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One-end finitely presented groups acting on the circle

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Abstract

We study possible one-end finitely presented subgroups of $\mathrm{Diff}_+^\omega(S^1)$, acting without finite orbits. Our main result, theorem 1, establishes that any such action possesses the so-called property $(\star)$, that allows one to make distortion-controlled expansion and is thus sufficient to conclude that the action is Lebesgue-ergodic. We also propose a path towards full characterization of such actions (conjectures 3–5).

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1. Introduction

This paper belongs to a series of papers devoted to a further understanding of finitely generated groups acting on the circle. One of the general goals here is a well-known question stated in the 1980s by Sullivan and Ghys (inspired by the Sullivan's dictionary and the Katok–Herman ergodicity result for one minimal diffeomorphism).

Conjecture 1. *If $G \subset \mathrm{Diff}^2(S^1)$ is a finitely generated group of circle diffeomorphisms and its action is minimal then it is Lebesgue-ergodic.*

This question in its full generality stays open even in the case of analytic actions.

Sullivan's exponential expansion strategy [25, 34] allows one to establish this conjecture under an additional assumption that for any point $x \in S^1$ there exists $g \in G$ such that $g'(x) > 1$. The obstacle to the application of this strategy is the presence of non-expandable points.

Definition 1. A point $x \in S^1$ is *non-expandable* (for the action of the group G), if for any $g \in G$ one has $|g'(x)| \leqslant 1$.

We denote the set of non-expandable points by $\mathrm{NE} = \mathrm{NE}(G)$.

In [8], to handle the difficulties arising from the presence of non-expandable points the following definition was introduced. We recall it for the case of analytic diffeomorphisms, the C^2 definition being slightly more complicated. Also, for simplicity, in this paper we will consider only the orientation-preserving actions (as otherwise, it suffices to pass to an index two subgroup).

Definition 2. Finitely generated group $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$ satisfies *property* ($\star$), if it acts minimally and $\forall x \in \mathrm{NE}\ \exists g \in G \setminus \{\mathrm{id}\} : \ g(x) = x$.

This property allows one to replace in a neighbourhood of a non-expandable point x the 'fast' expansion by a 'slow' one: iterating of g or g^{-1} till the point leaves the neighbourhood.

Assuming this property one still can expand with a uniform bound on the distortion; in particular, it is still sufficient to prove the ergodicity—we refer the reader to [8] for the details.

Note that it is not so easy for a finitely generated group of circle diffeomorphisms to act minimally, to have no common invariant measure and to possess non-expandable points at the same time. In particular, such an action is necessarily locally discrete (see details in section 2.4). However, there are examples of such actions: $\mathrm{PSL}(2, \mathrm{Z})$ and, more generally, Fuchsian groups corresponding to surfaces with cusps, as well as (for the smooth case) the Ghys–Sergiescu smooth realization [14] of the Thompson group T (and its minor modifications). Although all the known examples of such actions still satisfy property ($\star$), making it reasonable to attack the general conjecture 1 via the following.

Conjecture 2. *Any finitely generated group $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$ that acts minimally and has no invariant measure satisfies property ($\star$).*

Moreover, in [9] conjecture 2 was established for the case of a virtually free group G. Recall (see section 2.1 for details) that a finitely generated group can have one, two or infinitely many ends. It is quite natural to expect that the approach of [9] can be generalized to groups with infinitely many ends. As groups with two ends have Z as a finite index subgroup (and hence are not interesting to our consideration, as they are treated by the Katok–Herman result: see [17, 20]), it is interesting to concentrate on the only 'big' case left completely untouched: groups with one end.

Our main result, theorem 1, handles it under some additional assumptions:

Theorem 1. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$ be a finitely generated group with one end, acting minimally, preserving no measure on the circle and such that*

(i) G is finitely presented,
(ii) elements of G that are of finite order are of uniformly bounded order:

$$\exists C : \forall g \in G \qquad \mathrm{ord}(g) = +\infty \text{ or } \mathrm{ord}(g) \leqslant C.$$

Then G possesses property ($\star$).

This theorem already establishes the Lebesgue ergodicity for such groups. However, it turns out that the groups that satisfy property ($\star$) and have $\mathrm{NE} \neq \emptyset$ are rather special. In particular, they admit Markov partitions and their generic orbits can be decomposed in a finite union of trees [13]. This tree-likeness is very close to the statement of the following.

Conjecture 3. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$ be a finitely generated group that satisfies property ($\star$) and for which $\mathrm{NE} \neq \emptyset$. Then G is virtually free.*

This deduction from the tree-likeness, however, cannot be absolutely straightforward: recall that a cocompact Fuchsian group (having one end) also admits a Markov partition, as was shown by Bowen and Series [2]. However, the analogy between (orbits of) non-expandable points and cusps of the quotient surface (for non-cocompact Fuchsian groups) seems a good motivation for conjecture 3.

Note also that if conjecture 3 is established, it will immediately imply the following.

Conjecture 4. *Let G satisfy the assumptions of theorem 1. Then G has no non-expandable points.*

In its turn, the conclusion of conjecture 4 opens a path for the full characterization of such actions. Namely, there is an unpublished result by Deroin (private communication), saying that *if a finitely generated group of analytic orientation-preserving circle diffeomorphisms acts minimally, has no invariant measure, and its action is locally discrete and possesses no non-expandable points, then it is analytically conjugate to a finite cover of a cocompact Fuchsian group*. Joining this result with the conclusion of conjecture 4 leads to the following.

Conjecture 5. *Let G satisfy the assumptions of theorem 1. Then G is either locally non-discrete or is analytically conjugate to a finite cover of a cocompact Fuchsian group.*

As locally non-discrete groups are rather well understood (see, e.g., [23, 28–30]), the conclusion of conjecture 5 seems really to be a good characterization for such groups.

1.1. Outline of the proof and plan of the paper

Let us briefly outline the proof of theorem 1. Some of its ingredients (disjointness of images, distortion control, non-local discreteness arguments) are exactly the same as those used in [9] to handle the case of a free group action. However, some are completely different: on the one hand, we have to use completely new arguments to ensure the superlinear growth of sum of the derivatives (see section 3), and on the other hand the commutator technique does not ensure non-local discreteness immediately, as the chain of commutators can be broken by an appearance of two commuting maps (see section 4).

The proof of the theorem 1 is started by assuming the converse: that there exists a non-expandable point $\bar{x} \in S^1$, that is not fixed by any $g \in G \setminus \{\mathrm{id}\}$. Then, one can consider a sequence $x_n = f_n(\bar{x})$ of its closest images by maps $f_n \neq \mathrm{id}$ that are compositions of at most n generators.

We then prove that the sequence of lengths $l_n = |[\bar{x}, x_n]|$ decreases as $o(1/n)$. To do so, one notes that all the images $g([\bar{x}, x_n])$, where g ranges in the set of length $\leqslant n/2$ compositions of generators, are disjoint. Hence, the distortion control arguments allow one to bound l_n from above in terms of the sum $S(n/2)$ of derivatives $g'(\bar{x})$, where g ranges over the same set (see lemma 5). Finally (which is one of the key arguments of the proof), using a union of algebraic and distortion control methods, we show that S_n grows superlinearly (proposition 7).

Now, decrease of l_n as $o(1/n)$ implies that the maps f_n under an appropriate rescaling (to a scale much larger than l_n, but much smaller than $1/n$) become closer and closer to the identity (see lemma 4). The next step is to deduce from it that the group G cannot be locally discrete (proposition 8). Once this is proven, the proof of the theorem is concluded using standard arguments. Namely, that the group G is locally non-discrete implies via the Loray–Rebelo–Scherbakov–Nakai technique (see proposition 5) that the group G contains a local flow in its local closure.

In its turn, a local flow in the local closure of the group implies that the minimality assumption can be strengthened: we can map any point arbitrarily close to any other, keeping a uniform control over the derivative. However, this contradicts (see corollary 2) the existence of a non-expandable point $\bar{x}$: this point then can be mapped arbitrarily close to a hyperbolic repelling one, with the derivative bounded away from zero—and the composition with the expansion will then have arbitrarily large derivative at $\bar{x}$.

A fundamental tool that is used to show that G is not locally discrete is the Ghys commutator technique. Namely, if two analytic maps f_1, f_2 are sufficiently close to the identity on the unit disc $U_1^{\mathbb{C}}(0)$, their commutators $f_{n+1} = [f_n, f_{n-1}]$, $n \geqslant 2$ converge to id as $n \to \infty$.

Any two consecutive 'closest image' maps g_n and g_{n+1}, for n sufficiently large, can be rescaled to become simultaneously close to the identity. Hence, if in the corresponding sequence of commutators all the maps are different from identity (case (i) of the proof of proposition 8), we obtain the local non-discreteness of G by the Ghys argument. On the other side, if g_n always commutes with g_{n+1}, we show that these maps converge to id on some interval (case (ii) of the proof).

Finally, if $[f_n, f_{n+1}] \neq \mathrm{id}$, but the commutator sequence falls exactly to the identity at a given moment (case (iii) of the proof), considering three last nonidentity maps in this sequence gives us a relation $[f, g] = h$, $[g, h] = \mathrm{id}$. This relation is similar to the one we have in the affine group (for instance, for $f(x) = \lambda x$ and $g(x) = x + 1$). It turns out (see lemma 11) that the presence of three maps satisfying such a relation also implies the local non-discreteness of the group G (which is quite natural: in such an affine group we would have arbitrarily small translations). This concludes the proof of proposition 8 and thus of theorem 1.

2. Preliminaries

2.1. Algebra

We start with recalling the necessary definitions and facts from the geometric group theory. While we are working with our finitely generated group G, we choose and fix its system of generators $\mathcal{F}$. This defines us a right-invariant metric

$$d(g_1, g_2) = \min\{n \mid \exists \psi_1, \dots, \psi_n \ s.t.\ g_1 = \psi_1 \dots \psi_n g_2, \forall j \quad \psi_j \in \mathcal{F}\},$$

where for simplicity we assume that for each generator $\psi \in \mathcal{F}$ we also include in $\mathcal{F}$ its inverse ψ^{-1}.

This metric is in fact a restriction on G of the metric on a Cayley graph $\mathcal{C}$, obtained by joining each vertex $g \in G$ to all $\psi g \in G$, where $\psi \in \mathcal{F}$, by an edge of unit length.

Consider now in G radius n ball centred in the identity; by an abuse of notation, we will denote it by $B(n)$:

$$B(n) = B_n(\mathrm{id}) := \{g = \psi_1 \dots \psi_{n'} \mid n' \leqslant n, \forall j \leqslant n' \ \psi_j \in \mathcal{F}\}.$$

Also by a slight abuse of notation, further we will say that a set $X \subset G$ is *connected*, if the restriction on X of the Cayley graph $\mathcal{C}$ is connected, or, which is the same, any two points $g, g' \in X$ can be joined by a path $g_0 = g, g_1, \dots, g_n = g'$, where $d(g_i, g_{i+1}) = 1$ and $g_i \in X$ for all $i = 0, \dots, n-1$.

An *end* of the group G is, roughly speaking, a 'connected component at infinity'. A formal definition is the following.

Definition 3. An *end* of the group G is a sequence $(U_n)_{n=1}^{\infty}$, where each U_n is an unbounded connected component of $G \setminus B(n)$, and for any $n' < n$ holds $U_n \subset U_{n'}$.

It is known that any finitely generated group is either finite, or has one, two or infinitely many ends. A group with two ends is known to have Z as a finite index subgroup [18, satz V];

the structure of a group with infinitely many ends is described by the Stallings theorem [35, 36] or by Bass–Serre theory [33, 1]. Throughout this paper we will assume that the group G we are working with has one end; equivalently, for any n there is only one infinite connected component in $G \setminus B(n)$. We denote this component by $(G \setminus B(n))_\infty$. Note that the complement $G \setminus B(n)$ indeed can have finite connected components—corresponding to the so-called *dead ends* (see, e.g., [4, 5, 10, section IV.A.13]).

It is then natural to define a spherical layer of internal radius R and width r as

$$L(r, R) := (G \setminus B(R))_\infty \cap B(R + r).$$

We will need the following.

Definition 4. A group G with one end has the *connected spheres* property, if $\exists r : \forall R > 0$ $L(r, R)$ is connected.

The next proposition due to Gournay can be found in [16, theorem 2].

Proposition 1 (Gournay). *Assume that the group G with one end is finitely presented. Then it has the connected spheres property, and the width r can be chosen to be any number greater than twice the length of the longest defining relator.*

For the reader's convenience we reproduce here a sketch of its proof; we thank Lee Mosher for having explained it to us on MathOverflow and Valerio Capraro for providing a reference to Gournay's work.

Sketch of the proof. Let r be greater than twice the length of longest defining relator. Take any two points $g_1, g_2 \in L(r, R)$. On the one hand, as $g_1, g_2 \in B(R + r)$, they can both be joined to the identity by paths that stay inside $B(R + r)$. Joining these two paths one can find a path γ_1 that goes from g_1 to g_2 and stays inside $B(r + R)$.

On the other hand, as g_1 and g_2 both belong to the infinite connected component of $G \setminus B(R)$, they can be joined by a path γ_2 that stays in $G \setminus B(R)$.

Joining γ_1 and γ_2, we obtain a closed loop in the Cayley graph (see figure 1)—in other words, a relation. It can be decomposed as a product of the conjugates to the defining relations. Consider the corresponding Van Kampen diagram: a cellular decomposition of a disc, reading our relation along its boundary and with cells corresponding to the conjugates of the defining relations.

Let now $\overline{B}(R) := G \setminus (G \setminus B(R))_\infty$ be the radius R ball 'with filled holes', and consider the union U of γ_2 and the union of cells that intersect $\overline{B}(R)$. Note that the exterior component of ∂U consists of two curves, joining g_1 and g_2: γ_2 and another curve that we denote by γ_3.

We claim that γ_3 is the desired path. Indeed, each edge of γ_3 is either a boundary edge of cell that does not intersect $\overline{B}(R)$ or belongs to γ_2. In both cases, such an edge does not belong to $\overline{B}(R)$, and hence $\gamma_3 \subset G \setminus \overline{B}(R) = (G \setminus B(R))_\infty$.

On the other hand, any edge of γ_3 either belongs to γ_1 or is a boundary edge of an (outer) cell that intersects $\overline{B}(R)$. In both cases, such an edge is at the distance at most r from $\overline{B}(R)$. As γ_3 is a *connected* path in a connected set $(G \setminus B(R))_\infty$, whose boundary points belong to $B(R)$, this implies also that γ_3 lies in the r-neighbourhood of $B(R)$, that is, in $B(r + R)$.

Finally, one has $\gamma_3 \subset B(r + R) \cap (G \setminus B(R))_\infty = L(r, R)$. □

2.2. Centralizers for analytic circle diffeomorphisms

The main technical difficulty in the realization of the commutator strategy is the eventual appearance of commuting maps in the commutator chain. To handle it, we will need some

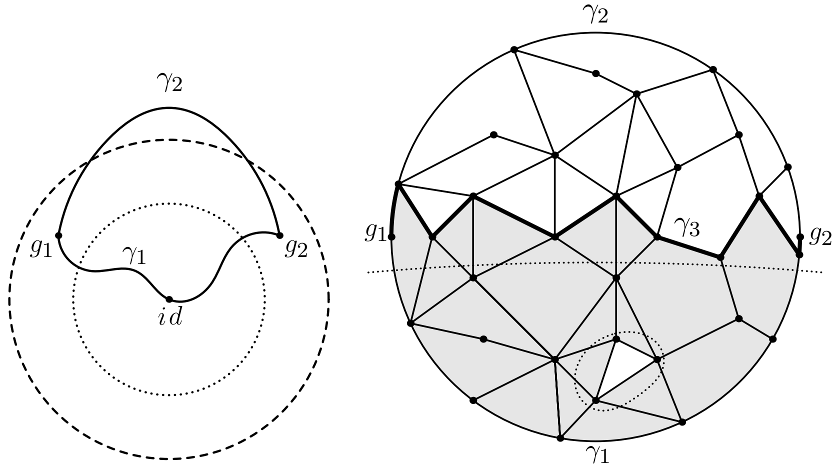

Figure 1. Left: closed loop corresponding to two points $g_1, g_2 \in L(r, R)$. Right: the corresponding Van Kampen diagram, the union U and the curve γ_3; dotted line shows the boundary of $\overline{B}(R)$.

well-known statements describing commuting analytic circle diffeomorphisms. We will start with an evident remark.

Lemma 1. *Let $f, g \in \mathrm{Diff}_+^\omega(\mathrm{S}^1)$ be two commuting circle diffeomorphisms, and assume that f is not of finite order. Then, their sets of periodic points satisfy $\mathrm{Per}(f) \subset \mathrm{Per}(g)$. In particular, if g is also of infinite order, then $\mathrm{Per}(f) = \mathrm{Per}(g)$. Finally, if g has a zero-rotation number, then g fixes all the periodic points of f.*

Proof. The first statement is an evident corollary of the fact that as the two maps commute, g preserves $\mathrm{Per}(f)$, which is a finite set. The second follows immediately, and for the third one it suffices to note that all the periodic points of a diffeomorphism with a zero-rotation number are fixed ones. □

Throughout this paper, we will use the following notation: for a diffeomorphism $g \in \mathrm{Diff}_+^\omega(\mathrm{S}^1)$ we denote by $Z(g)$ its centralizer, and for a point $p \in \mathrm{S}^1$ by $\mathrm{Stab}(p)$ its stabilizer:

$$Z(g) := \{f \in \mathrm{Diff}_+^\omega(\mathrm{S}^1) \mid fg = gf\}, \qquad \mathrm{Stab}(p) = \{f \in \mathrm{Diff}_+^\omega(\mathrm{S}^1) \mid f(p) = p\}.$$

If a group $G \subset \mathrm{Diff}_+^\omega(\mathrm{S}^1)$ is given, by $Z_G(g)$ and $\mathrm{Stab}_G(p)$ we denote, respectively, the centralizer in G and the stabilizer for the action of G:

$$Z_G(g) := \{f \in G \mid fg = gf\} = Z(g) \cap G,$$
$$\mathrm{Stab}_G(p) := \{f \in G \mid f(p) = p\} = \mathrm{Stab}(p) \cap G.$$

The next proposition is well known to specialists, but as we could not find a reference where it would be stated in exactly this way (the closest statement we have found was [12, theorem 1.3]), we provide it together with a proof (mostly following [12, 15]).

Proposition 2. *For any $g \in \mathrm{Diff}_+^\omega(\mathrm{S}^1)$ that is not of a finite order, its centralizer $Z(g) := \{f \in \mathrm{Diff}_+^\omega(\mathrm{S}^1) \mid fg = gf\}$ is an abelian group. Also, its part*

$$Z_0(g) := \{f \in Z(g) \mid \rho(f) = 0\},$$

formed by the maps with zero-rotation number, is either a cyclic group or a one-parametric group generated by an analytic vector field.

Before passing to its proof, let us recall an analogous statement, concerning the *germs* of analytic diffeomorphisms. The following description is immediate for the description of hyperbolic germs; for parabolic germs of the type $x \mapsto x + ax^2 + \cdots$ this statement can be found in [38, theorem 6.3], and for the general case of parabolic germs see [19, theorem 3].

Lemma 2 ([38, theorem 6.3], [19, theorem 3]). *For any non-trivial analytic germ* $\boldsymbol{g}$ *in the space* $\mathrm{Diff}_+^\omega(\mathbb{R}, 0)$ *of orientation-preserving analytic germs, its centralizer in* $\mathrm{Diff}_+^\omega(\mathbb{R}, 0)$ *is either a cyclic group or a one-parameter group generated by a germ of an analytic vector field.*

For the reader's convenience, we, however, recall here the (rough) ideas that are behind these statements (in particular, as similar ideas repeat afterwards), referring the reader to [19,38] for the complete proofs.

Sketch of the proof. For any parabolic germ $\boldsymbol{g} \in \mathrm{Diff}_+^\omega(\mathbb{R}, 0)$ of a form $\boldsymbol{g}(z) = z + az^{k+1} + \cdots$, the Leau–Fatou flower theorem (see, e.g., [3, theorem 2.12]) states that a (complex) neighbourhood of 0 is covered by $2k$ consecutive (intersecting) attraction and repulsion domains. For each of these domains, the space of orbits in this domain is $\mathbb{C}^*$, and passing to the Fatou coordinate transforms the map $\boldsymbol{g}$ to the map $w \mapsto w + 1$. Any germ $\boldsymbol{f}$ commuting with $\boldsymbol{g}$ thus descends for any of these $2k$ domains to an automorphism of the corresponding space of orbits, and the only automorphisms of $\mathbb{C}^*$ are multiplications by constants. Hence, in the Fatou coordinates $\boldsymbol{f}$ takes the form

$$w \mapsto w + t_j, \tag{1}$$

where $j = 1, \ldots, 2k$ is the number of the domain. The expansion of $\boldsymbol{f}$ then starts with $\boldsymbol{f}(z) = z + t_j a z^{k+1} + \cdots$, hence all the t_j's coincide. Thus, the question of describing the centralizer becomes the question of for which t the $2k$ maps defined on the attraction and repulsion domains by (1) with $t_j = t$ glue together in a single map. This immediately becomes the question of the group of symmetries of the corresponding Écale–Voronin module. This group is generically trivial, and is in all the cases either cyclic or generated by a vector field; it is then easy to obtain the desired description for the centralizer $\boldsymbol{Z}(\boldsymbol{g})$. As we have already said, we refer the reader to [19, 38] for the complete proofs. □

Proof of proposition 2. The case if the rotation number of g is irrational is easy: the map g is then (topologically) conjugate to an irrational rotation, and the only orientation-preserving homeomorphisms that commute with an irrational rotation are other rotations (as they should preserve the same unique invariant measure).

Consider now the case when the rotation number of g is rational. Let $P = \mathrm{Per}(g)$, $m = |P|$. Any map $f \in Z(g)$ preserves P, and thus cyclically permutes its points. It is easy to conclude (see [27, corollary 5.2], [12, lemma 2.2], [26, section 2.2.2]) that the restriction of the rotation number on $Z(g)$ is a homomorphism

$$\rho : Z(g) \to \mathbb{Z}/m\mathbb{Z} \subset S^1 = \mathbb{R}/\mathbb{Z}.$$

Thus, as the kernel of this map, $Z_0(g)$ is indeed a group. Let us proceed by establishing the second conclusion of the proposition; then, we will deduce from it the first one.

Our arguments from this moment will be quite close to the ones of [15, proof of proposition 3.8]. For any point $p \in P$ due to the last conclusion of lemma 1 we have an inclusion $Z_0(g) \subset \mathrm{Stab}(p)$. However, $g^m \in \mathrm{Stab}(p)$, and any diffeomorphism $f \in Z(g)$ commutes with g^m. Thus, for any point $p \in P$ we get an injective (due to the analyticity) homomorphism of $Z_0(g)$ to the centralizer of g^m in the space $\mathrm{Diff}_+^\omega(S^1, p)$ of analytic germs at p. Denote this centralizer by $\boldsymbol{Z}(g^m; p)$.

Due to the above description of the centralizers in the space of analytic germs, for each point $p \in P$ the centralizer $\boldsymbol{Z}(g^m; p)$ is either a cyclic group or a one-parametric group, corresponding to an analytic vector field. If at least one of these groups is cyclic, then $Z_0(g)$ is isomorphic to a subgroup of a cyclic group, and thus is cyclic.

Otherwise, for any point p we have an analytic vector field X_p, defined in a neighbourhood of p, such that the centralizer $\boldsymbol{Z}(g^m; p)$ coincides with the one-parametric group generated by X_p. Multiplying X_p by a constant, we can assume that for any p the map g^m is the time-one map of X_p. Also, extending the domain of definition of X_p with help of the dynamics of g^m, we can assume that X_p is defined on all the interval $I_p := (\mathcal{R}(p), \mathcal{R}^{-1}(p))$, where $\mathcal{R} : p \to P$ is the map associating with each $p \in P$ its nearest neighbour on the right (and hence $\mathcal{R}^{-1}(p)$ and $\mathcal{R}(p)$ are two nearest neighbours of the point p).

Any map $f \in Z_0(g)$ thus coincides on each interval I_p with the time t_p map of X_p for some $t_p \in \mathrm{R}$. Note that all the times t_p coincide. Indeed, for any two consecutive points $p, \mathcal{R}(p) \in P$ on the interval $J = (p, \mathcal{R}(p))$ acts a group, generated by f and g^m. Both t_p and $t_{\mathcal{R}(p)}$ can be seen as translation numbers of f after a change of variable that transforms g to $x \mapsto x + 1$ (using either X_p, or $X_{\mathcal{R}(p)}$ to make such a change). However, the translation number is well-defined and invariant under the lift of a one-periodical topological conjugacy, hence $t_p = t_{\mathcal{R}(p)}$. Thus, all the times $\{t_p\}_{p\in P}$ coincide.

This implies that the centralizer $Z_0(g)$ in this case is formed by the maps that are defined as time-t map on I_p:

$$\{f \mid \exists t \in \mathrm{R} : \quad \forall p \in P \quad f|_{I_p} = \exp(tX_p)\}. \tag{2}$$

The necessary condition for t to be realized as a time of a map f in (2) is that on each $(p, \mathcal{R}(p))$ the two maps, defined using X_p and using $X_{\mathcal{R}(p)}$, coincide:

$$\forall p \in P \qquad \exp(tX_p)|_{(p,\mathcal{R}(p))} = \exp(tX_{\mathcal{R}(p)})|_{(p,\mathcal{R}(p))}. \tag{3}$$

Hence, the group of possible times

$$T := \{t \in \mathrm{R} \text{ satisfying } (3)\}$$

is a *closed* subgroup of R (as condition (3) is closed). Hence, it is either a cyclic group, or it coincides with R. In the latter case, all the vector fields X_p are the restrictions of the same vector field X, for which g^m is its time-one map e^X, and $Z_0(g)$ is the one-parametric group generated by X. This concludes the proof of the second claim of the proposition: $Z_0(g)$ is either cyclic, or a one-parametric group generated by an analytic vector field.

To obtain the first claim of the proposition, let $f \in Z(g)$ be an element such that $\rho(f)$ generates the cyclic group $\rho(Z(g)) \subset \mathrm{Z}/m\mathrm{Z}$. Now note that f acts on $Z_0(g)$ by a continuous conjugacy (as such a conjugacy preserves the zero-rotation number). Moreover, this conjugacy fixes a non-trivial element $g^m \in Z_0(g)$. As the group $Z_0(g)$ is either cyclic or one-parametric, this suffices to conclude that this conjugacy is an identity map—and hence f commutes with $Z_0(g)$. However, due to the choice of f the group $Z(g)$ is generated by f and $Z_0(g)$; hence, the centralizer $Z(g)$ is abelian. □

Corollary 1. *If two diffeomorphisms $f, g \in \mathrm{Diff}_+^\omega(S^1)$ of infinite order commute, then $Z(f) = Z(g)$.*

Proof. As $Z(g)$ is abelian by proposition 2, and $f \in Z(g)$, we have $Z(f) \supset Z(g)$. In the same way, we have $Z(f) \subset Z(g)$. □

Finally, we will need the following statement (it uses the notion of local discreteness that we recall below, see definition 6):

Lemma 3 ([22, proposition 3.7]). *For any locally discrete finitely generated group* $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$*, the stabilizer* $\mathrm{Stab}_G(p) = \{f \in G \mid f(p) = p\}$ *of any point* $p \in \mathrm{S}^1$ *is a cyclic group.*

2.3. Distortion control and property (⋆)

We will need the following classical tool in the study of a one-dimensional dynamical system; we do not provide their proofs, referring the reader, for instance, to [8]. The following definition and bounds go back to the works of Denjoy [6], Schwartz [32] and Sacksteder [31]. Roughly speaking, one says how strongly the application of a map can change proportions for subsets of the given interval.

Definition 5. Given two intervals I, J and a C^1 map $g : I \to J$ which is a diffeomorphism onto its image, we define the *distortion coefficient* of g on I by

$$\varkappa(g; I) := \max_{x,y\in I} \left|\log\left(\frac{g'(x)}{g'(y)}\right)\right|.$$

The distortion coefficient is (as it follows immediately from the chain rule) subadditive under the composition, as well as invariant under the passage to the inverse map (with the appropriate change of the interval):

$$\varkappa(g, I) = \varkappa(g^{-1}, g(I)). \tag{4}$$

Finally (and most importantly), for C^2-maps the mean value theorem immediately implies an estimate

$$\varkappa(g, J) \leqslant C_{\{g\}}|J|,$$

where the constant $C_{\{g\}}$ depends only on the Diff^2-norm of g (to be more precise one can take $C_{\{g\}}$ as being the maximum of the absolute value of the derivative of the function $\log(g')$). Applying this, one obtains the following proposition, allowing one to give an upper bound for the distortion of a long composition, provided that the sum of the lengths of the intermediate images of the interval is controlled.

Proposition 3. *Let* $\mathcal{G}$ *be a subset of* $\mathrm{Diff}^2_+(\mathrm{S}^1)$ *that is bounded with respect to the* Diff^2*-norm. If* I *is an interval of the circle and* $g_1, \dots, g_n$ *are finitely many elements chosen from* $\mathcal{G}$*, then*

$$\varkappa(g_n \circ \ldots \circ g_1; I) \leqslant C_{\mathcal{G}} \sum_{i=0}^{n-1} |g_i \circ \cdots \circ g_1(I)|,$$

where the constant $C_{\mathcal{G}}$ *depends only on the set* $\mathcal{G}$*.*

This technique was further developed by Sullivan (see [25, 34, 37]), who noticed that knowing that the sum of the intermediate derivatives (for a long composition of 'simple' maps) at *one* point does not exceed some value S, suffices to give an upper bound for the distortion in a neighbourhood of this point of radius $\sim 1/S$:

Proposition 4 (Sullivan). *Under the assumptions of proposition 3, given a point* $x_0 \in \mathrm{S}^1$*, let us denote* $S := \sum_{i=0}^{n-1} f_i'(x_0)$*, where* $f_i = g_i \circ \ldots \circ g_1$ *and* $f_0 = \mathrm{id}$*. Then for every* $\delta \leqslant \log(2)/2C_{\mathcal{G}}S$*, one has*

$$\varkappa(f_n, U_{\delta/2}(x_0)) \leqslant 2C_{\mathcal{G}}S\delta,$$

where $U_{\delta/2}(x_0)$ *denotes the* $\delta/2$*-neighbourhood of* x_0*.*

The same estimates hold in a complex neighbourhood, provided that the maps g_i *are real-analytic and thus can be extended to a complex domain.*

2.4. Local (non)-discreteness, local flows and commutators

Recall first the definition of the local (non)-discreteness.

Definition 6. A group $G \subset \mathrm{Diff}^{\omega}(\mathrm{S}^1)$ is *(C^1-)locally discrete* if for any interval $I \subset \mathrm{S}^1$ the identity is a C^1-isolated point of $\{g|_I : g \in G\}$. Otherwise, it is said to be *(C^1-)locally non-discrete*.

The key reason, that relates it to our study, is the following: local non-discreteness implies the presence of local flows in the local closure of the group. Namely, the following proposition is in the spirit of results by Shcherbakov *et al* [11], Nakai [24], Rebelo [29, 30] and Loray and Rebelo [21]. (It has appeared in this form in [9, remark 2.9], though was definitely known to specialists before.)

Proposition 5. *Let I be an interval on which certain real-analytic non-trivial diffeomorphisms $f_k \in G$ are defined. Suppose that f_k converge to the identity in the C^1 topology on I, and let f be another C^{ω} diffeomorphism having a hyperbolic fixed point on I. Then there exists a nonzero C^1 change of coordinates $\phi : I_0 \to [-1, 2]$ on some subinterval $I_0 \subset I$ after which the pseudo-group G generated by the f_k's and f contains in its $C^1([0, 1], [-1, 2])$-closure a (local) translation subgroup:*

$$\overline{\left\{\phi \circ g \circ \phi^{-1}|_{[0,1]} \,\middle|\, g \in G\right\}} \supset \{x \mapsto x + s \mid s \in [-1, 1]\}.$$

Corollary 2. *Let $G \subset \mathrm{Diff}^{\omega}(\mathrm{S}^1)$ be a C^1-locally non-discrete finitely generated group, that has no invariant measure and that acts minimally on the circle. Then* $\mathrm{NE}(G) = \emptyset$.

Proof. The absence of an invariant measure implies (see [7, remark 14.17]) that at least one map $f \in G$ has a hyperbolic fixed point p on the circle. By minimality, conjugating g, one can find a hyperbolic fixed point of a map from G on an arbitrary interval $I \subset \mathrm{S}^1$. Applying proposition 5, one finds local flow in the local closure of the group. This allows one to conclude that the group has minimality with a control on the derivative: there exists a constant $C_{\min}$, such that for any $x, y \in \mathrm{S}^1$ and any $\varepsilon > 0$ there exists $g \in G$ such that

$$g(x) \in U_{\varepsilon}(y), \qquad \frac{1}{C_{\min}} < g'(x) < C_{\min}.$$

Indeed, such a minimality with control of the derivative clearly takes place on the interval I_0 from the conclusion of proposition 5; on the other hand, due to the minimality the circle can be covered with a finite number of images of this interval.

Now, as p is a hyperbolic fixed point of f (that we can assume to be a repelling one, otherwise replacing f with f^{-1}), for some n and for some neighbourhood U of p for any $y \in U$ one has $(f^n)'(y) > C_{\min}$. Any point $x \in \mathrm{S}^1$ can be sent into U by some g with $g'(x) > \frac{1}{C_{\min}}$, and we have

$$(f^n \circ g)'(x) = g'(x) \cdot (f^n)'(g(x)) > \frac{1}{C_{\min}} \cdot C_{\min} = 1.$$

Hence, there are no non-expandable points on the circle. □

The following commutator technique, due to Ghys, gives a way of showing that a group is not locally discrete: if one finds in G two maps sufficiently (locally) close to the identity, their commutators will (locally) converge to the identity. Let $U_r^{\mathbb{C}}(z)$ denote the complex radius r neighbourhood of point z.

Proposition 6 ([15, proposition 2.7]). *There exists $\varepsilon_0 > 0$ with the following property. Assume that the complex analytic local diffeomorphisms $f_1, f_2 : U_1^{\mathbb{C}}(0) \to \mathbb{C}$ are ε_0-close (in the C^0 topology) to the identity, and let the sequence f_k be defined by the recurrence relation*

$$f_{k+2} = [f_k, f_{k+1}], \qquad k = 1, 2, 3, \dots$$

Then all the maps f_k are defined on the disc $U_{1/2}^{\mathbb{C}}(0)$ of radius $1/2$, and f_k converges to the identity in the C^1 topology on $U_{1/2}^{\mathbb{C}}(0)$.

We will use the following immediate corollary of this proposition: if for two maps that are close to the identity their chain of commutators does not end up with the identity at a finite step, the group generated by them is not locally discrete, and hence (once it is sufficiently rich not to preserve any measure) does not possess non-expandable points.

2.5. Property ($\star$) and the maps close to the identity

To establish the property ($\star$) for any group G satisfying the assumptions of theorem 1, we will assume the contrary: that one of the non-expandable points $\bar{x}$ is not fixed by any map $g \in G \setminus \{\mathrm{id}\}$. The technique, repeating verbatim the arguments from [9], is then to consider for any $n \in \mathbb{N}$ among the images of $\bar{x}$ by maps from $g \in B(n) \setminus \{\mathrm{id}\}$ the one $x_n = g_n(\bar{x})$ that is closest to $\bar{x}$ on the right. The main idea is then to consider the corresponding maps g_n, note that due to non-expandability of $\bar{x}$ they become close to the identity in a well-chosen scale, and apply the technique of section 2.4 to deduce a contradiction.

A first remark (fully analogous to [9, lemma 3.7]) is that to find such a rescaling it suffices that the lengths $l_n := |[\bar{x}, x_n]|$ decrease as $o(1/n)$:

Lemma 4. *Let $\bar{x}$ be a non-expandable point for the action of a finitely generated group $G \subset \mathrm{Diff}_+^{\omega}(S^1)$, and $(g_n)_{n=1}^{\infty}$ be a sequence of maps, $g_n \in B(n)$, such that $l_n := |[\bar{x}, g_n(\bar{x})]| = o(1/n)$. Choose any sequence r_n such that $l_n = o(r_n)$, $\quad r_n = o(1/n)$. Then, the rescaled maps $\widetilde{g}_n$*

$$\widetilde{g}_n(y) = \frac{g_n(\bar{x} + r_n y) - \bar{x}}{r_n}$$

converge to the identity on $C^1([-1; 1])$ as well as in $C(U_1^{\mathbb{C}}(0))$.

Proof. Note first that due to the assumption $l_n = o(r_n)$ we have

$$\widetilde{g}_n(0) = \frac{l_n}{r_n} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, it suffices to check that the derivatives $\widetilde{g}_n'$ tend to 1 uniformly on the unit disc $U_1^{\mathbb{C}}(0)$; as the derivative is not changed by a linear change of variable, we want to check that

$$\max\{|g_n'(x) - 1| \mid x \in U_{r_n}^{\mathbb{C}}(\bar{x})\} \to 0, \qquad \text{as } n \to \infty.$$

Next, the point $\bar{x}$ is non-expandable. Hence, for any map $g \in B(n)$, representing it as a composition of at most n generators $g = \psi_n \dots \psi_1$, we see that the sum of intermediate derivatives at $\bar{x}$ does not exceed n:

$$\sum_{j=0}^{n-1} (\psi_j \dots \psi_1)'(\bar{x}) \leqslant \sum_{j=0}^{n-1} 1 = n.$$

Applying proposition 4, one sees that a distortion of any map $g \in B(n)$ admits a bound (uniform in g and in n) in the neighbourhood of radius $\sim \frac{1}{n}$:

$$\exists c : \; \forall n \, \forall r < c/n \qquad \varkappa(g_n, U_r(\bar{x})) \leqslant 2rnC_2,$$

where C_2 is the constant from proposition 4 (and the same estimate holds for the complex distortion in the complex neighbourhood $U_r^{\mathbb{C}}(\bar{x})$).

To show that the derivatives stay close to 1, consider the inverse map $g_n^{-1} \in B(n)$. On the one hand, $(g_n^{-1})'(\bar{x}) \leqslant 1$, as $\bar{x}$ is a non-expandable point. On the other hand, at the point $x_n := g_n(\bar{x})$ one has

$$(g_n^{-1})'(x_n) = (g_n^{-1})'(g_n(\bar{x})) = \frac{1}{g_n'(\bar{x})} \geqslant 1,$$

Since $|[\bar{x}, x_n]| = o(r_n)$ and $r_n = o(1/n)$, for all n sufficiently large we have $x_n \in U_{r_n}(\bar{x})$ and $r_n < c/n$. Control of distortion then implies that these two derivatives are close to each other:

$$\left| \log \frac{(g_n^{-1})'(x_n)}{(g_n^{-1})'(\bar{x})} \right| \leqslant C_2 n r_n.$$

Thus, as $r_n = o(1/n)$, both derivatives are $o(1)$-close to 1, and hence (due to the distortion control) the same holds for $g_n'(x)$ in $U_{r_n}^{\mathbb{C}}(\bar{x})$. □

Next step is to obtain a sufficient condition for the decrease speed of the lengths l_n:

Lemma 5 ([9], lemma 3.5). *Let $G \subset \mathrm{Diff}_+^{\omega}(S^1)$ be a finitely generated group, $\bar{x} \in S^1$ be a point that is not fixed by any $g \in G \setminus \{\mathrm{id}\}$, and for any $n \in \mathbb{N}$ consider the point $x_n = g_n(\bar{x})$ which is the closest to $\bar{x}$ to the right point of the set $X_n := \{g(\bar{x}) \mid g \in B(n) \setminus \{\mathrm{id}\}\}$. Then, the distances $l_n := |[\bar{x}, x_n]|$ satisfy*

$$l_n = O(1/S([n/2])),$$

where $S(m) := \sum_{g\in B(m)} g'(\bar{x})$.

As the proof coincides verbatim with the proof of [9, lemma 3.5], we repeat only its sketch.

Sketch of the proof. Let $J_n := [\bar{x}, x_n]$. A first easy remark is that all the images $g(J_n)$, where $g \in B_{[n/2]}$, are pairwise disjoint, as otherwise, if the left end of $g_1(J_n)$ belonged to $g_2(J_n)$, the image $(g_2^{-1} \circ g_1)(\bar{x})$ would be closer to $\bar{x}$ on the right than x_n while $g = g_2^{-1} \circ g_1$ would still belong to B_n.

Now, as these images are disjoint, the sum of their lengths does not exceed 1 (note that we consider the circle as R/Z and thus of length one and not 2π). Hence, the distortion control technique (see proposition 3) implies that the distortion of $g \in B([n/2])$ on J_n is uniformly bounded:

$$\exists C_1 : \forall n \in \mathbb{N} \quad \forall g \in B([n/2]) \qquad \frac{1}{C_1} g'(\bar{x}) < \frac{|g(J_n)|}{|J_n|} < C_1 \cdot g'(\bar{x}).$$

The length $g(J_n)$ can hence be estimated from below, uniformly in n and in $g \in B([n/2])$, as $\frac{1}{C_1} g'(\bar{x})|J_n|$, and the disjointness of such images implies

$$1 \geqslant \sum_{g\in B_{[n/2]}} |g(J_n)| \geqslant \frac{1}{C_1} |J_n| \sum_{g\in B_{[n/2]}} g'(\bar{x}).$$

Thus,

$$|J_n| \leqslant \frac{C_1}{\sum\limits_{g\in B_{[n/2]}} g'(\bar{x})} = O\left(\frac{1}{S([n/2])}\right),$$

where the last equality is the definition of $S(m)$. □

The key point in order to apply the above technique, hence, will be to establish a superlinear growth of the sums $S(n)$ of derivatives at the non-expandable point $\bar{x}$. We will establish it in section 3.

3. Superlinear growth of sums *S(n)*

This section is devoted to the proof of the following proposition, which, as we have just seen in the previous section, will allow us to apply the commutator technique to groups with one end.

Proposition 7. *Let $G \subset \mathrm{Diff}^{\omega}_{+}(\mathrm{S}^1)$ be a finitely presented group with one end, acting minimally and without an invariant measure, for the action of which the point $\bar{x}$ is non-expandable. Then, the sums $S(n) := \sum_{g\in B(n)} g'(\bar{x})$ grow superlinearly:*

$$\frac{S(n)}{n} \to \infty \qquad \text{as } n \to \infty.$$

Proof. Assume the contrary that there exists a constant C_2 and a sequence $n_k \to +\infty$ such that

$$\sum_{g\in B(n_k)} g'(\bar{x}) \leqslant C_2 \cdot n_k. \tag{5}$$

We will deduce from this that there exists a neighbourhood $U_\varepsilon(\bar{x})$ in which all the maps $f \in G$ have uniformly bounded distortion. This will immediately lead to a contradiction. Indeed, as we have already seen in the proof of corollary 2, the minimality and the absence of a preserved measure imply (via [7, remark 14.17]) that for any $\varepsilon > 0$ one can find a map $f \in G$ with a hyperbolic repelling fixed point $r \in U_\varepsilon(\bar{x})$. Then the sequence of iterations $\{f^m\}$ cannot be of uniformly bounded distortion on the interval $U_\varepsilon(\bar{x})$: otherwise, it would expand it to a length that exceeds the total length of the circle.

In section 2.1 we have fixed a finite system $\mathcal{F}$ of generators of G, which defines the Cayley graph $\mathcal{C}$ of G. Recall that due to Sullivan distortion arguments the distortion of a composition $g = \psi_n \circ \ldots \circ \psi_1$ can be estimated in a neighbourhood of $\bar{x}$ of radius $\sim \frac{1}{C_{\mathcal{F}} S}$ where $C_{\mathcal{F}}$ is a constant depending only on $\mathcal{F}$, and S is the sum of the intermediate derivatives: $S = \sum_{j=0}^{n-1} (\psi_j \circ \ldots \circ \psi_1)'(\bar{x})$. Hence, to obtain a contradiction it suffices to show that each element $g \in G$ can be represented as a product of generators in such a way that the sum of the intermediate derivatives stays uniformly bounded.

Define the function $\varphi(g) := g'(\bar{x})$ on G. First, we obtain an intermediate estimate, bounding the 'price' of going in G to infinity from an arbitrary initial point $f \in G$:

Lemma 6. *Let $G, \bar{x}$ be as in proposition 7. Then there exists a constant C_4 such that for any $f \in G$ there exists a path Γ_f in G from f to infinity for which the sum of values of φ along this path does not exceed C_4.*

Proof. Let $h_1 \in G$ be a map that has a hyperbolic attracting point a, and choose a closed interval I in its basin of attraction. Due to the minimality of the action and compactness of the circle, the circle can be covered by a finite number of images $h(I)$, where $h \in G$. Hence, for any point $y \in \mathrm{S}^1$ it can be mapped into I by an application of a map $h_2 \in G$, chosen from a finite set.

Now, for any $f \in G$, choose such h_2 that maps $y = f(\bar{x})$ to I and consider a piecewise geodesic path in the Cayley graph that passes through the points

$$f, h_2 f, h_1 h_2 f, h_1^2 h_2 f, \ldots, h_1^m h_2 f, \ldots.$$

The application of both h_1 and of h_2 can be decomposed into a bounded number of generators, thus it suffices to bound the sum of derivatives

$$\varphi(f) + \varphi(h_2 f) + \sum_{j=1}^{\infty} \varphi(h_1^j h_2 f) = f'(\bar{x}) + (h_2 f)'(\bar{x}) + \sum_{j=1}^{\infty} (h_1^j h_2 f)'(\bar{x}).$$

Due to the chain rule and the non-expandability of $\bar{x}$,

$$f'(\bar{x}) + (h_2 f)'(\bar{x}) + \sum_{j=1}^{\infty}(h_1^j h_2 f)'(\bar{x}) = f'(\bar{x}) + (h_2 f)'(\bar{x})$$

$$+(h_2 f)'(\bar{x}) \cdot \sum_{j=1}^{\infty}(h_1^j)'(h_2 f(\bar{x})) \leqslant 2 + \sum_{j=1}^{\infty}(h_1^j)'(h_2 f(\bar{x})). \tag{6}$$

As $h_2 f(\bar{x}) \in I$, and the closed interval I is a compact subset of the h_1-domain of attraction of a hyperbolic fixed point, the sum in the right-hand side of (6) converges exponentially and admits a uniform upper bound.

Finally, as the sum of derivatives along the chosen path converges, the derivatives tend to zero and hence the path escapes to infinity. This concludes the proof of the lemma. □

Now, let us estimate the sum of derivatives on (some) spherical layers:

Lemma 7. *Let $G, \bar{x}$ be as in proposition 7; let $r > 0$ be fixed, and assume that (5) holds. Then there exists C_5, for which for any R' there exists $R > R'$ such that*

$$\sum_{g\in L(r;R)} \varphi(g) \leqslant C_5. \tag{7}$$

Proof. Let any R' be given; we will find a spherical layer $L(r; R)$ with $R > R'$ satisfying (7). To do so, first note that spherical layers $L(r; R' + jr)$, $j = 0, 1, 2, \ldots$, are pairwise disjoint; also, while $R + r < n$, the layer $L(r, R)$ is a subset of $B(n)$. Hence, for any k such that $n_k > 2R'$, one has from (5) that

$$C_2 \cdot n_k \geqslant \sum_{g\in B(n_k)} g'(\bar{x}) \geqslant \sum_{j:\, R'+(j+1)r<n_k} \left(\sum_{g\in L(r;R'+jr)} \varphi(g) \right).$$

In the right-hand side we have a sum of $[\frac{n_k-R'-r}{r}]$ sums. By the pigeon-hole principle, for one of the indices j we have

$$\sum_{g\in L(r;R'+jr)} \varphi(g) \leqslant C_2 n_k \cdot \left[\frac{n_k - R' - r}{r}\right]^{-1}. \tag{8}$$

As $k \to \infty$ and thus $n_k \to \infty$, the right-hand side of (8) tends to $C_2 r$, in particular, it becomes less than $2C_2 r$ for some n_k. We thus have constructed a spherical layer $L(r; R)$ with the internal radius $R = R' + jr$, satisfying (7), and have shown that the internal radius can be chosen arbitrarily large. Thus, the proposition is proven. □

Joining these two lemmas, we obtain the following.

Lemma 8. *Let $G, \bar{x}$ be as in proposition 7, and assume that (5) holds. Then there exists a constant C_3 such that for any $f \in G$ there exists a path between the identity and f in the Cayley graph of G along which the sum of values of φ does not exceed C_3.*

Proof. As the group G is finitely presented, by proposition 1 there exists r such that any spherical layer of thickness r is connected. Take and fix any such r.

Now, for any $f \in G$ consider the paths Γ_{id} and Γ_f, going to the infinity from id and f, respectively, such that the sum of values of φ along each of these paths does not exceed C_4. Due to lemma 7, we can find $R > d(\mathrm{id}, f)$ such that

$$\sum_{g\in L(r;R)} \varphi(g) \leqslant C_5.$$

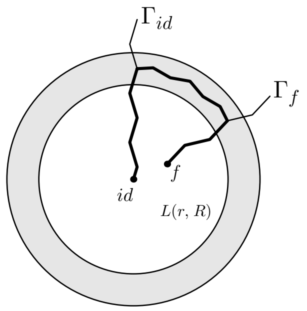

Figure 2. The path joining id and f.

The spherical layer $L(r; R)$ is intersected by both paths Γ_{id}, Γ_f, as they both start inside $B(R)$ and go to infinity. Cut these paths off once they intersect $L(r; R)$, and join the two intersection points by a path that stays inside $L(r; R)$; we can do it, as this spherical layer is connected due to the choice of r. The sum of values of φ along this joining segment hence does not exceed C_5.

Then, the sum of values of φ along the resulting path that joins id and f (see figure 2) does not exceed $2C_4 + C_5 =: C_3$. □

We are now ready to conclude the proof of proposition 7. Namely, as we have already mentioned, the conclusion of lemma 8 implies that for any $f \in G$ there exists a representation of f as a composition of generators (and their inverses) such that the sum of intermediate derivatives at $\bar{x}$ does not exceed C_3. Proposition 4 then implies that for any $\delta \leqslant \frac{\log(2)}{2C_{\mathcal{F}}C_3}$ the distortion $\varkappa(f; U_{\delta/2}(\bar{x}))$ is bounded uniformly in $f \in G$.

As we have already seen in the beginning of this proof, the presence of a hyperbolic fixed point contradicts such a uniform bound on the distortion. Thus, we have obtained a desired contradiction. □

4. Establishing property ($\star$): almost-identity maps and their commutators

We are now ready to prove theorem 1.

Proof. As we have mentioned before, the proof will go by contradiction. Namely, let $G \subset \text{Diff}_+^{\omega}(\mathrm{S}^1)$ satisfy the assumptions of theorem 1. Assume that there is a non-expandable point $\bar{x} \in \mathrm{S}^1$ that is not fixed by any $g \in G \setminus \{\text{id}\}$. Note that as $\text{NE} \neq \emptyset$, due to corollary 2 the group G should be locally discrete.

For any $n \in \mathrm{N}$, consider the map $g_n \in B_n \setminus \{\text{id}\}$ such that the image $x_n := g_n(\bar{x})$ is closest to $\bar{x}$ on the right among all the possible images by elements of $B_n \setminus \{\text{id}\}$. Denote also by J_n the interval $(\bar{x}; x_n)$ and by l_n its length.

Then, proposition 7 implies that the sums $S(n) := \sum_{g\in B(n)} g'(\bar{x})$ grow faster than linearly, and thus lemma 5 shows that $l_n = o(1/n)$. Also, choose and fix a sequence r_n such that $l_n = o(r_n)$, $\quad r_n = o(1/n)$; for instance, one can take $r_n = \sqrt{l_n/n}$. Then, the application of lemma 4 shows that the rescaled maps $\widetilde{g}_n(y) = \frac{g_n(\bar{x}+r_n y)-\bar{x}}{r_n}$ converge to the identity in the unit disc.

As we already mentioned in the outline, now the idea is to take a sequence of commutators starting with two consecutive g_n. Namely, for every n consider the sequence

$$f_{n,1} = g_n, \ \ f_{n,2} = g_{n+1}, \qquad f_{n,k+1} = [f_{n,k-1}, f_{n,k}], \ k > 1. \tag{9}$$

As both maps g_n and g_{n+1} are close to the identity in the r_n-scale around $\bar{x}$, Ghys' commutator technique (proposition 6) immediately provides us with the following.

Proposition 8. *In the above assumptions and notations, once n is sufficiently big, the sequence $\{f_{n,k}\}$ defined by (9) converges to the identity in the C^1-topology on some interval $I \subset S^1$.*

Now we have three possible cases.

(i) There exist arbitrary large n such that all the maps $f_{n,k}$ are different from identity;
(ii) for any sufficiently large n we have $[f_{n,1}, f_{n,2}] = [g_n, g_{n+1}] = \mathrm{id}$;
(iii) there exist arbitrary large n, for which $f_{n,3} \neq \mathrm{id}$, but for any sufficiently large n there exists k_n such that $f_{n,k_n} = \mathrm{id}$.

In all these cases we will show that the group G is not locally discrete, thus obtaining a contradiction with corollary 2, claiming such a discreteness, and hence concluding the proof of theorem 1. Case (i) is the most 'essential' but at the same time the simplest. Namely, in this case a sequence of commutators $f_{n,k} \neq \mathrm{id}$ converges to the identity on some interval $I \subset S^1$ due to proposition 8, and hence, the group is not locally discrete.

Before handling the two other cases, we will need a few remarks concerning the group G. Note that all the rotation numbers of elements of G are rational.

Lemma 9. *Assume that a finitely generated group G_1 is locally discrete. Then for any $g \in G_1$ its rotation number $\rho(g)$ is rational.*

Proof. Indeed, if there existed $g \in G_1$ with $\alpha = \rho(g) \notin \mathbb{Q}$, for the sequence $\frac{p_n}{q_n}$ of good rational approximations for α, a proposition by Herman [17, proposition 1.14, p 91] would imply that the powers g^{q_n} converge to the identity in the C^1–topology. □

Next, the maps $f_{n,k}$ are (once n is sufficiently big) not of a finite order:

Lemma 10. *For G and $f_{n,k}$ as above, there exists n_0 such that for any $n \geqslant n_0$ and any k, the diffeomorphism $f_{n,k}$ is not of finite order except if $f_{n,k} = \mathrm{id}$.*

Proof. Due to our assumption on group G, finite orders of its element do not exceed some constant C. Thus it suffices to show that either $f_{n,k} = \mathrm{id}$ or $f_{n,k}^j \neq \mathrm{id}$ for $j = 2, \ldots, C$. Indeed, recall that the estimates of proposition 6 imply that for any $\varepsilon > 0$ for all sufficiently big n and for all k, the rescaled map $\tilde{f}_{n,k}(y) := \frac{f_{n,k}(\bar{x}+r_n y)-\bar{x}}{r_n}$ is ε-close to the identity in $C^1([-1/2, 1/2], [-1, 1])$. Take now any point $y \in [-1/4, 1/4]$ for which $\tilde{f}_{n,k}(y) \neq y$, and consider the corresponding point $x = \bar{x} + r_n y$. Note that assuming that $\varepsilon > 0$ is sufficiently small all the iterations of $\tilde{f}_{n,k}^j(y)$, where $j = 1, 2, \ldots, C$, are defined and stay in $[-1/2, 1/2]$. This sequence is then monotonic, and hence the point y cannot be periodic for any period $j \leqslant C$; returning to the initial coordinates, we see that the point x is not periodic for $f_{n,k}$ for any period $j \leqslant C$, what implies the desired lower bound on the order of $f_{n,k}$. □

Let us choose and fix n_0 provided by lemma 10, and assume that we are in the case (ii). That is, that there exists $n_1 \geqslant n_0$ such that for all $n \geqslant n_1$ the maps g_n and g_{n+1} commute. As neither of them is of finite order, by lemma 1 they have the same set of periodic points. By induction, we obtain $\mathrm{Per}(g_n) = \mathrm{Per}(g_{n_1})$ for all $n \geqslant n_1$. Denote $P := \mathrm{Per}(g_{n_1})$, and let p be the point of P closest on the left to the point $\bar{x}$ (note that $\bar{x} \notin P$ due to our assumption). Taking $m = |P|$, we have $g_n^m(p) = p$ for any $n \geqslant n_1$.

Next, $\{g_n^m\}_{n \geqslant n_0}$ is a family of commuting maps. Lemma 3 then implies that these maps are generated by the same map $\bar{g} \in \mathrm{Stab}_G(p)$. But this is impossible as $g_n^m(\bar{x}) \to \bar{x}$ as $n \to \infty$, and if these maps were powers of some $\bar{g} \in G$, the images of $\bar{x}$ would not be able to intersect the interval between $\bar{g}(\bar{x})$ and $\bar{g}^{-1}(\bar{x})$.

Let us now pass to case (iii): take any $n \geqslant n_0$ such that $[g_n, g_{n+1}] = f_{n,3} \neq \mathrm{id}$. Let k be the minimal integer such that $f_{n,k} = \mathrm{id}$. Consider the last three maps before the sequence of commutators vanishes: let

$$f = f_{n,k_n-3}, \qquad g = f_{n,k_n-2}, \qquad h = f_{n,k_n-1}.$$

They satisfy the relations $[f, g] = h$, $[g, h] = \mathrm{id}$, and due to lemma 10 neither of these maps is of finite order. Now we have the following.

Lemma 11. *Let $f, g, h \neq \mathrm{id}$ be analytic, orientation-preserving circle diffeomorphisms that are not of finite order, satisfying the relations $[f, g] = h$, $[g, h] = \mathrm{id}$. Then the group G_1, generated by f and g, is not locally discrete.*

Proof. Note first that as $h = [f, g] = fgf^{-1}g^{-1}$ commutes with g, so does $hg = fgf^{-1}$. As g and fgf^{-1} are two commuting maps, neither of which is of finite order, their centralizers $Z(g)$ and $Z(fgf^{-1})$ coincide by corollary 1. Hence, the conjugating map

$$\mathrm{Conj}_f : a \mapsto faf^{-1}$$

is an automorphism of $Z(g)$: as an automorphism of G_1, it is an isomorphism between $Z(g)$ and $Z(\mathrm{Conj}_f(g)) = Z(g)$.

We can assume that g has a rational rotation number (otherwise, an application of lemma 9 concludes the proof) and hence that there exists a point p and an integer m such that $g^m(p) = p$. Consider now the zero-rotation number subgroup $Z_0(g)$ of the centralizer $Z(g)$. The conjugacy by f preserves the rotation number, hence Conj_f maps $Z_0(g)$ to itself. On the other hand, the last conclusion of lemma 1 implies that $Z_0(g) \subset \mathrm{Stab}_{G_1}(p)$: indeed, any $\xi \in Z_0(g)$ fixes all the periodic points of g, in particular, the point p.

Now, assume that the stabilizer $\mathrm{Stab}_{G_1}(p)$ is a cyclic group. Then, $Z_0(g)$ is a subgroup of a cyclic group, and hence also a cyclic group (infinite as it contains an element g^m of infinite order). Thus, Conj_f acts on it either identically, or by sending an element to its inverse. In both cases, its square Conj_f^2, which is a conjugacy by f^2, acts identically on $Z_0(g)$, and hence, f^2 commutes with g^m. Thus, we have

$$[f, f^2] = [f^2, g^m] = [g^m, g] = \mathrm{id}$$

a chain of commuting elements, neither of which is of finite order, which implies (due to corollary 1) that f and g commute. This gives us a contradiction (we assumed that $[f, g] = h \neq \mathrm{id}$) that came from our assumption that the stabilizer $\mathrm{Stab}_{G_1}(p)$ is cyclic. Hence, it is not, and applying lemma 3, we obtain that the group G_1 is not locally discrete, thus concluding the proof. □

Application of this proposition to f, g and h leads us to the contradiction in case (iii), thus concluding the proof of the theorem. □

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