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A 0-1 knapsack model for evaluating the possible Electoral College performance in two-party US presidential elections

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Abstract

The minimal fractions of the popular vote that could have elected a US President in the Electoral College in two-party elections in 1948–2004 are calculated by solving auxiliary knapsack problems. It is shown that under the rules of US presidential elections determined by Article 2 of the US Constitution, the values of these minimal fractions were within the range 16.072%–22.103%.

Keywords: Electoral College; Electoral vote; Integer linear programming; Knapsack problem; Popular vote

1. Introduction

In 1961, G. Polya proposed an elegant arithmetic approach for approximately calculating the minimal fraction of the popular vote that can elect a US President in the Electoral College [1] if (the electors of) only two presidential candidates receive all the cast votes. Though G. Polya's approach was mostly aimed at illustrating how a mathematical teacher can discuss this problem in a classroom at school, a part of his reasoning contains observations that are key to exactly solving this problem. At the same time, G. Polya's approach is based on assumptions that do not usually hold in US presidential elections [2]. Therefore, his calculations can give only a general idea about the size of the fraction under consideration, and, under his (unrealistic) assumptions, this fraction is approximately 22.08%.

In 1990, A. Barnett proposed a simple approach for calculating this fraction [3]. The idea of A. Barnett's approach is, in fact, identical to that proposed by G. Dantzig for solving the continuous 0-1 knapsack problem [4]. It consists of finding the fraction under consideration as a result of calculating the "prices per electoral vote" in all the states and in the District of Columbia (DC) and ordering these "prices" from the smallest to the largest. (For a state, this "price" is the ratio of the number of cast votes securing the winning of all the electoral votes in the state to the number of the electoral votes controlled by this state; the "price" for DC has the same meaning.) A pivotal state (or DC) in the ordered list of the states and DC determines a set of states controlling a majority of all the electoral votes (that are in play in the election) combined with sequentially non-decreasing "prices" (in each of these states, a bare majority of votes received by (the electors of) a US presidential candidate give all the electoral votes controlled by the state to this

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candidate). As mentioned in [3], this approach, generally, yields only an approximate solution to the problem under consideration.

In 2005, an approach for finding an exact solution to the problem under consideration in a general case, by reducing this problem to integer programming problems, was proposed by the author in [2]. Though this approach allows one to solve the problem for a multi-candidate race, for obvious reasons, the case in which (the electors of) only two major party candidates receive all the cast votes has remained the most interesting one for both scholars in the field and US voters.

The aim of this article is to show that if (the electors of) only two presidential candidates receive all the votes cast in a US presidential election, then

(a) under the "winner-take-all" principle of awarding electoral votes (currently exercised in 48 states and DC), and

(b) assuming that all the electoral votes are won by only one candidate in the state of Maine and by only one candidate in the state of Nebraska (as it has so far been in these only two states that do not follow the "winner-take-all" principle),

the minimal fraction of the popular vote to elect a US President in the Electoral College can be found by solving an auxiliary knapsack problem.

As shown in the article, under these three assumptions – which are assumed to hold throughout the rest of the article – and under the distributions of votes cast for state electors (and for those from DC since 1964) among the states (and among the states and DC since 1964) as they were in US presidential elections since 1948 through 2004, the fraction under consideration assumed values falling into the range 16.072%–22.103% in these elections.

To simplify the reasoning to follow, it is assumed that all the votes cast are those received by (the electors of) two presidential candidates, and a terminology from [2] is employed. Namely, states and the District of Columbia are called places, meaning parts of the country eligible to award electoral votes. One should bear in mind that the Union consisted of 48 states in 1948–1959, and 531 electoral votes were in play in the elections held during these years. Alaska and Hawaii joined the Union in 1959, and for the 1960 US presidential election, the number of Representatives in the House of Representatives was made equal to 437 so that the number of all the appointed electors in that election was equal to 537. (The number of Representatives in the House of Representatives has been 435 since 1912, with this exception in 1960, which was made to let the two newly admitted states cast electoral votes in the 1960 US presidential election.) In 1963, the District of Columbia was given three electoral votes so that since the 1964 US presidential election, the (maximal) number of the electoral votes in play has been 538, which has also been the actual number of all the appointed electors in each election since 1964 [5].

2. Basic result

Let us consider a US presidential election in which n places award electoral votes. It is obvious that the inequalities $48 \leq n \leq 51$ hold for the elections under consideration (since 1948 through 2004).

Let

$a_i \geq 3$ be the number of electors that place i appointed in the election, $i \in \overline{1, n}$,

b_i be the number of votes cast for the electors in place i, $i \in \overline{1, n}$,

x_i be a Boolean variable, $i \in \overline{1, n}$,

q be the number of electoral votes that are in play in the election,

$mmaj(q)$ be a number equal to the number of electors in the minimal majority of all the appointed electors in the election, and

h_i be a number equal to the minimal majority of votes cast for the electors in place i.

Here, for each state (except for Maine and Nebraska) and for DC, the numbers h_i are calculated according to the formula

$$h_i = \begin{cases} (b_i/2) + 1, & \text{if } b_i \text{ is an even number;} \\ (b_i + 1)/2, & \text{if } b_i \text{ is an odd number.} \end{cases}$$

For the state of Maine since 1969, and for the state of Nebraska since 1991, this formula can be used to calculate the minimal majority of votes in each congressional district only. If all the electoral votes are won by only one candidate in the state of Maine and by only one candidate in the state of Nebraska, the minimal majority of cast votes that the winner must receive in the state of Maine equals

$$h^{M}_{CD1} + h^{M}_{CD2},$$

whereas the minimal majority of cast votes that the winner must receive in the state of Nebraska equals

$$h^{N}_{CD1} + h^{N}_{CD2} + h^{N}_{CD3},$$

where h^{M}_{CDK} is the minimal majority of cast votes in congressional district K of the state of Maine, $K \in \overline{1,2}$, and h^{N}_{CDL} is the minimal majority of cast votes in congressional district L of the state of Nebraska, $L \in \overline{1,3}$.

Further, let us consider the following Boolean programming problem

$$\begin{aligned} &\sum_{i=1}^{n} a_i x_i \geq mmaj(q) \\ &\sum_{i=1}^{n} h_i x_i \to \min \\ &x_i \in \{0,1\}, \quad i \in \overline{1,n}. \end{aligned} \tag{1}$$

If $\{x_i^* = 1,\ i \in I,\ x_i^* = 0,\ i \in \overline{1,n} \setminus \{I\}\}$ is a solution to problem (1), then

$$\left(\sum_{i \in I} h_i\right) \Big/ \left(\sum_{i=1}^{n} b_i\right)$$

is the minimal fraction of the popular vote that could have elected a US President in the Electoral College in the election.

Let us now consider the knapsack problem

$$\begin{aligned} &\sum_{i=1}^{n} a_i x_i \leq q - mmaj(q) \\ &\sum_{i=1}^{n} h_i x_i \to \max \\ &x_i \in \{0,1\}, \quad i \in \overline{1,n}. \end{aligned} \tag{2}$$

Proposition 1. *The set of integers* $\{x_i^* = 1, i \in I_1, x_i^* = 0, i \in I_2, I_1, I_2 \subset \overline{1,n}, I_1 \cup I_2 = \overline{1,n}, I_1 \cap I_2 = \emptyset\}$ *is a solution to problem* (1) *if and only if the set of integers* $\{x_i^* = 0, i \in I_1, x_i^* = 1, i \in I_2\}$ *is a solution to problem* (2).

Proof. 1. From the equality $\sum_{i=1}^{n} a_i = q$ and the equality

$$\sum_{i \in J} a_i = q - \sum_{i \in \overline{1,n} \setminus \{J\}} a_i$$

for any $J \subset \overline{1,n}$, it stems that the set of integers $\{x_i^* = 1, i \in I_1, x_i^* = 0, i \in I_2, I_1, I_2 \subset \overline{1,n}, I_1 \cup I_2 = \overline{1,n}, I_1 \cap I_2 = \emptyset\}$ is a feasible solution to problem (1) if and only if the set of integers $\{x_I^* = 0, i \in I_1, x_i^* = 1, i \in I_2\}$ is a feasible solution to problem (2).

2. Let

$$\sum_{i \in I_1} h_i = \min_{J \in A} \sum_{i \in J} h_i, \tag{3}$$

where

$$A = \left\{ J \subset \overline{1,n} : \sum_{i \in J} a_i x_i \geq mmaj(q) \right\},$$

so that the set of integers $\{x_i^* = 1, i \in I_1, x_i^* = 0, i \in I_2, I_1, I_2 \subset \overline{1,n}, I_1 \cup I_2 = \overline{1,n}, I_1 \cap I_2 = \emptyset\}$ is a solution to problem (1).

If the inequality

$$\sum_{i \in Q \subset \overline{1,n}} h_i > \sum_{i \in I_2} h_i$$

holds for a set $Q \subset \overline{1,n}$ such that the inequality

$$\sum_{i \in Q} a_i x_i \le q - mmaj(q) \tag{4}$$

holds, then for $\overline{1,n} \setminus \{Q\} \in A$, the inequality

$$\sum_{i \in \overline{1,n} \setminus \{Q\}} h_i = \sum_{i=1}^{n} h_i - \sum_{i \in Q \subset \overline{1,n}} h_i < \sum_{i=1}^{n} h_i - \sum_{i \in I_2} h_i = \sum_{i \in I_1} h_i$$

should hold contradictory to equality (3) (since the inclusion $\overline{1,n} \setminus \{Q\} \in A$ holds) so that the inequality

$$\sum_{i \in Q \subset \overline{1,n}} h_i \le \sum_{i \in I_2} h_i,$$

holds for any $Q \subset \overline{1,n}$ such that the inequality (4) holds, and the set of integers $\{x_i^* = 0, i \in I_1, x_i^* = 1, i \in I_2, \}$ is a solution to problem (2).

Analogously, let

$$\sum_{i \in I_2} h_i = \max_{S \in B} \sum_{i \in S} h_i, \tag{5}$$

where

$$B = \left\{ S \subset \overline{1,n} : \sum_{i \in S} a_i x_i \le q - mmaj(q) \right\},$$

so that the set of integers $\{x_i^* = 0, i \in I_1, x_i^* = 1,\ i \in I_2, I_1, I_2 \subset \overline{1,n}, I_1 \cup I_2 = \overline{1,n}, I_1 \cap I_2 = \emptyset\}$ is a solution to problem (2). If the inequality

$$\sum_{i \in H \subset \overline{1,n}} h_i < \sum_{i \in I_1} h_i$$

holds for a set $H \subset \overline{1,n}$ such that the inequality

$$\sum_{i \in H} a_i x_i \ge mmaj(q) \tag{6}$$

holds, then for $\overline{1,n} \setminus \{H\} \in B$, the inequality

$$\sum_{i \in \overline{1,n} \setminus \{H\}} h_i = \sum_{i=1}^{n} h_i - \sum_{i \in H \subset \overline{1,n}} h_i > \sum_{i=1}^{n} h_i - \sum_{i \in I_1} h_i = \sum_{i \in I_2} h_i$$

should hold contradictory to equality (5) (since the inclusion $\overline{1,n} \setminus \{H\} \in B$ holds) so that the inequality

$$\sum_{i \in H \subset \overline{1,n}} h_i \ge \sum_{i \in I_1} h_i,$$

holds for any H such that the inequality (6) holds, and the set of integers $\{x_i^* = 1, i \in I_1, x_i^* = 0, i \in I_2, I_1, I_2 \subset \overline{1,n},\ I_1 \cup I_2 = \overline{1,n}, I_1 \cap I_2 = \emptyset\}$ is a solution to problem (1).

Proposition 1 is proved. □

Corollary. *Let*

$$\sum_{i\in I_1} h_i, \sum_{i\in I_2} h_i,$$

where $I_1, I_2 \subset \overline{1,n}$, $I_1 \cup I_2 = \overline{1,n}$, $I_1 \cap I_2 = \emptyset$, be the values of problems (1) and (2), respectively. Then the equality

$$\sum_{i\in I_1} h_i = \sum_{i=1}^{n} h_i - \sum_{i\in I_2} h_i$$

holds so that solving problem (1) is reducible to solving problem (2).

Proof. Let $h = (h_1, \ldots, h_n) \in R_+^n$, $e = (1, \ldots, 1) \in R_+^n$, and

$$M_1 = \left\{ x \in R_+^n : x_i \in \{0,1\},\ i \in \overline{1,n},\ \sum_{i=1}^{n} a_i x_i \geq mmaj(q) \right\},$$

$$M_2 = \left\{ y \in R_+^n : y_i \in \{0,1\},\ i \in \overline{1,n},\ \sum_{i=1}^{n} a_i y_i \leq q - mmaj(q) \right\}.$$

Since

$$\sum_{i\in I_1} h_i = \min_{x\in M_1} \langle h, x\rangle = \min_{x\in M_1} (\langle h, e\rangle - \langle h, y(x)\rangle),$$

where $y(x) = e - x$, the equalities

$$\sum_{i\in I_1} h_i = \min_{x\in M_1} \langle h, x\rangle = \langle h, e\rangle - \max_{x\in M_1} \langle h, y(x)\rangle = \sum_{i=1}^{n} h_i - \max_{y\in M_2} \langle h, y\rangle = \sum_{i=1}^{n} h_i - \sum_{i\in I_2} h_i$$

hold, since the mapping $y : M_1 \to M_2$, $y(x) = e - x$ establishes an isomorphism between the sets M_1 and M_2 so that finding the value of problem (1) is reducible to finding the value of problem (2). Corollary is proved. □

3. Calculations

The calculations of the minimal fractions of the popular vote to win the US Presidency in the Electoral College, assuming that all the cast votes were received by (the electors of) only two presidential candidates, were made for the last fifteen US presidential elections, held in 1948–2004. The following table illustrates the results obtained with the use of the code [6], which implements an algorithm proposed in [7] for solving the 0-1 knapsack problem [8]. David Leip's Atlas of US presidential elections [5] was a source for the numbers of votes cast in the states in 1948–2004.

	Contenders	q	$mmaj(q)$	fraction
1948	Dewey–Truman	531	266	16.072%
1952	Eisenh.–Steven.	531	266	17.547%
1956	Eisenh.–Steven.	531	266	17.455%
1960	Kennedy–Nixon	537	269	17.544%
1964	Johns.–Goldwat.	538	270	18.875%
1968	Nixon–Humphry	538	270	19.97%
1972	Nixon–McGov.	538	270	20.101%
1976	Carter–Ford	538	270	21.202%
1980	Reagan–Carter	538	270	21.348%
1984	Reag.–Mond.	538	270	21.53%
1988	Bush–Dukakis	538	270	21.506%
1992	Clinton–Bush	538	270	21.944%
1996	Clinton–Dole	538	270	22.103%
2000	Bush–Gore	538	270	21.107%
2004	Bush–Kerry	538	270	21.666%

It is important to notice that not all the places whose votes contributed to the formation of the minimal fractions of the popular vote under consideration were the same in optimal solutions to corresponding problems (1) in US presidential elections held in 1964–2004, when the number of all the electoral votes in play was 538. The states of California, Colorado, Connecticut, Florida, Georgia, Illinois, Iowa, Kansas, Louisiana, Maryland, Massachusetts, Minnesota, Missouri, New York, North Carolina, Oregon, Washington, and Wisconsin were off and on the list of the states whose votes contributed to the formation of the minimal fractions of the nationwide popular vote to elect a US President in the Electoral College in these years.

For instance, in the 1984 and 1988 elections – in which the minimal fractions of the popular vote that could have elected a US President in the Electoral College were almost the same – these fractions were formed by votes cast in the sets of states and DC consisting of 35 states and DC (set S_{1984}) and of 36 states and DC (set S_{1988}), respectively. Both sets included the states AK, AL, AR, AZ, CO, DE, GA, HI, IA, ID, KS, KY, LO, MD, ME, MS, MT, NC, ND, NE, NH, NV, OK, OR, RD, SC, SD, TN, UT, VI, VT, WV, WY and DC, with 222 electoral votes (e.v.) combined, whereas five more states, namely, CT (8 e.v.), IN (12 e.v.), MO (11 e.v.), NY (36 e.v.), and TX (29 e.v.)"competed" to contribute their votes to the minimal fractions of the cast votes. The set S_{1984} included IN and NY (with 48 e.v. combined), whereas the set S_{1988} included CT, MO, and TX (also with 48 e.v. combined).

4. Concluding remarks

1. As one can see from the table, G. Polya's approximate number (22,08%) is close to the numbers in the last eight elections, especially to the fractions under consideration in the 1992 and 1996 US presidential elections.

2. Calculations presented in the table were made under the following assumptions:

(a) in all US presidential elections from 1948 through 2004, all the places (states and DC) appointed all the electors that they were entitled to (which was the case in the actual elections),

(b) in all US presidential elections in the state of Maine since 1969 and in the state of Nebraska since 1991, all the electoral votes were won by only one presidential candidate (which was the case in the actual elections), and

(c) (the electors of) only two presidential candidates from major political parties received votes in the elections in each state (since 1948 through 2004) and in each state and in DC (since 1964 through 2004).

The rationale for assumption (c) in analyzing the problem under consideration is as follows: First, had all the candidates on the ballot in a state been able to receive a comparable number of votes with major party candidates, the minimal fraction under consideration would have decreased dramatically in all the considered elections. Yet, the number of votes received by non-major party candidates nationwide was significant (more than 5%) only in 1948, 1968, 1980, 1992, and 1996 US presidential elections. (The minimal fraction of votes that could have won all the electoral votes in a state or in DC – assuming that more than two presidential candidates received the cast votes – can be easily calculated by interested readers with the use of the formulae proposed in [2].)

Second, if one subtracted votes cast in a state for non-major party candidates from the state tally, this would only slightly change the value of the minimal fraction under consideration in a majority of the fifteen elections under consideration.

Third, in considering the worst-case scenarios of the Electoral College performance, it seems reasonable to focus on those close to real. An election in which eligible voters who decide to cast their votes are equally biased (or equally unbiased) towards only two major party candidates while insignificantly rewarding non-major party candidates on the ballot is certainly one such scenario. This scenario is in line with the widely proliferated viewpoint that under the Electoral College, only two major parties can really compete, and the 2004 election illustrates this viewpoint (though the 1968, 1992, and 1996 elections suggest that this is not necessarily the case). In any case, US presidential elections in which electors of only two candidates receive cast votes represent the best among all the worst-case scenarios of the distribution of votes among all (rather than between only two) presidential candidates in the race from the viewpoint of the number of voters whose votes can elect a US President in the Electoral College.

Finally, in all the considered hypothetical outcomes of US presidential elections held in 1948–2004, it was implicitly assumed that all the electoral votes were awarded according to the popular vote results in the states, in DC, and in congressional districts of the states of Maine and Nebraska. Such an assumption is substantial as it allows one to consider all the votes combined, cast throughout the country in a US presidential election (that can be recognized as legitimate votes) as the nationwide popular vote, which, generally, may not be the case.

Indeed, since in US presidential elections, American voters vote for slates of electors rather than for presidential candidates, the nationwide popular vote should be formed by (legitimate popular) votes in those states and DC that appoint electors according to the popular vote there. If, for instance, the legislature of a state finally decides to appoint its own electors after the voters have cast their votes, but the popular vote results have been contested (which could have happened in Florida in the 2000 US presidential election), the votes cast in the state should not be included in the nationwide popular vote tally, generally, even if the appointed (by the legislature) electors are those in favor of whom a part of the contested state (popular) votes were cast. In this case, *the nationwide popular will* and *the nationwide popular vote* do not coincide, since a part of electors in the Electoral College turn out to be appointed according to the will of the state legislature rather than according to the statewide popular vote—which was contested and was not determined, causing the intervention of the state legislature in the process of appointing the state electors.

As far as the author is aware, a possible difference between the nationwide popular will and the nationwide popular vote has never been discussed in scientific publications and in the media, apparently, because the above-mentioned scenario has never been put to a test. It seems, however, that such a possible difference, which was first referred to in [9], corresponds to the meaning of the nationwide popular vote under the Electoral College (though the nationwide popular vote tally does not have any constitutional grounds, despite the fact that it has been conducted since the 1824 US presidential election [2,5,10]).

At the same time, certain situations in which the popular will and the popular vote in a state may not coincide are well known. The so-called residual votes [11] – for instance, those in which voters mark the name of a particular candidate or a slate of electors of a particular candidate and write in the same name on the ballot, which are usually called "overvotes" – clearly express the will of these voters though this will cannot be reflected in the statewide popular vote tally.

Also, in the past US presidential elections in which voters could vote in favor of individual electors from different slates of electors, the nationwide popular will and the nationwide popular vote could be different. The 1960 US presidential election in Alabama can serve as an example of such a situation. As is known, the popular vote for a candidate in Alabama was considered equal to the largest number of votes received by any of the electors from the candidate's slate of electors, whereas votes cast by voters who did not vote for the electors who received the largest numbers in their slates of electors did not count at all [10].

3. As mentioned in the Introduction and pointed out in [3], the method proposed in [3] for solving problem (1) yields, generally, only an approximate solution to this problem. However, in certain situations, this method yields an exact solution to problem (1).

Proposition 2. *Let a_i, $i \in \overline{1,n}$ be integers such that the equality*

$$\sum_{i=1}^{k} a_i = mmaj(q) \tag{7}$$

holds for $1 < k < n$ and the inequalities

$$\frac{h_1}{a_1} \le \frac{h_2}{a_2} \le \cdots \le \frac{h_k}{a_k}, \tag{8}$$

hold, along with the inequalities

$$\frac{h_j}{a_j} \ge \frac{h_i}{a_i}, \quad i \in \overline{1,k},\ j \in \overline{k+1,n}. \tag{9}$$

Then the set of integers $\{x_i^* = 1, i \in \overline{1,k}, x_j^* = 0, j \in \overline{k+1,n}\}$ is an exact solution to problem (1).

Proof. Notice, first, that there is no number $j \in \overline{k+1,n}$ for which $a_j \ge a_i$, whereas $h_j < h_i$ for some $i \in \overline{1,k}$ in virtue of the obvious inequalities

$$\frac{h_j}{a_j} \le \frac{h_j}{a_i} < \frac{h_i}{a_i},$$

so the inequality $h_j \ge h_i$ should hold for those $j \in \overline{k+1,n}$ for which the inequality $a_j \ge a_i$ and inequalities (9) hold.

Further, let the inequalities

$$a_t + a_s \geq a_i, \quad a_t < a_i, \quad a_s < a_i, \quad i \in \overline{1,k},\ t,s \in \overline{k+1,n}$$

hold, along with the inequalities

$$\frac{h_s}{a_s} \geq \frac{h_i}{a_i}, \quad \frac{h_t}{a_t} \geq \frac{h_i}{a_i}, \quad i \in \overline{1,k}, t,s \in \overline{k+1,n}.$$

Then the inequalities

$$h_s + h_t \geq a_s \frac{h_i}{a_i} + a_t \frac{h_i}{a_i} = (a_s + a_t)\frac{h_i}{a_i} \geq h_i, \quad i \in \overline{1,k},$$

also hold.

Since for those $t \in \overline{k+1,n}$ for which the inequality $a_t < a_i$ holds for some $i \in \overline{1,k}$, equality (7) does not hold (if a_t is substituted for a_i in (7)), substituting any state or any combination of two states from the set with the numbers $\overline{k+1,n}$ for any state in the set of states with the numbers $\overline{1,k}$ (for which constraints (7)–(9) hold) cannot decrease the number of votes that secure the winning of the US Presidency in the Electoral College.

Finally, let $\{s_1,\ldots,s_l\} \subset \overline{k+1,n}$ be such that the inequality

$$a_{s_1} + \cdots + a_{s_l} \geq a_{\mu_1} + \cdots + a_{\mu_m}, \quad \mu_t \in \overline{1,k},\ t \in \overline{1,m}$$

holds, along with the inequalities

$$\frac{h_{s_j}}{a_{s_j}} \geq \frac{h_{\mu_i}}{a_{\mu_i}}, \quad j \in \overline{1,l}, i \in \overline{1,m},$$

and, for the sake of definiteness, let the inequalities

$$\frac{h_{\mu_m}}{a_{\mu_m}} \geq \frac{h_{\mu_{(m-1)}}}{a_{\mu_{(m-1)}}} \geq \cdots \geq \frac{h_{\mu_1}}{a_{\mu_1}}$$

also hold.

Since for any $a,b,c,d,e,f > 0$ such that

$$\frac{a}{b} \geq \frac{c}{d} \geq \frac{e}{f},$$

the inequalities

$$\frac{a}{b} \geq \frac{a+c}{b+d}, \qquad \frac{a}{b} \geq \frac{c}{d} \geq \frac{c+e}{d+f}, \qquad \frac{a}{b} \geq \frac{a+c+e}{b+d+f}$$

hold, the inequalities

$$\frac{h_{\mu_m}}{a_{\mu_m}} \geq \frac{h_{\mu_m} + h_{\mu_{(m-1)}}}{a_{\mu_m} + a_{\mu_{(m-1)}}} \geq \cdots \geq \frac{h_{\mu_m} + \cdots + h_{\mu_1}}{a_{\mu_m} + \cdots + a_{\mu_1}},$$

and, consequently, the inequalities

$$h_{s_1} + \cdots + h_{s_l} \geq (a_{s_1} + \cdots + a_{s_l})\frac{h_{\mu_m}}{a_{\mu_m}} \geq (a_{s_1} + \cdots + a_{s_l})\frac{h_{\mu_m} + \cdots + h_{\mu_1}}{a_{\mu_m} + \cdots + a_{\mu_1}} \geq h_{\mu_m} + \cdots + h_{\mu_1}$$

also hold.

Thus, substituting any combination of states from the set with the numbers $\overline{k+1,n}$ for any combination of states in the set of states with the numbers $\overline{1,k}$ (for which constraints (7)–(9) hold) cannot decrease the number of votes that secure the winning of the US Presidency in the Electoral College either.

This, in turn, means that forming the list of states according to the procedure proposed in [3] yields an exact solution to the problem under consideration if equality (7) and inequalities (8) and (9) hold. Proposition 2 is proved. □

The holding of relations (7)–(9) is sufficient for yielding an exact solution to problem (1) by the procedure proposed in [3] for solving problem (1), and other such sufficient conditions can be developed.

4. Let $a_1, a_2, \ldots, a_k$ be integers selected by the procedure suggested in [3] such that the inequalities

$$\sum_{i=1}^{k} a_i x_i < mmaj(q), \quad \frac{h_1}{a_1} \le \frac{h_2}{a_2} \le \cdots \le \frac{h_k}{a_k} \tag{10}$$

hold after choosing k places with $\sum_{i=1}^{k} h_i$ votes combined. Further, let the inequalities

$$\sum_{i=1}^{k} a_i + a_j \ge mmaj(q), \quad \frac{h_j}{a_j} \ge \frac{h_i}{a_i}, \quad i \in \overline{1,k},\ j \in \overline{k+1,n} \tag{11}$$

hold for any $a_j \in \{a_{k+1}, \ldots, a_n\}$. Finally, let the inequalities

$$\frac{h_{j^*}}{a_{j^*}} \le \frac{h_j}{a_j} \tag{12}$$

hold for for some $j^* \in \overline{k+1,n}$ and for all $j \in \overline{k+1,n} \setminus \{j^*\}$.

If the inequality

$$a_t < a_{j^*} \tag{13}$$

holds for some $t \in \overline{k+1,n} \setminus \{j^*\}$, along with the inequalities

$$h_t < h_{j^*}, \quad \frac{h_{j^*}}{a_{j^*}} < \frac{h_t}{a_t}, \tag{14}$$

choosing a place with the number j^* (according to the procedure suggested in [3]) leads to the inequality

$$\sum_{i=1}^{k} h_i + h_t < \sum_{i=1}^{k} h_i + h_{j^*},$$

whereas the inequality

$$\sum_{i=1}^{k} a_i + a_t \ge mmaj(q)$$

still holds. (If the inequalities $a_t \ge a_{j^*}$ and $h_t < h_{j^*}$ held for some $t \in \overline{k+1,n} \setminus \{j^*\}$, the inequalities

$$\frac{h_t}{a_t} \le \frac{h_t}{a_{j^*}} < \frac{h_{j^*}}{a_{j^*}}$$

would hold contradictory to inequalities (12).) This means that winning the US Presidency via the Electoral College in the places $\overline{1,k} \cup \{t\}$ can be achieved with a fraction of the popular vote that is smaller than that corresponding to winning the US Presidency via the Electoral College in the places $\overline{1,k} \cup \{j^*\}$, and this smaller fraction cannot be determined by the procedure proposed in [3].

Moreover, let the inequalities (10) and (11) hold, and let the inequalities

$$h_{j^*} \le h_j, \ \ j \in \overline{k+1,n} \tag{15}$$

hold for $j^* \in \overline{k+1,n}$.

One can easily be certain that choosing the number $j^* \in \overline{k+1,n}$ (i.e. choosing a place with the number j^* – among all the places with the numbers from the set $\overline{k+1,n}$ – with the lowest number of votes h_{j^*} that win all the electoral votes in this place) does not make the set of integers $\{x_i^* = 1, i \in \overline{1,k} \cup \{j^*\}, x_i^* = 0, i \in \overline{k+1,n} \setminus \{j^*\}\}$ an optimal solution to problem (1).

Indeed, if the inequalities

$$h_{j^{**}} < h_{i_1^*} + h_{i_2^*}, \quad a_{j^{**}} < a_{i_1^*} + a_{i_2^*}$$

and

$$\frac{h_{i_1^*}}{a_{i_1^*}} \le \frac{h_{i_2^*}}{a_{i_2^*}} \le \frac{h_{j^{**}}}{a_{j^{**}}}$$

hold for $i_1^*, i_2^* \in \overline{1,k}$, along with the inequality

$$\sum_{i \in \overline{1,k} \setminus \{i_1^*, i_2^*\}} a_i + a_{j^{**}} + a_{j^*} \ge mmaj(q)$$

for some $j^{**} \in \overline{k+1,n} \setminus \{j^*\}$, then substituting the place with the number j^{**} for the places with the numbers i_1^*, i_2^* leads to the inequality

$$\sum_{i \in \overline{1,k} \setminus \{i_1^*, i_2^*\}} h_i + h_{j^{**}} + h_{j^*} < \sum_{i=1}^{k} h_i + h_{j^*}.$$

This means that the procedure proposed in [3] may not yield an optimal solution to problem (1) even if place j^* with the lowest number h_j^* is chosen to occupy position $k+1$ on the list of places formed by the procedure (i.e. after the first k positions on the list for which the inequalities (10) and (11) hold have been chosen by the procedure).

The following example is illustrative of these statements.

Example. Let us consider a hypothetical US presidential election with the same allocation of electoral votes among the states as it was in the 2000 US presidential election. Further, let us choose the (hypothetical) distribution of the (popular) vote among the states and DC in the consideration to follow close to that from the 2000 election and in line with the official 2000 census data relating to the numbers of all eligible voters in the states and DC (from the viewpoint of the possibility to attain the chosen corresponding "prices" per electoral vote in the states and DC in principle).

Let $k = 22$, and let CA (54 e.v.), NY (33 e.v.), FL (25 e.v.), PA (23 e.v.), IL (22 e.v.), OH (21 e.v.), MI (18 e.v.), NC (14 e.v.), CT (8 e.v.), MS (7 e.v.), WV (5 e.v.), NM (5 e.v.), RI (4 e.v.), HI (4 e.v.), ID (4 e.v.), NV (4 e.v.), ND (3 e.v.), WY (3 e.v.), SD (3 e.v.), AK (3 e.v.), VT (3 e.v.), and DC (3 e.v.) form the first group of the places (states and DC), each with the "price" per electoral vote not exceeding 86,000, which control 269 electoral votes combined. Further, let the other group of the places, consisting of 29 states also controlling 269 electoral votes combined, each have at least the 90,000 "price" per electoral vote, except for one state for which this price equals 88,000 so that inequalities (11) hold for $i \in \overline{1,22}$ and $j \in \overline{23,51}$.

Then the procedure proposed in [3] puts the 22 places forming the first group in the first 22 positions on the ordered list of the 51 places (states and DC).

Finally, let NV be a place with the number i_1^*, NM be a place with the number i_2^*, OR (7 e.v.) be a place with the number j^{**}, and DE (3 e.v.) be a place with the number j^*, and let the equalities

$$h_{i_1^*} = 340{,}000 \qquad h_{i_2^*} = 430{,}000 \qquad h_{j^{**}} = 640{,}000 \qquad h_{j^*} = 270{,}000$$

hold, along with inequalities (15) for $k = 22$. Then the inequalities

$$640{,}000 = h_{j^{**}} < h_{i_1^*} + h_{i_2^*} = 340{,}000 + 430{,}000 = 770{,}000$$

and

$$7 = a_{j^{**}} < a_{i_1^*} + a_{i_2^*} = 4 + 5 = 9,$$

along with the inequalities

$$85{,}000 = \frac{h_{i_1^*}}{a_{i_1^*}} < \frac{h_{i_2^*}}{a_{i_2^*}} = 86{,}000 < 91{,}428 = \frac{h_{j^{**}}}{a_{j^{**}}},$$

the equality

$$\sum_{i\in\overline{1,22}\setminus\{i_1^*,i_2^*\}}^{22} a_i + a_{j^{**}} + a_{j^*} = (269-9)+7+3 = 270,$$

and the inequality

$$\sum_{i=1}^{22} a_i + a_{j^*} = 272 > 270,$$

also hold.

Further, the inequality

$$(H - 770{,}000) + 640{,}000 + 270{,}000 = \sum_{i\in\overline{1,22}\setminus\{i_1^*,i_2^*\}}^{22} h_i + h_{j^{**}} + h_{j^*} < \sum_{i=1}^{22} h_i + h_{j^*} = H + 270{,}000,$$

where $H = \sum_{i=1}^{22} h_i$, holds.

Thus, the 21 states and DC – to be chosen by the procedure proposed in [3] to occupy the first 22 places on the list – along with state j^* from the second group of the states with the smallest number h_j^*, form a feasible solution to problem (1) for the hypothetical election under consideration. However, this solution is not optimal.

Finally, let KY (8 e.v.) be a state for which the price per electoral vote equals 88,000. Then, according to the procedure proposed in [3], the state of Kentucky will occupy the 23rd place on the list, and the inequalities (11) and (12) hold for $i \in \overline{1,22}$ and $j \in \overline{23,51}$.

Since the inequalities

$$h_{j^{**}} = 640{,}000 < 88{,}000 \times 8 = 704{,}000 = h_{j^{***}}$$

and

$$a_{j^{**}} = 7 < 8 = a_{j^{***}}$$

hold, along with the inequality

$$91{,}428 = \frac{h_{j^{**}}}{a_{j^{**}}} > \frac{h_{j^{***}}}{a_{j^{***}}} = 88{,}000,$$

choosing a state to occupy the 23rd place on the list according to the procedure proposed in [3] does not lead to an optimal solution to problem (1) either, since the inequalities

$$277 = 269 + 8 = \sum_{i=1}^{22} a_i + a_{j^{***}} > \sum_{i=1}^{22} a_i + a_{j^{**}} = 276 > 270,$$

and

$$H + 704{,}000 = \sum_{i=1}^{22} h_i + h_{j^{***}} > \sum_{i=1}^{22} h_i + h_{j^{**}} = H + 640{,}000$$

hold.

The latter reasoning also illustrates that situations in which all inequalities (10)–(14) hold are possible (if $a_t = a_{j^{**}}$, $h_t = h_{j^{**}}$ and $a_{j^*} = a_{j^{***}}$, $h_{j^*} = h_{j^{***}}$ in the inequalities (12)–(14)).

In contrast, solving 0-1 knapsack problem (2) always leads to an optimal solution to problem (1).

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