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# New observables in topological instantonic field theories 

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#### Abstract

Instantonic theories are quantum field theories where all correlators are determined by integrals over the finite-dimensional space (space of generalized instantons). We consider novel geometrical observables in instantonic topological quantum mechanics that are strikingly different from standard evaluation observables. These observables allow jumps of special type for the trajectory (at the point of insertion of such observables). They do not (anti)commute with evaluation observables and raise the dimension of the space of allowed configurations, while the evaluation observables lower this dimension. We study these observables in geometric and operator formalisms. Simple examples are explicitly computed; they depend on the linking of points.

The new "arbitrary jump" observables may be used to construct correlation functions computing, e.g., the linking numbers of cycles, as we illustrate on Hopf fibration.

We expect that such observables could be generalized in an interesting way to instantonic topological theories in all dimensions.


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## 1. Introduction

Instantonic field theories were introduced and studied in [1-3]. These supersymmetric theories are defined by localization on instanton space. These $Q$-supersymmetric theories may be considered as an extension of Witten's topological theories [4] including all local observables (not necessarily $Q$-closed).

We consider a class of such theories, where $Q$ is the de Rham differential on the target manifold and fermions are identified with differentials. In particular, we study geometric topological Quantum Mechanics, where Hamiltonian is given by Lie derivative along the given vector field.

In this paper, we introduce a large class of $Q$-closed local observables in topological Quantum Mechanics.
One of the possible constructions is to associate observables to fibrations of the target space. Another possibility is to associate observables to cycles in the group of diffeomorphisms. All these observables do not commute with evaluation observables. We show that even the simplest observable of this type - corresponding to $U(1)$ fibrations - appears in natural problems of geometry.

We start this paper by quick reminder of the formalism of instantonic topological theories, evaluation and vector field observables in Section 2. This section is borrowed from [2].

Novel results start in Section 3, where we will address the question: How do we write down geometrical observables in topological quantum mechanics that do not commute with the evaluation observables and are non-zero in cohomologies?

[^0]One type of such observables (corresponding to diffeomorphisms that cannot be connected to identity) is well known. Are there any other observables that have geometrical meaning?

One way to find appropriate generalization is to note that nontrivial diffeomorphisms correspond just to zero cycles in the space of all diffeomorphisms, and we may generalize this to an arbitrary cycle in the space of diffeomorphisms.

Another generalization arises if we treat allowed diffeomorphisms as allowed jumps of the trajectory (at the prescribed time). Simple inspection shows that this may be generalized to jumps along compact fibers of an arbitrary fibration.

These two classes of generalizations have one common representative that we will study in detail in this paper it corresponds to the $U(1)$ fibration. From the point of view of cycles in diffeomorphisms, we study $S^{1}$ in this space, corresponding to the $U(1)$ action on the total space of the fibration.

We think that it is instructive to discuss such observable starting with the vector field that generates the $U(1)$ action. Vector field observables are $Q$-exact and seem to be irrelevant for the purpose of constructing nontrivial observables since they are zero in cohomology. However we may use them in construction of $\alpha$-jump operators, corresponding to the $U(1)$ rotation by the angle $\alpha$.

Still such operators are equivalent to unity. To get the novel operators we first supersymmetrize the space $S^{1}$ of angles $\alpha$ and construct a super-jump operator, parametrized by such superspace. That is, the super-jump operator turns out to be a differential form on $S^{1}$. It is easy to show that the integral of super-jump against a cycle is a $Q$-closed operator. In particular, the zero-cycle, corresponding to a point $\alpha$, gives the $\alpha$-jump itself. While the super-jump, corresponding to a fundamental cycle of the circle, has no reason to be trivial in cohomologies of $Q$. This operator will be denoted $K$ and is a prototype of the main object of study in this paper.

The above construction may be generalized as follows. Consider a finite-dimensional cycle $C$ in the group of diffeomorphism of the target $X$. Take the operator that pulls back the forms on $X$ to $C \times X$ along the diffeomorphism action on $X$ and then integrates over the cycle $C$.

In Section 4 we present the simplest example of correlation functions with the observable $K$ and justify our expectations that it is nontrivial in cohomologies and does not commute with evaluation observables. It follows from noncommutativity that the correlation functions may depend on order of times. Hence we may get worldsheet linking numbers.

In Section 5 we discuss integrated observables (integrated against time). These are commonly known as "descent observables" [5]. They correspond to deformations of $Q$ and hence of the Hamiltonian. Geometrically they correspond to counting intersections that may happen at arbitrary time.

We consider deformations $Q \rightarrow Q+\tau K$ and compare them to Novikov-Witten deformation $Q \rightarrow Q+\tau \omega$. Note that the cohomology of the former differential are just equivariant cohomology of the fibration.

We see that in case of Novikov-Witten deformations the higher differential corresponds to trajectory passing successively through cycles, while in case of deformation with $K$ the higher differential corresponds to trajectories with successive jumps. In Section 6 we present conclusions.

## 2. Sketch of geometric formalism in quantum mechanical instantonic theories

### 2.1. Idea of geometrical formalism (zero-dimensional instantonic field theory)

Let $X$ be a finite-dimensional manifold, $V_{X}$ a vector bundle over $X$, and $v$ a section of $V$. We will call it the defining vector field. Then

$$
\begin{equation*}
\langle F(x, \psi)\rangle=\int \mathrm{d} p_{a} \mathrm{~d} \pi_{a} \mathrm{~d} x^{i} \mathrm{~d} \psi^{i} \exp \left(i p_{a} v^{a}(x)-i \pi_{a} \partial_{j} v^{a} \psi^{j}\right) F(x, \psi)=\int_{\text {zeros of } v} \omega_{F} \tag{1}
\end{equation*}
$$

where $\omega_{F}$ denotes the differential form on $X$ corresponding to the function $F$ on the $\Pi T X$ (with even coordinates $x^{i}$ and odd coordinates $\psi^{i}$ ). The variables $p_{a}$ and $\pi_{a}$ correspond to the even and odd coordinates on $V$.

Let us now deform $v$. In other words, let

$$
\begin{equation*}
v_{\epsilon}=v_{0}+\epsilon^{\alpha} v_{\alpha} \tag{2}
\end{equation*}
$$

where $v_{0}$ and $v_{\alpha}$ are sections of $V$, and $\epsilon \in \mathbb{C}^{n}$ are (formal) deformation parameters.
Consider $X \times \mathbb{C}^{n}$, and call a projection to the first factor by $p r_{X}$ and call by $p r_{\epsilon}$ a projection to $\mathbb{C}^{n}$. The space of zeros of $v_{\epsilon}$ for all values of $\epsilon$ we call the extended instanton space $\mathcal{M}_{\text {ext }}$. Its immersion into $X \times \mathbb{C}^{n}$ we denote by $\iota: \mathcal{M}_{\text {ext }} \hookrightarrow{ }^{\iota} X \times \mathbb{C}^{n}$. The space $\mathcal{M}_{\text {ext }}$ is fibered over $\mathbb{C}^{n}$ with projection given by $p r_{\epsilon} \circ \iota$; the fibers $\mathcal{M}_{\epsilon}$ of this fibration are zeros of $v_{\epsilon}$ for given $\epsilon$.


Given a form $\omega_{F}$ on $X$ we may consider it as form on $X \times \mathbb{C}^{n}$ (it is just $p r_{X}^{*} \omega_{F}$ ). Now we restrict it to $\mathcal{M}_{\text {ext }}$ (so, we get $\iota^{*} p r_{X}^{*} \omega_{F}$ ), and integrate the resulting form against the fibers $\mathcal{M}_{\epsilon}$ of projection $p r_{\epsilon} \circ \iota$ (the operation of direct image $\left.\left(p r_{\epsilon} \circ \iota\right)_{*}\right)$. This way we get a form on the base $\mathbb{C}^{n}$. The whole operation corresponds to multiplying $\omega_{F}$ (considered as a form on $X \times \mathbb{C}^{n}$ ) by the $\delta$ form on $X \times \mathbb{C}^{n}$ that localizes to zeros of $v_{\epsilon}$ and integrating the result over the fiber. The integral representation of $\delta$-form on $X \times \mathbb{C}^{n}$ is built exactly as in Eq. (1) (we simply replace $X$ with $X \times \mathbb{C}^{n}$ there):

$$
\begin{equation*}
\int \mathrm{d} p_{a} \mathrm{~d} \pi_{a} \mathrm{~d} x^{i} \mathrm{~d} \psi^{i} \exp \left(i p_{a} v_{\epsilon}^{a}(x)-i \pi_{a} \partial_{j} v_{\epsilon}^{a} \psi^{j}-i \pi_{a} d \epsilon^{\alpha} v_{\alpha}^{a}\right) F(x, \psi)=\int_{\mathcal{M}_{\epsilon}} \omega_{F} \equiv \hat{\omega}_{F} \tag{3}
\end{equation*}
$$

In a more rigorous language

$$
\begin{equation*}
\hat{\omega}_{F} \equiv \int_{\mathcal{M}_{\epsilon}} \omega_{F}=\left(p r_{\epsilon} \circ \iota\right)_{*} \iota^{*} p r_{X}^{*} \omega_{F} . \tag{4}
\end{equation*}
$$

Acting with Lie derivative $\mathcal{L}_{\frac{\partial}{\partial \epsilon^{\alpha}}}$ we get $\mathcal{O}_{v_{\alpha}}$ observable, defined as

$$
\begin{equation*}
\mathcal{O}_{v_{\alpha}}=i p_{a} v_{\alpha}^{a}(x)-i \pi_{a} \partial_{j} v_{\alpha}^{a} \psi^{j} \tag{5}
\end{equation*}
$$

or acting with substitution $\iota \frac{\partial}{\partial \epsilon^{\alpha}}$ we get $\pi_{v_{\alpha}}$ observable:

$$
\begin{equation*}
\pi_{v_{\alpha}}=i \pi_{a} v_{\alpha}^{a} \tag{6}
\end{equation*}
$$

So $\hat{\omega}_{F}$ is a generating function for $\pi_{v_{\alpha}}$ and $\mathcal{O}_{v_{\alpha}}$ observables ${ }^{1}$ :

The main idea of the geometrical definition of correlators in an infinite-dimensional case is to consider an infinitedimensional version of the above statements as the definition of the generating function for the correlators.

### 2.2. Three points of view on instantonic quantum mechanics

### 2.2.1. Geometrical formulation of instantonic $Q M$

For geometrical definition of correlation function we need the following data: the space $X$, the differential form $\omega$ and the defining vector field together with its $\epsilon$ deformations. Quantum mechanics is a one-dimensional quantum field theory, so we consider a vector field on the space of parametrized paths $\gamma$ in the target space,

$$
\begin{equation*}
\gamma \in \operatorname{Maps}([0, T], X) \tag{8}
\end{equation*}
$$

with appropriate boundary conditions; say, $\gamma(T)=\gamma(0)$ for periodic maps or $\gamma(0) \in C_{\text {in }}$ and $\gamma(T) \in C_{\text {out }}$ (where $C_{\text {in/out }}$ are cycles in $X$ ).

The defining vector field $V_{0}$ gives a set of equations describing the evolution along the vector field $V_{0}$ on $X$ :

$$
\begin{equation*}
d X^{i}=d t V_{0}^{i}(X(t)) \tag{9}
\end{equation*}
$$

Local observables come from the evaluation map, namely,

$$
\begin{equation*}
\mathrm{ev}_{t}: \gamma \mapsto \gamma(t) \tag{10}
\end{equation*}
$$

So for any differential form $\omega$ on the target space we may consider its pullback to the space of parametrized paths, that we denote as $\omega(t)$ :

$$
\begin{equation*}
\omega(t)=\operatorname{ev}_{t}^{*} \omega \tag{11}
\end{equation*}
$$

and a general evaluation observable, corresponding to $\omega_{F}$ above, is a product of local evaluation observables at various times $\omega_{F}=\mathrm{ev}_{t_{1}}^{*} \omega_{1} \ldots \mathrm{ev}_{t_{m}}^{*} \omega_{m}$.

To define a deformation (9) we pick up vector fields $v_{\alpha}$ and put them at times $t_{\alpha}$ as

$$
\begin{equation*}
d X^{i}=d t V_{0}^{i}(X(t))+\sum_{\alpha} \epsilon_{\alpha} \delta\left(t-t_{\alpha}\right) v_{\alpha}^{i}(X(t)) \tag{12}
\end{equation*}
$$

then we may introduce local observables $\mathcal{O}_{v}$ and $\pi_{v}$. Note, that geometrically the deformation (12) corresponds to jump of the trajectory at $t=t_{\alpha}$ by diffeomorphism that is the flow along the vector field $v$ during the time $\epsilon_{\alpha}$, i.e. to $e^{\epsilon_{\alpha} \mathcal{L}_{v_{\alpha}}}$, where $\mathcal{L}$ is the Lie derivative on $X$.

We would like to stress that this already defines basic set of correlators

$$
\left\langle\pi_{v_{1}}\left(t_{1}\right) \ldots \pi_{v_{m}}\left(t_{m}\right) \mathcal{O}_{v_{m+1}}\left(t_{m+1}\right) \ldots \mathcal{O}_{v_{k}}\left(t_{k}\right) \omega_{F}\right\rangle
$$

in the theory in finite-dimensional terms.
We may define more local observables by fusing the generating ones, namely, given two local observables $\mathcal{O}_{1}$ and $\mathcal{O}_{2}$ we may define correlator of $\mathcal{O}_{1 * 2}\left(t_{1}\right)$ as follows:

$$
\begin{equation*}
\left\langle\mathcal{O}_{1 * 2}\left(t_{1}\right) \ldots\right\rangle=\lim _{t_{2} \rightarrow t_{1}+0}\left\langle\mathcal{O}_{1}\left(t_{1}\right) \mathcal{O}_{1}\left(t_{2}\right) \ldots\right\rangle \tag{13}
\end{equation*}
$$

[^1]
### 2.2.2. Functional integral representation

The language of functional integrals in a fashionable way to represent QFT, despite it is rarely rigorously defined. Therefore it is instructive to represent instantonic QM in this language. This representation is just the l.h.s. of Eq. (3), symbolically

$$
\int D X^{i}(t) D \psi^{i}(t) D p_{i}(t) D \pi_{i}(t) \mathrm{e}^{\int i p_{j}\left(\frac{\mathrm{~d} \mathrm{x}^{j}}{\mathrm{dt}}-v_{0}^{j}\right)-i \pi_{j}\left(\frac{\mathrm{~d} \chi^{j}}{\mathrm{dt}}-\partial_{k} v_{0}^{j} \psi^{k}\right) \mathrm{dt}} F_{1}(X, p, \psi, \pi)\left(t_{1}\right) \ldots F_{n}(X, p, \psi, \pi)\left(t_{n}\right)
$$

The measure is considered to be Beresin canonical supermeasure, one may hope that due to balance between bosons and fermions it is independent of the coordinate system taken.

In naive functional integral paradigm one should consider as local observables functions $F$ of $X, \psi, \pi$ and $p$. However, such functions do not give well-defined observables due to noncommutativity between $X$ and $p$ and non-anticommutativity between $\psi$ and $\pi$.

Fixing their order means that we have to construct these observables by fusing generating ones. And generating ones do have interpretation in geometric terms:

$$
\begin{align*}
X^{i}(t) & =\mathrm{ev}_{t}^{*} X^{i}  \tag{14}\\
\psi^{i}(t) & =\mathrm{ev}_{t}^{*} d X^{i}  \tag{15}\\
i p_{i}(t) & =\mathcal{O}_{\partial / \partial X^{i}}(t)  \tag{16}\\
i \pi_{i}(t) & =\pi_{\partial / \partial X^{i}}(t) \tag{17}
\end{align*}
$$

The supersymmetry generator $Q=d_{X}=p_{i} \psi^{i}$ is a de Rham differential, it acts as: $Q X^{i}=\psi^{i}$ and $Q \pi_{i}=p_{i}$.

### 2.2.3. Operator approach

The operator approach to quantum mechanics has historically been the first one [6]. In this approach we have a space of states $\mathscr{H}$, Hamiltonian $H$ and a set of local operators $\Phi_{i}$. The correlators are given by

$$
\begin{equation*}
\left\langle\Psi_{\text {out }}\right| \Phi_{n}\left(t_{n}\right) \ldots \Phi_{1}\left(t_{1}\right)\left|\Psi_{\text {in }}\right\rangle=\left\langle\Psi_{\text {out }}\right| \mathrm{e}^{-\left(T-t_{n}\right) H} \Phi_{n} \ldots \mathrm{e}^{-\left(t_{2}-t_{1}\right) H} \Phi_{1} \mathrm{e}^{-t_{1} H}\left|\Psi_{\text {in }}\right\rangle \tag{18}
\end{equation*}
$$

where $\left|\Psi_{\text {in }}\right\rangle \in \mathscr{H},\left\langle\Psi_{\text {out }}\right| \in \mathscr{H}^{*}, \Phi_{i}, H \in \operatorname{End}(\mathscr{H})$. In the physics of real world the space $\mathscr{H}$ is Hermitian and $H=i H_{\text {phys }}$, where $H_{\text {phys }}$ is Hermitian. However in the context of general one-dimensional QFT this condition may be omitted, for example, in statistical mechanics and in theories with complex Lagrangians.

Instantonic QM in operator approach is described as follows. The space of states is the space of differential forms on the target and Hamiltonian is just the Lie derivative along $V_{0}$, which is $Q$-exact.

In this correspondence the evaluation operators correspond to multiplication by differential forms while vector field operators $\mathcal{O}_{v}$ and $i \pi_{v}$ correspond to the Lie derivative and to operation of contraction with the vector field respectively (hence $\left\{d_{X}, \iota_{v}\right\}=\mathcal{O}_{v}$ is a Cartan formula). All operators we consider below have geometric meaning and correlation functions are solutions of particular geometric problems.

To relate the two approaches it is convenient to introduce a geometric basis on the space of wave forms. Consider a chain $C$ on the target $X$. Then we may write a corresponding $\delta$-form localized on this chain: $\delta_{C}$, roughly speaking this is a $\delta$-form in the directions orthogonal to the cycle [7]. ${ }^{2}$ The degree of this form is $\operatorname{deg} \delta_{C}=\operatorname{dim} X-\operatorname{dim} C$. There is a property: $d_{X} \delta_{C}=\delta_{\partial c}$. Cycles (i.e. chains without boundaries) correspond to closed forms and non-contractible cycles correspond to de Rham cohomologies of $X$. Taking $\left|\delta_{C}\right\rangle$ as ket-vectors we can define bra-vector as a chain itself, then the pairing is an intersection number:

$$
\begin{equation*}
\left\langle C_{1} \mid \delta_{C_{2}}\right\rangle=\int_{C_{1}} \delta_{C_{2}}=\text { intersection }\left(C_{1}, C_{2}\right) \tag{19}
\end{equation*}
$$

Therefore, if we compute correlator in the operator approach one may show that in general the position operator approach coincides with the geometrical one. For example, the evolution operator in the operator approach means that we take the incoming chain, deform it along the flow of the vector field (corresponding to Hamiltonian) and then intersect it with the outgoing chain. Thus, we compute the number of intersection points. However, if we consider the set of all preimages of these intersection points under the flow we restore the set of trajectories, starting on an incoming chain and ending on the outgoing one, as it should be in the geometrical approach.

This geometrical approach in a form described above is a bit naive, since intersection of chains is defined only if they are transversal to each other. This problem may be solved by "smoothening" of the incoming and outgoing chains, in particular, by replacing chains by smooth differential forms. Chains may be considered as limits of smooth differential forms (and intersection is computed by the integral of the wedge product). Therefore correlator in operator approach (with states given by smooth forms) always exist, and if the chain limit may be taken, it equals to the correlator in geometrical approach.

[^2]If we compactify time to a circle, then in the geometric approach we compute the number of periodic trajectories (subject to some additional requirements determined by observables). In order to compare it to the operator approach we have to cut the circle at some moment of time to get an interval, compute an operator on the space of differential forms, corresponding to this interval, and take a supertrace of it (by supertrace we mean weighting contributions of odd forms with the minus sign).

### 2.3. Searching for novel local geometric observables

As it is clear from the definition of evaluation observables, they form supercommutative algebra, that goes down to supercommutative algebra on cohomology.

Observables that correspond to vector fields are either not closed or exact, so they seem to produce nothing on the level of cohomology.

Looking at these observables one may even mistakenly conclude that all geometrical observables form a supercommutative structure.

However, it is well known that diffeomorphisms that cannot be deformed to identity provide an example of nonsupercommutative geometrical observable. We see that diffeomorphisms not connected to identity are not small deformations and in general do not allow an expansion in powers of small deformation parameters - thus, there is no simple expression for such observables in terms of the "fields" (14)-(17). But diffeomorphisms have a clear geometrical meaning and one can normally work with such observables in geometric formalism.

Below we will generalize this example. We will find many observables that have geometrical meaning, do not commute with evaluation observables, and decrease the degree of the wave-form. Thus, we study here a non-perturbative completion of evaluation and small-deformation observables, studied in [1-3].

## 3. Integrated super-jump operator and its generalizations

### 3.1. Super-jump operator

In this section we construct a new observable in operator formulation and then explain its geometrical meaning. Consider a jump operator, associated with a vector field $v$ on $X$ :

$$
\begin{equation*}
\operatorname{Jump}_{\epsilon v}=\mathrm{e}^{\epsilon \mathscr{L}_{v}} . \tag{20}
\end{equation*}
$$

Since $\mathcal{L}_{v}$ is $\left\{d_{X}, \iota_{v}\right\}$,

$$
\begin{equation*}
\text { Jump }-1=\left\{d_{X}, \ldots\right\} \tag{21}
\end{equation*}
$$

so we are not getting anything interesting.
In order to get something interesting we need to consider a super-jump operator

$$
\begin{equation*}
\operatorname{SJump}_{v}(\epsilon)=\mathrm{e}^{\epsilon \mathcal{L}_{v}+d \epsilon \iota_{v}} \tag{22}
\end{equation*}
$$

that is a differential form on the space of parameters $\epsilon$.
Note, that this operator is $d_{X}+d_{\epsilon}$ closed, therefore, being integrated along the cycle in the $\epsilon$-space it gives the $d_{X}$-closed operator (we remind that $Q=d_{X}$ ).

We may interpret Jump ${ }_{\epsilon v}$ for different $\epsilon$ as integrals of the super-jump operator against points (zero cycles) in $\epsilon$-space, corresponding to different values of $\epsilon$. Since $\epsilon$ space is connected, all of them are equivalent to zero jump, which also follows from Eq. (21).

Now it is clear how to get something more interesting - we just need to have the space of parameters with more nontrivial cycles.

The simplest choice is to consider the $\epsilon$-space being a circle. It means that the action of the vector field is lifted to the action of the circle, i.e. it has periodical trajectory with equal periods (that we may take to be 1 ), in other terms

$$
\begin{equation*}
\operatorname{Jump}_{v}=\mathrm{e}^{\mathscr{L}_{v}}=1 \tag{23}
\end{equation*}
$$

In this case the $\epsilon$-space has a nontrivial cycle - fundamental cycle, and we have a new operator $K_{v}$ defined as integral of the super-jump operator along this cycle

$$
\begin{equation*}
K_{v}=\int_{\epsilon \in S^{1}} \operatorname{SJump}_{v}(\epsilon)=\int_{S^{1}} \mathrm{~d} \epsilon \mathrm{e}^{\epsilon \mathcal{L}_{v}} \iota_{v} \tag{24}
\end{equation*}
$$

The geometrical meaning of insertion of $K_{v}$ at time $t_{K}$ is to allow trajectories that are the trajectories of the vector field $V_{0}$ everywhere outside $t_{K}$ (solving Eq. (9)) but they may have a jump at time $t_{K}$ along the orbit of the circle action.

Later we will see that operator $K$ is nontrivial in cohomology and does not supercommute with the evaluation observables. However, formulas above show that it is built out of observables associated to vector field - how could this
happen? The tricky point is that the operator $K$ is $Q$-closed in a nontrivial way. It is built using non-closed operator $\pi$, and the integrand in (24) is non-closed. However, the integral is closed since the vector field $v$ produces a circle action.

Let us make simple operator computations for the case where the target space is a circle itself, and $X$ is an angle on that circle. Then $K \equiv K_{\frac{\partial}{\partial X}}$ operator acting on degree 0 forms gives zero, and acting on degree one-form gives a number, which is an integral of this form over the circle. Now it is clear that $K$ acts nontrivially in cohomologies since it gives 1 when acting on delta-form $\delta(X) \psi$ (which can be nontrivial in cohomologies of $X$ ), but it gives zero if it acts on the vacuum $\mathbf{1}$ prior to $\delta(X) \psi$.

Thus we see that $K$ is $Q=d_{X}$ closed but not exact. In Section 4 we will use this for operator computations of correlation functions.

### 3.2. Generalization 1: projection operator

The above construction implies the following generalization. Consider a projection from the target $X$ to base manifold $B$ :

$$
\begin{equation*}
p r: X \rightarrow B . \tag{25}
\end{equation*}
$$

This defines a fibration and we assume that fibers are compact.
Define the operator $K_{\text {fib }}$ that acts on differential forms as follows: first integrate the differential form against fibers of $p r$ to get a form on $B$. Such operation is called $p r_{*}$ (the differentials transverse to fibers are identified with base differentials). Then take a pullback of the integrated form from $B$ back to $X$ (this we denote by $p r^{*}$ ), thus

$$
\begin{equation*}
K_{f i b} \omega=p r^{*} p r_{*} \omega=p r^{*} \int_{\text {fiber }} \omega \tag{26}
\end{equation*}
$$

Such an operation (anti)commutes with de Rham differential $d_{X}$ since both operations $p r^{*}$ and $p r_{*}$ (anti)commute with $d_{X}$ for compact fibers without boundary, thus it acts in cohomologies.

In quantum mechanics the evaluation observables correspond to multiplication of the wave function by some form (consider, e.g. a $\delta$-form), which obviously does not commute with integration of the wave function over the fiber.
Geometrical meaning In geometric formalism the insertion of $K_{\text {fib }}(t)$ has an effect of jump in the instanton solution at instant $t$ to any point on the fiber, containing the point $X(t)$. So, it is an arbitrary jump along the fiber. This definition tells what is the resulting instanton space (space of trajectories in case of QM ). Since all correlation functions are computed as integrals over instanton space, the definition is constructive.

### 3.3. Generalization 2: compact cycles in the group of diffeomorphisms of $X$

The example with the circle, described in Section 3.1 can be interpreted in terms of yet another construction. We may consider rotations along the arbitrary angle as a special one-dimensional cycle in the group of diffeomorphisms of $X$. It turns out that the construction above may be generalized to an arbitrary cycle in this group.

Indeed, consider the group Diff $X$ of diffeomorphisms of $X$, denote its action on $X$ by Act : (Diff $X) \times X \rightarrow X$. Choose a finite-dimensional compact cycle in diffeomorphisms: $C \subset \operatorname{Diff} X$.

Forms on $X$ may be pulled back to $\operatorname{Diff} X \times X$ and integrated against the cycle $C$. We may define the corresponding operator

$$
\begin{equation*}
K_{C} \omega=\int_{C} \text { Act }^{*} \omega \tag{27}
\end{equation*}
$$

In geometric formalism this construction corresponds to allowing such jumps that start- and end-points of the jump may be connected by a diffeomorphism in $C$. It is clear that the action of $K_{C}$ in $d_{X}$-cohomology is independent on the continuous deformations of $C$.

The simplest example of this construction is a point (i.e. zero-cycle) in the space of Diff $X$. This means that some fixed diffeomorphism is inserted. Such constructions were already studied in the literature under the name character-valued index [8,9]. The particular case of it for de Rham complex is known as Lefschetz number. Our jump constructions reduce then to twisting of the boundary conditions on the worldsheet circle used in these works.

### 3.4. Digression: cutting operator

Note that local observable in Hamiltonian language is an operator $V \rightarrow V$ where $V$ is a vector space $\left(V=\Omega^{\bullet}(X)\right.$ in our case). Any operator can be formally represented as an infinite sum of its matrix elements: $0=\sum C_{i j}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$.

Now observe that a simplest operator $K$ on $X=S^{1}$ can be represented as

$$
\begin{equation*}
K=\left|\delta_{X}\right\rangle\langle X| \tag{28}
\end{equation*}
$$

where $\left|\delta_{X}\right\rangle$ corresponds to unit wave function. This formula holds for arbitrary target if $K$ allows jumps to any point of the target. It is then natural to interpret such $K$ as cutting a time interval with free boundary conditions for both ends of the cut. This hints another possible generalization. Let us choose two cycles $C_{1}$ and $C_{2}$ on $X$ and consider the corresponding wave functions $\delta_{C_{1,2}}$ which are $\delta$-forms, corresponding to these cycles.

Consider the operator

$$
\begin{equation*}
K_{C_{1}, C_{2}}=\left|\delta_{C_{1}}\right\rangle\left\langle C_{2}\right| \tag{29}
\end{equation*}
$$

that cuts the time interval and creates particular boundary conditions. In geometric formalism it enforces the trajectory to pass through cycle $C_{2}$ and after passing it the trajectory jumps to arbitrary point of cycle $C_{1}$. It is easy to express $K_{C_{1}, C_{2}}$ in terms of $K$ (arbitrary jump to any point of $X$ ) and evaluation observables:

$$
\begin{equation*}
K_{C_{1}, C_{2}}=\delta_{C_{2}} K \delta_{C_{1}} \tag{30}
\end{equation*}
$$

the fermion degree of $K_{C_{1}, C_{2}}$ is $n_{f}\left(K_{C_{1}, C_{2}}\right)=\operatorname{deg} \delta_{C_{1}}+\operatorname{deg} \delta_{C_{2}}-\operatorname{dim} X=\operatorname{dim} X-\operatorname{dim} C_{1}-\operatorname{dim} C_{2}$. This operator obviously does not commute with evaluation observables.

## 4. Examples of geometrical computation of correlators with $K$

### 4.1. Correlator with one insertion of $K$

To have a simplest example, consider a quantum mechanics on the circle and take the target manifold to be also a circle. Recall that $K=K_{\frac{\partial}{\partial X}}$ corresponds to arbitrary jump on the circle.

Take an evaluation observable corresponding to one-form $\omega: \mathrm{ev}_{t_{2}}^{*} \omega=\omega\left(t_{2}\right) \psi\left(t_{2}\right)$ and compute

$$
\begin{equation*}
\left\langle K\left(t_{1}\right) \operatorname{ev}_{t_{2}}^{*} \omega\right\rangle=\int_{S^{1}} \omega \tag{31}
\end{equation*}
$$

Let us start with geometrical computation of $\left\langle K\left(t_{1}\right) \omega\left(t_{2}\right) \psi\left(t_{2}\right)\right\rangle$. Note, that the space of allowed trajectories is a space of constant maps - so it equals to $S^{1}$ and is compact. If $V_{0}=c$ (see (9) for definition of $V_{0}$ ) then the space of allowed trajectories is $X(t)=X\left(t_{1}\right)+c\left(t-t_{1}\right)$ and also equals to $S^{1}$ (being parametrized, say, by $\left.X\left(t_{1}\right)\right)$.

When we compute evaluation observable on this space we still get $\int_{S^{1}} \omega$ (it is independent of $c$ as we expected, because $\left.\int_{S^{1}} \omega\left(X_{1}+c\left(t_{2}-t_{1}\right)\right)=\int \omega\right)$. The example with non-zero $c$ shows that allowing a jump is really necessary, otherwise there are no solutions.

The operator computation for the same correlator gives $\operatorname{STr}(K \omega)$. Since the image of $K$ is only constants, the computation of $S T r$ reduces to multiplying 1 by $\omega$, acting with $K$ and projecting to constants. From the multiplication table (last paragraph in Section 3.1) it follows that the result is $\int_{S^{1}} \omega$.

### 4.2. Example with two $K$ observables and two evaluation observables

From the very beginning of topological theories there was a lot of confusion about the nature of topological observables $\int_{C_{i}} \mathcal{O}_{i}$, associated to cycles $C_{i}$ on the worldsheet. The original proposal of Witten implied that correlator should be independent under deformation of cycles in the same homology class. However, it was again Witten (in the Chern-Simons theory) who gave an example of correlators that are linking numbers. The resolution of the confusion is in different behavior of correlators of integrands

$$
\left\langle\mathcal{O}_{1}(x) \mathcal{O}_{2}(y)\right\rangle
$$

of the observables. If this correlator is smooth when $x$ and $y$ coincide, the correlator of topological observables really goes to homology of cycles $C_{i}$. However, if it is singular, the only allowed moves of cycle $C_{1}$ are in the complement to $C_{2}$ in the worldsheet, so we get a linking. This goes to dimension 1 of the worldsheet as follows. Correlator of observables associated to points (times) may be either smooth (supercommutative when points are interchanging their position) or not. In the latter case we have a one-dimensional linking, that is the dependence of the correlator on the order of points (usually linking is defined as a pairing between $d$-dimensional contractible cycles in $(2 d+1)$-dimensional space, in our case $d=0)$.

Since the operators $K$ do not commute with evaluation observables, we expect to get invariants, such as linking numbers, by computing the correlation functions. Consider two one-forms $\omega_{1}$ and $\omega_{2}$.

From the operator approach the linking is almost obvious since $K^{2}=0$ and $\omega_{1} \omega_{2}=0$ by the form degree considerations. Still, we would like to reproduce this result in geometrical way. Two $K$ operators geometrically split the circle in two intervals, each of these intervals may be mapped to its own point on $X$ (or a trajectory if $c \neq 0$ ), so when each of the intervals contains $\omega$, the answer is $\int_{S^{1}} \omega_{1} \int_{S^{1}} \omega_{2}$, and is zero otherwise. Taking $V_{0}$ to be non-zero does not really change the answer.

$$
\begin{equation*}
\left\langle K\left(t_{1}\right) K\left(t_{2}\right) \operatorname{ev}_{t_{3}}^{*} \omega_{1} \mathrm{ev}_{t_{4}}^{*} \omega_{2}\right\rangle=\int_{S^{1}} \omega_{1} \int_{S^{1}} \omega_{2} \operatorname{Link}\left(\left(t_{1}, t_{2}\right),\left(t_{3}, t_{4}\right)\right) \tag{32}
\end{equation*}
$$

where to define $\operatorname{Link}\left(\left(t_{1}, t_{2}\right),\left(t_{3}, t_{4}\right)\right)$ we fix an oriented paths connecting $\left(t_{1}, t_{2}\right)$ on $S^{1}$ and count intersections of it with points $t_{4}$ and $-t_{3}$ with signs, determined by the relative orientation. This gives the linking number.

## 5. Integrated observables

We considered above the observables that were placed at fixed times, so they corresponded to some geometrical event (like jump of prescribed type or passing through the chain of prescribed type) that happened at this particular moment. But there is an important class of problems where one is interested in geometrical event that happens at some (unspecified) time. To deal with such problems we integrate over time of insertion of observables, and these observables are called integrated observables.

It is instructive to compare the operator and geometric approaches to construction of such observables.
Let us begin with the most known example of integrated evaluation observable. Consider the evaluation observable $\mathrm{ev}^{*} \omega$, which is a form on both the time of evaluation and the instanton space: $\Omega\left(\mathbb{R}_{t} \times \mathcal{M}\right)$. Explicitly, having a space of instanton solutions $X(t, m)$, the evaluation observable equals to:

$$
\mathrm{ev}^{*} \omega(X, d X)=\omega\left(X(t, m), \frac{\partial X(t, m)}{\partial t} d t+\frac{\partial X(t, m)}{\partial m_{a}} d m_{a}\right)
$$

where $m_{a}$ stand for coordinates on the moduli space. The component of $\mathrm{ev}^{*} \omega$ that has zero degree along the space of times of evaluation is the fixed time evaluation observable $\mathrm{ev}_{t}^{*} \omega$ discussed above. The component containing $d t$ is the observable

$$
\operatorname{ev}_{t}^{*}\left(\iota_{\frac{\partial X}{\partial t}} \omega\right) d t=\operatorname{ev}_{t}^{*}\left(\iota_{V} \omega\right) d t
$$

and is known as a descent observable. This observable may be integrated against a subspace in the space of times. If there are no other observables (or if correlator is smooth in the sense described above) and if the space of times is a circle, one can integrate this observable against this circle (that is how we get integrated observables). However, if the space of times is an interval (or there are other observables such that the correlator is singular and we may integrate only along the interval of continuity) we meet the phenomena of boundary in the space of integration and it makes the meaning of integrated correlators more interesting - they correspond to deformations of the $Q$-operator.

In order to see this we consider the operator approach in general topological quantum mechanics.

### 5.1. Integrated observables and deformations of $Q$-operator

### 5.1.1. Topological quantum mechanics as a particular case of general topological quantum field theory

Consider a general topological field theory. In Atiyah formulation we should consider manifolds with boundary. Components of boundary are labeled as incoming and outgoing, and each component (incoming or outgoing) is associated to vector space $V_{i}^{\text {in }}$ or $V_{i}^{\text {out }}$ respectively. The main object in Atiyah formulation of TFT is a map that associates to any manifold with boundary a linear map $I$

$$
\begin{equation*}
I \in V_{1}^{\text {in }} \otimes \cdots \otimes V_{p}^{\text {in }} \rightarrow V_{1}^{\text {out }} \otimes \cdots \otimes V_{q}^{\text {out }} \tag{33}
\end{equation*}
$$

that factorizes under cutting manifold into pieces. Applying this formulation to quantum mechanics we consider intervals and associate the same vector spaces $V$ to both incoming and outgoing boundaries. According to Atiyah we should associate to an interval a linear operator

$$
U \in \operatorname{End}(V)
$$

such that

$$
U^{2}=U
$$

i.e. $I$ is a projector onto some space $V_{0}$; since correlators of all operators $\Phi$ are given by their restriction to $V_{0}$ : I $\Phi I$, we may start with $V=V_{0}$. This is nice but it is not exactly what we have in geometrical theories.

To include such theories in the formalism we need to extend Atiyah's formulation to Segal's one - namely, we have to replace manifolds by manifolds equipped with local geometrical data. By local data we mean the data on $X$ that uniquely determines the data on any piece of $X$, i.e. there is a map

$$
\operatorname{Cut}_{i}: \operatorname{Geom}(X) \rightarrow \operatorname{Geom}\left(X_{i}\right) .
$$

As an example of such data we may take metric or complex structure.
According to Segal, the main object is a map I from $\operatorname{Geom}(X)$ to the space (33), i.e.

$$
\begin{equation*}
I \in V_{1}^{\text {in }} \otimes \cdots \otimes V_{p}^{\text {in }} \otimes V_{1}^{\text {out }, *} \otimes \cdots \otimes V_{q}^{\text {out,* }} \otimes \operatorname{Funct}(\operatorname{Geom}(X)) \tag{34}
\end{equation*}
$$

such that for $X=X_{1} \cup X_{2}$

$$
\begin{equation*}
I(X)=C u t_{1}^{*} I(X) \cdot C u t_{2}^{*} I(X) \tag{35}
\end{equation*}
$$

here • stands for the natural contraction between vector spaces corresponding to boundaries that appear in cutting. In application to quantum mechanics (where we take the metric on time as a local geometrical data) it means that

$$
U\left(t_{1}\right) U\left(t_{2}\right)=U\left(t_{1}+t_{2}\right)
$$

where $t_{i}$ are lengths of the intervals. This equation is solved by

$$
U(t)=\exp (-t H)
$$

that is a well-known evolution process in operator formulation of quantum mechanics, where $H \in \operatorname{End}(V)$ is a Hamiltonian (in Euclidean signature).

In order to define topological theory we replace spaces $V$ and Funct $(\operatorname{Geom}(X))$ by complexes. For the space $V$ we may take the same space but with a differential $Q$ that squares to zero, while Funct $(X)$ has to be replaced by the space $\Omega(X)$ of all differential forms on the space of geometrical data, so that operator $d_{\text {Geom }}$ acts on it. The main condition for $I$ is the closeness of $I$ with respect to the total action:

$$
\begin{equation*}
\left(Q+d_{\text {Geom }}\right) I=0 \tag{36}
\end{equation*}
$$

together with factorization condition that looks exactly like (35) with space of functions being replaced by the space of differential forms on the geometrical data.

The universal solution to Eq. (36) in the case of quantum mechanics is given by

$$
\begin{equation*}
U(t, d t)=\exp \left(-\left[Q+d_{t}, t G\right]\right)=\exp (-t H-d t G), \quad \text { with } H=\{Q, G\} \tag{37}
\end{equation*}
$$

Now we may define observables, we will do it here for the case of manifold $X$ equipped with the Riemann metric. People use to study local observables, however, we will define here the notion of subspace observables as follows. Consider the subspace $Y$ of the worldsheet space $X$, and consider the $\epsilon$ tubular neighborhood of $Y$,

$$
Y_{\epsilon}=\{x \in X, \operatorname{dist}(x, Y)<\epsilon\}
$$

where $\operatorname{dist}(x, Y)$ is a distance between the point $x$ and the subspace $Y$, and we will take $\epsilon$ to be small enough.
Consider $I\left(X \backslash Y_{\epsilon}\right)$, it has additional boundary formed by points

$$
\Gamma\left(Y_{\epsilon}\right)=\{x \in X, \operatorname{dist}(x, Y)=\epsilon\}
$$

This boundary contains one component when dimension of $X$ is bigger than 1 , while it contains two components for onedimensional $X$. In the former case we will take the boundary to be incoming, while in the latter case we take one component to be incoming and the second - outgoing. Finally, let us take the state $v_{\epsilon}$ in the multidimensional case and the operator $\Phi_{\epsilon}$ in the one-dimensional case such that the $\epsilon \rightarrow 0$ limit of the contraction between $I$ and $v$ exists. So we define in the multidimensional case

$$
\begin{equation*}
I\left(X, O(Y)_{v}\right)=\lim _{\epsilon \rightarrow 0} I\left(X \backslash Y_{\epsilon}\right) v_{\epsilon} \tag{38}
\end{equation*}
$$

and in the one-dimensional case

$$
\begin{equation*}
I\left(X, O(P)_{\Phi}\right)=\lim _{\epsilon \rightarrow 0} I\left(X \backslash P_{\epsilon}\right) \cdot \Phi_{\epsilon} \tag{39}
\end{equation*}
$$

where $P$ is a point and $\cdot$ stands for the contraction between the operator and $V \times V^{*}$ associated to the two boundaries of the tubular neighborhood of the point $P$.

While peculiarities of the limit are rather interesting in the multidimensional case, in the one-dimensional case the situation is rather simple. Therefore, the generic correlator in quantum mechanics is given by a well-known formula

$$
\begin{equation*}
\langle o u t| U\left(T-t_{n}\right) \Phi_{n} \ldots U\left(t_{2}-t_{1}\right) \Phi_{1} U\left(t_{1}\right)|i n\rangle \tag{40}
\end{equation*}
$$

and the only difference in the topological quantum mechanical case is given by replacement of evolution operators $U(t)$ by their superanalogues.

This means that the generic correlator of local observables in quantum mechanics and the universal correlator on an interval equals to

$$
\begin{equation*}
I=\langle\text { out }| U\left(T-t_{n}, d T-d t_{n}\right) \Phi_{n} \ldots U\left(t_{2}-t_{1}, d t_{2}-d t_{1}\right) \Phi_{1} U\left(t_{1}, d t_{1}\right)|i n\rangle \tag{41}
\end{equation*}
$$

here $t_{i}$ are the positions of marked points on the interval of length $T$. One may show that

$$
\begin{equation*}
d I=0 \tag{42}
\end{equation*}
$$

for $Q$-closed operators and initial and final states. In particular, the zero-form component is independent of $t$ - that is the topologicity in strict sense. The topologicity for higher forms is not that obvious - it only means that integrals of $I$ along cycles do not depend on smooth deformations of these cycles.

### 5.2. Integrated observable

### 5.2.1. Integrated descent observable and non-Q-closeness of its integral

Now we are in position to give the universal definition of the integrated observable - (recall that in geometrical incarnation it stated that corresponding geometrical event happens at nonspecified moment) it means that we integrate the differential form (41) along the position of the marked point. Therefore, from the perspective of original quantum mechanics it corresponds to insertion of the operator

$$
\Phi_{i}^{(1)}=\left\{G, \Phi_{i}\right\}
$$

at point $t_{i}$ and integration of it along the time manifold. Symbolically, we may say that we study

$$
\left\langle\int_{X} \Phi_{i}^{(1)} \Phi_{1}\left(t_{1}\right) \ldots \Phi_{i-1}\left(t_{i-1}\right) \Phi_{i+1}\left(t_{i+1}\right) \ldots \Phi_{n}\left(t_{n}\right)\right\rangle .
$$

Such operator was introduced by Witten as descendant operator, since it obviously solves the descent equation

$$
\begin{equation*}
\left\{Q, \Phi^{(1)}\right\}=[H, \Phi]=\frac{d}{d t} \Phi \tag{43}
\end{equation*}
$$

where the last equality holds under correlator.
Naively, one may think that such observables preserve $Q$ - the naive argument goes as follows: Take $Q$-exact operator $\Phi_{1}=[Q, \Psi]$ and put it under correlator. Take $Q$ from $\Psi$ and act with it on $\Phi_{i}-$ it would give a total derivative. Suppose that we integrate along a compact time manifold without boundaries - then the integral of total derivative is zero.

Naiveness of this argument shows up already when we consider time manifold with boundaries - in this case total derivative results in action of operator $\Phi$ on boundary states. It makes us think that decoupling of $Q$-closed observable happens under additional condition that boundary states are annihilated by $\Phi_{i}$. Moreover, close inspection of the region of integration reveals another type of boundaries - when integrated operator hits operators, placed at fixed moments $t_{1}, \ldots t_{n}$. In this case the boundary contributions are expressed as commutators

$$
\left[\Phi_{i}, \Phi_{j}\right]
$$

### 5.2.2. Homological meaning of integrated observable

One may think that boundary contributions for integrated observables obstruct the homological interpretation of integrated observable. However, situation is simpler than one may expect: integrated observables correspond to deformations of $Q$-symmetry. In particular, consider deformation of $Q$ symmetry of the following form:

$$
Q_{\tau}=Q+\tau \Phi
$$

where we assume that

$$
\Phi^{2}=0
$$

If we keep the superpartner of the Hamiltonian - $G$ - intact we conclude that the zero degree component of the evolution operator changes as follows

$$
\exp (-t(H+\tau\{G, \Phi\}))=\exp (-t H)+\tau \int \mathrm{d} t_{1} \exp \left(-\left(t-t_{1}\right) H\right)\{G, \Phi\} \exp \left(-t_{1} H\right)+\cdots
$$

and one-form component is not changing (here we also assume that $G^{2}=0$ ) i.e. we just have the generating function for integrated observable with generating parameter $\tau$. Now we may easily interpret the boundary contributions - they correspond to the action of $Q_{\tau}$ on states and observables, and vanishing of boundary terms means that such states and observables are annihilated by the family of operators $\Phi$.

But this is not natural - rather one would expect that there is a family of operators $\Phi_{\tau}$ or a family of states annihilated by $Q_{\tau}$ :

$$
\begin{equation*}
(Q+\tau \Phi)\left(\left|i n_{0}\right\rangle+\tau\left|i n_{1}\right\rangle+\tau^{2}\left|i n_{2}\right\rangle+\cdots\right)=0 \tag{44}
\end{equation*}
$$

It is easy to show that taking into account the change of initial state $\left|i n_{1}\right\rangle$ we cancel the non-closeness of the integrated observable.

However, even this is not the end of the story - below we will show that there are obstructions in finding of such families. Moreover, these obstructions are also expressed in terms of integrated correlators.

### 5.2.3. Obstructions and integrated correlators

Consider the problem of construction of perturbative family of $Q_{\tau}$ closed states like in (44), modulus $Q_{\tau}$ exact states. Clearly, $\left|i n_{0}\right\rangle$ should be a representative of $Q$-cohomology class. What about $\left|i n_{1}\right\rangle$ ? It should be a solution to

$$
\begin{equation*}
Q\left|i n_{1}\right\rangle=\Phi\left|i n_{0}\right\rangle \tag{45}
\end{equation*}
$$

The right hand side of (45) is Q-closed while the equation itself states that stronger statement holds - it is $Q$-exact. The obstruction for this belongs in the cohomology class of the right hand side of (45), i.e. it is measured by

$$
\begin{equation*}
O b s t r_{1}=\left\langle\text { out }_{0}\right| \Phi\left|i n_{0}\right\rangle \tag{46}
\end{equation*}
$$

where $\left\langle o u t_{0}\right|$ is an element of the dual space of states representing a generic class of $Q$-cohomology. If the obstruction equals to zero we may proceed to the second order problem where we compute

$$
\begin{equation*}
Q\left|i n_{2}\right\rangle=\Phi\left|i n_{1}\right\rangle=\Phi Q^{-1} \Phi\left|i n_{0}\right\rangle \tag{47}
\end{equation*}
$$

and the second order obstruction equals to

$$
\begin{equation*}
O b s t r_{2}=\left\langle o u t_{0}\right| \Phi Q^{-1} \Phi\left|i n_{0}\right\rangle \tag{48}
\end{equation*}
$$

In the case of topological quantum mechanics there is a natural candidate for $Q^{-1}$, namely, let us take

$$
\begin{equation*}
h_{\mathrm{QM}}=\int_{0}^{+\infty} G \mathrm{~d} t \mathrm{e}^{-t H} \tag{49}
\end{equation*}
$$

If $H=\{Q, G\}$ satisfies the Hodge condition, i.e. it is positive definite outside the cohomology and vanishes on the cohomology, then the integral in the right hand side of (49) exists and

$$
\begin{equation*}
\left\{Q, h_{\mathrm{QM}}\right\}=1-\Pi, \tag{50}
\end{equation*}
$$

where $\Pi$ is the projector on the space of zero modes of $H$. It means that

$$
Q h_{Q M} \Phi\left|i n_{0}\right\rangle=\Phi\left|i n_{0}\right\rangle-\Pi \Phi\left|i n_{0}\right\rangle=\Phi\left|i n_{0}\right\rangle
$$

where the second equality holds when the first obstruction vanishes, so $h_{Q M}$ really works as $Q^{-1}$.
This construction is called Hodge construction since it was extensively studied on the example of de Rham cohomology of compact Riemann manifold. In this case

$$
G=d^{*} \quad \text { and } \quad H=\Delta
$$

such topological quantum mechanics is well known as $\mathcal{N}=1$ supersymmetric quantum mechanics.
It could be that $h_{Q M}$ may serve as $Q^{-1}$ even if Hodge condition is not satisfied. To see this we consider

$$
\begin{equation*}
Q \int_{0}^{+\infty} G \mathrm{~d} t \mathrm{e}^{-t H} \Phi\left|i n_{0}\right\rangle=\Phi\left|i n_{0}\right\rangle-\mathrm{e}^{-\infty H} \Phi\left|i n_{0}\right\rangle \tag{51}
\end{equation*}
$$

Therefore, in this case $h_{\mathrm{QM}}$ may work as inverse $Q$ if the limiting action of $\exp (-\infty H)$ on the state $\Phi\left|i n_{0}\right\rangle$ does not only exist but also equals to zero.

Interestingly enough this may happen in geometrical quantum mechanics where the Hamiltonian is the Lie derivative. In general, vector field may have limiting cycles (this may be cured by considering Morse vector field), and still the limiting action of the Morse flow may be non-vanishing. However, we will encounter below the example where everything works.

All this means that it is reasonable to consider the following correlator in topological quantum mechanics

$$
\begin{equation*}
\left\langle u_{0}\right| \Phi h_{Q M} \Phi\left|v_{0}\right\rangle=\int_{0}^{+\infty}\left\langle u_{0}\right| \Phi G \mathrm{~d} t \mathrm{e}^{-t H} \Phi\left|v_{0}\right\rangle \tag{52}
\end{equation*}
$$

that under condition discussed above leads to the second obstruction to solution of homological problem (44).
From the point of view of general topological quantum mechanics it is an integral over the space of metrics on an interval. Such object is often called an answer in topological gravity since we integrate against the space of metrics on a space-time, that is time in our case. From the point of view of geometrical topological theory it means that some geometrical event (given by the action of $\Phi$ on $\left|i n_{0}\right\rangle$ ) has happened at the beginning of time, then evolution took place until the second event happened (given by $\left.\left\langle o u t_{0}\right| \Phi\right)$.

### 5.3. Geometrical examples of deformation of $Q$

### 5.3.1. Massey operations

The first example of deformed operator $Q$ comes from evaluation observables. In this case we consider Witten-Novikov operator

$$
d+\tau \omega
$$

where differential form corresponds to evaluation observable associated to $\omega$.
Interestingly, obstructions (starting from the second one) that we mentioned above correspond to Massey operations. In particularly, it means that they may be computed in geometrical formulation of quantum mechanics, i.e. in terms of number of trajectories of the vector field passing through cycles (associated to differential form $\omega$ ). In this sense we see that higher obstructions are nothing but one-dimensional analogues of the celebrated Gromov-Witten invariants that compute the number of holomorphic curves passing through the prescribed set of cycles. We will discuss it in more details elsewhere, but it is not the main topic in the present paper - here we would like to concentrate on $K$ operators, that correspond to integrated jumps.

### 5.4. Equivariant cohomologies and jump operators

It turns out that geometrical problems associated to arbitrary jump operator $K$ arise in computation of equivariant cohomology.

Suppose that we have a $U(1)$ bundle $X$ with the base $Y$. One may study equivariant cohomology, i.e. cohomology on the space of $U(1)$ invariant forms with differential

$$
\begin{equation*}
Q_{e q}=d+\tau \iota_{v} \tag{53}
\end{equation*}
$$

where the vector field $v$ generates the $U(1)$ action. It is known that equivariant cohomology in the space of differential forms taking values in polynomials in $\tau$ are related to the cohomology of the base as follows: one has to substitute $\tau$ with the first Chern class of the bundle, i.e. with the class of curvature of the $U(1)$ connection.

In the case of integrated $K$ observable we should study the operator

$$
\begin{equation*}
Q_{d e f}=d+\tau K \tag{54}
\end{equation*}
$$

acting on the space of all differential forms. Since $K$ involves integration along the fiber it projects forms to invariant ones. It seems that people have missed the operator (54) since it is not differential operator, but we pay attention to it since it is geometrical.

Really, computation of obstructions for such new operator turns out to be an interesting geometrical problem in geometrical quantum mechanics. In particular, we may consider a Hopf bundle, that is a sphere $S^{3}$ fibered over a sphere $S^{2}$. Let us compute the second obstruction for deformation of the three-form that is a delta function on a point that we will call $P$. It is clear that the first obstruction vanishes. Really, the action of $K$ on the three-form gives a two-form that is a delta function on a fiber passing through this point. Since all two-cycles on a three-sphere are contractable the first obstruction vanishes.

The quantum mechanical expression for contraction provides a more detailed information on how this contraction happens. Really, consider as a Hamiltonian the special vector field $V_{0}$ on a three-sphere that leaves one point invariant and contracts the rest of the sphere to another point such that these fixed points of the vector fields do not coincide with the point $P$. The integral

$$
\int_{0}^{T} \exp (-t H) G \mathrm{~d} t \delta_{\text {Fiber }_{P}}
$$

is given by a one-form delta-form on an annulus formed by evolution lines of the special vector field $V_{0}$ that happens in time $T$ and that starts on the fiber passing through the point $P$. When $T$ goes to infinity this annulus tends to a disc (and the fiber passing through the point $P$ is its only boundary). It means that conditions of special homotopy (see (51) and below) hold.

Now we need to apply $K$ operator to it and intersect with the outcoming cycle. However, geometrically it is more convenient to apply $K$-operator to the outcoming cycle and intersect it with the disc.

Really, if we take another point $R$ as an outcoming cycle then the action of $K$ on it gives the delta function on the fiber passing through the point $Q$. Therefore, the second obstruction equals to intersection of the fiber passing through the point $Q$ and the disc, whose boundary is the fiber passing through $P$, i.e. it equals to linking number between fibers. This number equals to 1 for Hopf fibration.

Putting everything together, we get

Geometrically, the only trajectory contributing to the correlator looks as follows: it starts at point $P$, jumps along the fiber, then it moves along the trajectory of vector field over the disc toward the intersection with the second fiber. At this point trajectory jumps again to point $R$. That is how jump operators reveal themselves in computations in equivariant cohomology (really, in a problem equivalent to computation of equivariant cohomology).

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[^1]:    ${ }^{1}$ For definition of these observables it is sufficient to consider the formal neighborhood of zero in $\mathbb{C}^{n}$ rather than the full $\mathbb{C}^{n}$.

[^2]:    2 For example, on 2D plane $(x, y)$ a form $\delta(x)(\theta(y)-\theta(y-1)) d x$ corresponds to an interval $[(0,0),(0,1)]$.

