

Reflection equation algebra in braided geometry ¹

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Abstract

Braided geometry is a sort of the noncommutative geometry related to a braiding. The central role in this geometry is played by the reflection equation algebra associated with a braiding of the Hecke type. Using this algebra, we introduce braided versions of the Lie algebras $gl(n)$ and $sl(n)$. We further define braided analogs of the coadjoint orbits and the vector fields on a q -hyperboloid which is the simplest example of a "braided orbit". Besides, we present a braided version of the Cayley-Hamilton identity generalizing the result of Kantor and Trishin on the super-matrix characteristic identities.

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In memory of our friend Issai Kantor

1 Introduction

By braided geometry we understand a sort of the noncommutative geometry related to a braiding R , which is an invertible operator $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ obeying to the Yang-Baxter equation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

Here V is a vector space over the ground field $\mathbb{K} = \mathbb{C}$ (or \mathbb{R}).

In what follows we are dealing with braidings satisfying a second order equality

$$(qI - R)(q^{-1}I + R) = 0$$

where $q \in \mathbb{K}$ is assumed to be generic (in particular, this means $q^k \neq 1$ for any integer $k \geq 2$). Such a braiding is called a *Hecke symmetry*. It becomes an *involutive symmetry* in case $q = 1$.

Let $U_q(sl(n)) \rightarrow \text{End}(V)$ be the basic representation of the quantum group $U_q(sl(n))$, $\dim V = n$. Then the product of the image of the quantum universal R -matrix in the space $V^{\otimes 2}$ and the flip (transposition operator) is a Hecke symmetry. In what follows this Hecke symmetry will be referred to as a *standard* one. Note that its limit at $q \rightarrow 1$ is the usual flip. In this sense we can treat the standard Hecke symmetry as a deformation of the flip. In a similar way one can construct deformation of a super-flip. Besides, there are known Hecke symmetries which are neither deformations of a flip, nor of a super-flip (see the next section).

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With any Hecke symmetry we can associate (at least) two matrix algebras. One of them is called RTT (or Reshetikhin-Takhtajan-Faddeev) algebra (cf [17]), the other one is called Reflection Equation Algebra (REA)².

Both, the RTT and the RE algebras (supplied with a spectral parameter) are employed in constructing integrable dynamical models. Besides, the latter algebra possesses a number of remarkable properties which make it a very interesting object of the braided geometry.

First, the REA admits some quotients which can be regarded as noncommutative (braided) analogs of the coadjoint orbits. Moreover, it enables one to define a braided analog of the Lie bracket.

Second, for certain matrices with entries belonging to REA one can find the Cayley-Hamilton (CH) identities with central coefficients. These identities allow one to define a whole family of projective modules over the braided orbits. An attempt to develop a sort of related K-theory based on a "braided trace" instead of the usual one, which is traditionally employed in the classical K-theory, was made in [8]. Most of these objects are defined and studied for the REA related to the so called *even* Hecke symmetries (see the next section).

For the *general linear type* Hecke symmetries (further referred to as the $GL(p|r)$ type symmetries) we have established the *basic* CH identity. This identity is a q -extension of the characteristic identity for super-matrices discovered by I.Kantor and I.Trishin in [13]. Note that it is this remarkable paper which stimulated our interest in the REAs related to the $GL(p|r)$ type Hecke symmetries.

The present paper is a brief review of the main properties of the REA and their applications in braided geometry. In particular, we present a classification of the Hecke symmetries (section 2), define and compare the corresponding quantum matrix algebras (section 3), introduce braided analogs of the Lie algebras $gl(n)$ and $sl(n)$ (section 4), and consider braided version of the Cayley-Hamilton identity (section 5). We conclude the paper with an example of a q -hyperboloid for which we describe a braided analog of vector fields and describe their applications.

2 Classification of Hecke symmetries

Definition 2.1. Let V be a finite dimensional vector space over the field \mathbb{K} , $\dim_{\mathbb{K}} V = n$. An invertible operator $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ is called a *braiding* if it satisfies the quantum Yang-Baxter equation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

If, in addition, a braiding R obeys the relation

$$(qI - R)(q^{-1}I + R) = 0, \quad q \in \mathbb{K} \setminus 0$$

it is called a Hecke symmetry. If $q = 1$ a braiding R is called an involutive symmetry.

For any Hecke symmetry the braided analogs of the symmetric and skew-symmetric algebras of the space V can be constructed in the following way. Define the "R-symmetric" I_+ and "R-skew-symmetric" I_- subspaces of $V^{\otimes 2}$ by the relations

$$\begin{aligned} I_+ &:= \text{Im}(q^{-1}I + R) \quad \text{i.e.} \quad I_+ = \{q^{-1}x \otimes y + R(x \otimes y), \forall x, y \in V\} \\ I_- &:= \text{Im}(qI - R) \quad \text{i.e.} \quad I_- = \{qx \otimes y - R(x \otimes y), \forall x, y \in V\} \end{aligned}$$

²A method of constructing other matrix algebras by means of a pair of compatible braidings, one of them being the Hecke symmetry, is described in [11, 5].

Let $T(V)$ denotes the free tensor algebra of the space V , and $\langle I_{\pm} \rangle$ stands for a two-sided ideal in $T(V)$ generated by the space I_{\pm} . The quotients

$$\Lambda_+(V) := T(V)/\langle I_- \rangle \quad \text{and} \quad \Lambda_-(V) := T(V)/\langle I_+ \rangle$$

are called R -symmetric and R -skew-symmetric algebras of the space V respectively.

Noticing that $\Lambda_{\pm}(V)$ are quadratic graded algebras, we consider their Hilbert-Poincaré (HP) series

$$\mathcal{P}_+(t) := \sum_{k \geq 0} \dim \Lambda_+^k(V) t^k, \quad \mathcal{P}_-(t) := \sum_{k \geq 0} \dim \Lambda_-^k(V) t^k$$

where $\Lambda_{\pm}^k(V) \subset \Lambda_{\pm}(V)$ are homogeneous components of the degree k .

Theorem 2.2. *For generic values of q the following identity holds*

$$\mathcal{P}_+(t)\mathcal{P}_-(-t) = 1$$

Example 1. If $R = P$ where P is the usual flip then $\mathcal{P}_-(t) = (1+t)^n$ where $n = \dim V$.

Example 2. If $V = V_0 \oplus V_1$ is a super-space, $\text{sdim } V = (p|r)$ and R is the corresponding super-flip then $\mathcal{P}_-(t) = \frac{(1+t)^p}{(1-t)^r}$.

Definition 2.3. If $\mathcal{P}_-(t)$ (respectively $\mathcal{P}_+(t)$) is a monic polynomial then the space V and the braiding R are called *even* (respectively *odd*).

Theorem 2.4. *If R is even then the polynomial $\mathcal{P}_-(t)$ is reciprocal.*

Proposition 2.5. *For any V with $\dim V = n \geq 2$ there exists a Hecke symmetry $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ such that $\mathcal{P}_-(t) = 1 + nt + t^2$.*

Note that if $n > 2$ such a Hecke symmetry cannot be a deformation of the usual flip since the HP series are stable under the deformation.

Definition 2.6. Let R be an even Hecke symmetry. The degree $p = \deg \mathcal{P}_-(t)$ is called *the rank* of R .

Proposition 2.7. *Let p be an integer constrained by $2 \leq p \leq n = \dim V$. Then there exists an even Hecke symmetry $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ of the rank p .*

All even Hecke symmetries of rank 2 are classified (cf [4] and references therein).

Example 3. If $p = 3$, then $\mathcal{P}_-(t) = 1 + nt + nt^2 + t^3$.

Example 4. If $p = 4$, then $\mathcal{P}_-(t) = 1 + nt + rt^2 + nt^3 + t^4$.

Definition 2.8. Let $\mathcal{P}_-(t)$ be a rational function represented as a ratio of two coprime polynomials, the degrees of the numerator and denominator being p and r respectively. We shall say that the corresponding Hecke symmetry has *the bi-rank* $(p|r)$.

The bi-rank of a super-flip coincides with its super-dimension.

Theorem 2.9. [10] *Any Hecke symmetry has a certain bi-rank. In other words, its HP series $\mathcal{P}_-(t)$ is a rational function.*

Definition 2.10. A braiding R is called skew-invertible if there exists an operator $\Psi : V^{\otimes 2} \rightarrow V^{\otimes 2}$ such that

$$\text{Tr}_{(2)} R_{12} \Psi_{23} = P_{13}$$

where P_{13} is the flip transposing the first and the third spaces.

On fixing a basis $\{x_i\} \in V$ we get the corresponding basis $\{x_i \otimes x_j\}$ in the space $V^{\otimes 2}$. With respect to this basis the braiding R and the operator Ψ are represented by their matrices

$$R(x_i \otimes x_j) := x_k \otimes x_l R_{ij}^{kl}, \quad \Psi(x_i \otimes x_j) := x_k \otimes x_l \Psi_{ij}^{kl}$$

where the upper pair of indices enumerates columns of the matrix and the lower one — rows. In terms of matrices the relation in Definition 2.10 takes the form

$$\sum_{a,b=1}^n R_{ia}^{kb} \Psi_{bj}^{al} = \delta_i^l \delta_j^k$$

Theorem 2.11 ([2]). *If R is a skew-invertible symmetry then the numerator $N(t)$ of the rational function $\mathcal{P}_-(t)$ is a reciprocal polynomial, while the denominator $D(t)$ is a skew-reciprocal one (i.e $D(-t)$ is reciprocal).*

For any skew-invertible braiding the operators

$$B := \text{Tr}_{(1)} \Psi \quad (B_j^i = \Psi_{aj}^{ai}), \quad C := \text{Tr}_{(2)} \Psi \quad (C_j^i = \Psi_{ja}^{ia})$$

are well defined. They are analogs of the parity operators in super-spaces. They are used for the definition of R -traces associated with R in the spaces of endomorphisms. If $\text{End}(V) = V \otimes V^*$ is the space of left endomorphisms and the set $\{h_i^j\}$ is its natural basis (i.e $h_i^j(x_k) = \delta_k^j x_i$), then $\text{Tr}_R(h_i^j) = C_j^i$ (for the space of right endomorphisms there is a similar formula with B instead of C). Note that if R is a super-flip then Tr_R is nothing but the usual super-trace.

3 Quantum matrix algebras

The most famous quantum matrix algebra related to a Hecke symmetry R is the RTT-algebra. It is defined in following way. Let $T = \|t_i^j\|$, $1 \leq i, j \leq n = \dim V$ be a matrix with entries t_i^j . Then by definition an RTT algebra is generated by the unit and by the elements t_i^j subject to the relations

$$RT_1 T_2 = T_1 T_2 R, \quad \text{where } T_1 := T \otimes I, \quad T_2 := I \otimes T$$

or in a coordinate form

$$R_{ij}^{kl} t_k^m t_l^n = t_i^k t_j^l R_{kl}^{mn}$$

where the summation over repeated indices is always understood. Being equipped with the coproduct

$$\Delta(t_i^j) = t_i^k \otimes t_k^j$$

and a counit it becomes a bi-algebra. If in addition R is an even skew-invertible Hecke symmetry a group-like element $\det_q(T)$ (called *the quantum determinant*) can be defined. If it is central, the quotient of the RTT algebra over the ideal generated by $\det_q(T) - 1$ is a Hopf algebra. For the standard Hecke symmetry ("the standard case" in what follows) this quotient is denoted

$\mathbb{K}_q[SL(n)]$ and treated to be a quantum analog of the function space on the group $SL(n)$. Its restricted dual is just the quantum group (QG) $U_q(sl(n))$.

Another important quantum matrix algebra associated with R is the so-called Reflection Equation Algebra (REA). It is defined as follows. Let $L = \|l_i^j\|$, $1 \leq i, j \leq n = \dim V$ be a matrix with entries l_i^j (the lower index enumerates rows). Then the REA is generated by the unit and by the elements l_i^j subject to the relations

$$RL_1RL_1 - L_1RL_1R = 0$$

or, in coordinates

$$R_{i_1i_2}^{a_1b_2} l_{a_1}^{b_1} R_{b_1b_2}^{c_1j_2} l_{c_1}^{j_1} - l_{i_1}^{a_1} R_{a_1i_2}^{b_1c_2} l_{b_1}^{c_1} R_{c_1c_2}^{j_1j_2} = 0$$

This algebra has a braided bi-algebra structure and can be equipped with the coaction of the RTT algebra (in the standard case it can be also equipped with an action of the QG $U_q(sl(n))$). The term "braided" means that

$$\Delta(ab) = \Delta(a)\Delta(b) = (a_1 \otimes a_2)(b_1 \otimes b_2) := a_1 \tilde{b}_1 \otimes \tilde{a}_2 b_2$$

where $\Delta(a) = a_1 \otimes a_2$, $\tilde{d} \otimes \tilde{c} = R^{\text{End}(V)}(c \otimes d)$, and $R^{\text{End}(V)}$ is an extension of the initial Hecke symmetry R to the REA (see the next section and [7] for more detail).

If R is a skew-invertible even Hecke symmetry a group-like element $\overline{\det_q(L)}$ can be defined. Then the quotient of the REA over the ideal generated by $\overline{\det_q(L)} - 1$ has a braided Hopf structure [15]. It is a braided analog of function space on $SL(n)$. Denote it $\overline{\mathbb{K}_q[SL(n)]}$.

In the standard case $\mathbb{K}_q[SL(n)]$ and $\overline{\mathbb{K}_q[SL(n)]}$ arise from the quantization of two different Poisson structures on the group $SL(n)$. The algebra $\mathbb{K}_q[SL(n)]$ arises from the Sklyanin bracket, while $\overline{\mathbb{K}_q[SL(n)]}$ originates from the Semenov-Tian-Shansky (S-T-S) one. A universal description for both these algebras is given in [11, 5]. It is based on the use of a pair of compatible braidings, one of them being the Hecke symmetry.

However, the properties of $\mathbb{K}_q[SL(n)]$ and $\overline{\mathbb{K}_q[SL(n)]}$ differ drastically.

1. In the REA the elements³ $\text{Tr}_R L^k := \text{Tr}(C \cdot L^k)$ are central for any integer $k \geq 0$. In the RTT algebra their analogs are not central but form a commutative subalgebra (i.e they are in involution).
2. For the REA there is a Cayley-Hamilton identity in the classical form (see section 5). For the RTT algebra the "powers" of the matrix T coming in such a relation are defined in a more complicated way.
3. The REA has a further deformation which plays the role of the enveloping algebra of a "braided Lie algebra". We call this algebra *the modified REA* (mREA).

Definition 3.1. The mREA is an associative unital algebra, generated by the elements l_i^j subject to the following quadratic-linear relations

$$RL_1RL_1 - L_1RL_1R = \hbar(RL_1 - L_1R), \quad \hbar \in \mathbb{K} \setminus 0, \quad L := \|l_i^j\|$$

or, in components,

$$R_{i_1i_2}^{a_1b_2} l_{a_1}^{b_1} R_{b_1b_2}^{c_1j_2} l_{c_1}^{j_1} - l_{i_1}^{a_1} R_{a_1i_2}^{b_1c_2} l_{b_1}^{c_1} R_{c_1c_2}^{j_1j_2} = \hbar(R_{i_1i_2}^{j_1a} l_a^{j_2} - l_{i_1}^a R_{a i_2}^{j_1j_2})$$

³Emphasize that the notation Tr_R plays a double role. From one side, it stands for the R -trace applied to the matrix L or its powers. From the other side, when it is applied to an element of the space $\text{End}(V)$ it sends this element to the field \mathbb{K} as explained above.

Theorem 3.2 ([7]). *Let $R = R_q$ be a Hecke symmetry depending on $q \in \mathbb{K}$ (for $q = 1$ it becomes involutive). Then for a generic q the dimensions of homogeneous components of the RTT algebra and of the REA equal to those at $q = 1$, that is the dimensions are stable under the q -deformation.*

Moreover, there exists an analog of the PBW theorem for the mREA. Thus, in the standard case the mREA is a deformation algebra depending on 2 parameters \hbar and q . Its Poisson counterpart is a Poisson pencil generated by the linear Poisson-Lie bracket on $gl(n)^*$ and by a quadratic bracket which is an extension of the S-T-S one to the whole space $gl(n)^*$. We refer the reader to the paper [7] for description of this Poisson pencil and to [16] for details on S-T-S bracket on the group.

In the sequel we use the notation $\mathcal{L}(q, \hbar)$ for the mREA and $\mathcal{L}(q)$ for the REA.

Note that the element $l = \text{Tr}_R L := C_i^j l_j^i$ is central in the both algebras $\mathcal{L}(q, \hbar)$ and $\mathcal{L}(q)$. So, it is natural to introduce the quotients $\mathcal{SL}(q, \hbar) = \mathcal{L}(q, \hbar)/\langle l \rangle$ and $\mathcal{SL}(q) = \mathcal{L}(q)/\langle l \rangle$ over an ideal, generated by l .

In the standard case all these algebras can be endowed with an action of the QG $U_q(sl(n))$ so that

$$X(ab) = X_1(a) X_2(b), \quad a, b \in \mathcal{L}(q) \text{ or } \mathcal{L}(q, \hbar), \quad X \in U_q(sl(n)), \quad \Delta(X) = X_1 \otimes X_2$$

All the operators possessing this property are called *equivariant*.

4 mREA and braided Lie bracket

Let R be a skew-invertible braiding. We want to equip the space $\text{End}(V)$ (say, the left endomorphisms for the definiteness) with a structure of a generalized Lie algebra.

Let us first assume R to be involutive ($R^2 = 1$). Then there exists an extension of R up to

$$R^{\text{End}(V)} : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}$$

such that it is involutive and coordinated with the natural product

$$\mu : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)$$

as follows

$$R^{\text{End}(V)} \mu_{23} = \mu_{12} R_{23}^{\text{End}(V)} R_{12}^{\text{End}(V)}$$

Here the both sides of the equality should be applied to an element from $\text{End}(V)^{\otimes 3}$. This means that the result of applying the product μ does not depend on the position. The operators satisfying this property will be called R -invariant.

Let us set by definition

$$[\ , \] := \mu(I - R^{\text{End}(V)})$$

Theorem 4.1. *The following properties hold true:*

1. $R^{\text{End}(V)} [\ , \]_{23} = [\ , \]_{12} R_{23}^{\text{End}(V)} R_{12}^{\text{End}(V)}$ (the R -invariance of the bracket);
2. $[\ , \] R^{\text{End}(V)} = -[\ , \]$ (the R -skew-symmetry of the bracket);
3. $[\ , \] [\ , \]_{12} (I + R_{23}^{\text{End}(V)} R_{12}^{\text{End}(V)} + R_{12}^{\text{End}(V)} R_{23}^{\text{End}(V)}) = 0$ (the R -Jacobi identity).

Definition 4.2. A generalized Lie algebra, or R -Lie algebra, is a data

$$(V, R : V^{\otimes 2} \rightarrow V^{\otimes 2}, [,] : V^{\otimes 2} \rightarrow V)$$

satisfying the properties 1 – 3 above where $R^{\text{End}(V)}$ should be replaced by R .

The generalized Lie algebra defined above in the space $\text{End}(V)$ is denoted $gl(V_R)$.

Let $\text{Tr}_R : \text{End}(V) \rightarrow \mathbb{K}$ be the R -trace associated with a given skew-invertible involutive symmetry R (see section 2). The family of Tr_R -less elements of $gl(V_R)$ forms a generalized Lie subalgebra denoted $sl(V_R)$. For any generalized Lie algebra \mathfrak{g} its enveloping algebra can be defined in the natural way as the following quotient

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle \text{Im}(I - R - [,]) \rangle$$

It is a braided Hopf algebra, its coproduct being additive on the generators:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \forall X \in \mathfrak{g}.$$

We would like to extend this construction to the non-involutive case. However, in contrast to the involutive case, here we first define the "enveloping algebra" of a braided Lie bracket, and then the bracket itself. We consider the mREA $\mathcal{L}(q, \hbar)$ corresponding to a skew-invertible Hecke symmetry R as a proper analog of the enveloping algebra $U(gl(n))$.

In order to argue this point of view we restrict ourselves to the standard Hecke symmetries. The following statement is a corollary of the theorem 3.2.

Proposition 4.3. *The mREA corresponding to a standard Hecke symmetry is a two-parameter deformation of the commutative algebra $\text{Sym}(gl(n))$ and a one-parameter deformation of $U(gl(n)_\hbar)$ (the subscript \hbar means that the parameter \hbar stands as a multiplier at the usual $gl(n)$ Lie bracket).*

Any finite dimensional representation of $U(gl(n))$ can be deformed into an equivariant representation of the mREA $\mathcal{L}(q, \hbar)$.

Besides, a sort of the PBW theorem is valid for the mREA (cf [7]). Concerning simple algebras of the series B_n, C_n, D_n there exists no similar deformation of their enveloping algebras.

Now, we are able to define a braided analog of the Lie bracket arising from $\mathcal{L}(q, \hbar)$. Below we put $\hbar = 1$. The commutative relations among the generators l_i^j can be rewritten as follows (cf [7]):

$$l_i^j l_k^l - Q(l_i^j l_k^l) = \text{linear terms}$$

Denoting the right hand side of the above relation as $[l_i^j, l_k^l]$ we claim the following.

Theorem 4.4. *Consider the map $\mathcal{L}(q, 1) \rightarrow \text{End}(\mathcal{L}(q, 1)) : l_i^j \mapsto L_i^j$ where L_i^j is a linear operator acting on generators by the rule $L_i^j(l_k^l) := [l_i^j, l_k^l]$. This map is a representation of the algebra $\mathcal{L}(q, 1)$.*

We shall call the map $l_i^j \mapsto L_i^j$ the adjoint representation, L_i^j the adjoint operator and the operation $[,]$ the braided Lie bracket. Also, we denote the vector space $\text{Span}(l_i^j)$ endowed with the braided Lie bracket as $gl(V_R)$. Unfortunately, the properties listed in the theorem 4.1 (where R is assumed to be an involutive symmetry) are not valid in general case. Their analogs for the braided Lie bracket in question are more complicated (cf [7, 9] for detail).

In fact, we have identified the space $\text{Span}(l_i^j)$ with $\text{End}(V)$. The R -trace in the basis $\{l_i^j\} \in \text{End}(V)$ is $\text{Tr}_R l_i^j \sim \delta_i^j$ (up to a factor). It is not difficult to see that traceless elements of the

braided Lie algebra $gl(V_R)$ form a subalgebra $sl(V_R)$. Moreover, if $\text{Tr } C \neq 0$ there exists a natural projector $gl(V_R) \rightarrow sl(V_R)$ similar to the classical one.

However, the restriction of the adjoint representation of the algebra $gl(V_R)$ to the subalgebra $sl(V_R)$ is not in general a representation of the latter algebra. However, by a slight modification of this restriction (described in [7], section 6) we can get an analog of the adjoint representation of the algebra $sl(n)$.

Note that in the standard case the braided analogs of the $gl(n)$ and $sl(n)$ adjoint representations are equivariant. Also, note that by using the methods of [14] a map from $\mathcal{SL}(q, 1)$ to the QG $U_q(sl(n))$ (localized by the quantum Casimir element) can be constructed. Basing on this map it is possible to develop a representation theory of the algebra $\mathcal{SL}(q, 1)$ in the standard case.

5 Cayley-Hamilton identity

Let R be a skew-invertible Hecke symmetry and λ be a partition. Let us consider the corresponding REA or mREA and define a Schur function $s_\lambda(L)$ as follows

$$s_0 = 1, \quad s_\lambda(L) := (\text{Tr}_R)_{1\dots k}(L_{\bar{1}} \dots L_{\bar{k}} \rho_R(E_\alpha^\lambda)), \quad \lambda \vdash k, \quad k = 1, 2, \dots$$

Here

$$L_{\bar{1}} = L, \quad L_{\overline{k+1}} = R_{k \ k+1} L_{\bar{k}} R_{k \ k+1}^{-1}$$

E_α^λ is a primitive idempotent of the Hecke algebra $H_n(q)$ corresponding to a partition λ . The symbol ρ_R stands for the "local" representation of $H_n(q)$ associating the braidings $R_{k \ k+1}$ with the standard generators $\sigma_k \in H_n(q)$.

Theorem 5.1. *The elements $s_\lambda(L)$ are central in the algebra $\mathcal{L}(q)$.*

Theorem 5.2. [Cayley-Hamilton theorem] *Assume that R is an even Hecke symmetry of rank p . Then the matrix $L = \|l_i^j\|$ composed of the REA generators satisfies the following Cayley-Hamilton (CH) identity*

$$L^p - q s_{(1)}(L) L^{p-1} + q^2 s_{(1^2)}(L) L^{p-2} - \dots + (-q)^p s_{(1^p)}(L) I = 0$$

Here the partition (1^k) is represented by the one-column Young diagram of the height k .

Remark 5.3. In fact, this CH identity is the first one in a family of CH identities. It is called *the basic* identity. The other identities of the family deals with some extensions of the basic matrix L and they are called *the higher* identities.

Let us introduce formal elements $\mu_1, \mu_2, \dots, \mu_p$ such that

$$q^k s_{(1^k)}(L) = \sum_{1 \leq i_1 < \dots < i_k \leq p} \mu_{i_1} \dots \mu_{i_k}$$

Otherwise stated, μ_i are the roots of the equation

$$\mu^p - q s_{(1)}(L) \mu^{p-1} + q^2 s_{(1^2)}(L) \mu^{p-2} - \dots + (-q)^p s_{(1^p)}(L) I = 0$$

So, μ_i belong to an algebraic extension of the center of the algebra $\mathcal{L}(q)$. Then the CH identity factorizes into the product

$$\prod_{i=1}^p (L - \mu_i) = 0$$

Let us fix values of the elements μ_i and let $\mathcal{L}^\mu(q)$ denotes the quotient of the REA over the ideal generated by the set of elements

$$\left\{ q^k s_{(1^k)}(L) - \sum_{1 \leq i_1 < \dots < i_k \leq p} \mu_{i_1} \dots \mu_{i_k} \mid 1 \leq k \leq p \right\}$$

We assume the family $\{\mu_1, \dots, \mu_p\}$ to be generic (in particular, μ_i are pairwise distinct). Then the algebra $\mathcal{L}^\mu(q)$ is a braided analog of the coordinate algebra of a generic coadjoint orbit in $gl(n)^*$. This means that in the standard case this algebra arises from the quantization of a generic orbit in $gl(n)^*$.

Proposition 5.4. *The elements*

$$e_i(L) = \prod_{j \neq i} \frac{L - \mu_j}{\mu_i - \mu_j}$$

are pairwise orthogonal idempotents (projectors)

$$e_i(L)^2 = e_i(L), \quad e_i(L) e_j(L) = 0, \quad i \neq j, \quad \sum_{i=1}^p e_i(L) = I$$

Recall that the elements $\text{Tr}_R L^k$ are central in the algebra $\mathcal{L}(q)$.

Theorem 5.5. *In the algebra $\mathcal{L}^\mu(q)$ there is a spectral decomposition*

$$\text{Tr}_R L^k = \sum_{i=1}^p \mu_i^k d_i, \quad \text{where} \quad d_i = \text{Tr}_R e_i(L) = \prod_{j \neq i} \frac{q^{\mu_i} - q^{-1} \mu_j}{\mu_i - \mu_j} \tag{5.1}$$

Here Tr_R is normalized by the condition $\text{Tr}_R I = p_q = \frac{q^p - q^{-p}}{q - q^{-1}}$. The quantities d_i are called the quantum dimensions.

Using the idempotents $e_i(L)$ we can define quantum analogs of line bundles over the quantum orbits via projective modules in the spirit of the Serre-Swan’s approach.

Note that a CH identity is also valid for the algebra $\mathcal{L}(q, \hbar)$. For this algebra the formula similar to (5.1) takes place too, but μ_i become now the roots of the CH identity for $\mathcal{L}(q, \hbar)$ and the quantum dimensions should be modified as follows

$$d_i = \text{Tr}_R e_i(L) = \prod_{j \neq i} \frac{q^{\mu_i} - q^{-1} \mu_j - \hbar}{\mu_i - \mu_j}$$

In order to get the latter formula it suffices to replace the roots μ_i in (5.1) by $\mu_i - \frac{\hbar}{q - q^{-1}}$.

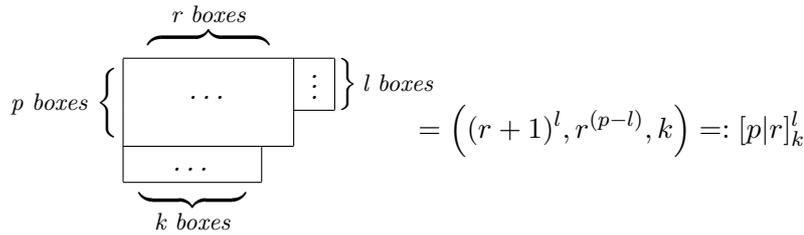
Formula (5.1) can be also considered as a parameterized relation between two families of central elements, namely, $\{\text{Tr}_R L^k\}$, $1 \leq k \leq p$, and the set of coefficients of the corresponding CH identity in the algebras $\mathcal{L}(q)$ or $\mathcal{L}(q, \hbar)$.

Now let us pass to the Hecke symmetries of $GL(p|r)$ type.

Theorem 5.6 ([5]). *Let now R be a skew-invertible Hecke symmetry of the bi-rank $(p|r)$. The matrix $L = \|l_i^j\|$ composed of the REA generators satisfies the following CH identity*

$$\sum_{i=0}^{p+r} L^{p+r-i} \sum_{k=\max\{0, i-r\}}^{\min\{i, p\}} (-1)^k q^{2k-i} s_{[p|r]_{i-k}^k}(L) = 0$$

where $[p|r]_{i-k}^k$ denotes the partition with the following Young diagram



This is a q -generalization of the Kantor-Trishin’s result [13].

Theorem 5.7 ([6]). *Being multiplied by $s_{[p|r]}$, the CH of Theorem 5.6 factorizes into the product*

$$\left(\sum_{k=0}^p (-q)^k L^{p-k} s_{[p|r]^k}(L) \right) \left(\sum_{l=0}^r q^{-l} L^{r-l} s_{[p|r]_l}(L) \right) = 0$$

Denoting the roots of the first and of the second factors as $\{\mu_i\}_{1 \leq i \leq p}$ and $\{\nu_j\}_{1 \leq j \leq r}$ respectively, we get the following parameterizations of the normalized coefficients of these factors

$$\frac{s_{[p|r]^k}(L)}{s_{[p|r]}(L)} \mapsto \frac{s_{[p|r]^k}(\mu, \nu)}{s_{[p|r]}(\mu, \nu)} := \sum_{1 \leq i_1 < \dots < i_p \leq p} q^{-k} \mu_{i_1} \dots \mu_{i_k} = e_k(q^{-1}\mu), \quad 1 \leq k \leq p \quad (5.2)$$

$$\frac{s_{[p|r]_l}(L)}{s_{[p|r]}(L)} \mapsto \frac{s_{[p|r]_l}(\mu, \nu)}{s_{[p|r]}(\mu, \nu)} := \sum_{1 \leq j_1 < \dots < j_l \leq r} (-q)^l \nu_{j_1} \dots \nu_{j_l} = e_l(-q\nu), \quad 1 \leq l \leq r \quad (5.3)$$

In terms of "even" roots μ_i and "odd" roots ν_j the CH identity factorizes as follows

$$(s_{[p|r]}(L))^2 \prod_{i=1}^p (L - \mu_i I) \prod_{j=1}^r (L - \nu_j I) = 0$$

Recently a formula similar to (5.1) has been obtained (the proof will be given in our forthcoming paper).

Theorem 5.8. *Let $\mathcal{L}(q)^{\mu, \nu}$ be the quotient algebra of the $GL(p|r)$ type REA by the relations (5.2), (5.3). In $\mathcal{L}(q)^{\mu, \nu}$ the following relations hold true*

$$Tr_R L^k = \sum_{i=1}^p d_i \mu_i^k + \sum_{j=1}^r \tilde{d}_j \nu_j^k$$

where

$$d_i := q^{-1} \prod_{s=1, s \neq i}^p \frac{\mu_i - q^{-2} \mu_s}{\mu_i - \mu_s} \prod_{j=1}^r \frac{\mu_i - q^2 \nu_j}{\mu_i - \nu_j}$$

$$\tilde{d}_j := -q \prod_{i=1}^p \frac{\nu_j - q^{-2} \mu_i}{\nu_j - \mu_i} \prod_{s=1, s \neq j}^r \frac{\nu_j - q^2 \nu_s}{\nu_j - \nu_s}$$

Similarly to the even case, if we replace μ_i (resp., ν_j) by $\mu_i - \frac{\hbar}{q-q^{-1}}$ (resp., $\nu_j - \frac{\hbar}{q-q^{-1}}$) we get the "quantum dimensions" d_i (resp., \tilde{d}_j) valid for the spectral decomposition in the algebra $\mathcal{L}(q, \hbar)$. In the modified formula μ_i and ν_j are regarded to be the roots of the CH identity for the matrix L composed of the generators of the algebra $\mathcal{L}(q, \hbar)$.

6 Example: q -Minkowski space and q -hyperboloid algebras

Let R be the standard Hecke symmetry in the case $n = 2$. In an appropriate basis $\{x, y\}$ of the two dimensional space V we get the following matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

which represents this Hecke symmetry in the basis $\{x \otimes x, x \otimes y, y \otimes x, y \otimes y\}$.

Also, we put $L = \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the system defining the mREA becomes

$$\begin{aligned} qab - q^{-1}ba &= \hbar b & q(bc - cb) &= ((q - q^{-1})a - \hbar)(d - a) \\ qca - q^{-1}ac &= \hbar c & q(cd - dc) &= c((q - q^{-1})a - \hbar) \\ ad - da &= 0 & q(db - bd) &= ((q - q^{-1})a - \hbar)b \end{aligned}$$

If $\hbar = 0$ this algebra is called the q -Minkowski space algebra [12]. For $\hbar \neq 0$ it is regarded to be a braided or q -analog of the enveloping algebra $U(\mathfrak{gl}(2)_{\hbar})$.

Changing the set of generators $\{a, d, b, c\}$ for $\{l, h, b, c\}$ where

$$l = q^{-1}a + qd, \quad h = a - d$$

we come to the relations

$$\begin{aligned} q^2hb - bh &= \hbar 2_q b - (q - q^{-1})lb, & bl &= lb \\ q^2ch - hc &= \hbar 2_q c - (q - q^{-1})lc, & cl &= lc \\ (q^2 + 1)(bc - cb) + (q^2 - 1)h^2 &= \hbar 2_q h - (q - q^{-1})lh, & hl &= lh \end{aligned}$$

In this basis of generators it is seen explicitly that the element $l = q^{-1}a + qd$ is central. Therefore, we can introduce the quotient algebra

$$\mathcal{SL}(q, \hbar) = \mathcal{L}(q, \hbar) / \langle l \rangle$$

The commutation relations among the independent generators of $\mathcal{SL}(q, \hbar)$ read

$$q^2hb - bh = \hbar 2_q b, \quad q^2ch - hc = \hbar 2_q c, \quad (q^2 + 1)(bc - cb) + (q^2 - 1)h^2 = \hbar 2_q h$$

Thus, we get a braided or q -counterpart of the algebra $U(\mathfrak{sl}(2)_{\hbar})$. But in contrast with the classical case, the algebra $\mathcal{SL}(q, \hbar)$ is not a subalgebra of $\mathcal{L}(q, \hbar)$.

Consider now the central element $\text{Tr}_q L^2 \in \mathcal{L}(q, \hbar)$. Its explicit form in the basis $\{l, h, b, c\}$ reads

$$\text{Tr}_q L^2 = q^{-1}bc + qcb + \frac{q}{1 + q^2}(h^2 + l^2)$$

Its image C in the algebra $\mathcal{SL}(q, \hbar)$ is

$$C = q^{-1}bc + qcb + \frac{q}{1 + q^2}h^2 \in \mathcal{SL}(q, \hbar) \tag{6.1}$$

This element is central in the algebra $\mathcal{SL}(q, \hbar)$. The quotient of this algebra over the ideal $\langle C - \alpha \rangle$, $\alpha \neq 0$, is called the quantum (or braided or q -)hyperboloid algebra

$$\mathcal{A}_{\alpha} := \mathcal{SL}(q, \hbar) / \langle C - \alpha \rangle$$

On the next step we introduce the braided Lie algebra $sl(V_R)$. But in the low-dimensional case in question we can simplify the construction of the corresponding braided Lie bracket. We only use the fact that this bracket is $U_q(sl(2))$ -covariant. We put $\mathcal{SL} = \text{Span}(b, h, c)$. Let us equip this space with an action of the QG $U_q(sl(2))$ and extend this action to the space $\mathcal{SL} \otimes \mathcal{SL}$ by using the coproduct of the QG. Then for a generic q the space $\mathcal{SL} \otimes \mathcal{SL}$ can be decomposed into a direct sum of irreducible $U_q(sl(2))$ submodules $\mathcal{SL} \otimes \mathcal{SL} = V_0 \oplus V_1 \oplus V_2$ where the subscript stands for the spin. Then the operator

$$[,] : \mathcal{SL} \otimes \mathcal{SL} \rightarrow \mathcal{SL}$$

is a $U_q(sl(2))$ morphism iff it is trivial on the components V_0 and V_2 and is an isomorphism between V_1 and \mathcal{SL} . By this property the bracket is uniquely defined up to a nonzero factor w .

Let us exhibit the multiplication table of this braided bracket:

$$\begin{aligned} [b, b] &= 0, & [b, h] &= -wb, & [b, c] &= w\frac{q}{2_q}h, & [h, b] &= wq^2b \\ [h, h] &= w(q^2 - 1), & [h, c] &= -wc, & [c, b] &= -w\frac{q}{2_q}h, & [c, h] &= wq^2c, & [c, c] &= 0 \end{aligned}$$

As can be easily seen, the bracket $[,]$ defines a representation of the algebra $\mathcal{SL}(q, 1)$ iff $w = \hbar(q^4 - q^2 + 1)^{-1}$.

Consider now q -analogs of adjoint operators. Being represented by matrices in the basis $\{b, h, c\}$ they have the form

$$B_q = w \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & \frac{q}{2_q} \\ 0 & 0 & 0 \end{pmatrix}, \quad H_q = w \begin{pmatrix} q^2 & 0 & 0 \\ 0 & q^2 - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C_q = w \begin{pmatrix} 0 & 0 & 0 \\ -\frac{q}{2_q} & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix} \quad (6.2)$$

Theorem 6.1. *The operators B_q, H_q, C_q satisfy the relation*

$$q^{-1}bC_q + \frac{hH_q}{2_q} + qcB_q = 0$$

Moreover, there exist extensions of these operators to the higher homogeneous components of the algebra $\mathcal{SL}(q)$ such that on each component we get a representation of the algebra $\mathcal{SL}(q, 1)$ and this relation is still valid for the extended operators B_q, H_q, C_q (we keep for them the same notations).

Now, we are able to define a braided analog of the space of vector fields on the classical hyperboloid. Let us consider the left \mathcal{A}_α -module generated by the operators B_q, H_q, C_q . It is possible to show that this \mathcal{A}_α -module is projective. It is natural to call it the "tangent module" on the q -hyperboloid in question. In a similar way the "cotangent module" on this q -hyperboloid can be introduced (it is isomorphic to the tangent one). Such braided geometrical structures on the quantum hyperboloid were first considered by one of the authors (D.G.) and P.Akueson (cf [1] and the references therein).

Note that the above vector fields are very useful for defining braided analogs of some operators of mathematical physics on the q -hyperboloid. Thus, the Laplace operator on a q -hyperboloid can be defined via the element (6.1) where the generators b, h, c should be replaced by braided vector fields B_q, H_q, C_q respectively. As a result, we get the following braided Laplace operator on the q -hyperboloid

$$q^{-1}B_qC_q + \frac{H_q^2}{2_q} + qC_qB_q$$

A braided version of the Dirac operator on the q -hyperboloid can be defined a similar way. Recently a braided analog of the Maxwell operator on the q -hyperboloid was constructed. Its construction will be published in [3].

It would be very interesting to generalize these constructions and results on other "braided varieties" of general type. The most intriguing problem is whether the module of the braided vector fields defined on these "varieties" possesses the properties analogous to those considered above.

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