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# On the Asymptotic Estimates of Solutions of Emden–Fowler Type Equations

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**Abstract**—Emden–Fowler type equations of arbitrary order are considered. The paper contains asymptotic estimates of nonoscillating continuable and noncontinuable solutions of such equations.

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## 1. INTRODUCTION

Consider the following equation:

$$\begin{gathered} y^{(n)} = p(x)|y|^{\sigma} \operatorname{sgn} y, \qquad n \geq 2, \quad \sigma > 1, \\ y = y(x), \quad p(x) \in C^0, \quad x, y \in \mathbb{R}^1, \quad p(x) \neq 0. \end{gathered} \tag{1}$$

For $n = 2$ and $p(x) = \pm x^{\beta}$, $x > 0$, $\beta = \text{const}$, this equation is known as the Emden–Fowler equation (see, for example, [1]), which occurs in the study of a number of physical processes.

**Definition 1.** The solution $y(x)$ of Eq. (1) is said to be *right-continuable* (*left-continuable*) if it is defined in a neighborhood of $+\infty$, $(-\infty)$.

**Definition 2.** The nontrivial solution $y(x)$ of Eq. (1) is said to be *right-oscillating* (*left-oscillating*) if, for any $x$ belonging to its domain, there exists a $\widetilde{x} > x$ ($\widetilde{x} < x$) such that $y(\widetilde{x}) = 0$.

By a *noncontinuable* (*nonoscillating*) solution in any direction we mean a solution that is not continuable (oscillating) in that direction.

In the present paper, we consider (right- or left-) nonoscillating continuable and noncontinuable solutions of Eq. (1) and present asymptotic estimates of such solutions as $x \to \pm\infty$ as well as of solutions tending to infinity as $x \to a \neq \pm\infty$.

The following theorem [2] establishes the existence of noncontinuable nonoscillating solutions of Eq. (1) for $p(x) > 0$.

**Theorem 1.** *If $p(x) > 0$, then, for any number $a$, there exists a right-noncontinuable solution $y(x)$ of Eq.* (1) *possessing the property*

$$\lim_{x \to a-0} |y^{(i)}(x)| = +\infty, \qquad 0 \leq i \leq n-1. \tag{2}$$

*A similar statement also holds for left-noncontinuable solutions of the equation under consideration.*

For the solutions indicated in Theorem 1, the following asymptotic estimates [3] are valid.

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**Theorem 2.** *If the solution $y(x)$ possesses property* (2) *for some $a$, then, in a left neighborhood of the point $x = a$, the following inequalities hold*:

$$b(a-x)^{n/(1-\sigma)} \le |y(x)| \le B(a-x)^{n/(1-\sigma)}, \qquad B, b = \text{const}, \quad B \ge b > 0. \tag{3}$$

*A similar assertion is also valid for left-noncontinuable solutions of the equation under consideration.*

These theorems were first obtained by V. A. Kondrat'ev and the author in 1980 and were jointly reported with the accompanying proofs at the Seminar on the Qualitative Theory of Differential Equations at Moscow State University; see also [3]. Closely-related results were obtained later in other papers (see, for example, [4]). Apparently, the proofs of these results are overburdened by many technical details and references to other papers. In what follows, we shall present a simple and straightforward proof of Theorem 2.

It is well known (see [4] as well as [2, Theorems 2 and 5]) that, under the condition

$$|p(x)| \ge cx^{-n}, \qquad c = \text{const} > 0, \quad x \ge x_0 > 0, \tag{4}$$

for $(-1)^n p(x) > 0$, Eq. (1) has nontrivial right-continuable nonoscillating solutions. For any such solution, the derivatives $y^{(i)}(x)$, $0 \le i \le n-1$, are monotone functions and the following condition holds:

$$y^{(i)}(x)\, y^{(i+1)}(x) < 0, \qquad \lim_{x\to+\infty} y^{(i)}(x) = 0, \quad 0 \le i \le n-1. \tag{5}$$

Theorem 3 (given below) contains asymptotic upper bounds for these solutions.

If $(-1)^n p(x) < 0$, then, under condition (4), Eq. (1) has no nontrivial right-continuable nonoscillating solutions (see [5] as well as [2, Theorems 3 and 4]).

**Theorem 3.** *If, in Eq.* (1)*, the following condition holds*:

$$(-1)^n p(x) \ge c_1 x^{-m}, \qquad x \ge x_0 > 0, \quad c_1, m = \text{const}, \quad c_1 > 0, \quad m \le n, \tag{6}$$

*then its right-continuable sign-preserving solutions satisfy the following estimates*:

- *if $n > m$, then*

$$|y(x)| \le Dx^{(m-n)/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \ge x_0; \tag{7}$$

- *if $n = m$, then*

$$|y(x)| \le D|\ln x|^{-1/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \ge x_0 + 1. \tag{8}$$

Using Theorem 3, we can obtain the following statement refining Theorem 5 from [6].

**Theorem 4.** *If, in Eq.* (1)*, the following condition holds*:

$$|p(x)| \ge c_1 x^{-n-\beta(\sigma-1)}, \qquad x \ge x_0 > 0, \quad c_1, \beta = \text{const}, \quad c_1 > 0, \quad 0 \le \beta \le n-1, \tag{9}$$

*then any right-continuable sign-preserving solution of this equation satisfies one of the following estimates*:

- *if*

$$\beta = 0, \qquad (-1)^n p(x) > 0, \tag{10}$$

*then*

$$|y(x)| \le D|\ln x|^{-1/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \ge x_0 + 1; \tag{11}$$

- *if*
$$k-1<\beta<k, \quad k\in\{1,\dots,n-1\}, \qquad (-1)^{n-k}p(x)>0, \tag{12}$$
*then*
$$|y(x)|\le Dx^{\beta}, \qquad D=\mathrm{const}>0, \quad x\ge x_0; \tag{13}$$

- *if*
$$\beta=k\in\{1,\dots,n-1\}, \qquad (-1)^{n-k}p(x)>0, \tag{14}$$
*then*
$$|y(x)|\le Dx^{\beta}|\ln x|^{-1/(\sigma-1)}, \qquad D=\mathrm{const}>0, \quad x\ge x_0+1; \tag{15}$$

- *if*
$$k-1<\beta\le k, \quad k\in\{1,\dots,n-1\}, \qquad (-1)^{n-k}p(x)<0, \tag{16}$$
*then*
$$|y(x)|\le Dx^{k-1}, \qquad D=\mathrm{const}>0, \quad x\ge x_0. \tag{17}$$

**Remark.** If $\beta=0$ and $(-1)^n p(x)<0$, then, as noted above, under condition (9), Eq. (1) has no right-continuable sign-preserving solutions.

The assertion contained in Theorems 3 and 4 can be carried over in a natural way to the left-continuable nonoscillating solutions of Eq. (1).

## 2. EXAMPLES

**Example 1.** Consider the equation
$$y^{(n)}=|y|^{\sigma}\operatorname{sgn} y, \qquad n\ge 2, \quad \sigma>1. \tag{18}$$

Let us show that it has a solution of the form (2). To be definite, we assume $a=1$ in (2). Here and elsewhere, we use the terminology adopted in [7] and [8]. The Newton polyhedron of this equation is the segment
$$[Q_1,Q_2], \qquad Q_1=(-n,1), \quad Q_2=(0,\sigma)$$
the normal to which is the vector $[1,\beta]$, $\beta=n(1-\sigma)^{-1}$. The reduced equation corresponding to this segment coincides with the complete equation (18). Following [7], we find its power-law solution $y(x)=c(-x+1)^{\beta}$, $x<1$. By substitution, we obtain
$$c=\big((-1)^n\beta(\beta-1)\cdots(\beta-n+1)\big)^{1/(\sigma-1)}.$$
Thus,
$$y(x)=c(-x+1)^{\beta}, \qquad x<1,$$
is a solution of Eq. (18) for which estimates (3) become equalities.

**Example 2.** Let us consider another example, more complicated than the previous one. Consider the following equation with variable coefficient:
$$y'''=(1+x^2)y^2. \tag{19}$$

Let us show that this equation has the solution $y(x)=60(-x)^{-3}(1+o(1))$ obviously satisfying estimates (3), where $a=0$.

The Newton polyhedron of this equation is a triangle with vertices $Q_1 = (-3, 1)$, $Q_2 = (0, 2)$, $Q_3 = (2, 2)$. Consider the reduced equation corresponding to the edge $[Q_1, Q_2]$:

$$y''' = y^2.$$

The power-law solution of this equation is of the form $y = -60x^{-3}$. Let us calculate the critical numbers of this solution (see [8, Sec. 1.4]). The first variation is of the form

$$\frac{d^3}{dx^3} - 2y,$$

and the corresponding characteristic equation is

$$k(k-1)(k-2) + 120 = 0.$$

The unique real root $k = -4$ of this equation is not critical, because it is less than the order of the solution of the reduced equation. Therefore, by Theorem 3.4 from [8], the complete equation (19) has a solution of the form of the convergent power series

$$y = -60x^{-3}\left(1 + \sum_{j=1}^{\infty} a_j x^j\right).$$

Obviously, this solution satisfies estimates (3) where $a = 0$.

The fact proved above can also be justified by using other considerations (such as those used in [9]). To do this, let us make, in (19), the transformation $y = (c+z)(-x)^{-3}$, $t = \ln(-x)$, $x < 0$, where $c$ is a constant to be defined later. We obtain the equation

$$z'''_t - 12z''_t + 47z'_t - 60(c+z) = -(c+z)^2(1 + e^{2t}).$$

Setting $c = 60$ and denoting

$$z_t^{(i)} = u_{i+1}, \quad 0 \le i \le 2, \qquad u = (u_1, u_2, u_3), \quad \|u\| = \sqrt{\sum_{1 \le j \le 3} u_j^2},$$

we obtain the following system of equations for the function $u(t)$ in a small neighborhood of zero:

$$\dot{u} = Au + F(u)f(t) + g(t), \qquad \|f(t)\| + \|g(t)\| \le D_1 e^{2t},$$
$$F(u) \in C^\infty, \quad F(0) = 0, \quad D_1 = \text{const} > 0.$$

For a small $\delta > 0$, this system of equations has the solution $u = u(t)$, $\|u(t)\| = o(e^{\delta t})$ as $t \to -\infty$. This is proved by using methods similar to to those used in [9, pp. 101–106]. Hence we find that Eq. (19) has the solution

$$y(x) = 60(-x)^{-3}(1 + o((-x)^\delta)), \qquad x < 0, \quad \delta > 0,$$

satisfying estimates (3).

**Example 3.** Consider the equation

$$y^{(4)} = x^{-3}y^2.$$

Its support is the segment $[Q_1, Q_2]$, $Q_1 = (-4, 1)$, $Q_2 = (-3, 2)$, the normal to which is the vector $n = (1, -1)$. Following [7], we search for the solution of the given equation in the form $y = cx^{-1}$. By substitution, we determine $c = 24$. Thus, this equation has the solution $y(x) = 24x^{-1}$ satisfying estimate (7), which now becomes an equality.

**Example 4.** Consider the equation

$$y^{(4)} = x^{-6}y^2. \tag{20}$$

This equation was studied in Theorem 4 and belongs to the case described in (14) (where $\beta = 2$). The support of the equation is the segment $[Q_1, Q_2]$, $Q_1 = (-4, 1)$, $Q_2 = (-6, 2)$. Let us make the transformation $y = x^2 z$, $t = \ln(x)$, after which Eq. (20) becomes

$$z_t^{(4)} + 2z_t''' - z_t'' - 2z_t' = z^2. \tag{21}$$

To the right edge of the Newton polyhedron of this equation corresponds the reduced equation $-2z_t' = z^2$ whose solution is of the form $z = z_0(t) = 2t^{-1}$. After the replacement $z = w + z_0(t)$, Eq. (21) takes the form

$$w_t^{(4)} + 2w_t''' - w_t'' - 2w_t' = w^2 + 4t^{-1}w + O(t^{-3}).$$

An analysis of this equation shows that it has the solution $w(t) = o(t^{-1-\delta})$, where $\delta > 0$ (no proof will be given here). This implies the existence of the following solution to Eq. (20):

$$y = 2x^2(\ln x)^{-1}(1 + o(\ln x)^{-\delta})$$

satisfying estimate (15).

**Example 5.** Consider the equation

$$y^{(4)} = -x^{-4-\beta}y^2, \qquad 1 < \beta \le 2. \tag{22}$$

This equation was studied in Theorem 4 and belongs to the case described in (16) (where $k = 2$). Consider any right-continuable positive solution $y(x)$ of this equation. It follows from (22) that $y'''(x)$ is a decreasing function. Its limit as $x \to +\infty$ cannot be negative. It also cannot be positive, because, in this case, for large values of $x$, this would lead to $y(x) \ge Dx^3$, $D = \text{const} > 0$, and $y^{(4)} \le -D_1$, $D_1 = \text{const} > 0$, which contradicts the assumption about the positive limit of $y'''(x)$. Therefore, this limit must be zero, and the function $y''(x)$ is increasing. Its limit as $x \to +\infty$ cannot also be positive. Indeed, in this case, the function $y(x)$ is increasing and, from (22), after integration on the interval $[x, +\infty)$, we obtain

$$y''' \ge Dx^{-3-\beta}y^2, \qquad D = \text{const} > 0.$$

But it follows from [2, Lemma 1] that, in this case, the solution $y(x)$ cannot be right-continuable. Thus, the limit $y''(x)$ must be zero; therefore, $y''(x) < 0$, and the function $y'(x)$ is decreasing. Hence $y(x) \le D_2 x$, $D_2 = \text{const} > 0$, i.e., the solution under consideration satisfies estimate (17).

The examples given above illustrate the sharpness of the results contained in the theorems stated above.

## 3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.** The proof was given in [2]. □

**Proof of Theorem 2.** Here and elsewhere, when it is necessary to show that a quantity is bounded by a constant, we shall use the so-called "universal" constant $D > 0$, assuming that $D + D = D$, $D^\beta = D$ ($\beta > 0$).

Obviously, it suffices to consider the solution $y(x)$ of Eq. (1) satisfying (2) for which, in some left neighborhood of the point $a$,

$$y^{(i)}(x) > 0, \qquad 0 \le i \le n-1. \tag{23}$$

Let $[x_0, a)$ be an interval in which inequalities (23) hold.

Multiplying both sides of Eq. (1) by $y'(x)$ and integrating it on the interval $[x_0, x)$, $x_0 \le x < a$, we obtain the inequality

$$y^{(n-1)}(x)y'(x) - y^{(n-1)}(x_0)y'(x_0) \ge D(y^{\sigma+1}(x) - y^{\sigma+1}(x_0)),$$

whence, obviously, it follows that, in the left neighborhood of the point $a$, the inequality

$$y^{(n-1)}(x)y'(x) \geq Dy^{\sigma+1}(x)$$

holds. Without loss of generality, we assume that this is so in the interval $[x_0, a)$. Again, let us multiply the resulting inequality by $y'(x)$ and integrate it on this interval. As a result, we obtain the inequality $y^{(n-2)}(y')^2 \geq Dy^{\sigma+2}$ (here and elsewhere, all the functions are taken at the point $x$). Arguing in the same way, we obtain the inequality

$$y' \geq Dy^{(\sigma+n-1)/n}.$$

Integrating the resulting inequality on the interval $[x, a)$ and taking into account (2), we obtain the right-hand side of the required estimate (3).

Let us now present an easy proof of the left-hand side of estimate (3), which is based on arguments from [4].

As a result of the transformation $z(t) = t^{n-1}y(a - 1/t)$, Eq. (1) becomes

$$z^{(n)} = \widetilde{p}(t)|z|^\sigma \operatorname{sgn} z, \qquad \widetilde{p}(t) = t^{-n-(n-1)\sigma-1}p\biggl(a - \frac{1}{t}\biggr). \tag{24}$$

At the same time, the solution $y(x)$ under study becomes the solution $z(t)$ of Eq. (24) defined for all $t \geq t_0$, $t_0 = (a - x_0)^{-1}$. In addition, a straightforward verification shows that $z^{(i)}(t) > 0$, $t \geq t_0$, $0 \leq i \leq n-1$, and $z^{(n-1)}(t) \to +\infty$ as $t \to +\infty$. This implies that, for $t \geq t_0$,

$$z(t) \leq Dt^{n-1}z^{(n-1)}(t). \tag{25}$$

Substituting (25) into Eq. (24) and denoting $u = z^{(n-1)}(t)$, we see that, for $t \geq t_0$, the function $u(t)$ satisfies the inequality $u' \leq Du^\sigma t^{-n-1}$. The integration of this inequality on the interval $[t, +\infty)$, $t \geq t_0$ yields the estimate

$$u(t) = z^{(n-1)}(t) \geq Dt^{n/(\sigma-1)}, \qquad t \geq t_0.$$

Hence, obviously,

$$z(t) \geq Dt^{n/(\sigma-1)+n-1}, \qquad t \geq t_0;$$

this yields the left-hand side of the required estimate (3). Theorem 2 is proved. □

In order to prove Theorem 3, we shall need the following lemmas.

**Lemma 1.** *If condition* (6)*, where* $m \leq 0$*, holds in Eq.* (1)*, then its right-continuable sign-preserving solutions satisfy estimate* (7).

**Lemma 2.** *If condition* (6)*, where* $m > 0$*, holds in Eq.* (1)*, then its right-continuable positive solutions satisfy the estimate*

$$\sum_{k=0}^{n-1}(-1)^{n-k}(y')^{n-k}(yx^{-1})^k \geq Dy^{\sigma+n-1}x^{-m}, \qquad D = \text{const} > 0. \tag{26}$$

Note that Lemma 1 is the assertion of Theorem 3 for $m \leq 0$.

**Proof of Lemma 1.** Let us prove (7) for $m = 0$. Without loss of generality, we shall consider only positive solutions of the form (5) of Eq. (1). Multiplying both sides of Eq. (1) by $(-1)^n y'$, integrating it on the interval $[x, x_1]$, $x_0 \leq x \leq x_1$, and taking into account (6), where $m = 0$, we can write

$$\begin{gathered} F(x_1) - F(x) - (-1)^n \int_x^{x_1} y^{(n-1)}y''\,dx \leq G_1(x_1) - G_1(x), \\ F(x) = (-1)^n y^{(n-1)}y', \qquad G_1(x) = Dy^{\sigma+1}. \end{gathered} \tag{27}$$

Letting $x_1 \to +\infty$ in (27), we obtain the inequality

$$(-1)^n y^{(n-1)} y' \geq D y^{\sigma+1}, \qquad D = \text{const} > 0. \tag{28}$$

Similarly, multiplying both sides of (28) by $y'$ and integrating the resulting inequality, we obtain

$$(-1)^n y^{(n-2)} y'' \geq D y^{\sigma+2}, \qquad D = \text{const} > 0.$$

Proceeding with the argument, we finally obtain the inequality

$$(-1)^n (y')^n \geq D y^{\sigma+n-1}, \qquad D = \text{const} > 0,$$

whose integration on the interval $[x_0, x]$ yields the required estimate (7), which here has the form

$$y(x) \leq D x^{-n/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0 > 0. \tag{29}$$

For $m = 0$, the lemma is proved.

Let us now fix a number $A$, $1 < A < (\sigma+1)/(\sigma-1)$, and define the following numbers:

$$m_0 = 0, \qquad m_k = n(A + \dots + A^k), \quad k = 1, 2, \dots . \tag{30}$$

We shall show that our lemma holds for $0 < -m \leq m_1$. We proceed just as for $m = 0$. Multiplying (1) by $(-1)^n y'$ and integrating, we obtain the inequality

$$\begin{gathered} F(x_1) - F(x) \leq G_2(x_1) - G_2(x), \\ F(x) = (-1)^n y^{(n-1)} y', \qquad G_2(x) = D y^{\sigma+1} x^{-m}. \end{gathered} \tag{31}$$

Now note that if (6) holds for $m < 0$, then it also holds for $m = 0$. Therefore, estimate (29) holds, whence

$$G_2(x) \leq D x^{\beta}, \qquad \beta = \frac{-n(\sigma+1)}{\sigma-1} + m_1 < 0, \quad D = \text{const} > 0, \quad x \geq x_0 > 0.$$

Letting $x_1 \to +\infty$ in (31), we obtain the inequality

$$(-1)^n y^{(n-1)} y' \geq D y^{\sigma+1} x^{-m}, \qquad D = \text{const} > 0. \tag{32}$$

Arguing as above, we finally obtain the inequality

$$(-1)^n (y')^n \geq D y^{\sigma+n-1} x^{-m}, \qquad D = \text{const} > 0, \tag{33}$$

whose integration on the interval $[x_0, x]$ yields the required estimate

$$y(x) \leq D x^{(-n+m)/(\sigma-1)}, \qquad D = \text{const} > 0, \quad x \geq x_0 > 0. \tag{34}$$

Thus, the assertion of Lemma 1 is proved for $0 < -m \leq m_1$.

Let us argue by induction and prove that this lemma holds for any $m \leq 0$. Suppose that the assertion of the lemma holds for $0 < -m \leq m_K$, $K \geq 1$; let us prove it for $m_K < -m \leq m_{K+1}$.

Multiplying (1) by $(-1)^n y'$ and integrating, we obtain inequality (31). If condition (6) holds for $-m > m_K$, then it also holds for $-m = m_K$. Therefore, estimate (34), where $-m = m_K$, is valid; whence

$$\begin{gathered} G_2(x) \leq D x^{\eta}, \qquad \eta = \frac{-(n+m_K)(\sigma+1)}{\sigma-1} + m_{K+1} < 0, \\ D = \text{const} > 0, \qquad x \geq x_0 > 0. \end{gathered}$$

Now, letting $x_1 \to +\infty$ in (31), we obtain inequality (32). Arguing as above, we obtain inequality (33) and, further, also estimate (34) for $m_K < -m \leq m_{K+1}$. This concludes the proof of Lemma 1. □

**Proof of Lemma 2.** Multiplying both sides of Eq. (1) by $(-1)^n y'$, integrating the resulting equation on the interval $[x, x_1]$, $x_0 \leq x \leq x_1$, and taking into account (6), where $m > 0$, we obtain

$$\begin{gathered} F_1(x_1) - F_1(x) \leq G_1(x_1) - G_1(x) + D \int_x^{x_1} y^{\sigma+1} x^{-m-1}\, dx, \\ F_1(x) = (-1)^n y^{(n-1)} y', \qquad G_1(x) = D y^{\sigma+1} x^{-m}, \qquad D = \text{const} > 0. \end{gathered} \tag{35}$$

Here have taken into account the inequality

$$(-1)^n \int_x^{x_1} y^{(n-1)} y''\, dx < 0.$$

Let us now multiply both sides of Eq. (1) by $yx^{-1}$ and integrate it on the same interval $[x, x_1]$, obtaining

$$\int_x^{x_1} y^{\sigma+1} x^{-m-1}\, dx \le D(H_1(x) - H_1(x_1)), \qquad H_1(x) = (-1)^{n-1} y^{(n-1)} y x^{-1}. \tag{36}$$

Here we have used the inequality

$$(-1)^n \int_x^{x_1} y^{(n-1)} (yx^{-1})'\, dx > 0.$$

Substituting (36) into (35) and letting $x_1 \to +\infty$, we obtain the inequality

$$F_1(x) + H_1(x) \ge D y^{\sigma+1} x^{-m}.$$

Arguing by induction in a similar way, it is easy to see that the following estimate holds for any $q \in \{0, 1, \dots, n-1\}$:

$$y^{(n-q)} \sum_{k=0}^{q} (-1)^{n-k} (y')^{q-k} (yx^{-1})^k \ge D y^{\sigma+q} x^{-m}. \tag{37}$$

Setting $q = n-1$ in (37), we obtain the required estimate (26). Lemma 2 is proved. □

**Proof of Theorem 3.** Without loss of generality, we shall only consider the positive solutions of Eq. (1).

For $m \le 0$, the assertion of the theorem is proved in Lemma 1. Consider the case $0 < m < n$. Then estimate (26) holds and, therefore, there exists a number $C > 0$ such that, for any $x \ge x_0$, either one of the two inequalities

$$yx^{-1} > C y^{(\sigma+n-1)/n} x^{-m/n}, \tag{38}$$

$$-y' > yx^{-1} \tag{39}$$

holds or both of these inequalities simultaneously hold. Note that if (39) holds, then, in view of (26), obviously, the following inequality also holds:

$$-y' > C y^{(\sigma+n-1)/n} x^{-m/n}. \tag{40}$$

By a linear change of the variable $y$, we can ensure that $C = 1$ in (38) and (40). Below we assume that this condition holds.

Obviously, if there exists a number $\widetilde{x} \ge x_0$ such that, for $x \ge \widetilde{x}$, inequality (38) (or (40)) holds, then estimate (7) is valid. Now consider the case in which there is a sequence of points

$$x_{j+1} > x_j \ge x_0, \qquad j = 1, 2, \dots$$

such that, for $x \in \Omega_{2q+1} = [x_{2q+1}, x_{2q+2})$, estimate (38) holds and, for $x \in \Omega_{2q+2} = [x_{2q+2}, x_{2q+3})$, estimate (39) holds. Here inequalities (38) and (39) become equalities at the points $x_{2q+2}$ and $x_{2q+3}$, respectively. The last condition obviously implies that $\lim_{j\to\infty} x_j = \infty$. For $x \in \Omega_{2q+1}$, estimate (7) obviously holds. Now consider the interval $\Omega_{2q+2}$.

First, let $a = (n-m)(\sigma-1)^{-1} < 1$. If $x \in \Omega_{2q+2}$, then it follows from (39) that

$$y < y_{2q+2} x_{2q+2} x^{-1}. \tag{41}$$

But $y_{2q+2} = (x_{2q+2})^{-a}$. Substituting this expression, where $a = (n-m)(\sigma-1)^{-1} < 1$, into (41), we obtain

$$y < (x_{2q+2})^{1-a} x^{-1} = \left(\frac{x_{2q+2}}{x}\right)^{1-a} x^{-a} \le x^{-a}$$

and, therefore, for $a < 1$, estimate (7) is valid for any $x \in \Omega_{2q+2}$ with $D = 1$.

Now let $a \geq 1$. For $x \in \Omega_{2q+2}$, integrating (40), we obtain

$$y < \left((y_{2q+2})^{(1-\sigma)/n} + \frac{1}{a}(x^{(n-m)/n} - x_{2q+2}^{(n-m)/n})\right)^{n/(1-\sigma)}. \qquad (42)$$

Substituting $y_{2q+2} = (x_{2q+2})^{-a}$ into (42), we obtain the inequality

$$y < \left(\left(1 - \frac{1}{a}\right)x_{2q+2}^{(n-m)/n} + \frac{1}{a}x^{(n-m)/n}\right)^{n/(1-\sigma)} \leq Dx^{(m-n)/(\sigma-1)}, \qquad D = a^{n/(\sigma-1)},$$

and hence estimate (7) is proved for any $x \in \Omega_{2q+2}$ with $D = a^{n/(\sigma-1)}$. Thus, Theorem 3 is proved for all $m < n$.

Now consider the last case in which $m = n$. If, at the point $x$, the inequality $-y' \leq yx^{-1}$ holds, then it follows from (26) that

$$-y' > B_1 y^\sigma x^{-1}, \qquad B_1 = \text{const} > 0. \qquad (43)$$

But if $-y' > yx^{-1}$, then, from (26), we obtain

$$-y' > B_2 y^{(\sigma+n-1)/n} x^{-1}, \qquad B_2 = \text{const} > 0. \qquad (44)$$

Here the numbers $B_1$, $B_2$ depend only on $n$ and the constant $D$ from (26). Without loss of generality, we can assume that $B_1 = B_2$ in (43) and (44). Let $x_1$ be such that, for $x \geq x_1$, we have $y(x) < 1$. Then (44) implies (43). Thus, for $x \geq x_1$, the solution $y(x)$ satisfies inequality (43) whose integration yields estimate (7). Theorem 3 is proved. □

**Proof of Theorem 4.** Under condition (10), estimate (11) was proved in Theorem 3. Now let $0 < \beta \leq n-1$. Let us find an integer $k \in \{1, \dots, n-1\}$ such that $k - 1 < \beta \leq k$. Note that, for large values of $x$, the functions $y^{(j)}(x)$, $j \in \{0, \dots, n-1\}$, are monotone. Let us prove that, as $x \to +\infty$, all the derivatives $y^{(j)}$, $k \leq j \leq n-1$, tend to zero. Let $j = n-1$ and

$$\lim_{x \to +\infty} y^{(n-1)}(x) = D, \qquad 0 < D \leq +\infty$$

(here and elsewhere we use the "universal" constant $D > 0$). If $p(x) > 0$, then this contradicts Lemma 1 from [2]. In the case $p(x) < 0$, for large values of $x$, we have

$$y \geq Dx^{n-1} \qquad \text{and} \qquad y^{(n)} \leq -Dx^{(n-1)\sigma - n - \beta(\sigma-1)};$$

hence we obtain $\lim_{x \to +\infty} y^{(n-1)}(x) = -\infty$, which cannot be true. Now let, as $x \to +\infty$,

$$y^{(m)} \to C, \quad 0 < C \leq +\infty, \qquad y^{(j)} \to 0, \quad 1 \leq k \leq m < j \leq n-1.$$

Then, for large values of $x$, the function $y(x)$ increases and, successively integrating Eq. (1) on the interval $[x, +\infty)$, we obtain by induction

$$|y^{(j)}| \geq D y^\sigma x^{-j-\beta(\sigma-1)}, \qquad j \in \{m+1, \dots, n-1\}.$$

If $y^{(m+1)}(x) > 0$, then this contradicts Lemma 1 from [2]. But if $y^{(m+1)}(x) < 0$, then, taking into account the fact that, for large values of $x$,

$$y \geq Dx^m \qquad \text{and} \qquad y^{(m+1)} \leq -Dx^{m\sigma - m - 1 - \beta(\sigma-1)},$$

we obtain $\lim_{x \to +\infty} y^{(m)}(x) = -\infty$, which cannot be true.

Thus, we have proved that, as $x \to +\infty$, all the derivatives $y^{(j)}$, $k \leq j \leq n-1$, tend to zero. In addition, for large values of $x$ the following condition holds:

$$y^{(j+1)}(x) y^{(j)}(x) < 0, \qquad k \leq j \leq n-1. \qquad (45)$$

It follows from this that $(-1)^{n-k} \operatorname{sgn} p(x) y^{(k)}(x) > 0$.

If condition (16) holds, then $y^{(k)}(x) < 0$, i.e., the function $y^{(k-1)}(x)$ will decrease, which obviously implies (17).

Now let conditions (12) or (14) hold. Then the inequality $y^{(k)}(x) > 0$ is valid. Further, if $\lim_{x\to+\infty} y^{(k-1)}(x) < +\infty$, then $y(x) \le Dx^{k-1}$, and the assertion of the theorem holds (in view of the inequalities $0 \le k-1 < \beta \le k$).

Now let $\lim_{x\to+\infty} y^{(k-1)}(x) = +\infty$. Then, for large values of $x$, all the functions $y^{(j)}(x)$, $0 \le j \le k-1$ will be positive. Below we assume that these conditions hold for $x \ge x_0$. In addition, $y^{(k)}(x)$ is a decreasing function tending to zero as $x \to +\infty$.

For large values of $x$, let us now show that the following estimate holds:

$$y^{(j)}(x) \ge \frac{xy^{(j+1)}(x)}{D}, \qquad 0 \le j \le k-1. \tag{46}$$

Obviously, for large values of $x$, in view of the decrease of the function $y^{(k)}(x)$ the following inequality holds:

$$y^{(k-1)}(x) = y^{(k-1)}(x_0) + \int_{x_0}^{x} y^{(k)}(t)\,dt \ge xy^{(k)}(x).$$

Thus, estimate (46) holds for $j = k-1$, $D = 1$.

Now assume that (46) holds for $0 < m \le j \le k-1$, $x \ge x_1 > 0$. Let us prove that this estimate also holds for $j = m-1$.

Let us show that, for some $D_1 > 0$, the following two inequalities hold:

$$y^{(m-1)}(x_1) > \frac{x_1 y^{(m)}(x_1)}{D_1}, \tag{47}$$

$$y^{(m)}(x) \ge \frac{y^{m}(x)}{D_1} + \frac{xy^{(m+1)}(x)}{D_1}, \qquad x \ge x_1. \tag{48}$$

The first of these inequalities is satisfied if

$$D_1 > x_1 y^{(m)}(x_1)(y^{(m-1)}(x_1))^{-1}.$$

The second inequality follows from (46) for $j = m$ and $D_1 \ge D + 1$.

From (47) and (48), it is easy to obtain the following estimate:

$$y^{(m-1)}(x) = y^{(m-1)}(x_1) + \int_{x_1}^{x} y^{(m)}(t)\,dt \ge \frac{xy^{(m)}(x)}{D_1}. \tag{49}$$

Indeed, it follows from (47) and (48) that the difference of the functions on the left-hand and right-hand sides of inequality (49) is positive at the point $x_1$, while the difference of their derivatives is nonnegative for $x \ge x_1$. It follows from inequality (49) that estimate (46) holds for $j = m-1$. Thus, estimate (46) is proved for all $0 \le j \le k-1$ and $x \ge x_1 > 0$. Hence we see that

$$y(x) \ge D_2 x^k y^{(k)}(x), \qquad D_2 = \mathrm{const} > 0. \tag{50}$$

Substituting inequality (50) into (1), denoting $z = y^{(k)}(x)$, and taking into account the inequality $(-1)^{n-k}p(x) > 0$, we obtain the equation

$$(-1)^{n-k} z^{(n-k)} = z^{\sigma} p_1(x),$$
$$p_1(x) \ge D_3 x^{-q}, \qquad q = n-k+(\beta-k)(\sigma-1),$$
$$D_3 = \mathrm{const} > 0, \qquad x \ge x_1 > 0.$$

Applying Theorem 3 to this equation, we obtain either

$$z = y^{(k)}(x) \le D_4 x^{\beta-k}, \qquad D_4 = \mathrm{const} > 0$$

if $k-1 < \beta < k$ or

$$z = y^{(k)}(x) \le D_4 (\ln x)^{-1/(\sigma-1)}, \qquad D_4 = \mathrm{const} > 0$$

if $\beta = k$. Integrating these inequalities, we obtain, respectively, estimates (13) and (15).

Theorem 4 is proved. □

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