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# Rogue waves in the basin of intermediate depth and the possibility of their formation due to the modulational instability

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Properties of rogue waves in the basin of intermediate depth are discussed in comparison with known properties of rogue waves in deep waters. Based on observations of rogue waves in the ocean of intermediate depth we demonstrate that the modulational instability can still play a significant role in their formation for basins of 20 m and larger depth. For basins of smaller depth, the influence of modulational instability is less probable. By using the rational solutions of the nonlinear Schrödinger equation (breathers), it is shown that the rogue wave packet becomes wider and contains more individual waves in intermediate rather than in deep waters, which is also confirmed by observations.

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Abnormally high waves unexpectedly appearing for a short time at the sea surface (rogue waves) represent the hot topic in science in the last two decades. The physical mechanisms of their formation are described in [1–6] and can be highlighted as: i) modulational instability (Benjamin–Feir instability), ii) geometrical and dispersive focusing, iii) wave-current and wave-bottom interaction, and iv) wind action. Initially known for water waves, later on rogue waves were discovered in different media; their numerous examples are given in the special issue of EPJ [7]. At the same time, in the last years the effect of modulational instability is considered to be the main mechanism of rogue wave formation in all media, and the nonlinear Schrödinger equation (NLS) and its generalizations – to be the basic model. Its rigorous solutions, so called Peregrine–Kuznetsov–Ma–Akhmediev breathers possess all rogue wave properties, such as sudden appearance and disappearance in the homogeneous wind wave field, abnormal amplification in 2–3 times within a short time interval [8–10]. Due to the integrability of the nonlinear Schrödinger equation it was possible to find solutions for super rogue waves, whose amplitude could be 10 times larger than of background waves [11, 12]. These waves were observed under laboratory conditions in wave flumes [12–18], in nonlinear fibres [19] and in multicomponent plasma [20]. It has been shown that modulational instability can result in rogue wave formation in random wave fields [21]. There have also been attempts to describe the formation of observed rogue waves in the deep sea by the nonlinear Schrödinger equation [15, 22, 23].

In the last years several catalogues and collections of observed rogue waves in the ocean have been created [24–28]. These collections include descriptions of rogue waves observed in different regions of the World Ocean both in its deep parts, in the coastal zone and at the coast [3, 26, 27]. Closer to the coast, the effect of modulational instability in the field of unidirectional waves vanishes [3, 29]. Therefore, in these regions, other mechanisms of rogue wave formation should dominate.

However, since measurements of waves are usually conducted in the point (time series record), which does not give enough information to the existing analytical models, the latter cannot describe the mechanism of each particular observed rogue wave. As the result, theory and measurements go "in parallel". In this regard the role of additional observation and, in particular, eye-witness reports starts to be very important, since it gives valuable information, which is needed to define the mechanism of the observed event. Thus, the formation of rogue waves off the south-western coast of Africa was explained as a result of wave interaction with the

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strong Agulhas current [30]. Rogue waves recorded in the Caribbean Sea in October 2005 and in the Mediterranean Sea (Louis Majesty accident) in March 2010 are associated with the abrupt change in wind direction and wave interaction with the swell [31, 32]. The model used by [32] is based on a coupled system of nonlinear Schrödinger equations, which allows modulational instability. There is no doubt that the criterion of modulation instability is fulfilled in the open sea. However, according to the recently published catalogue of rogue waves [26, 27], rogue waves are encountered in any part of the ocean, both in deep/shallow waters and along the coast. It is therefore interesting to estimate the number of rogue waves associated with modulational instability at different water depths.

Here, based on the data by [26, 27] at different water depths we try to find indirect proofs of feasibility of modulation instability mechanism. For this we analyze the events reported in [26, 27] and check if they meet the criterion of modulation instability. The considered data contain 22 rogue events, which occurred in 2006–2010 at the water depth less than 50 m. The reported rogue waves are satisfied to the amplitude criterion, according to which their heights are at least twice larger than the significant wave height $H_s$ (average of 1/3 of the highest waves). Wave periods are estimated from altimeter data following the algorithm suggested by [33].

The known criterion for modulational instability in the field of unidirectional water wave is

$$kh > 1.363, \tag{1}$$

where $h$ is the water depth and $k$ is the carrier wave number [29, 35, 39, 40]. In practice, the wave frequency $\omega$ is often known and the carrier wave number $k$ can be found from the linear dispersion relation

$$\omega = \sqrt{gk\tanh(kh)}. \tag{2}$$

Using the approximation formula for wave number (accuracy 1%) [34]

$$\begin{gathered} k^2 = \frac{\omega^2}{ghG(\alpha)} + \frac{\omega^4}{g^2}, \\ G = 1 + 0.6522\alpha + 0.4622\alpha^2 + 0.0864\alpha^4 + 0.0675\alpha^5, \\ \alpha = \omega^2 h/g, \end{gathered} \tag{3}$$

the modulational instability criterion can be expressed through measured characteristics $T$ and $h$

$$T < \sqrt{\frac{4\pi^2 h}{a_0 g}}, \tag{4}$$

where $a_0 \approx 1.195$ is found from

$$\sqrt{\frac{\alpha}{G(\alpha) + \alpha^2}} = 1.363. \tag{5}$$

Fig. 1 demonstrates the wave periods and water depths of 22 rogue waves occurred in 2006–2010 and

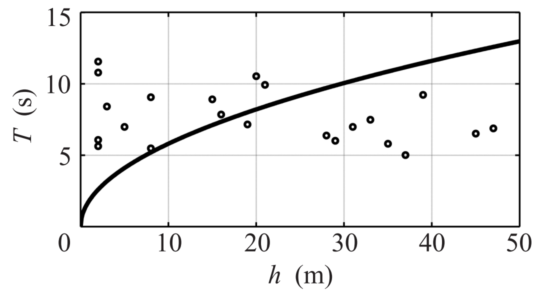


Fig. 1. Rogue wave period plotted against the water depth of their occurrence; black solid line corresponds to Eq. (4)

dashed line corresponds to Eq. (4). All data-points above the dashed line correspond to the modulationally stable waves and all data-points below the dashed line correspond to modulationally instable; reported rogue waves are distributed almost equally among them. At the same time, the possible definition of the critical water depth of about 20 m separating waves in deep and shallow waters based on the modulational instability criterion Eq. (1) also follows from Fig. 1. As known, the actual border between deep and shallow water depends on wind wave parameters and may be at different water depth. However, our analysis demonstrates that for all considered observed rogue events it only slightly deviates from 20 m. An increase in the rogue wave data could help to make this value more precise.

Fig. 2, where the dependency of the parameter $kh$ on the water depth is plotted, is even more indicative. The

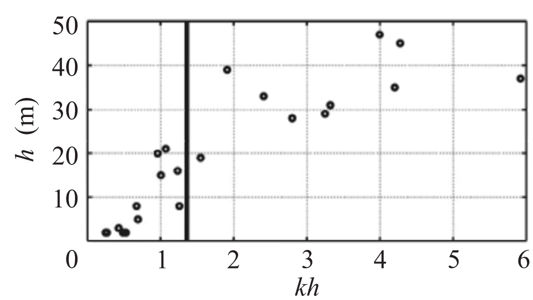


Fig. 2. Division of rogue events by the modulational instability criterion with respect to the water depth

black solid line here corresponds to the critical value of modulational instability criterion. It can be seen that waves are modulationally stable at water depths below 20 m; however, even for larger water depths (such as 40 m) the values of $kh$ are relatively small and are close to the critical value. At the same time, it is hard to find a similar criterion for the wave period, though most of waves with periods larger than 10 s (so called swell) are modulationally stable at water depths below 50 m. The mechanisms of formation of such modulationally stable long rogue waves are described in [5].

In addition, based on the available datacollection [26], the number of observed rogue waves in the wave packet can also be connected to the parameter $kh$. In most of cases (16 cases out of 22) only single rogue wave is observed for both deep and shallow water values of $kh$. Practically, all rogue wave groups (5 cases), a phenomenon known as "The Three Sisters", are observed in shallow water, which can be connected to the diminishing role of the dispersion, which leads to the enlargement of the wave packet as the whole.

The dynamics of weakly-nonlinear wave packets in fluid of an arbitrary depth is described by NLS equation [29, 35, 39, 40]

$$i\frac{\partial A}{\partial t} + \mu\frac{\partial^2 A}{\partial x^2} + \gamma|A|^2 A = 0, \tag{6}$$

where $A$ is a complex wave amplitude, and $\mu$ and $\gamma$ are coefficients of dispersion and nonlinearity respectively, and they have a meaning of the coefficients of the Taylor series of the nonlinear dispersion relation [35]. Note, that a variable-coefficient NLS equation can also be derived for a variable depth [36]. It is important that the wave frequency (wave period) does not change during wave propagation from deep water to shallow, while wave number changes according to the dispersion relation Eq. (2). That is why we base our analysis on wave frequency and corresponding parameter $\alpha$ (Eq. (3)) instead of $kh$. So, for a fixed central frequency $\omega$, both coefficients can be presented as a product of the corresponding coefficient of NLS equation in deep waters ($\mu_\infty$ and $\gamma_\infty$) and the correction, related to the finiteness of the water depth ($M$ and $G$):

$$\mu = \mu_\infty M(kh), \quad \mu_\infty = \frac{1}{2}\frac{\partial^2\omega}{\partial k^2} = -\frac{1}{8}\frac{g^2}{\omega^3}, \tag{7}$$

$$M = [\sigma - kh(1-\sigma^2)]^2 + 4k^2h^2\sigma^2(1-\sigma^2), \tag{8}$$

$$\gamma = \gamma_\infty G(kh), \quad \gamma_\infty = -\frac{\omega k^2}{2} = -\frac{\omega^5}{2g^2}, \tag{9}$$

$$G = \frac{1}{4\sigma^4}\left\{\frac{1}{c_{gr}^2 - gh}[4c_{ph}^2 + 4c_{ph}c_{gr}(1-\sigma^2) + \right.$$
$$\left. + gh(1-\sigma^2)^2] + \frac{1}{2\sigma^2}(9 - 10\sigma^2 + 9\sigma^4)\right\}, \tag{10}$$

here $\sigma = \tanh(kh)$, $c_{ph}$ and $c_{gr}$ are the phase and group velocities of the linear surface gravity waves for $\omega$ and $k = k(\omega)$, and $x$ is the running coordinate moving with the group velocity. The behavior of corrections $M$ and $G$ with respect to the parameter $\alpha$, defined in Eq. (3), is demonstrated in Fig. 3. The dispersion correction

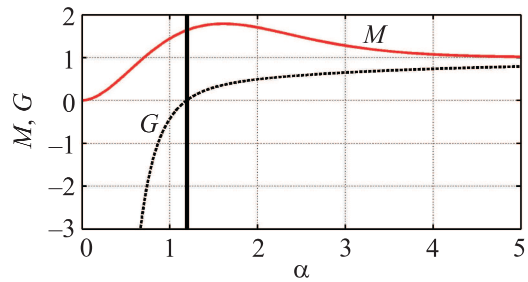


Fig. 3. Corrections to the coeffcient of dispersion $M$ and nonlinearity $G$ with respect to the parameter $\alpha$ (Eq. (3)), $\alpha \approx 1.195$ corresponds to the modulational instability limit

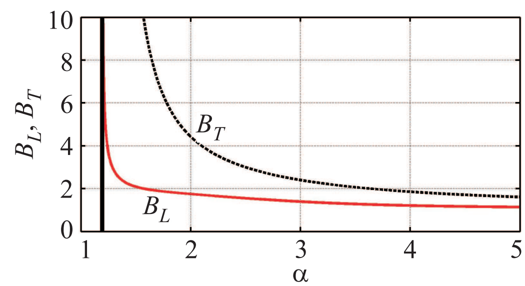


Fig. 4. Coefficients $B_L$ and $B_T$ versus parameter $\alpha$; $\alpha \approx 1.195$ corresponds to the modulational instability limit

changes nonmonotonically reaching its maximum value of 1.787 for $kh = 1.718$. It is equal to 1 in deep waters and is zero in shallow waters. The correction to nonlinearity coefficient changes monotonically and tends to 1 in deep waters and to minus infinity in shallow water. Note, that the wave becomes more linear while approaching the modulational instability limit: its nonlinearity decreses and its dispersion is still high and close to its maximum value. The dispersion coefficient $\mu$ is always negative, while $\sigma$ changes its sign from negative to positive passing though the critical value of $kh$ [35, 39, 40].

For the modulational instability regime (focused regime), Eq. (6) can be reduced to its canonical form

$$i\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial y^2} + 2|u|^2 u = 0, \tag{11}$$

where $y = kx$, $u = A\sqrt{\gamma/2\mu k^2}$, $\tau = -\mu k^2 t$.

The existence of rogue waves within NLS Eq. (11) have been intensively studied [3, 7, 37]. Here, the major role in demonstration of the rogue wave formation

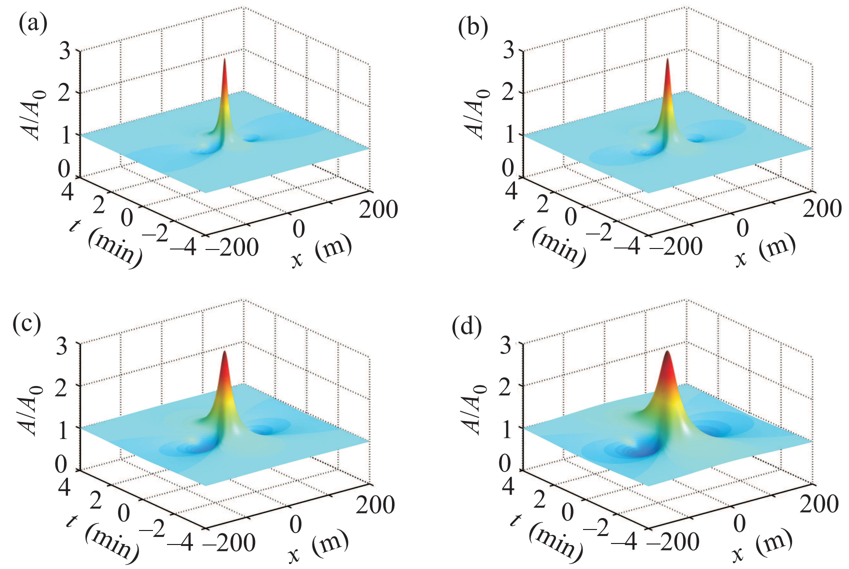


Fig. 5. The Peregrine breather at the background of the carrier wave with the period 6 s and amplitude $A_0 = 3$ m for (a) $\alpha = \infty$ ($kh = \infty$); (b) $\alpha = 5$ ($kh = 5$); (c) $\alpha = 1.93$ ($kh = 2$); (d) $\alpha = 1.48$ ($kh = 1.6$)

plays the family of rational or multi-rational solutions (breathers), which allow different shapes of rogue waves and significant rogue wave amplification against the background waves. One of the most famous prototypes of the rogue wave is the Peregrine breather (here expressed in original variables)

$$A(x,t) = A_0\left[-1 + \right.$$
$$\left. + \frac{4 - 8i\gamma A_0^2 t}{1 + 2\gamma A_0^2 x^2/\mu + 4\gamma^2 A_0^4 t^2}\right] \exp(-i\gamma A_0^2 t). \qquad (12)$$

Important breather characteristics, the number of waves in space $n_L$ and the number of waves in time $n_T$, can be presented as

$$n_L \sim kL \sim \frac{B_L}{\varepsilon}, \quad n_T \sim \omega T \sim \frac{B_T}{\varepsilon^2}, \quad \varepsilon = kA, \qquad (13)$$

where $\varepsilon$ is the wave steepness and coefficients $B_L$ and $B_T$ represent corrections for the finiteness of the water depth

$$B_L = \frac{1}{\sigma^2}\sqrt{\frac{M}{G}}, \quad B_T = \frac{1}{G^2\sigma^2}. \qquad (14)$$

Both coefficients are equal to 1 in the deep water and tend to infinity while approaching the limit of modulational instability (Eq. (5)) with an especially intensive increase in the duration of the breather (Fig. 4). It follows from Eq. (13) that even in deep waters the number of waves in time is larger than in space ($\varepsilon < 1$). However, the shallow water makes this difference even stronger. It is demonstrated in Fig. 5 for deep water ($\alpha = \infty$, $kh = \infty$) and for intermediate water ($\alpha = 5$, 1.93, 1.48 and the corresponding $kh = 5$, 2, and 1.6) conditions. The last value of $kh$ corresponds to the breather for the conditions of the famous 26-meter Draupner wave at the water depth of 70 m, whose record has been shown and analyzed in many works (see, for example [3, 6, 38]). The breather for the Draupner wave conditions (rather probable for offshore constructions) is wider than the one in deeper waters (3.5 times in duration and 2.5 times in length with respect to the level $A/A_0 = 2$), which should be accounted in prognostic simulations of extreme waves. Note that for a fixed wave period and amplitude, the number of individual waves within breather increases with a decrease in $kh$. As a result, the rogue event in the shallow water region contains more hazardous waves rather than in deep water region, which has been confirmed by the observations.

The main conclusions from this analysis are straightforward: i) it is shown that the modulational instability can still play an important role in rogue wave formation in intermediate waters up to 20 m water depth; and ii) the rogue wave packet in the basin of intermediate depth becomes wider and contains a larger number of individual waves rather than in deep water, which is also supported by observations. The same conclusions should be also valid for super rogue waves considered in [12] for deep water conditions.

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