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Nonlinear interaction of large-amplitude unidirectional waves in shallow water

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Abstract. Nonlinear interaction of long unidirectional waves is studied numerically in the framework of nonlinear shallow water theory in a basin of constant depth. The interaction of two initially separated unidirectional waves occurs only when the waves (transformed into the shock waves) overtake each other. It is demonstrated that the interaction of two large-amplitude wave crests results in the formation of one shock wave of triangular shape, which is qualitatively similar to the outcome of the nonlinear interaction of two weak-amplitude waves. The formation of shock waves from initially negative disturbances (wave troughs) is accompanied by the generation of reflected waves of negative polarity. These waves additionally influence the process of interaction. The interaction of waves of opposite polarities is possible only when the leading wave is negative.

Key words: nonlinear wave interaction, shallow water theory, unidirectional waves, Riemann waves, shock waves.

1. INTRODUCTION

Nonlinear interaction of unidirectional nonlinear waves is frequently observed in the nearshore region (Fig. 1). Typically waves of different amplitude approach the coast from the same offshore direction. Larger waves often overtake and absorb smaller ones. Interaction of unidirectional weakly nonlinear and dispersive shallow-water waves is usually studied in the framework of the Korteweg–de Vries (KdV) equation [1–4]. This fully integrable equation demonstrates the important role of solitary waves (solitons) in the nonlinear wave dynamics [5–7]. The interactions of solitons are elastic and do not lead to durable changes in their amplitudes in this framework. The wave field can be described by the superposition of cnoidal and

Fig. 1. Unidirectional nonlinear waves in the coastal zone of the Baltic Sea.

solitary waves by means of the nonlinear Fourier analysis [8]. Statistics of random waves in such a field differs from the Gaussian one whereas such fields support the formation of freak waves [5–14]. With an increase in the wave amplitude, the KdV equation no more exactly describes the wave motion: the interaction of solitary waves becomes inelastic and the wave amplitudes decrease due to the partial energy transfer to the oscillating components. An appropriate analytical model in this case is the extended Korteweg–de Vries (eKdV) equation, which can be integrated only asymptotically [15,16].

Solitary waves on the water surface usually exist only if their heights do not exceed 80% from the water depth [17]. This is why in the coastal zone, where the depth diminishes towards the shoreline, we usually observe nonlinearly deformed or even shock waves [18–20] (Fig. 1). Dispersive effects are significantly smaller in this area than in deeper areas; hence, nonlinear shallow water theory can serve as an adequate analytical model [1,4]. In this framework the propagation and transformation of a single wave in a basin of constant depth can be described in terms of Riemann waves with the subsequent formation of a shock wave [1,21–25]. The interaction of unidirectional weakly nonlinear shock waves is well described by the Burgers equation [26–28], which possesses a rigorous solution of the Cauchy problem. In this case the interaction of two shock waves leads to their merging and to the formation of one wave of a triangular shape. However, the formation of the shock wave from a large-amplitude Riemann wave differs from the analogous process in a weakly-nonlinear case [22,24–25] and should result in new features of shock wave interactions. These effects are studied in this paper.

2. MATHEMATICAL MODEL

Based on the above arguments, we assume that the interaction of two large-amplitude unidirectional waves is governed by nonlinear shallow water equations:

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial x}[Hu]=0, \tag{1}$$

$$\frac{\partial (uH)}{\partial t}+\frac{\partial}{\partial x}\left[Hu^2+\frac{gH^2}{2}\right]=0, \tag{2}$$

where $H=h+\eta$ is the total water flow depth, h is the unperturbed water depth, η is the water surface displacement, u is the depth-averaged horizontal water flow velocity, g is the gravity acceleration, x is the horizontal coordinate and t is time.

The unidirectional solution of Eqs. (1), (2) is represented by the so-called Riemann wave [22,24–25]

$$H(x,t)=H_0[x-V(H)t], \qquad u(x,t)=2\left(\sqrt{gH(x,t)}-\sqrt{gh}\right), \tag{3}$$

where $H_0(x)$ determines the initial water surface profile and

$$V=3\sqrt{gH}-2\sqrt{gh} \tag{4}$$

is the local speed of nonlinear wave propagation. The ratio between speeds of nonlinear Riemann waves V and linear wave propagation (wave celerity $c=\sqrt{gh}$) is shown in Fig. 2.

It can be seen in Fig. 2 that Riemann wave crests always propagate faster than c, and form a steep front at the face slope of the wave, while wave troughs always propagate slower than c, and form a steep front at the back slope of the wave. For very deep troughs there is a critical regime defined by

$$H<\frac{4h}{9}, \tag{5}$$

when a part of the wave propagates in the opposite direction. In this case the formation of the steep wave front occurs almost immediately and results in more pronounced nonlinear effects for wave troughs, rather than for wave crests [25].

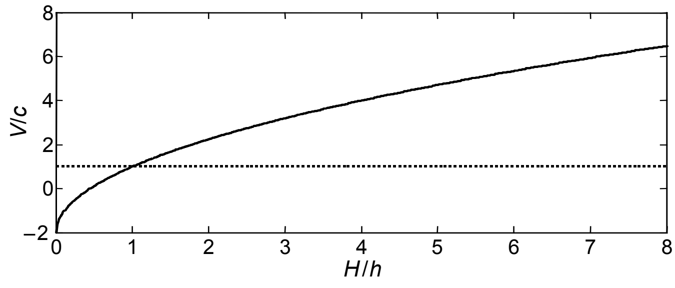

Fig. 2. Ratio of speeds of nonlinear and linear wave propagation. The dotted line corresponds to equal speeds.

Below we consider the nonlinear interaction of two large-amplitude waves for three cases: (i) interaction of unidirectional wave crests, (ii) interaction of unidirectional wave troughs and (iii) interaction of waves of different polarities. The study is performed numerically with the use of the Clawpack software package, which solves the hyperbolic equation using the finite volume method [29]. The numerical solution follows the mass conservation law with a high accuracy. In our case the variations of total mass were about 10^{-6}%. As the boundary condition we apply the Sommerfeld radiation condition. The spatial grid step is 30 m. Its refinement by 2–3 times leads to the difference in wave amplitudes of no more than 0.5%. The time step (60 s) has been chosen to satisfy the Courant–Friedrichs–Levy condition.

3. INTERACTION OF UNIDIRECTIONAL WAVE CRESTS

The nonlinear interaction has been studied for two wave crests of Gaussian shape

$$H_0(x) = h + A_1 \exp\left(-\frac{4x^2}{\lambda_1^2}\right) + A_2 \exp\left(-\frac{4(x-x_0)^2}{\lambda_2^2}\right) \tag{6}$$

in a basin of 1 m depth. Here A_i are initial wave heights, λ_i are the characteristic widths and x_0 is the distance between pulses. Several runs have been performed for different values of wave height and width. Figure 3 illustrates the interaction of two waves with heights of 0.9 and 0.8 m and widths of 0.9 and 2.8 km, respectively, separated by a 5.5 km long interval.

Both waves are characterized by very large amplitudes and transform into shock waves after about 20 min of their propagation (Fig. 3). The lagging wave, which is narrower and higher than the leading wave, transforms into a shock pulse after 3 min of its propagation and disperses sooner than the leading wave (Fig. 3). The speed of shock fronts exceeds the linear wave propagation but is less than the speed of the Riemann wave [1,26,30]. This difference provides the stabilization of the shock wave. As a result, the propagation of both shock waves is accompanied by a decrease in their heights and an increase in their lengths. In the weakly nonlinear case the formation of shock waves can be described analytically whereas the relevant solution predicts that the two shocks should merge into a triangle [26,30].

In the strongly nonlinear case this scenario is also realized although shock waves disperse during their propagation and their heights decrease to some extent before merging. This can be seen in Fig. 3 at time instants of 300 and 500 min.

Notice that the formation of the shock wave is accompanied by the formation of a weak-amplitude reflected wave of negative polarity (such a wave with an amplitude of 0.01 m can be observed at the time instants of 20 and 60 min in Fig. 3). This effect was predicted in [26] and observed experimentally in [31].

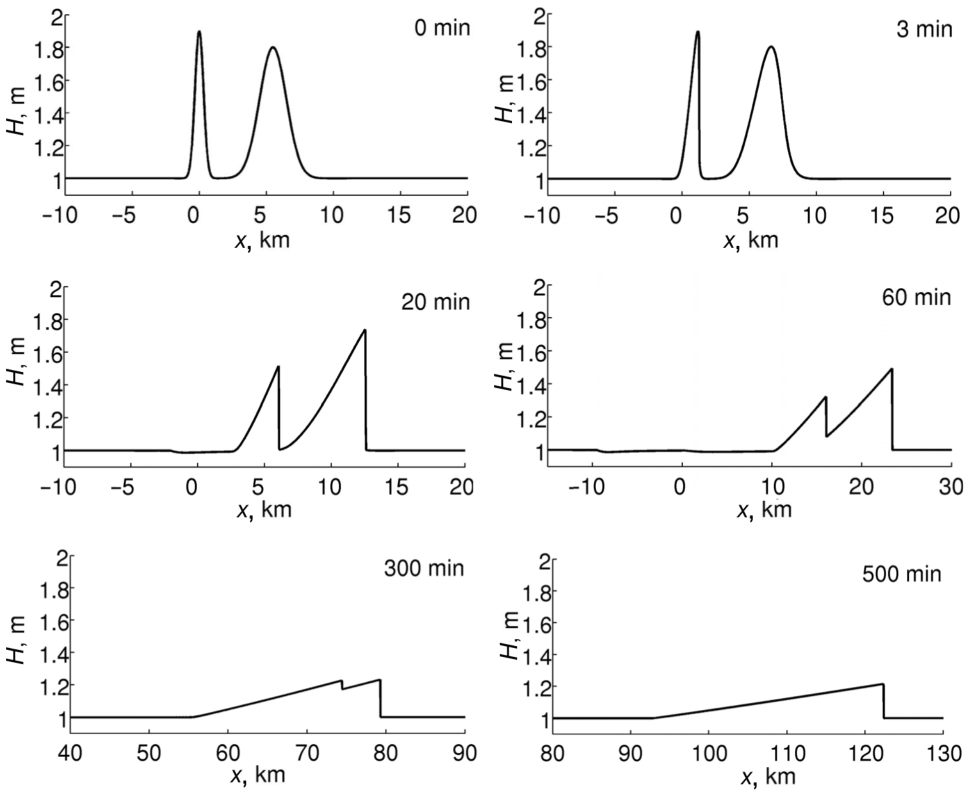

Fig. 3. Nonlinear interaction of two wave crests.

Thus, the process of interaction of strongly nonlinear waves of positive polarity is qualitatively the same as for weakly nonlinear waves except for the formation of a small reflected wave of negative polarity.

4. INTERACTION OF UNIDIRECTIONAL WAVE TROUGHS

Here we consider the interaction of two Gaussian waves of negative polarity, which correspond to the negative values of A_i in Eq. (6). Qualitatively, the interaction of weakly nonlinear waves does not depend on wave polarity and is the same for both positive and negative waves. New effects may be revealed for a strongly nonlinear case only. One of such effects is the formation of significant reflected waves, discussed in Section 2. That is why here we illustrate the interaction of identical strongly nonlinear waves of 0.9 m height and 0.9 km in width located 5 km away from each other (Fig. 4).

The process of formation of reflected waves is clearly visible in Fig. 4 at the time instant of 7 min. It results in a rapid wave attenuation of up to 30%.

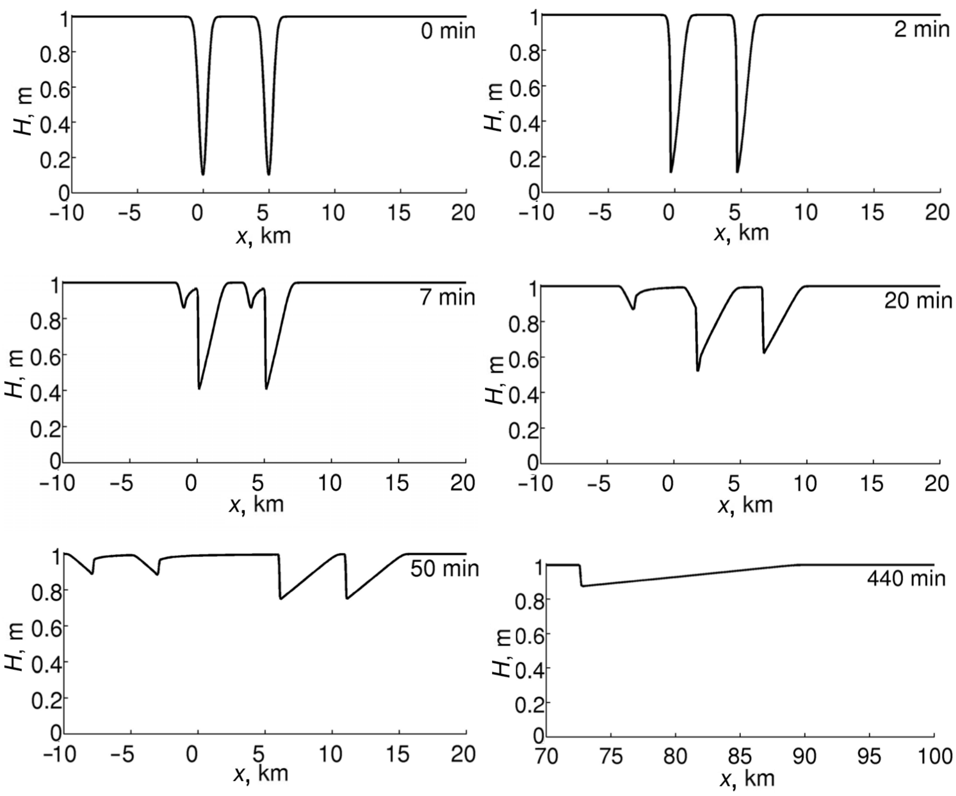

Fig. 4. Nonlinear interaction of two identical wave troughs.

The superposition of the lagging wave and the reflected wave, generated by the leading wave, leads to a short-time increase in the amplitude of the lagging wave, which can be seen at the instant of 20 min. Further on the nonlinear deformation of both waves evolves independently. Two pulses, propagating to the right, merge after 440 min, while left-going waves of smaller amplitudes are still separated by this time. Due to the negative polarity of waves, the shock is formed at the back-slope of both right- and left-going waves. Though during the interaction the waves transform in a different way (Fig. 4 at the time instant of 20 min), after the separation the two propagating right-going waves (the same for two reflecting left-going waves) have the same shape and amplitude (Fig. 4 at the time moment of 50 min).

If the waves are not identical, the wave field becomes more complicated (Figs 5 and 6). If the leading wave has a smaller (0.9 km) width than the 2.8 km long following wave (Fig. 5), it forms the shock and produces the reflected wave first and starts to propagate as a shock wave with a decrease in its amplitude.

As a result, when the lagging wave transforms into the shock wave, the amplitude of the leading wave is already noticeably smaller. The shock wave,

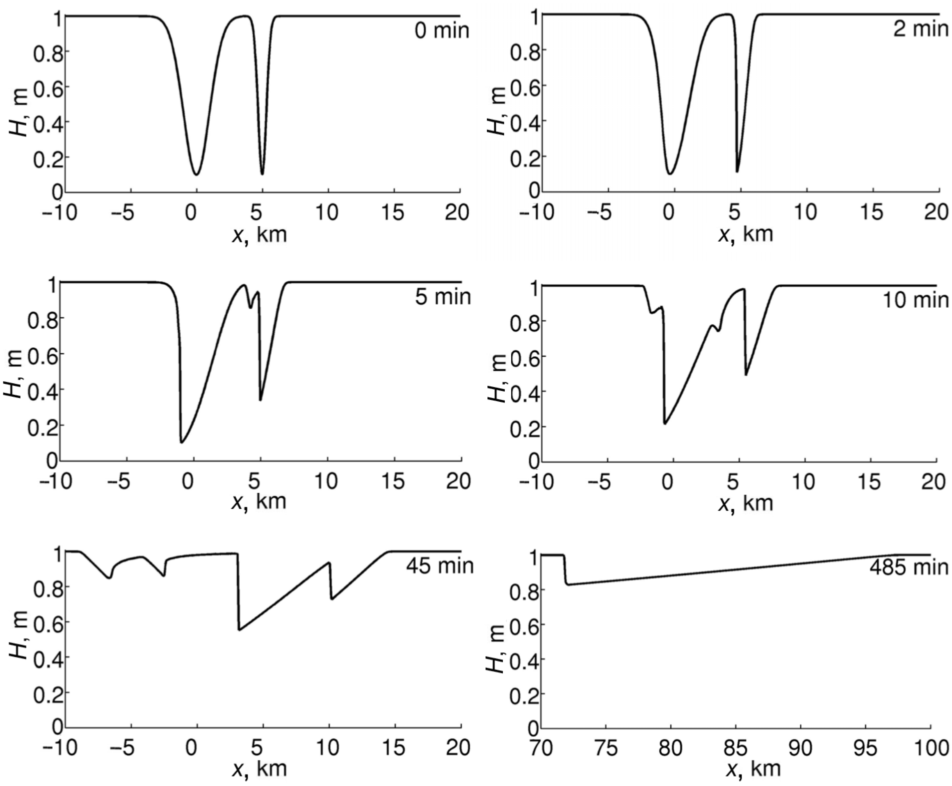

Fig. 5. Nonlinear interaction of two wave troughs of different width.

produced by the lagging wave is larger than for the leading wave and also larger than for the lagging wave of smaller width (Fig. 4). At the same time, due to the longer width of the lagging wave, the distance between shock wave fronts is larger than in the previous case (Fig. 4) and, as a result, it takes slightly longer time for the waves to merge.

Contrariwise, when the leading wave (2.8 km in width) is longer than the back one with the width of 0.9 km (Fig. 6), it preserves its height during a longer time and, as the result, overtaking of one wave by another occurs in a shorter time interval.

So, the interaction of two strongly nonlinear waves of negative polarity starts with a rapid decrease in the wave amplitude (by up to 30%) caused by the generation of reflected waves. After this phase, the waves continue their interaction following the weakly nonlinear scenario. It should be noted that the wave, reflected from the shock front of a narrow pulse, is shorter than the wave, reflected from the shock front of a wide pulse, and becomes a shock in a shorter time interval (similar to waves propagating to the right).

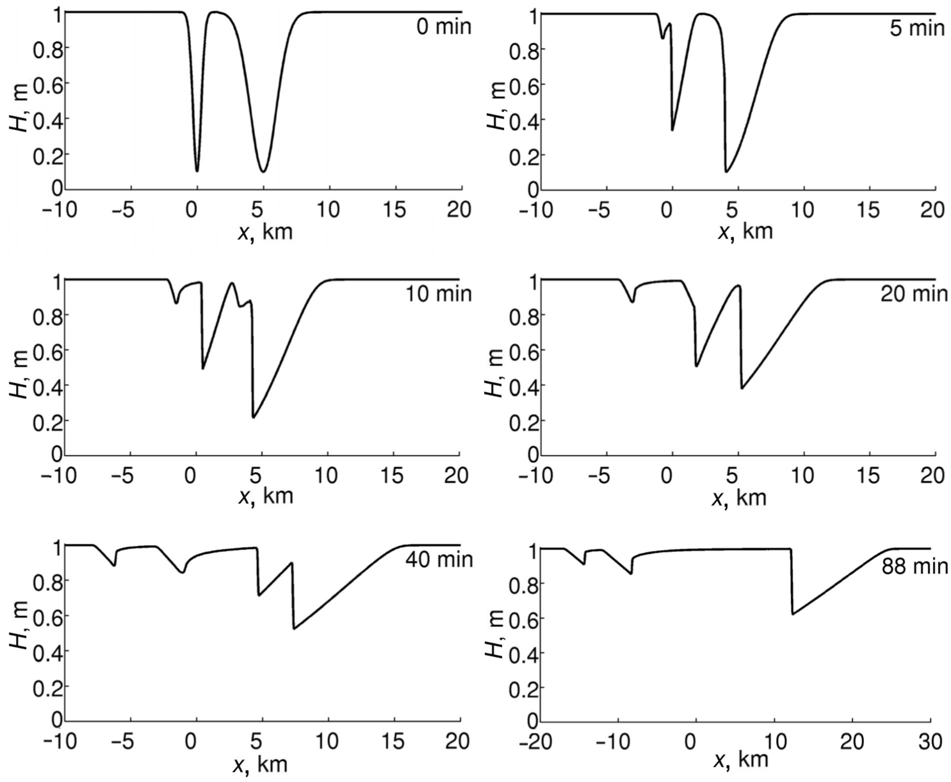

Fig. 6. Nonlinear interaction of two wave troughs of different width.

5. INTERACTION OF UNIDIRECTIONAL WAVES OF DIFFERENT POLARITIES

Another interesting case, reflecting qualitatively and quantitatively the difference in the propagation and transformation of wave crests and troughs, is the interaction of two waves of different polarities (Fig. 7). It can be seen that the wave crest and trough of the same amplitude 0.9 m and the same width 0.9 km behave differently. The wave trough steepens and transforms into a shock wave faster than the crest. As pointed out in Section 2, the nonlinearity is manifested stronger for the wave of negative polarity (trough) rather than for the crest. The shock trough produces the reflected wave, which starts its propagation to the left at the time instant of 5 min (Fig. 7). Then the wave crest is also transformed into a shock wave and produces another reflected wave (at 9 min in Fig. 7), which propagates to the left first, but has smaller amplitude than the one produced by the trough. When two shock waves of different polarities merge after 10 min of wave propagation, they generate one more reflected wave of negative polarity (0.08 m), which closes the sequence of three reflected wave troughs, propagating to the left (Fig. 8).

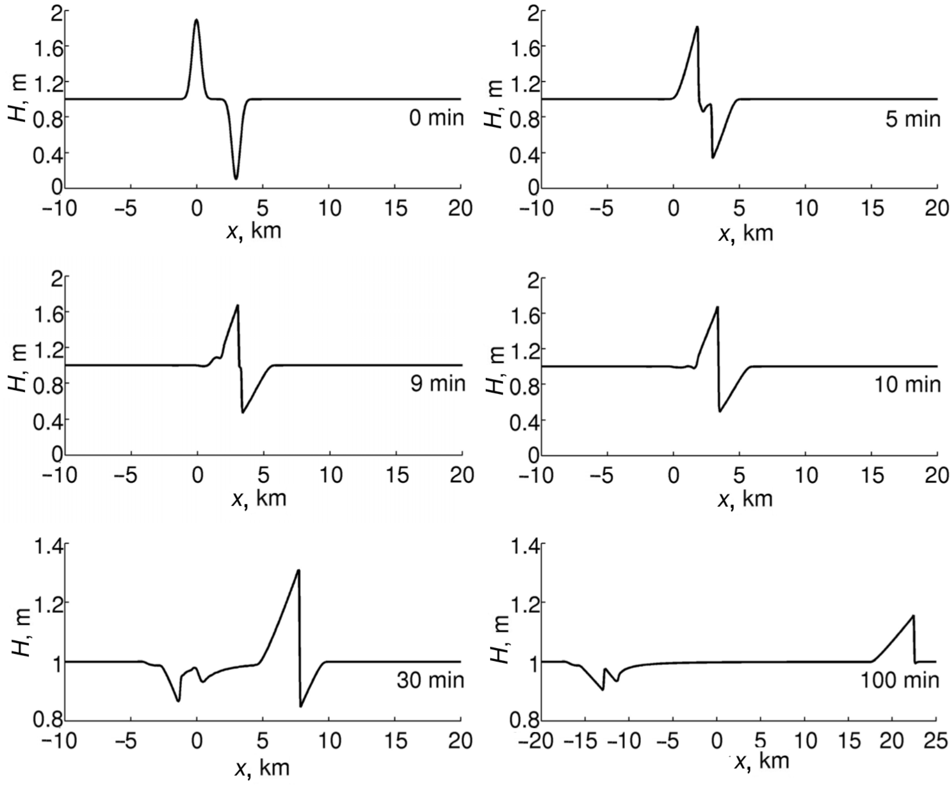

Fig. 7. Nonlinear interaction of wave trough and wave crest for time instants 0, 5, 9, 10, 30 and 100 min.

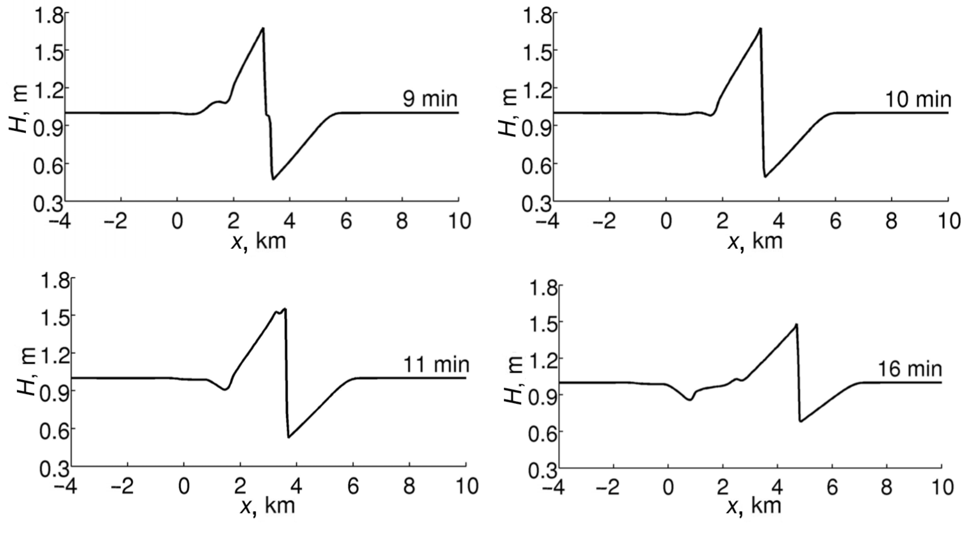

Fig. 8. Nonlinear interaction of wave through and wave crest for time instants 9, 10, 11 and 16 min.

The strong decrease of the amplitude of the trough due to the generation of the reflected wave results in the asymmetry of the right-going wave. The merged sign-variable shock wave transforms into the single wave crest after 100 min of propagation. This reflects the well-know feature that wave crests are more stable in shallow water than wave troughs.

In the reverse case, when the polarity of the leading wave is positive and that of the lagging wave is negative, the negative initial pulse never overtakes the positive one. However, the reflected waves appear and behave similarly to the previously discussed cases.

6. CONCLUSIONS

Interactions of two unidirectional large-amplitude Riemann waves in shallow water are studied in the framework of the nonlinear hyperbolic system that is solved numerically by the finite volume method. It is demonstrated that the generation of reflected waves during the shock wave formation strongly influences the interaction process. This influence is more pronounced for waves of negative polarity (troughs) providing an additional mechanism of water wave decay. For the initially equal wave amplitude and width, wave crests are more persistent for a longer time than wave troughs. Thus, the considered mechanism of nondispersive wave propagation leads to the same basic shape of shallow-water waves with higher crests and smaller troughs as predicted by the weakly nonlinear dispersive theory for cnoidal waves.

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Tugevalt mittelineaarsete madala vee lainete interaktsioonist

Ira Didenkulova, Efim Pelinovsky ja Artem Rodin

Pikkade tugevalt mittelineaarsete samasuunaliste madala vee lainete interaktsiooni on analüüsitud numbriliselt fikseeritud sügavusega vees levivate Riemanni lainete kontekstis mittelineaarse madala vee teooria raames. Interaktsioon leiab aset vaid siis, kui lööklaineteks muutunud lained jõuavad üksteisele järele. On näidatud, et kahe kõrge positiivse häirituse (laineharja) kohtumisel tekib üks kolmnurkse profiiliga lööklaine sarnaselt väiksemate lainete nõrgalt mittelineaarse interaktsiooniga. Negatiivsete häirituste (lainevagude) puhul kaasneb ühise lööklaine moodustumisega väiksemate lainevagude tekkimine, mis omakorda mõjutavad interaktsiooni käiku. On näidatud, et erineva polaarsusega häirituste interaktsioon on võimalik vaid siis, kui lainevagu jõuab laineharjale järele.