bandwidth.

1. NOTATIONS

We use the following notations: $\Delta(G)$ is the greatest vertex degree of a graph G; $N[x]$ is the set of all neighbors of a vertex x plus x; P_n is the simple path with n vertices; Sun_n is the graph with all vertex degrees not exceeding 3 obtained from a simple cycle with n vertices by adding n pairwise nonadjacent vertices each of which is adjacent to exactly one vertex in the cycle; $K_{p,q}$ is the complete bipartite graph with p vertices in one part and q vertices in the other part.

Given naturals a and b such that $a \leq b$, denote by $\overline{a,b}$ the set of naturals ranging from a to b.

2. $\mathbb{SH}$-BOUNDARY AND $\mathbb{M}$-CRITICAL GRAPH CLASSES FOR SOME GRAPH PROBLEMS

In this section, we consider two concrete classes of graphs. They are the set of all planar graphs $\mathcal{P}l$ and the class $\mathcal{T}$ consisting of all possible graphs whose each connected component having at least two vertices is homeomorphic to P_2 or $K_{1,3}$.

The proof of the fact that some graph class is a boundary class for some graph problem splits into two steps. It is first proved that the class is a limit class for the problem under consideration and then that it is minimal as a limit class. As a rule, the second step (i.e., the proof of minimality) is far more complicated than the first. However, in application to the pairs $(\mathcal{T}, \mathbb{SH})$ and $(\mathcal{P}l, \mathbb{M})$, there is some general argument making it possible to overcome this difficulty for a number of graph problems. It uses the notions of the cliquewidth [12] and treewidth [9] of a graph and the following three facts connected with these notions: for a class $\mathcal{X} \in \mathbb{SH}$ not including $\mathcal{T}$, the cliquewidth of every graph in $\mathcal{X}$ is at most some constant $C_1(\mathcal{X})$ [11], and for a minor closed class $\mathcal{Y}$ not including $\mathcal{P}l$, the treewidth of every graph in $\mathcal{Y}$ is at most some constant $C_2(\mathcal{Y})$ [19].

The following assertion holds for many graph problems: For every a priori given constant C, the problem can be solved in polynomial time in the class of graphs for each of which the cliquewidth (or treewidth) is at most C [8–10, 12, 16].

Theorem 1. *If the class $\mathcal{T}$ is $(\Pi, \mathbb{SH})$-limit and the problem Π is polynomially solvable on graphs with bounded cliquewidth then $\mathcal{T}$ is the only $(\Pi, \mathbb{SH})$-boundary class. If $\mathcal{P}l$ is Π-hard and Π is polynomially solvable on graphs with bounded treewidth then $\mathcal{P}l$ is the only $(\Pi, \mathbb{M})$-boundary class.*

Proof. We will prove only the first part of the theorem since the second is proved similarly.

Suppose that there is a $(\Pi, \mathbb{SH})$-boundary class $\mathcal{X}$ different from $\mathcal{T}$. Since $\mathcal{T}$ is $(\Pi, \mathbb{SH})$-limit, we have $\mathcal{T} \nsubseteq \mathcal{X}$. Therefore, for some graph $G \in \mathcal{T}$,

$$\mathcal{X} \subseteq \mathrm{Free}_s(\{G\}).$$

Hence, in every monotone decreasing sequence consisting of Π-elements of $\mathbb{SH}$ and converging to $\mathcal{X}$, there is an element contained in the class $\mathrm{Free}_s(\{G\})$. There exists a constant $C(G)$ such that the cliquewidth of any graph of class $\mathrm{Free}_s(\{G\})$ is at most $C(G)$. The problem Π is polynomially solvable on graphs with bounded cliquewidth, and hence in $\mathrm{Free}_s(\{G\})$; a contradiction to Theorem 1.

The proof of Theorem 1 is complete. □

The class $\mathcal{T}$ turns out to be $(\Pi, \mathbb{H})$-limit for many graph problems Π [6]. These include the independent set problem, the induced matching problem, and the dominating set problem. By analogy with the arguments in [6], for each of them, we can show the limitness of the class $\mathcal{T}$ also in the family $\mathbb{SH}$. The class $\mathcal{P}l$ is hard for these three problems [1]. Each of these problems is polynomially solvable on graphs of bounded cliquewidth [12, 15, 16] or bounded treewidth [9]; therefore, by Theorem 2, for each of them, the class $\mathcal{T}$ is the only boundary class in $\mathbb{SH}$, and the class $\mathcal{P}l$ is the only boundary class in $\mathbb{M}$.

3. AN $\mathbb{M}$-CRITICAL GRAPH CLASS FOR THE PROBLEM OF 2-ADDITIVE APPROXIMATION OF GRAPH BANDWIDTH

Definition 5. A *numeration of a graph* G is any injective mapping $f_G\colon V(G) \longrightarrow \overline{1, |V(G)|}$. The *width of* f_G is

$$\max_{(u,v)\in E(G)} |f_G(u) - f_G(v)|.$$

Definition 6. The *bandwidth* of G, denoted by $b(G)$, is the minimum of the lengths of the widths of all its possible numerations of G.

Definition 7. The *bandwidth problem* (briefly, the BW *problem*) for a given graph G and a number k consists in determining whether $b(G) \leq k$.

The BW problem is a classical NP-complete graph problem [1].

In this article, we consider the problem of the 2-additive approximation of the bandwidth of a given graph. The BW_{+2} *problem* for a given graph G and number k consists in determining whether $b(G) \leq k + 2$. We can give the following informal interpretations of both problems: the BW problem is concerned with the exact calculation of the bandwidth of a given graph G, and the BW_{+2} problem deals with the search for a number $b'(G)$ such that $b(G) \leq b'(G) \leq b(G) + 2$. The BW_{+2} problem is NP-complete in the class of all graphs. This follows from the NP-hardness for every $\epsilon > 0$ of the problem of the approximation of graph bandwidth with multiplicative error $1 + \epsilon$ [13] and the existence of an algorithm of complexity $O(2^{O(k)}n^{k+1})$ for the recognition of the validity of the inequality $b(G) \leq k$ for a given graph G with n vertices [21].

Lemma 1. *Let G be an arbitrary graph. If H is an induced subgraph of G then $b(H) \leq b(G)$. If $G_1, \dots, G_p$ are the connected components of G then $b(G) = \max(b(G_1), \dots, b(G_p))$.*

Proof. Consider an arbitrary optimal numeration f_G of G and the set $\{i \mid \exists v \in V(H),\ f_G(v) = i\}$. Order the elements of the latter increasingly. Let n_v be the number of $v \in V(H)$ in this ordering. Obviously,

$$|n_u - n_v| \leq |f_G(u) - f_G(v)|$$

for any adjacent vertices u and v in H. Consider the numeration of H in which each vertex $v \in V(H)$ gets the number n_v. Obviously, its width is at most $b(G)$. Hence, $b(H) \leq b(G)$.

Since each of the connected components $G_1, \dots, G_p$ is an induced subgraph in G, we have $b(G) \geq \max(b(G_1), \dots, b(G_p))$. Consider optimal numerations $f_{G_1}, \dots, f_{G_p}$ of the graphs $G_1, \dots, G_p$ respectively. For all $i \in \overline{1,p}$ and each $v \in V(G_i)$, add to $f_{G_i}(v)$ the number $\sum_{j=1}^{i-1} b(G_j)$. Such shifts generate a numeration of G of width $b(G) = \max(b(G_1), \dots, b(G_p))$. Hence, $b(G) = \max(b(G_1), \dots, b(G_p))$.

Lemma 1 is proved. □

Alongside the BW_{+2} problem, in this article, we also consider 1-caterpillars and cyclic 1-caterpillars.

Definition 8. A graph is called a 1-*caterpillar* if it can be obtained by adding to a simple path, called *supporting*, several pairwise nonadjacent vertices (possibly none) each of which is adjacent with exactly one vertex in the path.

Definition 9. A *cyclic* 1-*caterpillar* is a graph obtained by adding to a simple cycle, called *supporting*, of pairwise nonadjacent vertices (possibly none) each of which is adjacent exactly with one vertex in the cycle.

In [17], Miller proved

Lemma 2. *There exists an optimal numeration of a* 1-*caterpillar with supporting path $(v_1, \dots, v_k)$ such that, for each i, the numbers of the elements of $N[v_i] \setminus \{v_{i-1}, v_{i+1}\}$ coincides with the range of naturals*

$$\left[\sum_{j=1}^{i-1} |N[v_j] \setminus \{v_{j-1}, v_{j+1}\}| + 1,\ \sum_{j=1}^{i} |N[v_j] \setminus \{v_{j-1}, v_{j+1}\}|\right].$$

It follows from Lemma 2 that the result of the identification of an end of any simple path with any of the ends of the supporting path of an arbitrary 1-caterpillar has the same bandwidth as the 1-caterpillar itself.

Let a graph H be a cyclic 1-caterpillar with n vertices having exactly k vertices of degree 2. We will assume that $k > 0$; otherwise, H is a simple cycle and $b(H) = 2$. Remove from H all vertices belonging to balls of radius 2 centered at the vertices of H, each vertex being of degree greater than 2. In the resulting disjoint sum of simple paths, consider a path $P' \triangleq (u_1, \dots, u_s)$ of maximal length. Obviously, if a path P' is nonempty then the graph $H' \triangleq H \setminus P'$ is a 1-caterpillar. We will further prove that if $\Delta(H)$ and s are large enough (in terms of k) then

$$b(H') \le b(H) \le b(H') + 2.$$

In what follows, we will need some discrete function $\operatorname{num}(i, q)$ in which i is a natural argument and q is a natural parameter greater than or equal to 2. It is defined as follows: Remove from the set of naturals all numbers dividing by q. In the so-obtained sequence, put the ith term to be equal to $\operatorname{num}(i, q)$.

Lemma 3. *Given i,*

$$\operatorname{num}(i, q) \le i + 2\left\lfloor \frac{i+1}{q} \right\rfloor + 1.$$

For each i and j such that $|i - j| \le q$, we have $|\operatorname{num}(i, q) - \operatorname{num}(j, q)| \le q + 2$.

Proof. Obviously, the difference $\operatorname{num}(i + 1, q) - \operatorname{num}(i, q)$ is equal to 1 or 2 for all i. Moreover,

$$\operatorname{num}(i + 1, q) - \operatorname{num}(i, q) = 2$$

if and only if $\operatorname{num}(i, q) \equiv -1 \pmod q$. Such i's (for which $\operatorname{num}(i, q)$ is comparable with -1 modulo q) will be referred to as *growth points*. Obviously, each segment of the set of naturals containing q elements has either one or two growth points. Therefore,

$$|\operatorname{num}(i, q) - \operatorname{num}(j, q)| \le q + 2$$

for every i and j with $|i - j| \le q$. The segment $\overline{1, q-1}$ contains exactly one growth point. The segment $\overline{q, i}$ contains exactly $i - q + 1$ elements; thus, this segment contains at most

$$2\left(\left\lfloor \frac{i+1-q}{q} \right\rfloor + 1\right) = 2\left\lfloor \frac{i+1}{q} \right\rfloor$$

growth points. Hence, the segment $\overline{1, i}$ contains at most $2\lfloor (i + 1)/q\rfloor + 1$ growth points. Hence, for each i, we have

$$\operatorname{num}(i, q) \le i + 2\left\lfloor \frac{i+1}{q} \right\rfloor + 1.$$

Lemma 3 is proved. □

Lemma 4. *If $\Delta(H) \ge 8k + 4$ and $s \ge 8k + 10$ then* $b(H') \le b(H) \le b(H') + 2$.

Proof. Suppose that $\Delta(H) = \Delta$ and $b(H') = b$. Obviously, $n \le k(\Delta + s + 3)$. Since the graph H' contains $K_{1,\Delta}$ as an induced subgraph and $b(K_{1,\Delta}) \ge \lfloor \Delta/2\rfloor$, by Lemma 1, we have $b \ge \lfloor \Delta/2\rfloor > \Delta/2 - 1$. Check the inequality $2\lfloor n/b\rfloor + 3 \le s$. Indeed,

$$2\left\lfloor \frac{n}{b} \right\rfloor + 3 \le 2\frac{n}{b} + 3 < \frac{2k(\Delta + s + 3)}{\frac{\Delta}{2} - 1} + 3 = \frac{4k(\Delta + s + 3) + 3(\Delta - 2)}{\Delta - 2}.$$

Verify that the last number is at most s. To this end, it suffices to prove that

$$s(\Delta - 2 - 4k) > 4k\Delta + 3\Delta + 12k.$$

Since $\Delta \geq 4 + 8k$, we have $s(\Delta - 2 - 4k)/\Delta \geq s/2$. Clearly, $4k + 3 + 12k/\Delta \leq 4k + 5$, and hence $s/2 \geq 4k + 3 + 12k/\Delta$ since $s \geq 8k + 10$.

Let i^* be the maximal number such that $\mathrm{num}(i^*, b) \leq n$. By construction of $\mathrm{num}(i, q)$, we have $\mathrm{num}(i^*, b) \geq n - 2$. Check that $i^* + s \geq n$. Indeed, $s \geq 2\lfloor n/b \rfloor + 3$. Show that $i^* \geq n - 2\lfloor n/b \rfloor - 3$. Suppose that $i^* < n - 2\lfloor n/b \rfloor - 3$. By Lemma 3,

$$\mathrm{num}(i^*, b) \leq i^* + 2\left\lfloor \frac{i^* + 1}{b} \right\rfloor + 1.$$

The right-hand side is less than

$$n - 2\lfloor n/b \rfloor - 2 + 2\lfloor n/b \rfloor = n - 2;$$

a contradiction to the choice of i^*.

Consider a subpath $P \triangleq (u_1, \dots)$ in P' such that the graph $H^* \triangleq H \setminus V(P)$ contains i^* vertices. Such a subpath necessarily exists because $i^* + s \geq n$. Clearly, H^* is a 1-caterpillar. By Lemma 2, we can assume that, in some optimal numeration f_{H^*} of H^*, the neighbor of u_1 not belonging to P has number 1, while the neighbor of the second end of P not belonging to P has number i^*. Also by Lemma 2, we have

$$b(H^*) = b(H').$$

Given $i \in \overline{1, i^*}$, assign to the vertex of H^* with number i the number $\mathrm{num}(i, b)$. Each element of

$$\overline{1, n} \setminus \bigcup_{i=1}^{i^*} \{\mathrm{num}(i, b)\}$$

either divides by b or lies in $\{n - 1, n\}$ by the definition of $\mathrm{num}(i, q)$ and the inequality $\mathrm{num}(i^*, b) \geq n - 2$. For each $i \in \overline{1, \lfloor n/b \rfloor}$, assign to the ith vertex of P (counting from u_1) the number ib. To the remaining vertices of P, injectively assign the numbers in the set

$$\overline{1, n} \ \setminus \left(\bigcup_{i=1}^{i^*} \{\mathrm{num}(i, b)\} \cup \bigcup_{i=1}^{\lfloor n/b \rfloor} \{ib\} \right)$$

(if there are some). Obviously, we have constructed a numeration of H. Its width is at most $b + 2$ by Lemma 3. By Lemma 1, $b \leq b(H)$.

The proof of Lemma 4 is complete. □

Given a family $\mathbb{F} \in \{\mathbb{H}, \mathbb{SH}, \mathbb{M}\}$ and a class $\mathcal{X}$, refer as the $\mathbb{F}$-*closure of* $\mathcal{X}$ to the set of all graphs that are induced subgraphs of graphs in $\mathcal{X}$ (if $\mathbb{F} = \mathbb{H}$) or subgraphs of graphs in $\mathcal{X}$ (if $\mathbb{F} = \mathbb{SH}$), or minors of graphs in $\mathcal{X}$ (if $\mathbb{F} = \mathbb{M}$). Denote the $\mathbb{M}$-closure of the class of all cyclic 1-caterpillars by $\mathcal{CC}$.

Theorem 2. *The class* $\mathcal{CC}$ *is* $(\mathrm{BW}_{+2}, \mathbb{M})$-*critical.*

Proof. It is known that, for every $\epsilon > 0$, the problem of the approximation of graph bandwidth with multiplicative error $2 - \epsilon$ is NP-hard in the class of all cyclic 1-caterpillars [13]. This and the result of [21] imply that the class $\mathcal{CC}$ is BW_{+2}-hard. Prove that each of its proper minor closed subclasses is BW_{+2}-easy, which implies the theorem.

Let G' be an arbitrary graph in $\mathcal{CC}$. There exists a cyclic 1-caterpillar G for which G' is a minor. Denote the number of vertices in G by n. Clearly,

$$\mathcal{CC} \cap \mathrm{Free}_m(\{G'\}) \subseteq \mathcal{CC} \cap \mathrm{Free}_m(\{G\}).$$

Let H be an arbitrary graph in $\mathcal{CC} \cap \mathrm{Free}_m(\{G\})$. Each of its components is either a cyclic 1-caterpillar or a 1-caterpillar, and there is at most one component of the first type. The BW problem is polynomially solvable in the class of 1-caterpillars [17]. By this and Lemma 1, we may assume that H is isomorphic to a cyclic 1-caterpillar. The number of vertices in H each of which has degree at least 3 does not exceed $2n$; otherwise, the graph Sun_{2n} is a minor of H. The last is impossible since any cyclic 1-caterpillar with at most n vertices is a minor in Sun_{2n}.

It is not hard to propose a numeration of H with width bounded by some linear function of $\Delta(H)$. To this end, take an arbitrary cyclic vertex of H and assign number 1 to it. With respect to this vertex, we can define the notions of right and left vertices of H. For enumerating the left vertices, we will use only even numbers, and for enumerating the right vertices, only odd numbers. Therefore, if $\Delta(H) \leq 16n + 3$ then the bandwidth of H is bounded by some linear function of n. If $\Delta(H) \geq 16n + 4$ and the distance between any two vertices of H each of which has degree at least 3 is at most $16n + 9$ then the supporting cycle of H has at most $2(16n + 9)n$ vertices. It is known that, for any fixed number C, the BW problem is polynomially solvable in the class of graphs for which there are at most C vertices such that each edge of the graph is incident to one of these vertices [14]. Owing to this and a result from [21], we may assume that $\Delta(H) \geq 16n + 4$ and the distance between two cyclic vertices of H each of which has degree at least 3 is at least $16n + 10$. By Lemma 4, from the graph H we can construct a 1-caterpillar having bandwidth from $b(H)$ to $b(H) + 2$. This and a result from [17] imply the validity of the theorem. □

Note that, at present, for the BW and BW_{+2} problems in the families $\mathbb{H}$ and $\mathbb{SH}$, there are known neither boundary classes nor minimal hard classes. It is only possible to observe that the $\mathbb{H}$-closure of the set of all 1-caterpillars (denoted by $\mathcal{C}$) is as a (BW, $\mathbb{H}$)-limit class as a (BW_{+2}, $\mathbb{H}$)-limit class. A similar result also holds for the family $\mathbb{SH}$.

Lemma 5. *For each of the problems* BW *and* BW_{+2}*, the class* $\mathcal{C}$ *is limit in the family* $\mathbb{H}$.

Proof. Observe that the BW and BW_{+2} problems are NP-complete in the class of cyclic 1-caterpillars. For the BW problem, this was proved in [18], and for the BW_{+2} problem, this follows from [13, 21]. Since further arguments for the BW and BW_{+2} problems are quite similar, we will consider only the BW problem.

The set of all cyclic 1-caterpillars is infinite and countable. Enumerate the elements of this set and obtain some sequence $G_1, G_2, \ldots$. Put $\mathcal{X}_0$ to be equal to $\mathcal{CC}$. This class is BW-hard [18]. For each $i > 0$, denote the set $\mathcal{X}_{i-1} \cap \mathrm{Free}(\{G_i\})$ by $\mathcal{X}_i$. Show that the class $\mathcal{X}_i$ is BW-hard for each i. Indeed, the set $\mathcal{X}_{i-1} \setminus \mathcal{X}_i$ consists of cyclic 1-caterpillars for each of which the graph G_i is an induced subgraph. Consequently, for all such cyclic 1-caterpillars, the length of a sporting cycle is fixed. Due to this and a result from [14], for each i, the BW problem is polynomially solvable in $\mathcal{X}_{i-1} \setminus \mathcal{X}_i$. This implies that the class $\mathcal{X}_i$ is BW-hard for each i. By this fact, the chain of inclusions $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$, and the equality $\bigcap_{i=1}^{\infty} \mathcal{X}_i = \mathcal{C}$, $\mathcal{C}$ is a (BW,$\mathbb{H}$)-limit class.

Lemma 5 is proved. □

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