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Wold Decomposition in Banach Spaces

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Abstract—We propose a natural analog of the Wold decomposition in the case of a linear noninvertible isometry V in a Banach space X. We obtain a criterion for the existence of such a decomposition. In a reflective space, this criterion is reduced to the existence of the linear projection $P\colon X \to VX$ with unit norm. Separately, we discuss the problem of the Wold decomposition for the isometry V_φ induced by an epimorphism φ of a compact set H in the space of continuous functions $C(H)$. We present a detailed study of the mapping $z \to z^m$ of the circle $|z| = 1$ with an integer $m \geq 2$.

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Key words: *Wold decomposition, linear noninvertible isometry, Banach space, reflexive space, unitary operator, completely nonunitary isometry, one-sided shift.*

INTRODUCTION

It is well known [1]–[3] that a linear isometry V in Hilbert space X generates the so-called *Wold decomposition*. We mean the decomposition of X into an infinite orthogonal direct sum of the form

$$X = M \otimes K, \qquad K = \bigoplus_0^\infty V^n R,$$

where R is the orthogonal complement of the image VX, the subspaces M and K are invariant with respect to V, the restriction of V to M is unitary, and the restriction of V to K does not have eigenvectors. Moreover, the operator V is not unitary in any invariant subspace $L \subset K$. In fact, the Wold decomposition allows one to reduce studying arbitrary isometries in Hilbert space to studying unitary operators and *completely nonunitary* isometries with the specific structure of a *one-sided shift*.

The present paper contains conditions (see Theorem 2.1) necessary and sufficient for the realization of such a decomposition in Banach spaces. In this case, the usual Hilbert orthogonality is replaced by the orthogonality in the sense of Birkhoff. The usual component of the corresponding criterion is the existence of at least one linear orthogonal projection operator $P\colon X \to VX$ (with norm 1). We show (Theorem 2.6) that, in a reflexive space, the existence of the orthoprojection on VX is necessary and sufficient for the Wold decomposition to take place. But if the linear isometry V acts in an *arbitrary* Banach space X, then the problem of the existence of such a decomposition depends not only to the properties of the operator V, but also to the choice of an appropriate element P from the set $\mathscr{P}_V$ of all possible orthogonal projection operators on the subspace VX.

We especially consider the case of the situation in which the field of action is the space $C(H)$ of continuous scalar functions $x(\xi)$ on the Hausdorff compact set H and the isometry $V = V_\varphi$ in $C(H)$ is induced by the continuous surjection $\varphi\colon H \to H$, i.e., $(Vx)(\xi) = x(\varphi\xi)$ for $\xi \in H$, which is interesting from the standpoint of the functionally analytic approach to problems of topological dynamics. In particular, we show (see Sec. 3) that the mapping $\varphi\colon z \to z^m$ of the circle $S\colon |z| = 1$ with an integer $m \geq 2$ (more precisely, the induced isometry $V = V_\varphi$) generates the Wold decomposition in $C(S)$ under an appropriate choice of the projection operator $P \in \mathscr{P}_V$. At the same time, for the case $m = 2$, we present an orthogonal projection operator $P' \in \mathscr{P}_V$ such that the pair of operators (V, P') *does not generate* the Wold decomposition of the space $C(S)$.

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1. PRELIMINARY STATEMENTS

Let V be an isometric linear operator in real or complex Banach space X. In other words, let $\|Vx\| = \|x\|$ for $x \in X$. We stress that all the subsequent results are meaningful only if $VX \neq X$, and thus, the operator V is not a unitary operator. Following [4] and [5], we say that the subspace $X_1 \subset X$ *is orthogonal* to the subspace $X_2 \subset X$ and write $X_1 \perp X_2$ if $\|x_1\| \leq \|x_1 + x_2\|$ for any $x_1 \in X_1$ and $x_2 \in X_2$. The linear projection operator $P\colon X \to X_1$ with norm 1 is said to be *orthogonal*. In the non-Hilbert case, for a given subspace $X_1 \subset X$, it is known that such a projection operator does not always exist (this problem was discussed in the book [5]).

In what follows, we start from the fact that the linear orthogonal projection $P\colon X \to VX$ exists. If $R = (I - P)X$, where $I = \mathrm{id}$ in X, then the subspaces $V^n R$ with $n \geq 1$ are orthogonal to the subspace R. Since the operator V is isometric, this also implies that $V^n R \perp V^m R$ for all $n > m$. We define a bounded linear operator T in X by the relation $T = V^{-1}P$, where V^{-1} is the left inverse operator isometrically taking VX to X. It is clear that $\|T\| = 1$, $TV = I$, and $VT = P$. We set $P_0 = I$ and

$$P_n = V^n T^n, \qquad n \geq 1. \tag{1}$$

We see that

$$\|P_n\| = 1, \qquad P_n = V P_{n-1} T, \qquad \text{and} \qquad P_n^2 = P_n.$$

The last relation is obtained from (1) by using the identity $TV = I$. Since $TX = X$, we have $P_n X = V^n X$, and the operators P_n are the orthogonal projection operators on the subspaces $V^n X$. We have the following decompositions into the direct sum:

$$X = VX \otimes R, \qquad V^m X = V^{m+1} X \otimes V^m R,$$

and, as a consequence, the decomposition

$$X = V^n X \oplus V^{n-1} R \oplus \cdots \oplus VR \oplus R,$$

where

$$V^n X \perp V^m R \quad \text{for } n > m \geq 0 \qquad \text{and} \qquad V^m R \perp V^k R \quad \text{for } n-1 \geq m > k \geq 0.$$

The sequence $\{P_n\}$ is a decreasing chain of commuting projection operators, i.e., $P_n P_m = P_m P_n = P_n$ for $n \geq m$. The image of the projection operator $Q_n = P_n - P_{n+1}$ coincides with the subspace $V^n R$, and we have the relation

$$I = P_n + \sum_{m=0}^{n-1} Q_m. \tag{2}$$

By the letters s and w we denote the strong and the weak convergence of elements or of bounded linear operators in Banach spaces. We use the symbols Im and Ker to denote the image and the kernel of linear operators. We also set

$$M = \bigcap_1^\infty \operatorname{Im} V^n, \qquad K = \left(\bigcap_1^\infty \operatorname{Ker} P_n \right)^c,$$

where $\operatorname{Im} V^n = \operatorname{Im} P_n$ and $(\,\cdot\,)^c$ is the strong closure operation for sets in X. The inclusions

$$\operatorname{Im} P_n \subset \operatorname{Im} P_m \qquad \text{and} \qquad \operatorname{Ker} P_n \supset \operatorname{Ker} P_m$$

are obvious for $n > m$. Now we fix some elementary properties of the subspaces M and K.

Lemma 1.1. *The following assertions hold*:

(a) $K = \{x \in X : s\text{-}\lim_{n\to\infty} P_n x = 0\}$ *and* $M \cap K = \{0\}$;

(b) *if* $s\text{-}\lim_{n\to\infty} P_n x = x_0$, *then* $x_0 \in M$;

(c) $M \oplus K = \{x \in X : \exists\ s\text{-}\lim_{n\to\infty} P_n x\}$;

(d) *M is orthogonal to K.*

Proof. Let K_1 be the union over $n \geq 1$ of the system of sets $\operatorname{Ker} P_n$. The inclusion $\operatorname{Ker} P_n \supset \operatorname{Ker} P_m$ for $n > m$ ensures the strong convergence $P_n x \to 0$ as $n \to \infty$ on the lineal K_1. If the relations $\|P_n\| = 1$ are taken into account, then this inclusion ensures the strong convergence on its closure $K = (K_1)^c$. Conversely, if $p_n x \xrightarrow{s} 0$ for some $x \in X$, then $(I - P_n)x \xrightarrow{s} x$. But

$$x - P_n x \in \operatorname{Ker} P_m \qquad \text{for} \quad m > n;$$

therefore, $x \in K$. Property (b) follows from the identity $P_m P_n = P_n$ for $n > m$. Further, $P_n x = x$ on M, and hence $M \cap K = \{0\}$ and the projection operators P_n strongly converge on $M \oplus K$ as $n \to \infty$. On the other hand, if $P_n x \xrightarrow{s} x_0$, then $x_0 \in M$ and $x - x_0 \in K$, because $P_n(x - x_0) = P_n x - x_0$, and hence $P_n(x - x_0) \xrightarrow{s} 0$. Finally, for elements $x \in M$ and $x_1 \in K$, the relation

$$s\text{-}\lim_{n\to\infty} P_n(x + x_1) = x$$

holds. Since $\|P_n(x + x_1)\| \leq \|x + x_1\|$, this implies that $\|x\| \leq \|x + x_1\|$ and hence $M \perp K$. The proof of the lemma is complete. □

Remark 1.2. It follows from the proof of the lemma that assertions (a)–(c) remain valid if, instead of the strong convergence of the elements $P_n x$, we consider their weak convergence.

The subspace $M \oplus K$ is closed and, in general, does not coincide with X. But we assume that the limit

$$w\text{-}\lim_{n\to\infty} P_n = P_\infty \tag{3}$$

exists, i.e., that the projection operators P_n converge weakly on X. Since $P_m P_\infty = P_\infty$ and $\|P_m\| = 1$ for all $m \geq 1$, we see that P_∞ is the orthogonal projection operator on the subspace M and $\operatorname{Ker} P_\infty = K$. Now, in (2), we pass to the limit as $n \to \infty$ and obtain the partition of unity converging in the weak operator topology,

$$I = P_\infty + \sum_{m=0}^{\infty} Q_m \tag{4}$$

or (in terms of direct sums) the decomposition

$$X = M \oplus K, \qquad K = \bigoplus_{0}^{\infty} V^n R \tag{5}$$

of the space X. This decomposition with $M = \operatorname{Im} P_\infty$, $K = \operatorname{Ker} P_\infty$, and $R = \operatorname{Ker} P$ is orthogonal in the sense that $M \perp K$ and

$$\bigoplus_{m}^{\infty} V^n R \perp \bigoplus_{0}^{m-1} V^n R$$

for any $m \geq 1$. In this case, $VK \subset K$, $VM = M$, and the restriction $V|_M$ is a unitary operator. By definition, M contains each subspace X_1 satisfying the condition $VX_1 = X_1$. Thus, M is the maximal invariant subspace in X on which the operator V is unitary or, which is the same, invertible. In particular, all the eigenvectors of the operator V belong to M.

By analogy with the Hilbert case, relations (5) will be called the *Wold decomposition*. Of course, we have $K = \{0\}$ and $M = X$ if the operator $V\colon X \to X$ is initially a unitary operator. By Lemma 1.1 (c), the decomposition into the direct sum

$$X = M \oplus K \tag{6}$$

is equivalent to the strong convergence of the projection operators P_n in (3) and hence is equivalent to (5). Now we can formulate the following statement.

Remark 1.3. Relations (3)–(6) are equivalent, and the weak convergence of the projection operators P_n in (3) implies their strong convergence. The same also holds for partial sums in the partition of unity (4).

In Hilbert space (see [1]–[3]), the convergence of the projection operators P_n holds for any arbitrary isometry V, and, in this case, the restriction $V|_K$ with a special structure is called the *one-sided shift* with generating subspace R. In an arbitrary Banach space X, the decomposition (5) for a given isometry V depends on the (possibly nonunique) choice of the orthoprojection P on the image $\operatorname{Im} V$ and, respectively, on the choice of the generating subspace $R = \operatorname{Ker} P$. In what follows, we let $\mathscr{P}_V$ denote the set of all linear orthogonal projections on $\operatorname{Im} V$. In general, this set *may be empty*, and then it does not make sense to speak about the Wold decomposition. But if the set $\mathscr{P}_V$ is not empty, then it is closed in the weak operator topology, bounded with respect to the norm, and convex.

It is desirable to find conditions under which the pair of operators consisting of the isometry V and the projection operator $P \in \mathscr{P}_V$ generates the Wold decomposition of the space X. In this case, because of the equivalence between relations (3) and (5), it is more convenient to speak about the conditions of weak convergence for the sequence of projection operators P_n. We note that the adjoint projection operators P_n^* in X^* form a decreasing commuting chain, because

$$P_n^* P_m^* = P_m^* P_n^* = p_n^* \qquad \text{for} \quad n \geq m.$$

Moreover, we have $\|P_n^*\| = 1$ for $n \geq 1$.

The *annihilator*

$$B^0 = \{y \in X^* : (x, y) = 0 \ \forall \, x \in B\}$$

of an arbitrary set $B \subset X$ is a weakly* closed subspace in the dual space X^*. In what follows, we use the *standard relations* several times (see, e.g., [4, Chap. 6]) between the kernel and the image of a bounded linear operator $A \colon X \to X$ and its adjoint $A^* \colon X^* \to X^*$, namely,

(i) $(\operatorname{Im} A)^0 = \operatorname{Ker} A^*$;

(ii) $(\operatorname{Ker} A)^0 = \operatorname{Im} A^*$ if the image $\operatorname{Im} A$ is closed.

In particular,

$$\operatorname{Im} P^* = (\operatorname{Ker} P)^0 = R^0 \qquad \text{and} \qquad \operatorname{Im} V^* = (\operatorname{Ker} V)^0 = X^*.$$

We show that the restriction of the operator V^* to the subspace $R^0 \subset X^*$ is isometric. For any functional $y \in R^0$ and $\varepsilon > 0$, we find an element $x \in X$ such that $|(x, y)| > (1 - \varepsilon)\|x\| \cdot \|y\|$. Since $(x, y) = (Px, y)$ and $\|x\| \geq \|Px\|$, we have

$$|(Px, y)| > (1 - \varepsilon)\|Px\| \cdot \|y\|.$$

If $Px = Vx_1$, then $\|Px\| = \|x_1\|$, $(Vx_1, y) = (x_1, V^*y)$, and

$$|(x_1, V^*y)| > (1 - \varepsilon)\|x_1\| \cdot \|y\|,$$

which implies the estimate $\|V^*y\| \geq \|y\|$. But $\|V^*\| = \|V\| = 1$, and therefore, $\|V^*y\| = \|y\|$, as was to be proved.

Thus, the operator V^* isometrically maps R^0 onto X^*. We set $U = T^*$, where $T = V^{-1}P$ is the operator from formula (1); then we have

$$\operatorname{Im} U = (\operatorname{Ker} T)^0 = (\operatorname{Ker} P)^0 = R^0.$$

Next, $TV = I$, and if $I^* = \mathrm{id}$ in X^*, then $V^*U = I^*$, and hence the operator U isometrically maps X^* onto R^0. Passing to the adjoint operators in (1), we obtain

$$P_n^* = U^n (V^*)^n, \qquad n \geq 1.$$

Since $V^*X^* = X^*$, we have $\operatorname{Im} P_n^* = \operatorname{Im} U^n$. Now we set

$$N = \bigcap_1^\infty \operatorname{Im} U^n.$$

Since $\operatorname{Im} P_n^* = (\operatorname{Ker} P_n)^0$, we have $N = K^0$. We see that the subspaces $N \subset X^*$ and $K \subset X$ depend on the operator V and on the projection operator $P \in \mathscr{P}_V$, while the subspace $M \subset X$ depends only on V. We note that the isometry V cannot generate the Wold decomposition in X if the linear orthogonal projection does not exist on the subspace $M(V)$.

2. MAIN RESULTS

It turns out that the criterion for the Wold decomposition to exist for the isometry V and the orthogonal projection operator $P \in \mathscr{P}_V$ can be formulated in terms of the "critical" subspaces

$$M = M(V) \qquad \text{and} \qquad N = N(V, P).$$

We say that M *separates* N if $(x, y_1) \neq (x, y_2)$ for any distinct functionals $y_1, y_2 \in N$ and an appropriate element $x \in M$. A more convenient (equivalent) statement is: M separates N if $M^0 \cap N = \{0\}$. We define the lineal Y in the dual space X^* as the union over $n \geq 1$ of the expanding system of sets $\operatorname{Ker} P_n^*$ and consider its annihilator

$$Y^0 = \{x \in X : (x, y) = 0 \ \forall y \in Y\}.$$

Since $(\operatorname{Im} P_n)^0 = \operatorname{Ker} P_n^*$, we have

$$(\operatorname{Ker} P_n^*)^0 = (\operatorname{Im} P_n)^c = \operatorname{Im} P_n \qquad \text{and} \qquad Y^0 = M.$$

The well-known rule $(Y^0)^0 = (Y)^a$, where $(\,\cdot\,)^a$ is the operation of the weak* closure of sets in X^*, leads to the formula

$$M^0 = \left(\bigcup_1^\infty \operatorname{Ker} P_n^*\right)^a. \tag{7}$$

Now we formulate the main result of this paper.

Theorem 2.1. *Suppose that, for the linear isometry V, the linear orthogonal projection P on $\operatorname{Im} V$ exists in a Banach space X. Then the pair of operators (V, P) generates the Wold decomposition* (5) *if and only if $M(V)$ separates $N(V, P)$.*

This theorem follows from a more general statement. We consider the sequence $\{T_n\}$ of bounded linear operators in X such that $\|T_n\| \leq \text{const}$ and $T_n T_m = T_m T_n = T_n$ for $n > m$. We set

$$\mathscr{M} = \bigcap_1^\infty F(T_n), \qquad \mathscr{N} = \bigcap_1^\infty F(T_n^*).$$

From now on, $F(\,\cdot\,)$ is the set of fixed vectors of the corresponding operator.

Lemma 2.2. *The following conditions are equivalent*:

(a) *the operators T_n weakly converge as $n \to \infty$;*

(b) *the operators T_n strongly converge as $n \to \infty$;*

(c) *$\mathscr{M}$ separates $\mathscr{N}$.*

Proof. Clearly, (b) ⇒ (a). For arbitrary distinct functionals $y_1, y_2 \in \mathscr{N}$, we find an element $x \in X$ such that $(x, y_1) \neq (x, y_2)$. Then we have

$$(x, y_i) = (x, T_n^* y_i) = (T_n x, y_i) \qquad \text{for } n \geq 1 \text{ and } i = 1, 2.$$

If $w\text{-}\lim_{n\to\infty} T_n = T_\infty$, then $(T_\infty x, y_1) \neq (T_\infty x, y_2)$. Since

$$\mathop{w\text{-}\lim}_{n\to\infty} T_m T_n x = T_m T_\infty x \qquad \text{and} \qquad T_m T_n = T_n \quad \text{for } n > m,$$

we have $T_m T_\infty x = T_\infty x$ for all $m \geq 1$; hence $T_\infty x \in \mathscr{M}$ and the implication (a) ⇒ (c) has been proved.

Now let

$$D = \bigcup_{1}^{\infty} \mathrm{Im}(I - T_n), \qquad G = \{x \in X : \exists \ s\text{-}\lim_{n\to\infty} T_n x\}.$$

Since $\|T_n\| \le \mathrm{const}$, we see that G is a closed subspace in X, and since

$$T_n(I - T_m) = 0 \qquad \text{for} \quad n > m$$

we have $s\text{-}\lim_{n\to\infty} T_n x = 0$ on D and $G \supset D$. Next,

$$\mathrm{Im}(I - T_n)^0 = \mathrm{Ker}(I^* - T_n^*) = F(T_n^*) \qquad \text{and} \qquad D^0 = \mathscr{N};$$

therefore, $G^0 \subset \mathscr{N}$. On the other hand, $G \supset \mathscr{M}$, and hence $G^0 \subset \mathscr{M}^0$. But, by the assumption of item (c), we have $\mathscr{M}^0 \cap \mathscr{N} = \{0\}$ and hence $G^0 = \{0\}$ and $G = X$. Thus, we have proved the implication (c) $\Rightarrow$ (b), and the proof of the lemma is complete. □

Remark 2.3. This lemma can be obtained from the general version of the operator ergodic theorem established by Sato [6]. But the corresponding argument is not shorter than the above argument.

Now we return to Theorem 2.1. The Wold decomposition (5) is equivalent to relation (3) for the projection operators P_n; therefore, it suffices to use Lemma 2.2 with $T_n = P_n$, $\mathscr{M} = M(V)$, and $\mathscr{N} = N(V, P)$. The theorem is proved.

The problem of convergence (3) of the projection operators P_n is often reduced to the mere comparison between the dimensions of the subspaces $M = M(V)$ and $N = N(V, P)$.

Lemma 2.4. *An arbitrary linear isometry V in X and an arbitrary orthogonal projection operator $P \in \mathscr{P}_V$ satisfy the following conditions*:

(a) $\dim M \le \dim N$;

(b) *if M separates N, then* $\dim M = \dim N$;

(c) *if* $\dim M < \infty$ *and* $\dim M = \dim N$*, then M separates N.*

Proof. Since $K^0 = N$, we have $\mathrm{codim}\, K = \dim N$. But $M \cap K = \{0\}$, hence $\mathrm{codim}\, K \ge \dim M$ and statement (a) has been proved. Next, since M separates N, it follows from Theorem 2.1 that the projection operators P_n weakly converge to P_∞ as $n \to \infty$, and hence $P_n^* \to P_\infty^*$ in the weak* operator topology. Since

$$P_m^* P_\infty^* = P_\infty^*, \quad \|P_m^*\| = 1 \qquad \text{for} \quad m \ge 1,$$

we see that P_∞^* is an orthogonal projection operator on the subspace N. Thus, $M = \mathrm{Im}\, P_\infty$ and $N = \mathrm{Im}\, P_\infty^*$, which implies that $\dim M = \dim N$. Finally, if $\dim M < \infty$ and $\dim M = \dim N$, then, taking the relation $\mathrm{codim}\, K = \dim M$ into account, we have

$$X = M \oplus K \qquad \text{and} \qquad M^0 \cap N = \{0\}.$$

Thus, M separates N, and the proof of the lemma is complete. □

The subspace $M = M(V)$ contains all the eigenvectors of the isometric operator V, and, in particular, $M \supset F(V)$, where $F(V) = \mathrm{Ker}(I - V)$. Similarly, the subspace $N = N(V, P)$ contains all the eigenvectors of the isometric operator U, and, in particular, $N \supset F(U)$. We use the identity $V^* U = I^*$ to show that N contains only those eigenvectors of the operator V^* that, first, correspond to the eigenvalues λ such that $|\lambda| = 1$ and, second, belong to the annihilator R^0. We show that the problem of the Wold decomposition for the pair of operators (V, P) is related to the possible inclusion $N \subset F(V^*)$, where $F(V^*) = \mathrm{Ker}(I^* - V^*)$.

Lemma 2.5. *If M separates N, then the conditions $N \subset F(V^*)$ and $M = F(V)$ are equivalent.*

Proof. The identities $P_n = V^nT^n$ and $TV = I$ imply the formula

$$P_n(I-V) = P_n - P_{n-1} + (I-V)P_{n-1}.$$

Since $N = K^0$ and $\mathrm{Im}(I-V)^0 = F(V^*)$, we have $N^0 = K$ and $\mathrm{Im}(I-V) \subset F(V^*)^0$. Now the condition $N \subset F(V^*)$ ensures the embedding $\mathrm{Im}(I-V) \subset K$, and thus, it ensures the convergence of the operators $P_n(I-V) \xrightarrow{s} 0$ as $n \to \infty$. Since $M^0 \cap N = \{0\}$, Theorem 2.1 guarantees (see Remark 1.3) the relation $P_n - P_{n-1} \xrightarrow{s} 0$ and hence the strong convergence $(I-V)P_{n-1} \to 0$. But $P_{n-1}x = x$ on M, and hence $M = F(V)$.

Conversely, if $M = F(V)$ and M separates N, then we have

$$(I-V)P_{n-1} \xrightarrow{s} 0, \quad P_n - P_{n-1} \xrightarrow{s} 0 \qquad \text{as} \quad n \to \infty.$$

But, in this case, we also have $P_n(I-V) \xrightarrow{s} 0$, which implies that $\mathrm{Im}(I-V) \subset K$. Passing to the annihilators, we obtain the inclusion $N \subset F(V^*)$. The proof of the lemma is complete. □

It turns out that, in a reflexive space X, the condition stating that "M separates N" is satisfied automatically.

Theorem 2.6. *Suppose that, for the linear isometry V, the linear orthogonal projection P on $\mathrm{Im}\, V$ exists in a reflexive Banach space X. Then the pair of operators (V, P) generates the Wold decomposition of the space X.*

Proof. By Theorem 2.1, it suffices to show that $M(V)$ separates $N(V, P)$. In the dual space X^*, the weak* closure of a convex set coincides with its strong closure. Therefore, relation (7) becomes

$$M^0 = \left(\bigcup_1^\infty \mathrm{Ker}\, P_n^*\right)^c.$$

The same argument as in the proof of item (a) in Lemma 1.1 allows us to prove that the sequence of projection operators P_n^* strongly converges to zero on the subspace M^0. But we have $P_n^* y = y$ on N for $n \ge 1$, and hence $M^0 \cap N = \{0\}$. The proof of the theorem is complete. □

According to Remark 1.3, the statement of the theorem is equivalent to the convergence (3) of the projection operators P_n. In the reflexive space, the convergence of a decreasing sequence of uniformly bounded commuting projection operators was proved even by Lorch [7]. This allows us to obtain a short alternative proof of Theorem 2.6, although the above proof also does not look too complicated. On the other hand, Lorch's result can easily be obtained as a special case of Lemma 2.2 if we act in the same way as in the proof of Theorem 2.6.

Moreover (see [5, Theorem 4.1.10]), each complemented subspace of a reflexive space is the image of the projection operator with the minimal possible norm. Thus, in Theorem 2.6, it suffices to assume that the projection $P_\varepsilon \colon X \to VX$ such that $\|P_\varepsilon\| \le 1 + \varepsilon$ exists for any $\varepsilon > 0$. In this connection, we also point out Pelczynski's classical result [8]: if V is an isometry in $X = l_p$, $1 \le p < \infty$, then the orthogonal projection operator $P \colon X \to VX$ exists. Meanwhile [5, Corollary 4.2.2], in smooth Banach spaces, one of which is the space l_p for $p > 1$, there is at most one orthoprojection on each subspace. So we have the following statement.

Theorem 2.7. *An arbitrary isometry in l_p, $1 < p < \infty$, generates a single Wold decomposition.*

Wold decompositions of the form (5) plays an important role in constructing (see [1]–[3]) the non-classical spectral theory of operators in Hilbert space based on the Szökefalvi-Nagy–Foiaş functional model. Of course, a constructive Banach version of such a theory would be of great interest. From this standpoint, it is worth pointing out the paper [9], where the Halmos results [2, Problem 116] describing the commutant of the one-sided shift of multiplicity 1 in Hilbert space were generalized to the Banach case.

3. CASE $X = C(H)$

Linear isometries naturally appear in topological dynamics. If φ is a continuous noninvertible mapping of a Hausdorff compact set H *onto itself*, then the operator $V = V_\varphi$ acting according to the law

$$(Vx)(\xi) = x(\varphi\xi) \qquad \text{for} \quad \xi \in H \tag{8}$$

is isometric in the space $C(H)$, equipped with sup-norm, of real or complex continuous functions $x(\xi)$ on H. The behavior of the iterations V^n in $C(H)$ is closely related to the dynamical properties of the semicascade $\{\varphi^n\}$, $n \ge 0$, on H. It would be useful to find out under what conditions on the epimorphism φ the operator $V = V_\varphi$ generates or does not generate the Wold decomposition of the space $C(H)$ depending on the choice of the projection operator $P \in \mathscr{P}_V$. In connection with this undoubtedly interesting problem, at this point, we restrict ourselves to several general remarks about the existence of linear orthogonal projections on the image $\operatorname{Im} V$ and to a detailed analysis of a meaningful example.

We note that the operator $V = V_\varphi$ of the form (8) realizes an algebraic homomorphism of the Banach algebra $C(H)$ and is a Markov operator, i.e., $V1 = 1$ and $Vx \ge 0$ for $x \ge 0$. In our case, it is more convenient to use the equivalent definition of the Markov operator, which says that the conditions $V1 = 1$ and $\|V\| = 1$ are satisfied. For each $n \ge 1$, the equivalent relation

$$\xi_1 \sim \xi_2 \qquad \text{if} \quad \varphi^n\xi_1 = \varphi^n\xi_2,$$

induces a decomposition of H into closed sets of the form $\xi^{(n)} = \varphi^{-n}\varphi^n\xi$, $\xi \in H$, and $\xi \in \xi^{(n)}$. The image $\operatorname{Im} V^n$ of the operator V^n consists of all continuous functions constant on each of the sets $\xi^{(n)}$ and is a closed subalgebra in $C(H)$ containing constants and even, in the complex situation, complex conjugate elements. Both the intersection of all $\operatorname{Im} V^n$ and the subalgebra $M(V)$ have the same properties. If the compact set H is metrizable, then, according to the results obtained in [10] and [11], a sufficient (but not necessary!) condition for an orthogonal (Markov) projection operator on the subspace $\operatorname{Im} V$ to exist in $C(H)$ is the condition that the mapping φ is open. Indeed, in the paper [10], it was proved that a linear operator $T\colon C(H) \to C(H)$ with the properties $\|T\| = 1$ and $TV = 1$ exists for an open surjection $\varphi\colon H \to H$, but then VT is a Markov projection operator on $\operatorname{Im} V$. In the general situation, the set $\mathscr{P}_V$ of such projection operators can be empty, for example (see [11]), for the following mapping φ of the interval $[0, 3]$ into itself:

$$\varphi\xi = \begin{cases} \xi, & \xi \in [0,1]; \\ 1, & \xi \in [1,2]; \\ 2\xi - 3, & \xi \in [2,3]. \end{cases}$$

For the example promised above, we consider the linear extension mapping of the unit circle S. In the multiplicative notation, this mapping is given by the formula

$$\varphi_m z = z^m, \qquad |z| = 1,$$

and in the additive notation, it is given by the relation

$$\varphi_m\theta = m\theta \mid \operatorname{mod} 2\pi.$$

Here $z = e^{i\theta}$ and m is a positive integer different 1. As is known [12], the iterations φ_m demonstrate a very complicated chaotic dynamics on S. We show that the isometry $V = V_\varphi$ corresponding to the mapping $\varphi = \varphi_m$ generates or does not generate the Wold decomposition of the space $C(S)$ depending on the choice of the Markov projection $P \in \mathscr{P}_V$.

The set AF of functions with absolutely converging Fourier series

$$x(z) = \sum_{k=-\infty}^{\infty} a(k) z^k$$

is strongly dense in $C(S)$. One of the Markov projection operators on $\operatorname{Im} V$ can be taken in the form

$$(Px)(z) = m^{-1} \sum_{j=0}^{m-1} x(z\nu_j), \tag{9}$$

where $x \in C(S)$ and ν_j are the mth roots of 1. For the functions $x(z)$ of class AF, we find that

$$(Px)(z) = \sum_{k=-\infty}^{\infty} a(km) z^{km}.$$

We inductively use the identity $P_n = VP_{n-1}V^{-1}P$ for projection operators of the form (1) along with the relations $V\colon z^k \to z^{km}$ and $V^{-1}\colon z^{km} \to z^m$ and obtain

$$(P_n x)(z) = \sum_{k=-\infty}^{\infty} a(km^n) z^{km^n}, \qquad x \in AF.$$

Since $\|x\| \le \sum_{k=-\infty}^{\infty}$ for $x \in AF$ and $\|P_n\| = 1$ for $n \ge 1$, we have

$$s\text{-}\lim_{n\to\infty} P_n x = \text{const} \qquad \text{for all } x \in C(S).$$

Thus (see Remark 1.3), we have proved that the isometry V and a projection operator P of the form (9) generate the Wold decomposition in $C(S)$.

To complete the picture, we find out how the critical subspaces $M = M(V)$ and $N = N(V,P)$ are organized in this case. By Theorem 2.1, M separates N. Since

$$P_n 1 = 1 \quad \text{for } n \ge 1 \qquad \text{and} \qquad s\text{-}\lim_{n\to\infty} P_n x = \text{const} \quad \text{on } C(S),$$

we have $M = \{\text{const}\}$. Therefore, $\dim M = 1$ and, by Lemma 2.4 (b), the subspace N is one-dimensional. We note that

$$P_n = m^{-n} \sum_{k=0}^{b(m,n)} \Phi(2\pi k m^{-n}),$$

where $b(m,n) = m^n - 1$ and $\Phi(\theta)$ is the shift operator in $C(S)$ acting according to the rule

$$\Phi(\theta)\colon x(z) \to x(ze^{i\theta}).$$

The corresponding operator $\Phi^*(\theta)$ in the dual space $C^*(S)$ has the form

$$\Phi^*(\theta)\mu = \mu_\theta, \qquad \text{where} \quad \mu_\theta(E) = \mu(e^{-i\theta}E)$$

for an arbitrary measure $\mu \in C^*(S)$ and a Borel set $E \subset S$. Thus,

$$P_n^*\mu = m^{-n} \sum_{k=0}^{b(m,n)} \Phi^*(2\pi k m^{-n})\mu, \qquad \mu \in C^*(S),$$

and hence $P_n^*\gamma = \gamma$ for the Lebesgue measure γ on S and any $n \ge 1$. Therefore, the subspace $N(V,P)$ consists of measures that are multiples of γ.

Now we construct (only for the case $m = 2$) the Markov projection operator $P \in \mathscr{P}_V$ with the property that the pair of operators (V,P) does not generate the Wold decomposition of the space $C(S)$. In this case, we start from the additive notation for the linear extension mapping $\varphi_2\colon S \to S$. In this situation, the set $\mathscr{P}_V$ is, in fact, described by Lloyd [11, Example 2] and consists of projection operators P_s acting on the elements $x \in C(S)$ by the rule

$$(P_s x)(\theta) = s(\theta)x(\theta) + s(\theta+\pi)x(\theta+\pi).$$

Here $s(\theta)$ is an arbitrary continuous function on S satisfying the conditions

$$0 \le s(\theta) \le 1 \qquad \text{and} \qquad s(\theta) + s(\theta+\pi) = 1 \quad \text{for } \theta \in S.$$

In particular, the projection operator (9) for $m = 2$ is associated with the function $s(\theta) \equiv 1/2$.

We set $s(\theta) = \cos^2 3\theta/2$, then the operator $T_s = V^{-1}P_s$ in $C(S)$ is given by the formula

$$(T_s x)(\theta) = s\left(\frac{\theta}{2}\right)x\left(\frac{\theta}{2}\right) + s\left(\frac{\theta}{2}+\pi\right)x\left(\frac{\theta}{2}+\pi\right),$$

and the adjoint operator $U_s = T_s^*$ acts on the Dirac measures $\delta(\theta) \in C^*(S)$ as follows:

$$U_s\delta(\theta) = s\left(\frac{\theta}{2}\right)\delta\left(\frac{\theta}{2}\right) + s\left(\frac{\theta}{2}+\pi\right)\delta\left(\frac{\theta}{2}+\pi\right).$$

For example,

$$U_s\delta(0) = s(0)\delta(0) + s(\pi)\delta(\pi) = \delta(0),$$

because $s(0) = 1$ and $s(\pi) = 0$. On the other hand,

$$U_s\left(\delta\left(\frac{2\pi}{3}\right) + \delta\left(\frac{4\pi}{3}\right)\right) = s\left(\frac{\pi}{3}\right)\delta\left(\frac{\pi}{3}\right) + s\left(\frac{4\pi}{3}\right)\delta\left(\frac{4\pi}{3}\right) + s\left(\frac{2\pi}{3}\right)\delta\left(\frac{2\pi}{3}\right) + s\left(\frac{5\pi}{3}\right)\delta\left(\frac{5\pi}{3}\right)$$
$$= \delta\left(\frac{2\pi}{3}\right) + \delta\left(\frac{4\pi}{3}\right),$$

because $s(2\pi/3) = s(4\pi/3) = 1$ and $s(\pi/3) = s(5\pi/3) = 0$. The subspace $N = N(V, P_s)$ contains all the invariant vectors of the operator U_s, and hence $\dim N \geq 2$. But $M(V) = \{\text{const}\}$; therefore, $\dim M = 1$ and, by Lemma 2.4 (b), the subspace M does not separate N. Now Theorem 2.1 allows us to conclude that the operator pair (V, P_s) *does not generate* the Wold decomposition of the space $C(S)$.

CONCLUSION

The Wold decomposition gives useful information about the structure of the isometric linear operator V acting in an arbitrary Banach space X. From this standpoint, special attention must be paid to the class of isometries $V_\varphi\colon C(H) \to C(H)$ induced by continuous mappings φ of the Hausdorff compact set H on itself. It would be interesting to understand under what topological conditions on the compact set H and the epimorphism φ the operator V_φ generates the Wold decomposition of the space of continuous functions $C(H)$ and to find out how the presence or absence of such a decomposition is related to the dynamics of the semicascade $\{\varphi^n\}$, $n \geq 0$, on H. These topics can be studied in subsequent papers.

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