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ON MAPPINGS OF BOUNDED VARIATION

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ABSTRACT. We present the properties of mappings of bounded variation defined on a subset of the real line with values in metric and normed spaces and show that major aspects of the theory of real-valued functions of bounded variation remains valid in this case. In particular, we prove the structure theorem and obtain the continuity properties of these mappings as well as jump formulas for the variation. We establish the existence of Lipschitz continuous geodesic paths and prove an analog of the well-known Helly selection principle. For normed space-valued smooth mappings we obtain the usual integral formula for the variation without the completeness assumption on the space of values. As an application of our theory we show that compact set-valued mappings (= multifunctions) of bounded variation admit regular selections of bounded variation.

1. INTRODUCTION

The aim of this paper is to obtain the properties of mappings of bounded variation (BV, for short) in the classical sense of Camille Jordan. Consider a BV-mapping $f : E \to X$ defined on the nonempty subset E of the real line $\mathbb{R}$ with values in the metric (or normed) space X. If $X = \mathbb{R}$ and E is a closed bounded closed interval $[a, b]$ or an open interval $]a, b[$, the theory of BV-functions is well established and known (for instance, Natanson [19], Ch. 8, or Folland [9], Ch. 3). In particular, $f : E \to \mathbb{R}$ is a BV-function if and only if it is the difference of two bounded nondecreasing functions (the Jordan decomposition). This criterion reduces the theory of BV-functions to that of bounded nondecreasing functions, and the main results follow immediately. However, for metric or normed space-valued mappings the BV-theory seems to be less known (see, however, Schwartz [20], Ch. 4, Sec. 9 and

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Barbu [2], Ch. 1, Sec. 2). If X is a metric space, then the Jordan criterion is inapplicable, yet we will show that the major aspects of the theory of real-valued BV-functions remains valid in this case. Our decomposition theorem reads as follows (Sec. 3): $f : E \to X$ is a BV-mapping if and only if it is a composition of a bounded nondecreasing function $\varphi : E \to \mathbb{R}$ and an X-valued mapping defined on the image of φ and satisfying the Lipschitz condition with the Lipschitz constant ≤ 1 (a particular case of this result was outlined by Federer [8], Sec. 2.5.16). Of course, the theory of metric space-valued BV-mappings is poorer than that of real-valued BV-functions, but we point out that no special structure of the domain E such as connectedness (closed and open intervals, etc.) is needed to obtain the usual properties of BV-mappings. In this way, we establish the continuity properties of BV-mappings and present the "intuitively clear" relations between the total variation of a mapping on the whole of E and its variation on E without a limit point (Secs. 4 and 5).

With the decomposition theorem at hand, in the case of the compact metric space X, we prove that there always exist Lipschitzian geodesic paths between two points of X if there is at least one path of finite length connecting these points (Sec. 6) and that any infinite family of paths in X of uniformly bounded length (variation) admits a sequence which converges pointwise to a BV-mapping (the Helly selection principle, see Sec. 7).

In Sec. 8 we obtain additional properties of BV-mappings with values in normed vector spaces. In particular, we prove that the total variation of a continuously differentiable mapping is equal to the integral of the norm of its derivative without assuming that the normed vector space under consideration is complete.

Finally, in Sec. 9 we consider the set-valued mappings of bounded variation, which play an important role in optimal control theory, especially for nonlinear systems with relaxed controls like impulse controls, and in the theory of differential inclusions (see Lee and Markus [13], Sec. 4.2, Aubin and Cellina [1], Ch. 2, and Mordukhovich [17]). We show that if a set-valued mapping from a closed bounded interval into nonempty subsets of a Banach space is of bounded variation and its graph is compact, then it admits a selection of bounded variation.

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2. Notation, definition, and main properties

Throughout this paper we use the following notation: $\emptyset \neq E \subset \mathbb{R}$, $E_t^- = E \cap]-\infty, t]$, and $E_t^+ = E \cap [t, \infty[$ if $t \in E$, $E_a^b = E \cap [a, b]$ if a, $b \in E$, $a \leq b$ (where $[a, b] \subset \mathbb{R}$ is a closed interval), X is a metric space with a fixed metric (or distance function) $d = d(\cdot\,,\cdot)$, X^E is the set of all mappings $f : E \to X$ from E into X. If $f \in X^E$, we denote by $f(E)$ the image of f in X, and by $\omega(f,E) = \sup\{\,d(f(t),f(s)) : t,\, s \in E\,\}$ the oscillation of f on E (or the diameter of the image $f(E)$). Given two mappings $f : E \to X$ and $\varphi : E_1 \to E$, their composition $f \circ \varphi : E_1 \to X$ is defined as usual by $(f \circ \varphi)(\tau) = f(\varphi(\tau))$ for all $\tau \in E_1$.

Definition. We denote by

$$\mathcal{T}(E) = \big\{\, T = \{t_i\}_{i=0}^m \subset E : m \in \mathbb{N} \cup \{0\},\ t_{i-1} \le t_i,\ i = 1, \ldots, m \,\big\}$$

the set of all partitions of E by finite ordered collections of points in E. Given a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$ and a mapping $f : E \to X$, we set

$$V(f,T) = \sum_{i=1}^{m} d\big(f(t_i), f(t_{i-1})\big),$$

and extend it to the whole E by the formula

$$V(f,E) = \sup\{\, V(f,T) : T \in \mathcal{T}(E) \,\}.$$

The quantity $V(f,E) \in [\,0,\infty]$ is called the *total variation* of f over E. If $V(f,E) < \infty$, a mapping f is called a *bounded variation mapping* (BV-mapping, for short). The set of all BV-mappings from E into X is denoted by $\mathcal{V}(E;X)$. If $\varnothing \ne A \subset E$, we set $V(f,A) = V(f|_A, A)$, where $f|_A$ is the restriction of f to A, and we set $\mathcal{T}(\varnothing) = \varnothing$ and $V(f,\varnothing) = 0$ (so that $\sup\varnothing = 0$). The functional $V : X^E \times 2^E \to \mathbb{R}_0^+ \cup \{\infty\}$ is called a *variation*.

The above definition is classical and is due to C. Jordan (cf. [20], Ch.4, Sec. 9). Note that this definition is also suitable for mappings defined on any linearly ordered set E. A number of results of this paper are valid in the case where $\le$ is a linear ordering on E.

The following assertions constitute the main (almost axiomatic) properties of the variation and are easy to prove, and so we omit the proofs (cf. [6]).

General properties of the variation. Let $f : E \to X$ be an arbitrary mapping. We have

- (P1) *minimality*: if $t,\, s \in E$, then $d(f(t),f(s)) \le \omega(f,E) \le V(f,E)$;
- (P2) *monotonicity*: if $A \subset B \subset E$, then $\mathcal{T}(A) \subset \mathcal{T}(B)$ and $V(f,A) \le V(f,B)$;
- (P3) *additivity*: if $t \in E$, then $V(f,E) = V(f,E_t^-) + V(f,E_t^+)$;
- (P4) the *change of a variable*: if $E_1 \subset \mathbb{R}$ and $\varphi : E_1 \to E$ is a (not necessarily strictly) monotone function, then $V(f,\varphi(E_1)) = V(f \circ \varphi, E_1)$;
- (P5) *regularity*: $V(f,E) = \sup\{\, V(f,E_a^b) : a,\, b \in E,\ a \le b \,\}$;

(P6) the *limit properties*: let $s = \sup E \in \mathbb{R} \cup \{\infty\}$, and let $i = \inf E \in \mathbb{R} \cup \{-\infty\}$, then

($P6_1$) if $s \notin E$, then $V(f, E) = \lim_{E \ni t \to s} V(f, E_t^-)$,

($P6_2$) if $i \notin E$, then $V(f, E) = \lim_{E \ni t \to i} V(f, E_t^+)$,

($P6_3$) if $s \notin E$ and $i \notin E$, then, in addition to ($P6_1$) and ($P6_2$), we have

$$V(f, E) = \lim_{\substack{E \ni a \to i \\ E \ni b \to s}} V(f, E_a^b) = \lim_{E \ni b \to s} \lim_{E \ni a \to i} V(f, E_a^b) = \\ = \lim_{E \ni a \to i} \lim_{E \ni b \to s} V(f, E_a^b);$$

(P7) *lower semicontinuity*: if the sequence of mappings $\{f_n\}_{n=1}^{\infty} \subset X^E$ converges pointwise to f as $n \to \infty$ (i.e., $\lim_{n \to \infty} d(f_n(t), f(t)) = 0$ for all $t \in E$), then $V(f, E) \leq \liminf_{n \to \infty} V(f_n, E)$.

A few remarks are in order. Note that we do not assume the boundedness of $V(f, E)$. Property (P3) is not valid if $t \notin E$ (for example, $E = [-1, 1] \setminus \{0\}$, $X = \mathbb{R}$, $f : E \to X$, $f = 0$ on $[-1, 0[$, $f = 1$ on $]0, 1]$, and $t = 0$; see Sec. 5). If $s \in E$, then $V(f, E) = V(f, E_s^-)$ so that ($P6_1$) is not true in general (consider $f : [0, 1] \to \mathbb{R}$, $f = 0$ on $[0, 1[$ and $f(1) = 1$), and analogously for ($P6_2$) and ($P6_3$). Property (P7) holds if the pointwise convergence of f_n is replaced by a weaker condition: $\liminf_{n \to \infty} d(f_n(t), f(t)) = 0$ for all $t \in E$. However, the inequality $\leq$ in (P7) cannot be replaced by an equality and lim inf cannot be replaced by lim even if the convergence of f_n to f is uniform (in fact, $f_n(t) = (\sin(2\pi n t))/n$, $t \in [0, 1]$, converges uniformly to $f \equiv 0$, but $V(f_n, [0, 1]) = 4$).

Property (P1) implies that a BV-mapping is a bounded mapping in the sense that its image has a finite diameter. The following proposition is a refinement of this property (cf. [7], Ch. 7, Sec. 6, Problem 3 in the case $E = [a, b]$).

Proposition 2.1. *If $f \in \mathcal{V}(E; X)$, then the image $f(E)$ is totally bounded in X and separable. If, in addition, X is complete, then $f(E)$ is precompact (i.e., the closure of $f(E)$ in X is compact).*

Proof. In order to prove that $f(E)$ is totally bounded, we have to show that for any $\varepsilon > 0$ the set $f(E)$ can be covered by a finite number of balls from X of radii ε centered at $f(E)$. On the contrary, suppose that for some $\varepsilon > 0$ there is no cover of this kind. Consider a sequence $\{x_n\}_{n=0}^{\infty} \subset E$ which is given inductively as follows: if $t_0 \in E$ is fixed, set $x_0 = f(t_0)$, and if x_0, $x_1, \dots, x_{n-1} \in f(E)$ are already chosen, take $x_n \in f(E) \setminus \bigcup_{j=1}^{n-1} B_\varepsilon(x_j)$, where $B_\varepsilon(x_j) = \{y \in X : d(y, x_j) < \varepsilon\}$. Let $x_n = f(t_n)$ for some $t_n \in E$, $n \in \mathbb{N}$. Since $d(x_n, x_k) \geq \varepsilon$ for $n \neq k$, we have $t_n \neq t_k$. Without loss of generality, we can suppose that $t_{n-1} < t_n$ for all $n \in \mathbb{N}$. Then, for

$T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we have

$$V(f, E) \geq V(f, T) = \sum_{i=1}^{m} d\big(f(t_i), f(t_{i-1})\big) = \sum_{i=1}^{m} d(x_i, x_{i-1}) \geq m\varepsilon.$$

By virtue of the arbitrariness of $m \in \mathbb{N}$, we infer that $V(f, E) = \infty$; this is a contradiction.

A totally bounded set in a metric space is known to be separable, and precompact if the metric space is complete. $\square$

Since continuous mappings from the closed interval $[a, b]$ into X play an important role in theory (they are called *paths* in X), we are going to recall an equivalent definition of the variation $V(f, [a, b])$ for (one-sided) continuous mappings $f : [a, b] \to X$. We denote by

$$\mathcal{T}_a^b = \big\{ T = \{t_i\}_{i=0}^m \subset [a, b] : m \in \mathbb{N},\ a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b \big\}$$

the set of all partitions of $[a, b]$ containing the points a and b and we set

$$V_a^b(f) = \sup\{ V(f, T) : T \in \mathcal{T}_a^b \}.$$

Obviously, $V_a^b(f) = V(f, [a, b])$. If $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$, then we define the *fineness* of T by $\lambda(T) = \max_{1 \leq i \leq m}(t_i - t_{i-1})$, and we set $\mathcal{B}_\delta = \big\{ T \in \mathcal{T}_a^b : \lambda(T) \leq \delta \big\}$ if $\delta > 0$. We define a *filter base* in $\mathcal{T}_a^b$ to be the set $\{ \mathcal{B}_\delta : \delta > 0 \}$, which is denoted by $\lambda(T) \to 0$.

Theorem 2.2. *If the mapping $f : [a, b] \to X$ is continuous from the right on $[a, b[$ or continuous from the left on $]a, b]$, then we have the following equalities:*

(a) $V_a^b(f) = \lim_{\lambda(T) \to 0} V(f, T)$,
(b) $V_a^b(f) = \lim_{\lambda(T) \to 0} \Omega(f, T)$,

where $\Omega(f, T) = \sum_{i=1}^{m} \omega\big(f, [t_{i-1}, t_i]\big)$ *if* $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$.

(Note that we do not suppose $V_a^b(f)$ to be finite, and that the assumption of one-sided continuity of f is essential for the validity of both (a) *and* (b).*)*

Proof. Although this is a well-known fact (cf. [19], Ch. VIII, Sec. 5, Theorem 2 or [20], Ch. 4, Sec. 9) for completeness we recall the proof of this assertion here.

(a) Let f be continuous from the right on $[a, b[$ (a similar argument holds for f which is continuous from the left on $]a, b]$).

1. First, note that if $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$ and $t_{k-1} \leq t \leq t_k$ for some $1 \leq k \leq m$, then

$$V(f, T \cup \{t\}) \leq V(f, T) + 2d\big(f(t_k), f(t)\big). \tag{2.1}$$

Indeed, we have

$$\begin{aligned}V(f,T\cup\{t\}) &= \sum_{i=1}^{k-1} d(f(t_i),f(t_{i-1})) + d(f(t),f(t_{k-1})) + \\ &+ d(f(t),f(t_k)) + \sum_{i=k+1}^{m} d(f(t_i),f(t_{i-1})) = \\ &= V(f,T) + d(f(t),f(t_{k-1})) + d(f(t_k),f(t)) - \\ &- d(f(t_k),f(t_{k-1})), \end{aligned} \tag{2.2}$$

so that if we take into account the triangle inequality

$$d\big(f(t),f(t_{k-1})\big) \le d\big(f(t_k),f(t_{k-1})\big) + d\big(f(t_k),f(t)\big),$$

then we get (2.1).

2. To prove (a), we have to show that

for any $\varepsilon > 0$ such that $\varepsilon < V_a^b(f)$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $V(f,T) \ge \varepsilon$ for all $T \in \mathcal{T}_a^b$ with $\lambda(T) \le \delta$.

(In other words, we will prove that $V_a^b(f) \le \lim_{\delta\to+0} \inf_{T\in\mathcal{B}_\delta} V(f,T)$.)

Let $\varepsilon_1 \in \mathbb{R}$ be such that $\varepsilon < \varepsilon_1 < V_a^b(f)$. Then there exists a partition $\xi = \{\xi_j\}_{j=0}^n \in \mathcal{T}_a^b$ such that $V(f,\xi) \ge \varepsilon_1$. Since, in particular, f is continuous from the right at each point ξ_j, $j = 1, \ldots, n-1$, $\delta > 0$ can be chosen such that

(i) $d(f(t),f(\xi_j)) \le (\varepsilon_1 - \varepsilon)/2n$ for all $j = 1, \ldots, n-1$ and all $t \in [\,\xi_j, \xi_j + \delta\,]$;

(ii) $\delta < \min\{\,\xi_j - \xi_{j-1} : 1 \le j \le n\,\}$, so that if $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$ and $\lambda(T) \le \delta$, then every closed interval $[\,t_{i-1}, t_i\,]$ contains *at most* one point ξ_j.

Let us show now that $V(f,T) \ge \varepsilon$ if $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$ and $\lambda(T) \le \delta$. Indeed, for such a partition T we have, due to (P2),

$$\varepsilon_1 \le V(f,\xi) \le V(f,T\cup\xi) = V(f,T\cup\{\xi_j\}_{j=1}^{n-1}).$$

If, by virtue of (*ii*), we assume that $t_{k-1} \le \xi_1 \le t_k$ for some $1 \le k \le m$, then, from (2.1) with T replaced by $T\cup\{\xi_j\}_{j=2}^{n-1}$ and $t = \xi_1$, we have

$$V(f,T\cup\{\xi_j\}_{j=1}^{n-1}) \le V(f,T\cup\{\xi_j\}_{j=2}^{n-1}) + 2d(f(t_k),f(\xi_1)).$$

Noting that $|t_k - \xi_1| \le t_k - t_{k-1} \le \lambda(T) \le \delta$, i.e., $t_k \in [\,\xi_1, \xi_1 + \delta\,]$, by virtue of (*i*), we have $d\big(f(t_k),f(\xi_1)\big) \le (\varepsilon_1 - \varepsilon)/2n$ so that

$$\varepsilon_1 \le V(f,T\cup\{\xi_j\}_{j=2}^{n-1}) + 2(\varepsilon_1 - \varepsilon)/2n.$$

Now a similar argument with $\xi_2, \ldots, \xi_{n-1}$ instead of ξ_1 gives

$$\varepsilon_1 \leq V(f,T) + 2(n-1)(\varepsilon_1 - \varepsilon)/2n \leq V(f,T) + \varepsilon_1 - \varepsilon;$$

this implies the inequality $V(f,T) \geq \varepsilon$.

(b) If $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$, then by virtue of (P1) we have

$$d(f(t_i), f(t_{i-1})) \leq \omega(f, [t_{i-1}, t_i]) \leq V_{t_{i-1}}^{t_i}(f),$$

so that summing up over $i = 1, \ldots, m$, and taking into account (P3), we obtain the inequalities

$$V(f,T) \leq \Omega(f,T) \leq V_a^b(f), \quad T \in \mathcal{T}_a^b.$$

Passing to the limit as $\lambda(T) \to 0$, we obtain (b). □

Remark. Theorem 2.2 will not be used until Sec. 8.

3. Structure theorem

Definition. A mapping $f : E \to X$ is of *locally bounded variation* (in the notation, $f \in \mathcal{V}_{\mathrm{loc}}(E;X)$) if $V(f, E_a^b) < \infty$ for all a, $b \in E$, $a \leq b$. Clearly, $\mathcal{V}(E;X) \subset \mathcal{V}_{\mathrm{loc}}(E;X)$.

A mapping $f : E \to X$ is *Lipschitzian* if there exists a number $C \in \mathbb{R}_0^+$ such that $d(f(t), f(s)) \leq C|t-s|$ for all t, $s \in E$. The minimal number C satisfying the above inequality is called the *Lipschitz constant* of f and is denoted by $\mathrm{Lip}(f)$.

A mapping $g : E \to X$ is *naturalized* if $V(g, E_a^b) = b - a$ for all a, $b \in E$, $a \leq b$. Clearly, the naturalized mapping $g : E \to X$ is of locally bounded variation and is Lipschitzian with $\mathrm{Lip}(g) \leq 1$ since, by virtue of (P1), we have

$$d\big(g(t), g(s)\big) \leq V(g, E_t^s) = s - t, \quad t,\ s \in E,\ t \leq s.$$

The main result of this section is the following

Theorem 3.1. *A mapping $f : E \to X$ is of locally bounded variation (resp., is a BV-mapping) if and only if there are a nondecreasing (resp., a bounded nondecreasing) function $\varphi : E \to \mathbb{R}$ and a naturalized mapping $g : \varphi(E) \to X$ (and, hence, g is Lipschitzian with* $\mathrm{Lip}(g) \leq 1$*) such that $f = g \circ \varphi$ on E.*

The proof of this theorem is contained in the following two lemmas. The first lemma (sufficiency) gives a large number of examples of mappings of (locally) bounded variation.

Lemma 3.2. *If $\varphi : E \to \mathbb{R}$ is monotone, $g : \varphi(E) \to X$ is Lipschitzian, and $f = g \circ \varphi$, then $f \in \mathcal{V}_{\mathrm{loc}}(E;X)$. If, in addition, φ is bounded, then $f \in \mathcal{V}(E;X)$.*

Proof. Suppose that φ is nondecreasing. Since

$$\varphi(E\cap[a,b]) = \varphi(E)\cap[\varphi(a),\varphi(b)], \quad a,\, b\in E,\, a\le b, \tag{3.1}$$

by virtue of (P4), we have

$$V(f,E_a^b) = V(g\circ\varphi, E_a^b) = V(g,\varphi(E_a^b)) = V(g,\varphi(E)_{\varphi(a)}^{\varphi(b)}).$$

If $T=\{t_i\}_{i=0}^m$ is a partition of the set in (3.1), then

$$V(g,T)\le \mathrm{Lip}(g)\sum_{i=1}^m (t_i-t_{i-1}) \le \mathrm{Lip}(g)\cdot(\varphi(b)-\varphi(a)),$$

so that

$$V(f,E_a^b)\le \mathrm{Lip}(g)\cdot(\varphi(b)-\varphi(a)) < \infty.$$

Now property (P5) and the monotonicity of φ yield

$$V(f,E)\le \mathrm{Lip}(g)\cdot\Big(\sup_{t\in E}\varphi(t) - \inf_{t\in E}\varphi(t)\Big) = \mathrm{Lip}(g)\cdot\omega(\varphi,E).$$

If, in addition, φ is bounded ($\omega(\varphi,E)<\infty$), then $f\in\mathcal{V}(E;X)$.

If φ is nonincreasing, the proof is analogous. □

Remarks. (a) If the mapping g in Lemma 3.2 is naturalized, then, in addition, we have

$$V(f,E_a^b) = |\varphi(b)-\varphi(a)|, \quad a,\, b\in E,\, a\le b, \quad \text{and} \quad V(f,E)=\omega(\varphi,E).$$

In particular, if $f: E\to\mathbb{R}$ is monotone (resp., bounded and monotone), then, setting $X=\mathbb{R}$, $\varphi=f$ and $g(s)=s$ if $s\in f(E)$ in Lemma 3.2, we have $f\in\mathcal{V}_{\mathrm{loc}}(E;\mathbb{R})$ (resp., $f\in\mathcal{V}(E;\mathbb{R})$).

(b) If $f: E\to X$ is Lipschitzian (and E is bounded resp.), then $f\in\mathcal{V}_{\mathrm{loc}}(E;X)$ (resp., $f\in\mathcal{V}(E;X)$) and

$$V(f,E_a^b)\le \mathrm{Lip}(f)\cdot(b-a), \quad a,\, b\in E,\, a\le b,$$
$$\text{resp.,}\ V(f,E)\le \mathrm{Lip}(f)\cdot(\sup E - \inf E).$$

To this end, it suffices to consider $\varphi(t)=t$ for $t\in E$ and $g=f$ in Lemma 3.2.

The second lemma (necessity) gives the canonical decomposition of a mapping of (locally) bounded variation.

Lemma 3.3. *Let $f\in\mathcal{V}_{\mathrm{loc}}(E;X)$. Then there exist a nondecreasing function $\varphi: E\to\mathbb{R}$ and a naturalized mapping $g: E_1=\varphi(E)\to X$ such that*

(a) *$f=g\circ\varphi$ on E;*
(b) *$g(E_1)=f(E)$ in X;*
(c) *$V(g,E_1)=V(f,E)$ in $[0,\infty]$.*

If, moreover, $f\in\mathcal{V}(E;X)$, then, in addition, the function φ is bounded and the values in (c) *are finite.*

Proof. Fix a point $a \in E$, and set

$$\varphi(t) = \begin{cases} V(f, E_a^t) & \text{if } t \in E_a^+, \\ -V(f, E_t^a) & \text{if } t \in E_a^-, \end{cases} \qquad t \in E.$$

The function $\varphi : E \to \mathbb{R}$ is well defined, nondecreasing by (P2), and $\varphi(a) = 0$. If $\tau \in E_1$, we denote by $\varphi^{-1}(\tau) = \{t \in E : \varphi(t) = \tau\}$ the inverse image of the one-point set $\{\tau\}$ for the function φ. We define the mapping $g : E_1 \to X$ as follows: if $\tau \in E_1$, we set

$$g(\tau) = f(t) \quad \text{for any point} \quad t \in \varphi^{-1}(\tau). \tag{3.2}$$

This is correct, i.e., $f(t)$ is one and the same element of X for all $t \in \varphi^{-1}(\tau)$ since, by virtue of (P1) and (P3), we have

$$d\big(f(s), f(t)\big) \le V(f, E_t^s) = \varphi(s) - \varphi(t), \quad t \in E,\ s \in E_t^+; \tag{3.3}$$

indeed, if t, $s \in \varphi^{-1}(\tau)$, $t \le s$, then $\varphi(t) = \tau = \varphi(s)$, so that (3.3) implies $f(t) = f(s)$.

Now, the representation of f in (a) follows from (3.2), since if $t \in E$, then $\tau = \varphi(t) \in E_1$ and $t \in \varphi^{-1}(\tau)$, so that (3.2) yields $f(t) = g(\tau) = g(\varphi(t)) = (g \circ \varphi)(t)$. Item (b) follows from (a), and item (c) follows from (P4) and (a).

It remains to prove that g is naturalized. Taking into account (3.1), we have

$$E_1 \cap [0, \tau] = \varphi(E \cap [a, t]), \quad 0 \le \tau \in E_1, \quad t \in \varphi^{-1}(\tau),$$

so that applying (P4), we have

$$V(g, (E_1)_0^\tau) = V(g, \varphi(E_a^t)) = V(g \circ \varphi, E_a^t) = V(f, E_a^t) = \varphi(t) = \tau.$$

Similarly,

$$V(g, (E_1)_\tau^0) = -\tau \quad \text{if} \quad 0 \ge \tau \in E_1.$$

Hence, if α, $\beta \in E_1$, $0 \le \alpha \le \beta$, then, by virtue of (P3), we have

$$V(g, (E_1)_\alpha^\beta) = V(g, (E_1)_0^\beta) - V(g, (E_1)_0^\alpha) = \beta - \alpha.$$

The cases where $\alpha \le 0 \le \beta$ and $\alpha \le \beta \le 0$ are completely analogous. □

Remarks.

(a) Note that the mapping g in the proof of Lemma 3.3 satisfies the following property: if α, $\beta \in E_1$, and $t \in \varphi^{-1}(\alpha)$, $s \in \varphi^{-1}(\beta)$, then

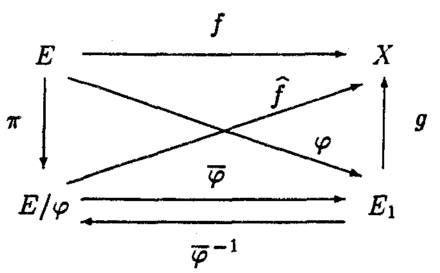
$$d\big(g(\alpha), g(\beta)\big) = d\big(g(\varphi(t)), g(\varphi(s))\big) = d\big(f(t), f(s)\big).$$

(b) In the case where $\varphi : E \to E_1$ is strictly increasing, it is a bijection, so that the equality $f = g \circ \varphi$ on E is equivalent to the equality $g = f \circ \varphi^{-1}$ on E_1, where $\varphi^{-1} : E_1 \to E$ is the inverse function of φ.

(c) If, in addition, $f \in \mathcal{V}(E;X)$, then $|\varphi(t)| \le V(f,E)$, $t \in E$, so that the function φ is bounded and the two values in Lemma 3.3(c) are finite. In this case, one can replace φ by the function $E \ni t \mapsto V(f, E_t^-) \in \mathbb{R}_0^+$.

(d) If $f : E \to \mathbb{R}$, then the Jordan decomposition is valid as was mentioned in the Introduction: $f \in \mathcal{V}_{\rm loc}(E;\mathbb{R})$ (resp., $f \in \mathcal{V}(E;\mathbb{R})$) if and only if f is the difference of two nondecreasing (resp., bounded nondecreasing) functions on E. The sufficiency is clear from the fact that a monotone (resp., bounded monotone) function is of locally BV (resp., BV) and that the difference of two functions of this kind is again of locally BV (resp., BV). The necessity follows from the equality $f = \varphi - (\varphi - f)$ with $\varphi(t) = V(f, E_t^-)$, $t \in E$, and (3.3) since, if t, $s \in E$, $t \le s$, then $f(s) - f(t) \le \varphi(s) - \varphi(t)$, or, equivalently, $(\varphi - f)(t) \le (\varphi - f)(s)$.

Now we shall briefly present an algebraic aspect in the construction of a naturalized mapping. The arguments in Theorem 3.1 (necessity) go back to the *factorization* of a mapping. We describe these aspects here. The mapping $\varphi : E \to E_1$ induces an equivalence relation on E as follows: $t \sim s$ in $E \iff \varphi(t) = \varphi(s)$ in E_1. Let $\bar{t} = \{ s \in E : s \sim t \}$ be the equivalence class of $t \in E$ in the quotient set E/φ, and let $\pi : E \to E/\varphi$ be the canonical surjection given by $\pi(t) = \bar{t}$ for $t \in E$. We have $\bar{t} = \varphi^{-1}(\tau)$, where $\tau = \varphi(t) \in \varphi(E)$, and if we set $\overline{\varphi}(\bar{t}) = \varphi(s)$ for any $s \in \bar{t}$, or $\overline{\varphi} \circ \pi = \varphi$, then the mapping $\overline{\varphi} : E/\varphi \to E_1$ is well defined and is called the *factorization* of φ; in other words, we set $\overline{\varphi}(\varphi^{-1}(\tau)) = \tau$ for all $\tau \in \varphi(E)$, so that $\overline{\varphi}$ is (always) injective. In general, $\overline{\varphi}$ is not surjective, but if φ is surjective, as is the case in Lemma 3.3, then so is $\overline{\varphi}$, and, therefore, the bijection $\overline{\varphi}$ has the inverse $\overline{\varphi}^{-1} : E_1 \to E/\varphi$ given by $\overline{\varphi}^{-1}(\tau) = \varphi^{-1}(\tau)$ if $\tau \in E_1 = \varphi(E)$.



Given a mapping $f : E \to X$, let $\widehat{f} : E/\varphi \to X$ be defined by $\widehat{f}(\bar{t}) = f(s)$ if $\bar{t} \in E/\varphi$ and $s \in \bar{t}$ (see the diagram), so that $\widehat{f} \circ \pi = f$. The mapping $\widehat{f}$ is well defined only if t, $s \in E$ and $t \sim s$ imply $f(t) = f(s)$ in X, which is satisfied in Lemma 3.3 due to (3.3). Now, if $g = \widehat{f} \circ \overline{\varphi}^{-1} : E_1 \to X$, then it follows that $\widehat{f} = g \circ \overline{\varphi}$, and we have the "characteristic" representation

$$f = \widehat{f} \circ \pi = (g \circ \overline{\varphi}) \circ \pi = g \circ (\overline{\varphi} \circ \pi) = g \circ \varphi.$$

Comparing the decompositions $f = \widehat{f} \circ \pi = \widehat{f} \circ \overline{\varphi}^{-1} \circ \varphi$ and $g = \widehat{f} \circ \overline{\varphi}^{-1}$, we infer that (a) the "interesting" properties of f contained in $\widehat{f}$ are preserved by g (see Lemma 3.3(b) and (c)); (b) g "simplifies" f (in particular, the cardinality $\#(E)$ of E is not less than the cardinality $\#(E_1) = \#(E/\varphi)$); (c) practically, the mapping g is more "valuable" than f (since it is more interesting and simpler!).

Finally, note that in the case where $\varphi : E \to E_1$ is strictly increasing, it is a bijection, so that $E/\varphi = E$, $\overline{\varphi} = \varphi$, $\widehat{f} = f$ and $g = f \circ \varphi^{-1}$.

4. Continuity properties

We now turn to continuity properties of BV-mappings which, at the same time, will result (in Sec. 5) in the intuitively clear formulas relating the total variation of a mapping to its variation on the set from which a limit point is removed.

In this section and in the next one $f : E \to X$ is a fixed BV-mapping and the function $\varphi : E \to \mathbb{R}$ is defined by $\varphi(t) = V(f, E_t^-)$ for $t \in E$.

Theorem 4.1.

(a) *f is continuous from the right at the point $t \in E$, $t \neq \sup E$ (resp., from the left at the point $t \in E$, $t \neq \inf E$) if and only if the function φ has this property*;

(b) *f is continuous on E outside, possibly, of a countable subset of E.*

Proof. (a) We only consider the case of continuity from the right. If $t \in E$, $t \neq \sup E$, is an isolated (i.e., not limit) point of the set E_t^+, the assertion is obvious. Hence, in the rest of the proof we assume that t is a limit point of the set E_t^+.

Sufficiency follows from (3.3), which holds for all $s \in E_t^+$.

Necessity. It suffices to prove that for every $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon) \in E$ with $t_0 > t$ such that

$$\varphi(s) - \varphi(t) \leq d(f(s), f(t)) + \varepsilon \quad \forall\, s \in E_t^{t_0}. \tag{4.1}$$

Since $V(f, E_t^+) \leq V(f, E) < \infty$, by (P2), for $\varepsilon > 0$ there exists a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E_t^+)$ with $t < t_0$ such that

$$V(f, E_t^+) \leq d(f(t_0), f(t)) + V(f, T) + \varepsilon.$$

Noting that, actually, $T \in \mathcal{T}(E_{t_0}^+)$ and applying (P1) and (P3), we have, for all $s \in E_t^{t_0}$,

$$\begin{aligned} V(f, E_t^+) &\leq d(f(t_0), f(s)) + d(f(s), f(t)) + V(f, E_{t_0}^+) + \varepsilon \leq \\ &\leq V(f, E_s^{t_0}) + d(f(s), f(t)) + V(f, E_{t_0}^+) + \varepsilon = \\ &= V(f, E_s^+) + d(f(s), f(t)) + \varepsilon, \end{aligned}$$

and again applying (P3), we have

$$V(f,E_s^-)-V(f,E_t^-)=V(f,E_t^+)-V(f,E_s^+)\le d(f(s),f(t))+\varepsilon;$$

this is (4.1).

(b) This assertion follows from the fact that a nondecreasing function on E has at most a countable number of points of discontinuity and that, by (a), the sets of discontinuity points of f and the nondecreasing function φ are the same.

Note that a similar theorem holds for the mappings $f\in\mathcal{V}_{\mathrm{loc}}(E;X)$. □

Theorem 4.2. *Let $t\in E$ be a limit point of the set E_t^- (resp., E_t^+). Then*

(a) $V(f,E_t^-)-V(f,E_t^-\setminus t)=\varphi(t)-\varphi(t-)=\lim_{E\ni s\to t-0}V(f,E_s^t)$ *(resp.,* $V(f,E_t^+)-V(f,E_t^+\setminus t)=\varphi(t+)-\varphi(t)=\lim_{E\ni s\to t+0}V(f,E_t^s)$ *);*

(b) *f is continuous from the left (resp., from the right) at the point t if and only if* $V(f,E_t^-)=V(f,E_t^-\setminus t)$ *(resp.,* $V(f,E_t^+)=V(f,E_t^+\setminus t)$ *); here* $\varphi(t\pm)=\lim_{E\ni s\to t\pm 0}\varphi(s)\in\mathbb{R}_0^+$.

Proof. Note, first of all, that the limit $\varphi(t-)$ (resp., $\varphi(t+)$) exists since it is equal to $\sup\{\varphi(s)\,:\,s\in E_t^-\setminus t\}$ and the function φ is bounded and nondecreasing on E.

(a) Property (P6_1) (resp., (P6_2)) with the set $E_t^-\setminus t$ (resp., $E_t^+\setminus t$) instead of E implies (resp., by virtue of (P3), implies)

$$V(f,E_t^-\setminus t)=\lim_{E\ni s\to t-0}V(f,(E_t^-\setminus t)_s^-)=\lim_{E\ni s\to t-0}V(f,E_s^-)=\varphi(t-)$$

$$\text{(resp., } V(f,E_t^+\setminus t)=\lim_{E\ni s\to t+0}V(f,(E_t^+\setminus t)_s^+)=\lim_{E\ni s\to t+0}V(f,E_s^+)=$$
$$=V(f,E)-\lim_{E\ni s\to t+0}V(f,E_s^-)=V(f,E_t^+)+\varphi(t)-\varphi(t+)\text{).}$$

Thus, if we again apply (P3), (a) follows from the equations

$$\varphi(t)-\varphi(t-)=\lim_{E\ni s\to t-0}(V(f,E_t^-)-V(f,E_s^-))=\lim_{E\ni s\to t-0}V(f,E_s^t)$$

$$\text{(resp., }\varphi(t+)-\varphi(t)=\lim_{E\ni s\to t+0}(V(f,E_t^+)-V(f,E_s^+))=\lim_{E\ni s\to t+0}V(f,E_t^s)\text{).}$$

(b) follows from (a) and Theorem 4.1(a). □

Theorem 4.3. *Let $t\in E$ be a limit point of each of the sets E_t^- and E_t^+. Then*

(a) $V(f,E)=V(f,E_t^-\setminus t)+V(f,E_t^+\setminus t)+(\varphi(t+)-\varphi(t-))$, *and f is continuous at t if and only if* $V(f,E)=V(f,E_t^-\setminus t)+V(f,E_t^+\setminus t)$;

(b) $V(f,E) = V(f,E\setminus t) + (\varphi(t+) - \varphi(t-)) - \lim_{\substack{E\ni a\to t-0\\ E\ni b\to t+0}} V(f,E_a^b\setminus t)$, *and if f is continuous at t, then $V(f,E) = V(f,E\setminus t)$; in general, the converse statement is false.*

Proof. (a) We can obtain the equality in (a) by adding the equalities from Theorem 4.2(a) and applying (P3), and the assertion in (a) follows from Theorem 4.2(a), the equality in (a), and the inequalities $\varphi(t-) \le \varphi(t) \le \varphi(t+)$.

(b) We give the proof in steps for clarity.

(1) To prove the equality in (b), we note that for all a, $b \in E$ such that $a < t < b$, we have, due to (P3),

$$\begin{aligned} V(f,E) - V(f,E\setminus t) &= (V(f,E_a^-) + V(f,E_a^b) + V(f,E_b^+)) - \\ &- (V(f,E_a^-) + V(f,E_a^b\setminus t) + V(f,E_b^+)) = \\ &= (\varphi(b) - \varphi(a)) - V(f,E_a^b\setminus t). \end{aligned} \tag{4.2}$$

The equality in (b) follows if we take into account the fact that

$$\varphi(b) - \varphi(a) \to \varphi(t+) - \varphi(t-) = (\varphi(t) - \varphi(t-)) + (\varphi(t+) - \varphi(t)), \tag{4.3}$$

$$V(f,E_a^b\setminus t) \to \inf\{\, V(f,E_a^b\setminus t) : a\in E_t^-,\, b\in E_t^+,\, a<t<b\,\} \in \mathbb{R}_0^+ \tag{4.4}$$

as $E\ni a\to t-0$ and $E\ni b\to t+0$.

(2) Before proving the second part of (b), we show (not assuming the continuity of f at t) that, on the one hand,

$$d(f(b),f(a)) \le V(f,E_a^b\setminus t) \quad \forall\, a\in E_t^-, \forall\, b\in E_t^+,\ a<t<b, \tag{4.5}$$

and, on the other hand,

for every $\varepsilon > 0$ there exist $a_0 = a_0(\varepsilon)$, $b_0 = b_0(\varepsilon) \in E$, $a_0 < t < b_0$, such that

$$V(f,E_a^b\setminus t) \le d(f(b),f(a)) + \varepsilon \quad \forall\, a\in E_{a_0}^t\setminus t, \quad \forall\, b\in E_t^{b_0}\setminus t. \tag{4.6}$$

Inequality (4.5) is a consequence of (P1). To prove (4.6), fix $\varepsilon > 0$, and, by the definition of the variation $V(f,E\setminus t)$ which is $\le V(f,E) < \infty$, consider a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E\setminus t)$ such that

$$t_0 \le t_1 \le \cdots \le t_{k-1} < t < t_k \le \cdots \le t_{m-1} \le t_m \quad \text{for some} \quad 1\le k\le m$$

and

$$V(f,E\setminus t) \le V(f,T) + \varepsilon = \sum_{i=1}^m d(f(t_i),f(t_{i-1})) + \varepsilon.$$

Setting $T_1 = \{t_i\}_{i=0}^{k-1}$, $T_2 = \{t_i\}_{i=k}^{m}$, $a_0 = t_{k-1}$, $b_0 = t_k$, and noting that, actually, $T_1 \cup \{a\} \in \mathcal{T}(E_a^-)$ and $\{b\} \cup T_2 \in \mathcal{T}(E_b^+)$, we have the following calculations for all a, $b \in E$ such that $a_0 \le a < t < b \le b_0$:

$$\begin{aligned} V(f, E \setminus t) &\le V(f, T_1) + d(f(t_k), f(t_{k-1})) + V(f, T_2) + \varepsilon \le \\ &\le V(f, T_1) + d(f(a), f(t_{k-1})) + d(f(b), f(a)) + \\ &+ d(f(t_k), f(b)) + V(f, T_2) + \varepsilon \le \\ &\le V(f, E_a^-) + d(f(b), f(a)) + V(f, E_b^+) + \varepsilon, \end{aligned}$$

so that now (4.6) follows from (4.2).

(3) Assuming the continuity of f at t, we obtain the assertion in (b) from the equality in (b) since limit (4.3) is zero by Theorem 4.1(a) and limit (4.4) is zero by (4.6).

(4) The converse assertion in (b) is false: consider, for instance, $E = [-1, 1]$, $f : E \to \mathbb{R}$, $f = -1$ on $[-1, 0[$, $f(0) = 0$, $f = 1$ on $]0, 1]$, and $t = 0$. □

5. Jump formulas

Up till now we have made no assumptions concerning the metric space X under consideration. In this section we assume that X is a *complete* metric space, $f : E \to X$ is a fixed BV-mapping, and the function $\varphi : E \to \mathbb{R}$ is still given by $\varphi(t) = V(f, E_t^-)$ for all $t \in E$. We are going to obtain relations between the total variation of the BV-mapping on E with its variation on E without a (deleted) limit point.

Lemma 5.1. (a) *At every point $t \in E$, which is a limit point of the set E_t^+ (resp., E_t^-), there exists a limit (in the metric d) from the right $f(t+) = \lim_{E \ni s \to t+0} f(s) \in X$ (resp., from the left $f(t-) = \lim_{E \ni s \to t-0} f(s) \in X$).*

(b) *Moreover, if $\sup E \in (\mathbb{R} \setminus E) \cup \{\infty\}$ (resp., if $\inf E \in (\mathbb{R} \setminus E) \cup \{-\infty\}$), then there exists a limit in X of $f(s)$ as $E \ni s \to \sup E$ (resp., as $E \ni s \to \inf E$).*

Proof. (a) Consider the case of a limit from the right. As was already mentioned at the beginning of the proof of Theorem 4.2, the limit $\varphi(t+)$ exists in $\mathbb{R}_0^+$, so that the function φ satisfies the Cauchy criterion of the existence of $\varphi(t+)$. In view of (3.3), it follows that the Cauchy criterion for the existence of the limit $f(t+)$ is valid in the *complete* metric space X.

(b) As in (a), this follows from the existence of the corresponding limit of the function $\varphi(s)$ which, by virtue of ($\dot{\text{P}}$6), is equal to $V(f, E) < \infty$ (resp., zero). □

Now the jumps of the function φ in Theorem 4.2(a) and Theorem 4.3 can be considered as the jumps of the desired mapping f.

Lemma 5.2. *The following formulas hold:*
(a) *if $t \in E$ is a limit point of the set E_t^-, then*

$$\varphi(t) - \varphi(t-) = d(f(t), f(t-));$$

(b) *if $t \in E$ is a limit point of the set E_t^+, then*

$$\varphi(t+) - \varphi(t) = d(f(t+), f(t));$$

(c) *if $t \in E$ is a limit point of the sets E_t^- and E_t^+, then*

$$\lim_{\substack{E \ni a \to t-0 \\ E \ni b \to t+0}} V(f, E_a^b \setminus t) = d(f(t+), f(t-));$$

here $f(t+)$ and $f(t-)$ are as in Lemma 5.1.

Proof. Since (a) is analogous to (b), we prove only (b) and (c).

(b) can be obtained by passing to the limit as $E \ni s \to t+0$ in (3.3) and (4.1) if we take into account the arbitrariness of $\varepsilon > 0$ in the second limit;

(c) can be obtained by passing to the limit as $E \ni a \to t-0$, $E \ni b \to t+0$ in (4.5) and (4.6) if we take into account the arbitrariness of $\varepsilon > 0$ in the second limit. □

The formulas in Theorems 4.2(a) and 4.3 assume the most applicable form.

Theorem 5.3. *Let X be a complete metric space, and let $f : E \to X$ be a BV-mapping. Then we have*

(a) *if $t \in E$ is a limit point of the set E_t^-, then*
$V(f, E_t^-) = V(f, E_t^- \setminus t) + d(f(t), f(t-))$;
(b) *if $t \in E$ is a limit point of the set E_t^+, then*
$V(f, E_t^+) = V(f, E_t^+ \setminus t) + d(f(t+), f(t))$;
(c) *if $t \in E$ is a limit point of each of the sets E_t^- and E_t^+, then*

$$V(f, E) = V(f, E_t^- \setminus t) + V(f, E_t^+ \setminus t) + d(f(t), f(t-)) + d(f(t+), f(t)),$$

and, moreover,

$$V(f, E) = V(f, E \setminus t) + d(f(t), f(t-)) + d(f(t+), f(t)) - d(f(t+), f(t-)), \tag{5.1}$$

$$V(f, E \setminus t) = V(f, E_t^- \setminus t) + V(f, E_t^+ \setminus t) + d(f(t+), f(t-)).$$

Proof. It is obvious from the above considerations. □

It is interesting to study the formulas in Theorem 5.3 even in the case of the closed interval $E = [a, b]$.

Corollary 5.4. *Let X be a complete metric space and $f \in \mathcal{V}([a, b]; X)$. Then we have*

(a) $V(f,[a,t[) = V_a^t(f) - d(f(t), f(t-)),\ t \in]a,b]$;

(b) $V(f,]t,b]) = V_t^b(f) - d(f(t+), f(t)),\ t \in [a,b[$.

If $t \in]a,b[$, *then*

(c) $V_a^b(f) = V(f,[a,t[) + V(f,]t,b]) + d(f(t), f(t-)) + d(f(t+), f(t))$;

(d) $V(f,[a,b]\setminus t) = V_a^b(f) - d(f(t), f(t-)) - d(f(t+), f(t)) + d(f(t+), f(t-))$;

(e) $V(f,[a,b]\setminus t) = V(f,[a,t[) + V(f,]t,b]) + d(f(t+), f(t-))$;

and

(f) $V(f,]a,b[) = V_a^b(f) - d(f(a+), f(a)) - d(f(b), f(b-))$.

Remark. Note that the equalities in Theorem 5.3 are some kind of "limit forms" of more elementary equalities. Consider, for instance, formula (5.1). If we have a set $T \subset E$ of $m+2$ points $t_0 \le t_1 \le \cdots \le t_{k-1} \le t \le t_k \le \cdots \le t_{m-1} \le t_m$, then, obviously (see the equality (2.2))

$$V(f,T) = V(f,T\setminus t) + d(f(t), f(t_{k-1})) + d(f(t_k), f(t)) - d(f(t_k), f(t_{k-1})).$$

This equality has a "limit" in the form of equality (5.1).

6. Geodesic paths

We denote by $C([a,b];X)$ the set of all continuous mappings from $[a,b]$ into the metric space X. A *path* in X is a continuous mapping $f:[a,b]\to X$; its *trajectory* is the image $f([a,b])$ which, as is well known, is a compact subset of X. The domain $[a,b]$ of f is said to be a *set of parameters* of (on) the path; in this case we also say that the path is *parametrized* by the closed interval $[a,b]$. The *length* of the path $f:[a,b]\to X$ is its total variation $V_a^b(f)$. Two points x, $y \in X$ are said to be *connected by a path* in X if there exists a path $f:[a,b]\to X$ such that $f(a)=x$ and $f(b)=y$; in this case we say that f is a *path between* x *and* y.

Theorem 6.1. *Let $K \subset X$ be a compact subset. If the points x, $y \in K$ can be connected by a path in K of finite length, then there exists a Lipschitzian path in K between x and y of minimal length (such a path is called a geodesic path between x and y).*

Proof. The theorem is trivial if $x = y$. Hence we suppose that $x \ne y$. Since any path $f:[a,b]\to X$ can be replaced by a path of the same length (and the same trajectory) and the set of parameters $[0,1]$ (see (P4)), it suffices to restrict our consideration to paths defined on $[0,1]$. Thus, consider the set of paths in K defined on $[0,1]$, and connecting the points x and y:

$$W(x,y) = \{ f \in C([0,1];K) : f(0)=x,\ f(1)=y \},$$

and set

$$\ell = \inf\{ V_0^1(f) : f \in W(x,y) \}.$$

By the assumption, $W(x,y)$ contains a path f_0 of finite length, so that $\ell \le V_0^1(f_0)$ is finite. On the other hand, for any $f \in W(x,y)$ we have, by virtue of (P1),

$$V_0^1(f) \ge d(f(0), f(1)) = d(x,y) > 0, \tag{6.1}$$

so that $\ell \ge d(x,y)$. Since $\ell < \infty$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset W(x,y)$ such that

$$\lim_{n\to\infty} \ell_n = \ell, \quad \text{where} \quad \ell_n = V_0^1(f_n) > 0 \quad \text{by (6.1)}.$$

The existence of the latter limit implies that if $L = \sup_{n\in\mathbb{N}} \ell_n$, then L is finite > 0, so that the sequence $\{f_n\}$ is of uniformly bounded variation. By Lemma 3.3, for any $n \in \mathbb{N}$ there exists a naturalized path $g_n : [0, \ell_n] \to X$ with the properties

$$d(g_n(\alpha), g_n(\beta)) \le |\alpha - \beta|, \quad \alpha, \beta \in [0, \ell_n],$$

$$f_n = g_n \circ \varphi_n \quad \text{on} \quad [0,1], \quad \text{where} \quad \varphi_n(t) = V_0^t(f_n), \ t \in [0,1],$$

and, in particular, $g_n(0) = f_n(0) = x$, $g_n(\ell_n) = f_n(1) = y$, $g_n([0,\ell_n]) = f_n([0,1]) \subset K$ and $V_0^{\ell_n}(g_n) = V_0^1(f_n) = \ell_n$. If we set $h_n(\tau) = g_n(\tau\ell_n)$, $\tau \in [0,1]$, then we have

$$h_n \in W(x,y),$$

$$V_0^1(h_n) = \ell_n \to \ell \quad \text{as} \quad n \to \infty \quad \text{(by (P4))},$$

$$d(h_n(\alpha), h_n(\beta)) \le \ell_n|\alpha - \beta| \le L|\alpha - \beta|, \quad \alpha, \beta \in [0,1].$$

It follows that the sequence $\{h_n\}_{n=1}^{\infty} \subset \mathcal{C}([0,1];K)$ is equicontinuous, so that by the Ascoli-Arzelà theorem (cf. [9], p. 131, Theorem (4.44)), this sequence has a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \sup_{\tau\in[0,1]} d(h_{n_k}(\tau), h(\tau)) = 0 \quad \text{for some} \quad h \in \mathcal{C}([0,1];K).$$

Obviously, $h \in W(x,y)$, and h is Lipschitzian with $\mathrm{Lip}(h) \le L$. From (P7) we infer that

$$V_0^1(h) \le \liminf_{k\to\infty} V_0^1(h_{n_k}) = \lim_{k\to\infty} \ell_{n_k} = \ell.$$

It remains to note that from the definition of ℓ we have $\ell \le V_0^1(h)$, so that $\ell = V_0^1(h)$; this completes the proof. □

7. Helly selection principle

The main result of this section is the following analog of the classical *E. Helly selection principle.*

Theorem 7.1. *Let X be a compact metric space and $\mathcal{F} \subset C([a,b];X)$ be an infinite family of continuous mappings of uniformly bounded variation, i.e., $\sup_{f\in\mathcal{F}} V_a^b(f) < \infty$. Then there exists a sequence of mappings $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ which converges pointwise on $[a,b]$ to a BV-mapping $f:[a,b]\to X$.*

Proof. By Theorem 3.1, any mapping $f \in \mathcal{F}$ can be written in the form $f = g_f \circ \varphi_f$ on $[a,b]$, where $\varphi_f(t) = V_a^t(f)$, $a \le t \le b$, and $g_f : [0,\ell_f] \to X$ is a Lipschitzian mapping with $\mathrm{Lip}(g_f) \le 1$ and $\ell_f = V_a^b(f)$. Note that φ_f is nondecreasing, nonnegative, and $\varphi_f(a) = 0$, and, since f is continuous, φ_f is also continuous, so that $\varphi_f([a,b]) = [0,\ell_f]$. The family of nondecreasing functions $\{\varphi_f : f \in \mathcal{F}\}$ is infinite and uniformly bounded (since $\omega(\varphi_f,[a,b]) = \varphi_f(b) = V_a^b(f)$), and, hence, it contains a sequence of functions φ_n corresponding to the decomposition $f_n = g_n \circ \varphi_n$ which converges pointwise on $[a,b]$ to the nondecreasing function $\varphi : [a,b] \to \mathbb{R}$ (cf. [19], Ch. 8, Sec. 4, Lemma 2). Let $\ell = V_a^b(\varphi) = \varphi(b)$. Then $0 \le \ell < \infty$, and, if $\ell_n = V_a^b(\varphi_n) = \varphi_n(b)$, then $\ell_n \to \ell$ as $n \to \infty$. If some $\ell_n \ge \ell$, then we consider g_n only on the closed interval $[0,\ell]$, and if some $\ell_n < \ell$, then we extend g_n to $]\ell_n,\ell]$ by setting $g_n(\tau) = g_n(\ell_n)$ for all $\tau \in]\ell_n,\ell]$. Then, by the Ascoli-Arzelà theorem, the sequence of Lipschitzian mappings $g_n : [0,\ell] \to X$ with $\mathrm{Lip}(g_n) \le 1$ has a uniformly convergent subsequence $\{g_{n_k}\}_{k=1}^{\infty}$. Let g be the uniform limit of $\{g_{n_k}\}$. Then $g:[0,\ell]\to X$ is Lipschitzian with $\mathrm{Lip}(g) \le 1$, so that, by virtue of Lemma 3.2, $f = g \circ \varphi$ is a BV-mapping on $[a,b]$. Now, if $t \in [a,b]$, we have

$$\begin{aligned} d(f_{n_k}(t), f(t)) &= d((g_{n_k} \circ \varphi_{n_k})(t), (g\circ\varphi)(t)) \le \\ &\le d(g_{n_k}(\varphi_{n_k}(t)), g_{n_k}(\varphi(t))) + d(g_{n_k}(\varphi(t)), g(\varphi(t))) \le \\ &\le |\varphi_{n_k}(t) - \varphi(t)| + d(g_{n_k}(\varphi(t)), g(\varphi(t))) \end{aligned}$$

with the right-hand side tending to zero as $k \to \infty$. Thus, f_{n_k} converges pointwise on $[a,b]$ to f. This completes the proof. □

Remark. As can be seen from the proof of Theorem 7.1, the assumption of continuity of the family $\mathcal{F}$ seems to be indispensable for the validity of this theorem. However, we do not know the exact counterexample. Note that if $X = \mathbb{R}$ the continuity of $\mathcal{F}$ is redundant (see [19], Ch. 8, Sec. 4, the Helly theorem) since the Jordan decomposition takes place in this case. On the other hand, the continuity of the family $\mathcal{F}$ does not, in general, imply that the resulting BV-mapping f is continuous, as the following simple example shows: the sequence $f_n : [0,2] \to \mathbb{R}$ defined by $f_n(t) = t^n$ if $t \in [0,1]$ and by $f_n(t) = (2-t)^n$ if $t \in [1,2]$ converges pointwise as $n \to \infty$ to $f = 0$ on $[0,2] \setminus \{1\}$ and $f(1) = 1$.

8. Normed space-valued BV-mappings

In this section we assume that X is a normed vector space over the field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$ with the norm $\|\cdot\|$, and, as usual, $\emptyset \neq E \subset \mathbb{R}$. Actually, X^E becomes a vector space (over $\mathbb{K}$) with respect to the pointwise operations

$$(f+g)(t) = f(t) + g(t), \quad (cf)(t) = cf(t), \quad f,\, g \in X^E, \quad c \in \mathbb{K}, \quad t \in E.$$

Proposition 8.1. *The functional $V(\,\cdot\,, E) : X^E \to [0,\infty]$ has the following properties*:

(a) $V(f+g, E) \le V(f,E) + V(g,E)$, $f, g \in X^E$ (*subadditivity*);
(b) $V(cf, E) = |c| V(f,E)$, $f \in X^E$, $c \in \mathbb{K}$ (*homogeneity*); (*hence, $V(\,\cdot\,,E)$ is convex*: *if f, $g \in X^E$ and $\alpha \in [0,1]$, then*

$$V(\alpha f + (1-\alpha) g, E) \le \alpha V(f,E) + (1-\alpha) V(g,E)\,);$$

(c) *if $\|f\|_V^* = \|f(a)\| + V(f,E)$ for $f \in X^E$ and fixed $a \in E$, then $\|\cdot\|_V^* : X^E \to [0,\infty]$ is a pseudonorm on X^E (i.e., it satisfies the axioms of norm and possibly takes infinite values)*;
(d) *$V(\,\cdot\,,E)$ is sequentially continuous on X^E with respect to $\|\cdot\|_V^*$, i.e., if $\{f_n\}_{n=1}^\infty \subset X^E$, $f \in X^E$, and $\lim_{n\to\infty} \|f_n - f\|_V^* = 0$, then $\lim_{n\to\infty} V(f_n, E) = V(f,E)$, and if, moreover, $\{f_n\}_{n=1}^\infty \subset \mathcal{V}(E;X)$, then $\sup_{n\in\mathbb{N}} V(f_n,E) < \infty$ and $f \in \mathcal{V}(E;X)$.*

Proof. Items (a) and (b) are obvious. (c) By virtue of (a) and (b), it suffices to verify that

$$\text{if } f \in X^E \text{ and } \|f\|_V^* = 0, \text{ then } f(t) = 0 \text{ for all } t \in E.$$

This follows immediately from $\|f(t)\| \le \|f\|_V^*$ for all $f \in X^E$ and $t \in E$, since, due to (P1), we have $\|f(t)\| - \|f(a)\| \le \|f(t) - f(a)\| \le V(f,E)$ by the triangle inequality.

(d) Let $\|f_n - f\|_V^* \to 0$, $n \to \infty$. Then $\|f_n(t) - f(t)\| \le \|f_n - f\|_V^* \to 0$ as $n \to \infty$ for all $t \in E$, so that $V(f,E) \le \liminf_{n\to\infty} V(f_n, E)$ by (P7). On the other hand,

$$V(f_n, E) \le V(f_n - f, E) + V(f,E) \le \|f_n - f\|_V^* + V(f,E), \quad n \in \mathbb{N}, \tag{8.1}$$

whence

$$\limsup_{n\to\infty} V(f_n,E) \le \lim_{n\to\infty} \|f_n - f\|_V^* + V(f,E) = V(f,E).$$

Hence, $V(f,E) = \lim_{n\to\infty} V(f_n,E)$.

Now suppose that $f_n \in \mathcal{V}(E;X)$, $n \in \mathbb{N}$. From (8.1) with $f = f_k$ we have as n, $k \to \infty$:

$$|V(f_n, E) - V(f_k, E)| \le \|f_n - f_k\|_V^* \le \|f_n - f\|_V^* + \|f - f_k\|_V^* \to 0,$$

so that $\{V(f_n, E)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and, hence, it is bounded and convergent. The inclusion $f \in \mathcal{V}(E;X)$ is then obvious. □

Proposition 8.2. *The restriction of* $\|\cdot\|_V^*$ *to* $\mathcal{V}(E;X)$ *is a norm on* $\mathcal{V}(E;X)$ *and* $V(\cdot, E)$ *is a continuous functional on* $\mathcal{V}(E;X)$. *If, in addition,* X *is a Banach space, then* $\mathcal{V}(E;X)$ *is also a Banach space with respect to* $\|\cdot\|_V^*$.

It suffices to prove that $\mathcal{V}(E;X)$ is complete. This is a consequence of the following general observation.

Proposition 8.3. *Let* $\|\cdot\|^*$ *be a pseudonorm on* X^E *and let* $\mathcal{B}(E;X)$ *be the set of all* $f \in X^E$ *such that* $\|f\|^* < \infty$. *Suppose that the following two conditions hold*:

(a) *if a sequence* $\{f_n\}_{n=1}^{\infty} \subset X^E$ *converges pointwise on* E *to* $f \in X^E$ *as* $n \to \infty$, *then* $\|f\|^* \le \liminf_{n\to\infty} \|f_n\|^*$;

(b) *every* $\|\cdot\|^*$*-Cauchy sequence in* $\mathcal{B}(E;X)$ *has a pointwise convergent subsequence.*

Then $\|\cdot\|^*$ *is a norm on* $\mathcal{B}(E;X)$, *and* $\mathcal{B}(E;X)$ *is a Banach space with* $\|\cdot\|^*$.

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{B}(E;X)$. By (b), there exist a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and a mapping $f \in X^E$ such that $f_{n_k} \to f$ pointwise on E as $k \to \infty$. Hence, for all $n \in \mathbb{N}$, the sequence $f_n - f_{n_k}$ converges pointwise to $f_n - f$ as $k \to \infty$, so that (a) yields

$$\|f_n - f\|^* \le \liminf_{k\to\infty} \|f_n - f_{n_k}\|^* = \lim_{k\to\infty} \|f_n - f_{n_k}\|^* \quad \forall\, n \in \mathbb{N}.$$

Thus, using the fact that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, we have

$$\limsup_{n\to\infty} \|f_n - f\|^* \le \lim_{n\to\infty} \lim_{k\to\infty} \|f_n - f_{n_k}\|^* = 0,$$

so that $\|f_n - f\|^* \to 0$ as $n \to \infty$. It follows that $f \in \mathcal{B}(E;X)$ since $\|f_{n_0} - f\|^* \le 1$ for some $n_0 \in \mathbb{N}$, and therefore

$$\|f\|^* \le \|f - f_{n_0}\|^* + \|f_{n_0}\|^* \le 1 + \|f_{n_0}\|^* < \infty. \quad \square$$

Remark. Note that condition (b) is satisfied provided that X is complete and $\|f(t)\| \to 0$ as $\|f\|^* \to 0$ for all $t \in E$, i.e., $\forall\, t \in E\ \forall\, \varepsilon > 0\ \exists\, \delta = \delta(t, \varepsilon) > 0\ \forall\, f \in X^E$, $\|f\|^* \le \delta \Longrightarrow \|f(t)\| \le \varepsilon$. Indeed, if $\{f_n\}_{n=1}^{\infty} \subset X^E$ is a Cauchy sequence, then, given $t \in E$ and $\varepsilon > 0$, there exists $N = N(\delta) \in \mathbb{N}$ such that $\|f_n - f_m\|^* \le \delta$ for all n, $m \ge N$, so that $\|f_n(t) - f_m(t)\| \le \varepsilon$. It follows that for all $t \in E$ the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is Cauchy in X, and hence, it is convergent.

With this remark and Proposition 8.3, we obtain the completeness of $\mathcal{V}(E;X)$. Note that this proposition can be applied for the proof of the completeness of a large number of normed function spaces (for instance, in the theory of Lebesgue integrable functions condition (a) follows from the Fatou lemma).

Let $I \subset \mathbb{R}$ be a connected interval (open, closed or half-closed, bounded or unbounded). Recall the *mean value theorem* (cf. [4], Ch. I, Sec. 2.3).

Theorem 8.4. *Let $f : I \to X$ be a continuous mapping for which the (norm) right derivative $f'_+(t) \in X$ exists for all $t \in I \setminus Q$, where $Q \subset I$ is at most countable. Then, for all a, $b \in I$, $a < b$, and $t_0 \in I \setminus Q$, we have*

(a) $\|f(b) - f(a)\| \le (b-a)\sup\{\|f'_+(t)\| : t \in]\,a,b\,[\setminus Q\}$;
(b) $\|f(b) - f(a) - f'_+(t_0)(b-a)\| \le (b-a)\sup_{t\in]\,a,b\,[\setminus Q}\|f'_+(t) - f'_+(t_0)\|$.

A similar theorem holds in the case of the left derivative $f'_-(t)$.

Corollary 8.5. *Under the assumptions of Theorem 8.4,*

(a) *if, for all a, $b \in I$, $a < b$, the right derivative f'_+ is bounded on $]\,a,b\,[\setminus Q$ (by a constant depending on a and b), then $f \in \mathcal{V}_{\mathrm{loc}}(I;X)$;*
(b) *if f'_+ is bounded on $I \setminus Q$ and I is bounded, then $f \in \mathcal{V}(I;X)$.*

Proof. Apply Theorem 8.4(a) and Lemma 3.2 with $\varphi(t) = t$, $t \in I \setminus Q$. □

Lemma 8.6. *If $f \in \mathcal{V}([\,a,b\,];X)$, then, for all $h \in]\,0, b-a\,]$, we have*

$$\int_a^{b-h} \left\| \frac{f(t+h) - f(t)}{h} \right\| dt = \int_{a+h}^{b} \left\| \frac{f(t) - f(t-h)}{h} \right\| dt \le V_a^b(f).$$

Proof. Let $h \in [\,0, b-a\,]$. Since we have $a \le t \le t+h \le b$ for $t \in [\,a, b-h\,]$, (P1), (P3), and (P2), imply that

$$\|f(t+h) - f(t)\| \le V_t^{t+h}(f) = V_a^{t+h}(f) - V_a^t(f) \le V_a^b(f).$$

It follows that the function $[\,a, b-h\,] \ni t \mapsto \|f(t+h) - f(t)\| \in \mathbb{R}_0^+$ is bounded and continuous almost everywhere (due to Theorem 4.1(b)), so that it is Riemann integrable on $[\,a, b-h\,]$ due to the Lebesgue criterion. Now it suffices to integrate the first inequality above:

$$\int_a^{b-h} \|f(t+h) - f(t)\|\, dt \le \int_{a+h}^{b} V_a^t(f)\, dt - \int_a^{b-h} V_a^t(f)\, dt \le$$

$$\le \int_{b-h}^{b} V_a^t(f)\, dt \le hV_a^b(f). \quad \square$$

Theorem 8.7.

(a) *If the mapping* $f \in \mathcal{C}([a,b];X)$ *has a right derivative* $f'_+ : [a,b[\to X$ *(defined arbitrarily at* $t = b$*) such that* $\|f'_+(t)\|$ *is Riemann integrable as a function of* $t \in [a,b]$*, then* $f \in \mathcal{V}([a,b];X)$ *and*

$$V_a^b(f) \le \int_a^b \|f'_+(t)\|\, dt.$$

(A similar assertion holds in the case of the left derivative $f'_- :]a,b] \to X$*.)*

(b) *If* $f \in \mathcal{C}^1([a,b];X)$ *(i.e.,* $f : [a,b] \to X$ *is continuously differentiable on* $[a,b]$*), then* $f \in \mathcal{V}([a,b];X)$ *and*

$$V_a^b(f) = \int_a^b \|f'(t)\|\, dt.$$

(Note that in (b) *we do not assume the normed vector space* X *to be* complete.)

Proof.

(a) Let $T = \{t_i\}_{i=0}^m \in \mathcal{T}_a^b$ be a partition of $[a,b]$. Applying Theorem 8.4(a) to f on $[t_{i-1}, t_i]$, we have

$$\|f(t_i) - f(t_{i-1})\| \le (t_i - t_{i-1}) \sup\{\|f'_+(t)\| : t \in]t_{i-1}, t_i[\},$$

which, after the summing over $i = 1, \dots, m$, implies

$$V(f,T) \le \sum_{i=1}^m \sup\{\|f'_+(t)\| : t \in [t_{i-1}, t_i]\} \cdot (t_i - t_{i-1}).$$

Now we pass to the limit as $\lambda(T) \to 0$ in the latter inequality: the left-hand side tends to $V_a^b(f)$ by Theorem 2.2, and the right-hand side, which is the upper Darboux sum of $t \mapsto \|f'_+(t)\|$ corresponding to T, tends to $\int_a^b \|f'_+(t)\|\, dt$ by the classical Darboux theorem. (Note that (a) remains valid if we replace $[a,b[$ in (a) by $[a,b[\setminus Q$, where Q is a finite subset of $[a,b[$.)

(b) The derivative $f' : [a,b] \to X$ of f is continuous, and, hence, its norm $t \mapsto \|f'(t)\|$ is also continuous and Riemann integrable, so by virtue of (a), we have

$$V_a^b(f) \le \int_a^b \|f'(t)\|\, dt.$$

Let us prove the converse inequality. It will follow at once from Lemma 8.6 if we show that

$$\lim_{h\to+0}\int_a^{b-h}\left\|\frac{f(t+h)-f(t)}{h}\right\|dt=\int_a^b\|f'(t)\|\,dt.$$

We fix $h\in]\,0,b-a\,[$. Applying Theorem 8.4(b) for every t in $[\,a,b-h\,]$ to the mapping $[\,0,h\,]\ni s\mapsto f(t+s)\in X$, we have

$$\|f(t+h)-f(t)-f'(t)h\|\le h\sup_{s\in]\,0,h\,[}\|f'(t+s)-f'(t)\|.$$

Thus,

$$\left|\int_a^{b-h}\left\|\frac{f(t+h)-f(t)}{h}\right\|dt-\int_a^b\|f'(t)\|\,dt\right|=$$

$$=\left|\int_a^{b-h}\left(\left\|\frac{f(t+h)-f(t)}{h}\right\|-\|f'(t)\|\right)dt-\int_{b-h}^{b}\|f'(t)\|\,dt\right|\le$$

$$\le\int_a^{b-h}\left\|\frac{f(t+h)-f(t)}{h}-f'(t)\right\|dt+\int_{b-h}^{b}\|f'(t)\|\,dt\le$$

$$\le\int_a^{b-h}\sup_{s\in]\,0,h\,[}\|f'(t+s)-f'(t)\|\,dt+h\sup_{t\in[\,b-h,b]}\|f'(t)\|.$$

The latter two terms tend to zero as $h\to+0$ due to the uniform continuity and boundedness of the derivative f' on $[\,a,b\,]$. □

Remark. The almost everywhere differentiability and weak differentiability of vector-valued (namely, reflexive Banach space-valued) absolutely continuous mappings and mappings of bounded variation were treated by Komura [12], Barbu [2], Ch. 1, Sec. 2, and Barbu and Precupanu [3], Ch. 1, Sec. 3.

9. BV-SELECTIONS OF BV-SET-VALUED MAPPINGS

Before stating and proving the main result of this section (Theorem 9.1), we recall some definitions (see Castaing and Valadier [5], Ch. 2, Sec. 1, and Aubin and Cellina [1], Ch. 1, Secs. 1, 5).

Let A, $B\subset X$ be two *nonempty* subsets of the metric space (X,d). The *excess of A over B* is defined by

$$e(A,B)=\sup_{x\in A}\operatorname{dist}(x,B)=\sup_{x\in A}\inf_{y\in B}d(x,y)\in[\,0,\infty\,],$$

and the *Hausdorff distance between A and B* is defined by

$$d_H(A,B) = \max\{\, e(A,B), e(B,A)\,\}.$$

Since $e(A,B)=0$ if and only if A is contained in the closure of B, and since $e(A,B) \le e(A,C) + e(C,B)$ for nonempty $C \subset X$, it follows that d_H is a *pseudometric* on the set of all nonempty closed subsets of X, i.e., d_H satisfies the usual axioms of a metric and possibly takes infinite values. The mapping d_H is a *metric* on the set of all nonempty closed bounded subsets of X, and, hence, on the set of all nonempty compact subsets of X and, if X is bounded, also on the set of all nonempty closed subsets of X.

Let E and X be two metric spaces, 2^X be the set of all subsets of X, and let $\dot{2}^X = 2^X \setminus \{\emptyset\}$. A *set-valued mapping* from E into X is a mapping $F: E \to 2^X$, so that we have $F(t) \subset X$ for every $t \in E$. The *graph of F* is the set $\mathrm{Gr}(F) = \{\,(t,x) \in E \times X : x \in F(t)\,\}$ and the *range of F* is the set $R(F) = \bigcup_{t\in E} F(t)$, so that $R(F) \subset X$.

The set-valued mapping $F: E \to \dot{2}^X$ is said to be

(a) *upper semicontinuous* (*u.s.c.*) *at $t_0 \in E$* if, for any neighborhood $\mathcal{N}(F(t_0))$ of the set $F(t_0)$, there exists a neighborhood $\mathcal{N}(t_0)$ of t_0 such that for every $t \in \mathcal{N}(t_0)$ we have $F(t) \subset \mathcal{N}(F(t_0))$;
(b) *lower semicontinuous* (*l.s.c.*) *at $t_0 \in E$* if, for any $x_0 \in F(t_0)$ and any neighborhood $\mathcal{N}(x_0)$ of x_0, there exists a neighborhood $\mathcal{N}(t_0)$ of t_0 such that for every $t \in \mathcal{N}(t_0)$ we have $F(t) \cap \mathcal{N}(x_0) \ne \emptyset$ (this is equivalent to: for any sequence t_n converging to t_0 in E and any $x_0 \in F(t_0)$ there exists a sequence $x_n \in F(t_n)$ which converges to x_0 in X);
(c) *continuous at $t_0 \in E$* if it is both u.s.c. and l.s.c. at t_0;
(d) *Hausdorff continuous at $t_0 \in E$* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $t \in E$ and $d_E(t,t_0) \le \delta$, then $d_H(F(t),F(t_0)) \le \varepsilon$;
(e) *u.s.c. on E* (resp., *l.s.c.*, *continuous* or *Hausdorff continuous on E*) if it is so at every $t_0 \in E$;
(f) *Lipschitz continuous on E* if $d_H(F(t),F(s)) \le L d_E(t,s)$ for some $L \ge 0$ and for all t, $s \in E$. The minimal L of this kind is called the *Lipschitz constant* of F and is denoted by $\mathrm{Lip}(F)$;
(g) *compact-valued* if $F(t)$ is a compact subset of X for every $t \in E$;
(h) *compact* if its graph $\mathrm{Gr}(F)$ is compact in $E \times X$ (hence, F is compact-valued, but not vice versa);
(i) *of bounded variation on $E = [a,b] \subset \mathbb{R}$* if

$$V_a^b(F) = \sup\{\, V_H(F,T) : T \in \mathcal{T}_a^b \,\} < \infty,$$

where

$$V_H(F,T)=\sum_{i=1}^{m} d_H\big(F(t_i),F(t_{i-1})\big),\quad T=\{t_i\}_{i=0}^{m}\in\mathcal{T}_a^b.$$

The mapping $f: E\to X$ is said to be a (*regular*) *selection* of the set-valued mapping $F: E\to \dot{2}^X$ if $f(t)\in F(t)$ for all $t\in E$.

It is known that a compact-valued set-valued mapping $F: E\to\dot{2}^X$ is continuous on E if and only if it is Hausdorff continuous on E (cf. [1], Ch. 1, Sec. 5, Corollary 1).

In what follows, we assume that $E=[a,b]\subset\mathbb{R}$.

Theorem 9.1.

(a) *Let X be a Banach space with the norm $\|\cdot\|$, and let $F:[a,b]\to\dot{2}^X$ be a compact continuous set-valued mapping of bounded variation on $[a,b]$. Then, for any $t_0\in[a,b]$ and any $x_0\in F(t_0)$, there exists a continuous mapping $f:[a,b]\to X$ of bounded variation on $[a,b]$ such that*

$$f(t_0)=x_0,\quad f(t)\in F(t)\ \text{for all}\ t\in[a,b],\quad\text{and}\quad V_a^b(f)\le V_a^b(F). \tag{9.1}$$

(b) *If, in addition, the range $R(F)$ of F is contained in a convex compact subset of X, and, in particular, if $\dim X<\infty$, then the assumption of continuity of F can be omitted, so that $f\in\mathcal{V}([a,b];X)$ and f satisfies* (9.1).

The following lemma, used in the proof of Theorem 9.1, is itself interesting.

Lemma 9.2. *Let X be a Banach space with norm $\|\cdot\|$, and let $F:[a,b]\to\dot{2}^X$ be a compact Lipschitz continuous set-valued mapping on $[a,b]$. Then, for any $t_0\in[a,b]$ and any $x_0\in F(t_0)$, there exists a Lipschitzian mapping $f:[a,b]\to X$ such that*

$$f(t_0)=x_0,\ f(t)\in F(t)\ \text{for all}\ t\in[a,b],\ \text{and}\ \mathrm{Lip}(f)\le\mathrm{Lip}(F). \tag{9.2}$$

Proof. Consider a sequence $\{T_n\}_{n=1}^\infty\subset\mathcal{T}_a^b$ of partitions of $[a,b]$ such that $t_0\in T_n$ for all $n\in\mathbb{N}$ and $\lambda(T_n)\to 0$ as $n\to\infty$. In other words, for $n\in\mathbb{N}$, we have

$$T_n=\{\{t_i^n\}_{i=0}^n\subset[a,b]: a=t_0^n<t_1^n<\cdots<t_{n-1}^n<t_n^n=b\},$$

$t_0=t_{k(n)}^n$ for some $k(n)\in\{0,1,\dots,n\}$, and $\lim\limits_{n\to\infty}\max\limits_{1\le i\le n}(t_i^n-t_{i-1}^n)=0$. We are going to define $x_i^n\in F(t_i^n)$ inductively as follows. Let $a<t_0<b$, and let $n\in\mathbb{N}$.

(a) Set $x_{k(n)}^n=x_0$.

(b) If $i \in \{1, \ldots, k(n)\}$ and if $x_i^n \in F(t_i^n)$ is already chosen, fix $x_{i-1}^n \in F(t_{i-1}^n)$ such that $\|x_i^n - x_{i-1}^n\| = \operatorname{dist}(x_i^n, F(t_{i-1}^n))$.

(c) If $i \in \{k(n)+1, \ldots, n\}$ and if $x_{i-1}^n \in F(t_{i-1}^n)$ is already chosen, fix $x_i^n \in F(t_i^n)$ such that $\|x_{i-1}^n - x_i^n\| = \operatorname{dist}(x_{i-1}^n, F(t_i^n))$.

If $t_0 = a$, so that $k(n) = 0$, then we use only (a) and (c) to define x_i^n, and if $t_0 = b$, so that $k(n) = n$, then we define x_i^n as in (a) and (b).

We note at once that by virtue of (b) and (c), we have

$$\|x_i^n - x_{i-1}^n\| \le d_H(F(t_i^n), F(t_{i-1}^n)) \le \operatorname{Lip}(F) \cdot (t_i^n - t_{i-1}^n). \tag{9.3}$$

Now we define a sequence of mappings $f_n : [a, b] \to X$, $n \in \mathbb{N}$, as follows:

$$f_n(t) = x_{i-1}^n + \frac{t - t_{i-1}^n}{t_i^n - t_{i-1}^n}(x_i^n - x_{i-1}^n), \quad t \in [t_{i-1}^n, t_i^n], \quad i = 1, \ldots, n. \tag{9.4}$$

Obviously, $f_n(t_0) = x_0$ for all $n \in \mathbb{N}$.

First, we note that $\{f_n\}_{n=1}^\infty$ is uniformly Lipschitzian on $[a, b]$ with $\operatorname{Lip}(f_n) \le \operatorname{Lip}(F)$. Indeed, t, $s \in [t_{i-1}^n, t_i^n]$; then, due to (9.4) and (9.3), we have

$$\|f_n(t) - f_n(s)\| \le \frac{|t - s|}{t_i^n - t_{i-1}^n}\|x_i^n - x_{i-1}^n\| \le \operatorname{Lip}(F) \cdot |t - s|,$$

and, hence, the assertion follows, so that $\{f_n\}_{n=1}^\infty$ is equicontinuous on $[a, b]$.

Second, for every $t \in [a, b]$ the sequence $\{f_n(t)\}_{n=1}^\infty$ is precompact in X. Indeed, given $t \in [a, b]$, for every $n \in \mathbb{N}$ there exists $i(n) \in \{1, \ldots, n\}$ such that $t_{i(n)-1}^n \le t \le t_{i(n)}^n$, and therefore, since $\lambda(T_n) \to 0$ as $n \to \infty$, we find that $t_{i(n)-1}^n$ and $t_{i(n)}^n$ tend to t as $n \to \infty$. Thus, (9.3) implies

$$\left\| \frac{t - t_{i(n)-1}^n}{t_{i(n)}^n - t_{i(n)-1}^n}(x_{i(n)}^n - x_{i(n)-1}^n) \right\| \le \operatorname{Lip}(F) \cdot |t - t_{i(n)-1}^n| \to 0, \quad n \to \infty.$$

Since F is compact and $x_{i(n)-1}^n \in F(t_{i(n)-1}^n)$, there exists a subsequence of $(t_{i(n)-1}^n, x_{i(n)-1}^n) \in \operatorname{Gr}(F)$ (still denoted by the same symbol) which converges in $[a, b] \times X$ to the point $(\tau, x) \in \operatorname{Gr}(F)$. However, $t_{i(n)-1}^n \to t$ as $n \to \infty$, and therefore, $\tau = t$, so that $(t, x) \in \operatorname{Gr}(F)$ or $x \in F(t)$. It follows that $f_n(t) \to x$ in X as $n \to \infty$, where $x \in F(t)$, and therefore, $\{f_n(t)\}_{n=1}^\infty$ is precompact in X.

By the Ascoli-Arzelà theorem, $\{f_n\}_{n=1}^\infty$ is precompact in $C([a, b]; X)$, and, hence, there exists a subsequence of $\{f_n\}_{n=1}^\infty$ which uniformly converges on $[a, b]$ to the mapping $f : [a, b] \to X$. Clearly, f is Lipschitzian and f satisfies the properties in (9.2). $\square$

Proof of Theorem 9.1.

(a) By virtue of Theorem 3.1 or Lemma 3.3, we have the decomposition $F = G \circ \varphi$ on $[a,b]$, where $[a,b] \ni t \mapsto \varphi(t) = V_a^t(F) \in \mathbb{R}_0^+$ is a bounded nondecreasing continuous function and $G : [0,\ell] = \varphi([a,b]) \to \dot{2}^X$ is a Lipschitz continuous set-valued mapping such that $\ell = V_a^b(F) < \infty$ and $\mathrm{Lip}(G) \le 1$. Since F is compact, G is compact as well. Indeed, if $\{(\tau_n, y_n)\}_{n=1}^\infty \subset \mathrm{Gr}(G)$, then $\tau_n = \varphi(t_n)$ for some $t_n \in [a,b]$ and $y_n \in G(\varphi(t_n)) = F(t_n)$, so that $S = \{(t_n, y_n)\}_{n=1}^\infty \subset \mathrm{Gr}(F)$. Hence, there exists a subsequence of S (denoted by the same symbol) such that $t_n \to t$ and $y_n \to y$ as $n \to \infty$, where $y \in F(t)$. We set $\tau = \varphi(t)$. Since φ is continuous, (τ_n, y_n) converges to $(\tau, y) \in \mathrm{Gr}(G)$. Now, noting that $x_0 \in F(t_0) = G(\tau_0)$ with $\tau_0 = \varphi(t_0)$, by Lemma 9.2 we find a Lipschitzian mapping $g : [0,\ell] \to X$ such that $g(\tau_0) = x_0$, $g(\tau) \in G(\tau)$ for all $\tau \in [0,\ell]$ and $\mathrm{Lip}(g) \le \mathrm{Lip}(G) \le 1$.

We set $f = g \circ \varphi$. Then $f : [a,b] \to X$ is continuous, $f \in \mathcal{V}([a,b];X)$ by Lemma 3.2, $f(t_0) = x_0$, and $f(t) = g(\varphi(t)) \in G(\varphi(t)) = F(t)$ for all t. Finally, by virtue of (P4) and the inequality in Remark (b) on p. 268, we infer that

$$V_a^b(f) = V_a^b(g \circ \varphi) = V_0^\ell(g) \le \ell \cdot \mathrm{Lip}(g) \le \ell = V_a^b(F).$$

(b) Suppose now that $R(F) \subset K$, where K is a convex compact subset of X. We start, as in the proof of Lemma 9.2, up to formula (9.4). Note that the mappings $f_n : [a,b] \to X$ defined there are continuous and that by virtue of (P3), Theorem 8.7(b), and the first inequality in (9.3), we have

$$V_a^b(f_n) = \sum_{i=1}^n V_{t_{i-1}^n}^{t_i^n}(f_n) = \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \|f_n'(t)\| \, dt =$$

$$= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} \frac{\|x_i^n - x_{i-1}^n\|}{t_i^n - t_{i-1}^n} \, dt = \sum_{i=1}^n \|x_i^n - x_{i-1}^n\| \le$$

$$\le \sum_{i=1}^n d_H(F(t_i^n), F(t_{i-1}^n)) = V_H(F, T_n) \le V_a^b(F) < \infty, \quad n \in \mathbb{N}.$$

Since F is compact, $R(F)$ is compact in X, and, by our assumption, the convex hull $\mathrm{co}\, R(F)$ of $R(F)$ is contained in K. Since the image of every f_n is contained in $\mathrm{co}\, R(F)$, and, hence, in K, we can apply Helly's selection principle (Theorem 7.1): there exists a subsequence of $\{f_n\}_{n=1}^\infty$ (still denoted by the same symbol) which converges pointwise on $[a,b]$ to a BV-mapping $f : [a,b] \to X$. The first two properties in (9.1) are then clear, and it suffices to note that by virtue of (P7), we have

$$V_a^b(f) \le \liminf_{n\to\infty} V_a^b(f_n) \le V_a^b(F).$$

Finally, if dim $X < \infty$, then the convex hull co $R(F)$ of the compact set $R(F)$ is compact and convex (cf. [1], Ch. 0, Sec. 5, Proposition 5), so that we can set $K = \text{co } R(F)$.

This completes the proof. □

Remarks. Continuous selections of convex-valued set-valued mappings under very general conditions are known to exist due to Michael [14]–[16] (see also Castaing and Valadier [5], Ch. 3, Sec. 2, and Aubin and Cellina [1], Ch. 1, Secs. 6–14). In Sec. 9 we treated the nonconvex case: Theorem 9.1 generalizes the results of Hermes [10], Kikuchi, and Tonita [11] to the infinite-dimensional case without convexity. Lemma 9.2 is known; see Mordukhovich [17], Supplement, Theorem 1.8, where he also proves, in a different way, the existence of a continuous selection under the conditions of Theorem 9.1(a). However, we prove, in addition, that there are continuous selections of bounded variation. Part (b) of Theorem 9.1 without the continuity assumption of F is new. It should be noted that continuous selections of compact set-valued mappings cannot exist if (a) $F : [a, b] \to \dot{2}^{\mathbb{R}^2}$ is continuous only (Hermes [10] and Aubin and Cellina [1], Ch. 1, Sec. 6), or (b) $F : E \to \dot{2}^X$ is Lipschitz continuous, dim $E > 1$, and dim $X < \infty$ (Hermes [10] and Nadler [18]).

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