J. Fluid Mech. (2014), vol. 748, pp. 416–432.
doi:10.1017/jfm.2014.197

Nonlinear wave run-up in bays of arbitrary cross-section: generalization of the Carrier–Greenspan approach

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(Received 1 March 2013; revised 18 March 2014; accepted 6 April 2014)

We present an exact analytical solution of the nonlinear shallow water theory for wave run-up in inclined channels of arbitrary cross-section, which generalizes previous studies on wave run-up for a plane beach and channels of parabolic cross-section. The solution is found using a hodograph-type transform, which extends the well-known Carrier–Greenspan transform for wave run-up on a plane beach. As a result, the nonlinear shallow water equations are reduced to a single one-dimensional linear wave equation for an auxiliary function and all physical variables can be expressed in terms of this function by purely algebraic formulas. In the special case of a U-shaped channel this equation coincides with a spherically symmetric wave equation in space, whose dimension is defined by the channel cross-section and can be fractional. As an example, the run-up of a sinusoidal wave on a beach is considered for channels of several different cross-sections and the influence of the cross-section on wave run-up characteristics is studied.

Key words: coastal engineering, solitary waves, surface gravity waves

1. Introduction

The dynamics of sea waves climbing the coast is a excellent manifestation of strong nonlinear effects in ocean physics. Understanding this process allows us to forecast inundation caused by storms, hurricanes and tsunamis. The calculation of wave run-up is usually based on a system of nonlinear shallow water equations, which are often enhanced by taking into account wave breaking, dispersion and friction. In general,

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such systems can only be analysed numerically. In many common physical situations, wave breaking, dissipation and dispersion can be neglected, particularly when the offshore wave heights are small and the waves are long.

If we further idealize the problem by considering a plane beach of constant slope, the run-up of long waves parallel to the beach can then be modelled by a system of one-dimensional nonlinear hyperbolic shallow water equations. The latter can be solved explicitly by the hodograph transformation suggested by Carrier & Greenspan (1958). Following this approach, a number of various wave shapes (solitons, cnoidal waves, Gaussian and Lorentz pulses, N-waves and Riemann waves) have already been studied (Carrier & Greenspan 1958; Spielvogel 1975; Pedersen & Gjevik 1983; Synolakis 1987; Synolakis, Deb & Skjelbreia 1988; Synolakis 1991; Pelinovsky & Mazova 1992; Tadepalli & Synolakis 1994; Brocchini 1998; Brocchini & Gentile 2001; Carrier, Wu & Yeh 2003; Kânoğlu 2004; Tinti & Tonini 2005; Kânoğlu & Synolakis 2006; Antuono & Brocchini 2007; Didenkulova, Kurkin & Pelinovsky 2007; Antuono & Brocchini 2008; Madsen & Fuhrman 2008; Didenkulova 2009; Antuono & Brocchini 2010). The main application of these studies has been in estimating tsunami heights on land (Synolakis 1987; Didenkulova, Pelinovsky & Soomere 2008). More recently, statistical effects caused by the irregular character of the incident wave and a randomly varying depth have also been taken into account (Denissenko *et al.* 2011; Didenkulova, Pelinovsky & Sergeeva 2011; Dutykh, Labart & Mitsotakis 2011). We would also like to point out recent studies of the resonance effects in the nearshore zone (Stefanakis, Dias & Dutykh 2011; Ezersky, Abcha & Pelinovsky 2013).

Bathymetries of coastal zones often have pronounced 2D geometries, resulting in wave focusing and strong variation of the run-up height along the coast, as can be observed in the catastrophic events of the 2004 tsunami in Sumatra (Synolakis & Bernard 2006) and the 2011 Tohoku tsunami in Japan (Choi *et al.* 2012). Tsunami waves are amplified dramatically in long and narrow bays, as demonstrated in numerous videos from 2011 (Fritz *et al.* 2012). We could only find analytical solutions of the linear shallow water equations in the work of Sammarco & Renzi (2008) and Renzi & Sammarco (2012).

However, for long and narrow bays, relative to the characteristic scale of the forcing waves, the water flow can be considered uniform over the cross-section. This allows us to introduce averaged variables and reduce the two-dimensional shallow water equations to a one-dimensional set. The case of a linearly inclined bay of parabolic cross-section has been studied, since in this case the shallow water equations have a relatively simple analytical solution (Zahibo *et al.* 2006; Choi *et al.* 2008; Didenkulova & Pelinovsky 2009, 2011*a*). As expected from physical considerations, the run-up height in such bays is considerably larger than on a plane beach. Nonlinear waves have been shown to propagate in inclined bays of parabolic cross-section without inner reflections from the bottom slope, but with reflections only from the shoreline, which leads to significant wave intensification on the beach.

In this study we present a novel analytical description of the wave run-up in inclined bays of arbitrary, not necessarily parabolic-like cross-section. Our solution presents a generalization of the Carrier–Greenspan method – hence all previously known solutions (the plane beach and inclined channel of parabolic cross-section) are special cases of our solution. Our method allows us to analyse more complicated bottom geometries and estimate tsunami heights for them. Our results help explain why bays focus tsunami energy, and can be used as benchmarks for numerical solutions.

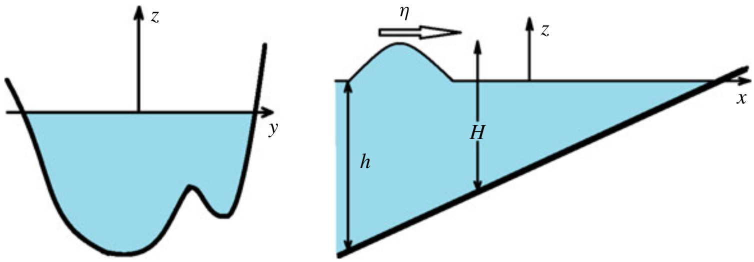

FIGURE 1. (Colour online) Cross-section and longitudinal projection of a long bay.

2. Generalized Carrier–Greenspan approach

Let us consider nonlinear wave propagation in a long and narrow bay, taking into account a uniform distribution of the incompressible and inviscid water flow over the cross-section (figure 1)

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(uS) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = g\frac{\mathrm{d}h}{\mathrm{d}x}. \tag{2.2}$$

Here $u(x, t)$ is the flow velocity averaged over the cross-section, $H(x, t) = \eta(x, t) + h(x)$ and $h(x)$ are respectively the perturbed (total) and unperturbed water depth along the main axis of the bay (the x-axis), $\eta(x, t)$ is the water displacement along the x-axis, $S(H)$ or $S(x)$ is the variable cross-section of the bay and g is the acceleration due to gravity. We impose the physically natural assumption that $S(H)$ is a monotonic function of H, and therefore $\mathrm{d}S/\mathrm{d}H \neq 0$.

Here we consider a linearly inclined bay with the x-axis directed onshore (figure 1)

$$h(x) = -\alpha x, \tag{2.3}$$

and an arbitrary cross-section $S(H)$.

Assuming that the water flow $u(H)$ is a function of the total depth only, it follows from (2.1) and (2.2) for a flat bottom that

$$u(H) = \pm \int \sqrt{\frac{g}{S}\frac{\mathrm{d}S}{\mathrm{d}H}}\,\mathrm{d}H, \tag{2.4}$$

(the $\pm$ sign in (2.4) defines the direction of wave propagation). For the Riemann invariants (Didenkulova & Pelinovsky 2011*b*) we then have

$$J_{\pm} = u \pm \int \sqrt{\frac{g}{S}\frac{\mathrm{d}S}{\mathrm{d}H}}\,\mathrm{d}H. \tag{2.5}$$

By (2.5), the system of (2.1) and (2.2) can be reduced to

$$\frac{\partial J_{\pm}}{\partial t} + c_{\pm}\frac{\partial J_{\pm}}{\partial x} = -g\alpha, \tag{2.6}$$

where

$$c_{\pm} = u \pm \sqrt{gS\frac{\mathrm{d}H}{\mathrm{d}S}}. \tag{2.7}$$

Substituting

$$I_{\pm} = J_{\pm} + g\alpha t, \tag{2.8}$$

in (2.6) yields

$$\frac{\partial I_\pm}{\partial t} + c_\pm \frac{\partial I_\pm}{\partial x} = 0. \tag{2.9}$$

The flow velocity can now be expressed in terms of modified Riemann invariants $I_\pm$ as

$$u = \frac{I_+ + I_-}{2} - g\alpha t. \tag{2.10}$$

The total water depth cannot be found directly at this step and will be addressed later.

The system of (2.9) can be written in the form

$$\frac{\partial(I_\pm, x)}{\partial(t, x)} + c_\pm \frac{\partial(t, I_\pm)}{\partial(t, x)} = 0, \tag{2.11}$$

where $\partial(A, B)/\partial(t, x) = (\partial A/\partial t)(\partial B/\partial x) - (\partial A/\partial x)(\partial B/\partial t)$ as usual denotes the Jacobian.

Apply the hodograph transformation to (2.11). Multiplying (2.11) by the non-zero Jacobian $\partial(t, x)/\partial(I_+, I_-) \neq 0$ (zero Jacobian corresponds to the wave breaking, which will be discussed later), we obtain the following system

$$\frac{\partial(I_\pm, x)}{\partial(I_+, I_-)} + c_\pm \frac{\partial(t, I_\pm)}{\partial(I_+, I_-)} = 0, \tag{2.12}$$

which can also be written in the form

$$\frac{\partial x}{\partial I_\mp} - c_\pm \frac{\partial t}{\partial I_\mp} = 0. \tag{2.13}$$

Eliminating x in (2.13), we obtain the second-order PDE

$$(c_- - c_+)\frac{\partial^2 t}{\partial I_+ \partial I_-} + \frac{\partial c_-}{\partial I_-}\frac{\partial t}{\partial I_+} - \frac{\partial c_+}{\partial I_+}\frac{\partial t}{\partial I_-} = 0 \tag{2.14}$$

for the time t. Now, following Carrier–Greenspan, it is convenient to introduce new variables

$$\lambda = \frac{I_+ + I_-}{2}, \quad \sigma = \frac{I_+ - I_-}{2}. \tag{2.15a,b}$$

Equation (2.14) then transforms into

$$(c_- - c_+)\left(\frac{\partial^2 t}{\partial \lambda^2} - \frac{\partial^2 t}{\partial \sigma^2}\right) + \frac{\partial(c_- - c_+)}{\partial \lambda}\frac{\partial t}{\partial \lambda} - \frac{\partial(c_- - c_+)}{\partial \sigma}\frac{\partial t}{\partial \sigma} \\ + \frac{\partial(c_- + c_+)}{\partial \lambda}\frac{\partial t}{\partial \sigma} - \frac{\partial(c_- + c_+)}{\partial \sigma}\frac{\partial t}{\partial \lambda} = 0. \tag{2.16}$$

The coefficients in (2.16) can be found from (2.7), (2.10) and (2.15). We have

$$c_+ + c_- = 2\lambda - 2g\alpha t, \tag{2.17}$$

$$c_+ - c_- = 2\sqrt{gS\frac{\mathrm{d}H}{\mathrm{d}S}} = F(\sigma), \quad \sigma = \int_0^H \sqrt{\frac{g}{S}\frac{\mathrm{d}S}{\mathrm{d}H'}}\mathrm{d}H', \tag{2.18}$$

and (2.16) simplifies to read

$$\left(\frac{\partial^2 t}{\partial \lambda^2} - \frac{\partial^2 t}{\partial \sigma^2}\right) - \frac{1}{F(\sigma)}\left(2 + \frac{\partial F}{\partial \sigma}\right)\frac{\partial t}{\partial \sigma} = 0. \tag{2.19}$$

Thus, we have obtained a linear second-order wave equation with a variable coefficient, $F(\sigma)$, determined by the shape of the bay cross-section. The main difference from previously considered cases of a plane beach and a parabolic cross-section (Carrier & Greenspan 1958; Choi *et al.* 2008; Didenkulova & Pelinovsky 2011*a*) is that the function $F(\sigma)$ does not have a general analytical expression.

Taking into account the expression for u found from (2.4) and (2.10):

$$u = \lambda - g\alpha t. \tag{2.20}$$

Equation (2.19) can be rewritten for the flow velocity as

$$\left(\frac{\partial^2 u}{\partial \lambda^2} - \frac{\partial^2 u}{\partial \sigma^2}\right) - \frac{1}{F(\sigma)}\left(2 + \frac{\partial F}{\partial \sigma}\right)\frac{\partial u}{\partial \sigma} = 0. \tag{2.21}$$

So, if we find a solution $u(\sigma, \lambda)$ of (2.21), then by (2.18) and (2.20) we also find H and t. What is missing so far is an expression for x. In order to find it, we use the difference between the two branches of (2.13):

$$\frac{\partial x}{\partial \sigma} = \frac{\partial x}{\partial I_+} - \frac{\partial x}{\partial I_-} = -c_+\frac{\partial t}{\partial I_-} + c_-\frac{\partial t}{\partial I_+} = \frac{c_+ + c_-}{2}\frac{\partial t}{\partial \sigma} + \frac{c_- - c_+}{2}\frac{\partial t}{\partial \lambda}. \tag{2.22}$$

Equation (2.22) can be simplified using (2.17) and (2.18) to read

$$\frac{\partial x}{\partial \sigma} = (\lambda - g\alpha t)\frac{\partial t}{\partial \sigma} - \frac{F(\sigma)}{2}\frac{\partial t}{\partial \lambda}, \tag{2.23}$$

and then by (2.20) we get

$$g\alpha\frac{\partial x}{\partial \sigma} = -u\frac{\partial u}{\partial \sigma} - \frac{F(\sigma)}{2} + \frac{F(\sigma)}{2}\frac{\partial u}{\partial \lambda}. \tag{2.24}$$

Equation (2.24) can be integrated if we introduce the substitution

$$u = \frac{1}{F(\sigma)}\frac{\partial \Phi}{\partial \sigma}, \tag{2.25}$$

and then find the desired expression for the coordinate x

$$2g\alpha x = -\int_0^{\sigma} F(\sigma')\mathrm{d}\sigma' + \frac{\partial \Phi}{\partial \lambda} - u^2. \tag{2.26}$$

Substituting (2.25) into (2.21), we finally obtain the linear variable-coefficient wave equation

$$\left(\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2}\right) - W(\sigma)\frac{\partial \Phi}{\partial \sigma} = 0, \quad W(\sigma) = \frac{2 - \mathrm{d}F/\mathrm{d}\sigma}{F(\sigma)}. \tag{2.27}$$

All original physical variables (the water flow velocity u, total wave depth along the main bay axis H, time t and coordinate x) can be found from the wavefunction $\Phi(\sigma, \lambda)$, using (2.18), (2.20), (2.25) and (2.26).

We recall that the water displacement is assumed uniform over the cross-section and the water surface is bounded by the bay geometry. Formally, the water displacement can be found from the definition of $H(x, t)$ using (2.18) and (2.26). We have

$$\eta(x, t) = H(x, t) - h(x) = H(\sigma) - \frac{1}{2g}\int_0^{\sigma} F(\sigma')\mathrm{d}\sigma' + \frac{1}{2g}\frac{\partial \Phi}{\partial \lambda} - \frac{u^2}{2g}, \tag{2.28}$$

where $H(\sigma)$ can be found as an inverse function of the right-hand side of (2.18). For the final statement of the problem, we should define boundary conditions at $\sigma = 0$ (moving shoreline) and at infinity $\sigma \to \infty$ (far offshore). At $\sigma = 0$ it is natural to assume that the wavefield should be bounded – that is,

$$\Phi(\sigma = 0, \lambda) < \infty, \quad \frac{\partial \Phi}{\partial \sigma}(\sigma = 0, \lambda) = 0, \tag{2.29}$$

and at infinity, where the water depth tends to infinity, it should tend to zero.

In the case of tsunami run-up it is common (see e.g. the popular Okada model (Okada 1985)) to assume zero initial velocity – that is,

$$H(x, t = 0) = H_0(x), \quad u(x, t = 0) = 0. \tag{2.30a,b}$$

It follows from (2.20) and (2.30) that at the initial time ($t = 0$) we have $\lambda = 0$ and, therefore, the corresponding initial conditions for $\Phi(\sigma, \lambda)$ can be found from (2.25) and (2.26) by

$$\Phi(\sigma, \lambda = 0) = 0, \quad \frac{\partial \Phi}{\partial \lambda}(\sigma, \lambda = 0) = \Psi(\sigma). \tag{2.31}$$

We note that the case of nonzero initial velocity (i.e. $u(x; t = 0) \neq 0$) requires certain modifications. In particular, in order to pass from (2.30) to (2.31) a linearization step is needed. This can be done similarly to the case of a plane beach, as considered in Kânoğlu & Synolakis (2006).

Equation (2.31) combined with the wave equation (2.27), (2.18), (2.20), (2.25) and (2.26) completely determine the mathematical problem of wave dynamics in an inclined bay. Of course, due to initial conditions (2.30) or (2.31) the wavefield represents two waves propagating in opposite (onshore and offshore) directions. We consider the one that describes the dynamics of an onshore (incoming) wave climbing the beach and then reflecting from it. It will be discussed later how we determine this wave.

These formulas generalize the classical Carrier–Greenspan approach developed for wave run-up on a plane beach (Carrier & Greenspan 1958) to a more general case of inclined bays of arbitrary cross-section. This is our main theoretical result. Note that the classical Carrier and Greenspan case follows from our (2.27) when $W(\sigma) = 1/\sigma^{-}$. It can be shown that all conclusions for the classical Carrier–Greenspan approach also remain valid for nonlinear waves in long bays.

Our approach goes as follows. We first solve the wave equation (2.27) on the semi-axis $\sigma \geqslant 0$, with the fixed boundary $\sigma = 0$ determining the moving shoreline. Note that this is a significant simplification over the original problem, which has to be solved in a domain with an unknown boundary (moving shoreline). This conclusion is general and independent of the bay configuration.

We note next that the 'linear' approximation for computing extreme run-up characteristics (maximum and minimum run-up heights and maximum run-up and run-down velocities) also remains valid. This approach is based upon the fact that the linearized shallow-water equations ((2.1) and (2.2)) can be reduced to the same wave equation (2.27), where variables σ and λ are proportional to x and t respectively, and do not contain nonlinear corrections (secondary terms in (2.20) and (2.26)). If the wavefield (incident wave) is determined far from the coast (where it is linear) then the initial and boundary conditions in the linear and nonlinear problems coincide. Hence, the solutions of the wave equation (2.27) for the function Φ in both the linear and nonlinear problems also coincide, although the physical meaning of variables (σ, λ) differ. As a result, the extreme values of the wavefield at the moving boundary $\sigma = 0$ (wave run-up) in both the linear and nonlinear problems coincide, although the 'physical' wavefields differ. This means that by computing the maximum water displacement and velocity at $x = 0$ (fixed shoreline) in the framework of the linear shallow water theory, we can find the maximum run-up height and shoreline velocity within the nonlinear theory. This conclusion was known for long-wave run-up on a plane beach (Pelinovsky & Mazova 1992; Didenkulova 2009) and it is now rigorously proved for inclined bays. The analogy between linear and nonlinear solutions indicated above can also be expressed in simple formulas for characteristics of the nonlinear wavefield at $\sigma = 0$ (i.e. the shoreline (Lagrangian) velocity $u(t)$ and the vertical shoreline motion $r(t)$). The latter, in turn, are related to corresponding solutions of the linear shallow-water equations at $x = 0$ (i.e. the vertical water displacement $R(t)$, the horizontal water displacement $X(t) = R(t)/\alpha$ and the flow velocity $U(t)$) and can be derived from (2.20) and (2.26) by setting $\sigma = 0$:

$$u(t) = U\left(t + \frac{u}{g\alpha}\right), \quad r(t) = R\left(t + \frac{u}{g\alpha}\right) - \frac{u^2}{2g}. \tag{2.32a,b}$$

Equation (2.32) clearly demonstrates the nonlinear deformation of the wave profile, which is similar to the nonlinear transformation of Riemann waves in nonlinear acoustics (Rudenko & Soluyan 1977). This coincidence of extreme characteristics in linear and nonlinear theories directly follows from (2.32), as shown by Synolakis (1987) and Pelinovsky & Mazova (1992) for a plane beach problem. Here we find that this run-up invariance holds for our generalized bathymetry as well. It should be noted that even though the equations in (2.32) are universal for bays of arbitrary cross-sections, the magnitudes of their run-up characteristics are different in bays of different cross-sections, and depend on particular configurations. Our last conclusion following from (2.32) is related to the validity of our results due to wave breaking. The Lagrangian velocity of the shoreline, defined by the first equation in (2.32), remains a single-valued function of time only as long as

$$Br = \frac{\max(\mathrm{d}U/\mathrm{d}t)}{g\alpha} \leqslant 1. \tag{2.33}$$

Equation (2.33) can be obtained by differentiating the first equation in (2.32) with respect to time (see Didenkulova 2009). From the mathematically point of view, the condition $Br = 1$ corresponds to the so-called gradient catastrophe in hyperbolic equations, and in physical terms it identifies wave breaking. It is remarkable that the breaking criterion in the nonlinear theory (2.33) can be found from the characteristics of the linear wavefield.

Due to the simple mechanical relation $U = (1/\alpha)\mathrm{d}R/\mathrm{d}t$ at $x = 0$ between the linear water displacement R and the flow velocity U, the breaking criterion can be rewritten in the more convenient form

$$Br = \frac{\max(\mathrm{d}U/\mathrm{d}t)}{g\alpha} = \frac{\max(\mathrm{d}^2R/\mathrm{d}t^2)}{g\alpha^2} = 1. \tag{2.34}$$

We emphasize that the breaking criterion in the form of (2.34) is independent of the bay geometry. It can also be obtained from the applicability condition of the hodograph transformation, which states that the Jacobian of the transformation should not vanish. i.e.

$$\frac{\partial(t, x)}{\partial(I_+, I_-)} = (c_+ - c_-)\frac{\partial t}{\partial I_+}\frac{\partial t}{\partial I_-} = \frac{F(\sigma)}{4g^2\alpha^2}\left[\left(1 - \frac{\partial u}{\partial \lambda}\right)^2 - \left(\frac{\partial u}{\partial \sigma}\right)^2\right] \neq 0. \tag{2.35}$$

As expected, it follows from (2.35) that the Jacobian vanishes at the shoreline $\sigma = 0$, where hyperbolicity of the (initial) shallow water equations breaks down. In a small neighbourhood of this point the Jacobian tends to zero if $\partial u/\partial\lambda \to 1$ and $\partial u/\partial\sigma \to 0$. We have not been able to prove in general that $\partial u/\partial\sigma \to 0$ at the moving shoreline, but have demonstrated this for the case of U-shaped bays (see below after (3.12)). The condition $\partial u/\partial\lambda = 1$ is equivalent to (2.34).

3. Nonlinear wave run-up in U-shaped bays

Let us apply the developed theory to nonlinear waves in a bay with a cross-section described by the power law

$$z \sim |y|^m, \tag{3.1}$$

with an arbitrary positive exponent m. In particular, the case of $m \to \infty$ corresponds to a bay of rectangular cross-section, considered in numerous papers on the run-up problem for a plane beach cited in the Introduction. The case $m = 2$ (a bay of parabolic cross-section) is studied in Didenkulova & Pelinovsky (2009); Didenkulova & Pelinovsky (2011*a*). The case of (3.1) is more general and leads to the new results discussed below. In the case of (3.1), $S \sim H^{(m+1)/m}$ and the function $F(\sigma)$ becomes explicit

$$F(\sigma) = \frac{m}{m+1}\sigma, \quad \sigma(H) = 2\sqrt{\frac{m+1}{m}gH}. \tag{3.2a,b}$$

It follows from (3.2) that the variable σ is proportional to the square root of the total water depth and has a better physical meaning for a U-shaped bay geometry. The wave equation (2.27) for U-shaped bays also simplifies to read

$$\left(\frac{\partial^2\Phi}{\partial\lambda^2} - \frac{\partial^2\Phi}{\partial\sigma^2}\right) - \frac{m+2}{m\sigma}\frac{\partial\Phi}{\partial\sigma} = 0. \tag{3.3}$$

Consider in (3.3) two particular cases of m: (i) the radially symmetric wave equation ($m \to \infty$) corresponding to the problem of wave run-up on a plane beach, and (ii) the spherically symmetric wave equation ($m = 2$) corresponding to the problem of wave run-up in inclined bays of parabolic cross-section. Both equations have been completely investigated in mathematical physics. Their Green's functions are known and thus it is possible to find exact solutions for run-up of nonlinear waves of different shapes, including the Korteweg–de Vries soliton (see, e.g. Synolakis 1987; Carrier *et al.* 2003). Much less is known about the wave equation for other values of the parameter m. Note that (3.3) can be rewritten as an n-dimensional spherically symmetric equation

$$\left(\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2}\right) - \frac{n-1}{\sigma}\frac{\partial \Phi}{\partial \sigma} = 0, \tag{3.4}$$

where

$$n = 2\left(1 + \frac{1}{m}\right) \quad \text{or} \quad m = \frac{2}{n-2}. \tag{3.5}$$

As we pointed out above, the radially symmetric case of (3.3) ($n = 2$, $m = \infty$), corresponding to a plane beach, is studied in Carrier & Greenspan (1958); Synolakis (1987); Didenkulova & Pelinovsky (2008); the spherically symmetric case ($n = 3$, $m = 2$) for waves in channels of parabolic cross-section is studied in Choi *et al.* (2008) and Didenkulova & Pelinovsky (2009); Didenkulova & Pelinovsky (2011*a*). As follows from (3.5), the space dimension n increases when m decreases. In particular, the case of $n = 4$ corresponds to $m = 1$ and describes wave run-up in the inclined channel of triangular cross-section; for $n = 5$ we have a V-shaped (convex) bay with slope $m = 2/3$, and so on. It is important that wave solutions in even and odd spaces are fundamentally different (Morse & Feshbach 1953; Courant & Hilbert 1989). In odd spaces the Green's function is always finite, while in even spaces it always has a decaying tail. We find it quite interesting and even surprising that these fundamental mathematical physics results of a linear nature also play an important role in the nonlinear problem of sea wave run-up in bays and channels, as will be demonstrated below.

For the power-law cross-section (3.1) many formulas obtained above can be simplified. In particular, integral expressions for x and η are

$$2g\alpha x = -\frac{m}{m+1}\frac{\sigma^2}{2} + \frac{\partial \Phi}{\partial \lambda} - u^2, \tag{3.6}$$

$$\eta = \frac{1}{2g}\frac{\partial \Phi}{\partial \lambda} - \frac{u^2}{2g}. \tag{3.7}$$

The bounded solution of (3.3) for an arbitrary value of parameter m is

$$\Phi = \frac{A_0}{\sigma^{1/m}} J_{1/m}\left(\frac{\omega\sigma}{g\alpha}\right) \sin\left(\frac{\omega\lambda}{g\alpha} + \varphi\right), \tag{3.8}$$

where A_0 and φ are two arbitrary constants and J is a Bessel function of the first kind. The solution (3.8) satisfies both boundary conditions at $\sigma = 0$ and $\sigma \to \infty$. It also satisfies the initial condition (2.31) if $\varphi = 0$.

Far from the shoreline, the wave has small amplitude and can be considered linear. The solution (3.8) can then be rewritten by using asymptotics of the Bessel function and neglecting all nonlinear terms in (3.6) and (3.7). That is,

$$\eta\,(x \to -\infty, t) = A\left\{\sin\left(\omega\,[t+\tau] - \frac{\pi}{2m} - \frac{\pi}{4}\right) + \sin\left(\omega\,[t-\tau] + \frac{\pi}{2m} + \frac{\pi}{4}\right)\right\}, \tag{3.9}$$

$$A(x) = \frac{\alpha A_0 \sqrt{\omega}}{2\sqrt{2\pi}\,(\alpha g)^{(2m+1)/m}\,\tau^{((m+2)/2m)}}, \qquad \tau\,(x) = \sqrt{\frac{4|x|}{g\alpha}\left(\frac{m+1}{m}\right)}. \tag{3.10a,b}$$

Here $\tau(x)$ has the meaning of the travel time from the shoreline to the current position. Equation (3.9) describes a standing wave formed due to full reflection from the shore. Its amplitude decreases as $|x|$ (or the travel time τ from the shore) increases. The solution (3.9) represents the superposition of two waves propagating in onshore (first term) and offshore (second term) directions. In fact, the initial (Cauchy) problem (2.31) can be now reformulated as the boundary problem at a fixed distance L from the shoreline, assuming that the parameters of the incident wave (its amplitude and phase) at this point are known. With no loss of generality we can insert the phase in (3.9) again. Such a boundary condition better corresponds to the physics of tsunami waves approaching the coast and can be recorded at a fixed point by a buoy. This will be used in all the examples below. Strictly speaking, it is more natural to state such a boundary condition at a point where the inclined beach meets the flat bottom. As shown in Antuono & Brocchini (2007), the nonlinear transformation at this joining point results in stronger nonlinear shoreline dynamics. However, if this joining point is located far from the shoreline, the nonlinear effects are weak and can be neglected in the first approximation.

The reflected wave passing through the fixed point $x=-L$ has the following phase shift with respect to the incident wave

$$\Psi = 2\omega\tau - \frac{\pi}{2} - \frac{\pi}{m}. \tag{3.11}$$

The physical meaning of the first term in (3.11) is the travel time to the shore and back to the initial position. The other two terms are related to the specifics of the wave reflection from the shore, which corresponds to the reflection from what they call a singular point in mathematics. In the case of a plane beach ($m\rightarrow\infty$), the additional phase shift $\pi/2$ is equivalent to the Hilbert transform (Huang, Shen & Long 1999). As a result, the reflected wave has a different shape from that of a non-monochromatic incident wave (Pelinovsky & Mazova 1992). For an inclined channel of parabolic cross-section ($m=2$) the phase shift is equal to π, so the wave inverts (Didenkulova & Pelinovsky 2011*a*). For $m=2/3$ there is an additional phase shift 2π, which means that the shapes of incident and reflected waves coincide. We emphasize that all these conclusions are obtained within the nonlinear theory. So, a wave initially linear and monochromatic becomes strongly nonlinear and non-monochromatic near the shore, but once reflected from the shore it eventually reverts to linear and monochromatic.

The fixed amplitude A and phase φ of the incident monochromatic wave completely determines the solution by (3.8). By (2.25) and (3.2), for the velocity field we have

$$u(\sigma,\lambda) = \frac{m+1}{m\sigma}\frac{\partial\Phi}{\partial\sigma}. \tag{3.12}$$

Note that

$$\frac{\partial u}{\partial\sigma} = A_0\frac{m+1}{m}\left(\frac{\omega}{g\alpha}\right)^2 \sigma^{-(m+1)/m} J_{(2m+1)/m}\left(\frac{\omega\sigma}{g\alpha}\right)\sin\left(\frac{\omega\lambda}{g\alpha}+\varphi\right). \tag{3.13}$$

Taking into account the asymptotic behaviour of the Bessel function for $\sigma\rightarrow 0$, we obtain that $\partial u/\partial\sigma \sim \sigma \rightarrow 0$ at the moving shoreline. This proves the validity of the wave breaking condition $\partial u/\partial\lambda = 1$ in (2.35) for the case of U-shaped bays.

For the point of the moving shoreline $\sigma=0$, (3.12) can be simplified. To this end, we use the asymptotic expansion of the Bessel function and substitute the constant A_0

from (3.10) into (3.12) and finally get

$$u(\lambda, \sigma = 0) = -2A \frac{\omega\sqrt{\pi}}{\alpha \Gamma\left(\dfrac{m+1}{m}\right)} \left(\frac{\omega\tau}{2}\right)^{(m+2)/2m} \sin\left(\frac{\omega\lambda}{g\alpha} + \varphi\right). \tag{3.14}$$

Similarly, it follows from (3.7) for $\sigma \to 0$ that

$$\eta(\lambda, \sigma = 0) = A \frac{2\sqrt{\pi}}{\Gamma\left(\dfrac{m+1}{m}\right)} \left(\frac{\omega\tau}{2}\right)^{(m+2)/2m} \cos\left(\frac{\omega\lambda}{g\alpha} + \varphi\right) - \frac{u^2(\lambda, \sigma = 0)}{2g}, \tag{3.15}$$

and, therefore, for the maximal run-up height we have

$$R = A \frac{2\sqrt{\pi}}{\Gamma\left(\dfrac{m+1}{m}\right)} \left(\frac{\omega\tau}{2}\right)^{(m+2)/2m}. \tag{3.16}$$

In the limiting case of a plane beach or channel of rectangular shape ($m \to \infty$) this equation simplifies to read (Carrier & Greenspan 1958; Synolakis 1987; Madsen & Fuhrman 2008; Didenkulova 2009)

$$R = A\sqrt{2\pi\omega\tau}, \quad \tau = \sqrt{\frac{4L}{g\alpha}}. \tag{3.17a,b}$$

In the case of a channel of parabolic cross-section ($m = 2$), (3.16) transforms (Didenkulova & Pelinovsky 2011*a*) into

$$R = 2A\omega\tau, \quad \tau = \sqrt{\frac{6L}{g\alpha}}. \tag{3.18}$$

The wave amplification as a function of $\omega\tau$ for different m is shown in figure 2. Note that (3.16) is valid only for relatively large $\omega\tau$, where the asymptotic expansions of the Bessel function are valid. Therefore in this limiting case the amplification (the ratio of the wave run-up height to the initial wave amplitude) is larger for narrower channels (smaller m), which is quite natural since the channel cross-section is smaller for smaller m. However, as follows from figure 2, for each fixed $\omega\tau$, there is no monotonic dependence between the wave amplification and the parameter m. It can be explained as follows. In narrower channels, for relatively small $\omega\tau$, a greater velocity of the water flow results in a higher run-up at short distance. Note that the travel time τ also depends on the shape coefficient m, which also influences the magnitude of the wave amplification.

Using the Fourier superposition of the elementary solution (3.8), we can also calculate the maximum run-up height of an arbitrary wave. Consider the incident wave at a distance L from the shore, given by the Fourier integral

$$\eta_{in}(t, L) = \int_{-\infty}^{\infty} A(\omega, L) \exp(\mathrm{i}\omega t)\, \mathrm{d}\omega, \tag{3.19}$$

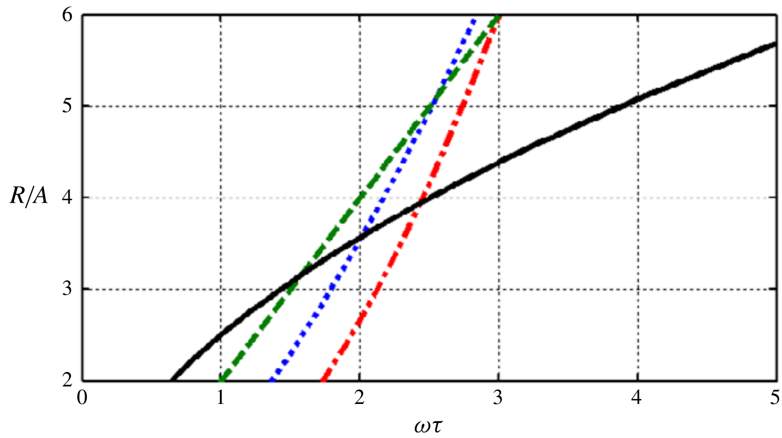

FIGURE 2. (Colour online) Wave amplification as a function of the parameter $\omega\tau$ for an almost plane beach with $m=100$ (black solid), $m=2$ (green dashed), $m=1$ (blue dotted) and $m=2/3$ (red dash-dotted).

where $A(\omega)$ is a complex amplitude taken to satisfy $A(-\omega)=A^*(\omega)$ in order to make the wavefield real valued. In this case water oscillations at the beach will also be in the form of the Fourier integral, which can in turn be found by (3.15). Thus we have

$$\eta(\lambda,\sigma=0)=\frac{2\sqrt{\pi}}{\Gamma\left(\dfrac{m+1}{m}\right)}\left(\frac{\tau}{2}\right)^{((m+2)/2m)}\int_{-\infty}^{+\infty}|\omega|^{((m+2)/2m)}A(\omega)\exp\left(\mathrm{i}\left[\omega\left(\frac{\lambda}{g\alpha}-\tau\right)\right.\right.$$
$$\left.\left.-\left(\frac{\pi}{2m}+\frac{\pi}{4}\right)\operatorname{sign}(\omega)\right]\right)\mathrm{d}\omega-\frac{u^2(\lambda,\sigma=0)}{2g}, \tag{3.20}$$

where

$$u(\lambda,\sigma=0)=g\frac{\mathrm{d}}{\mathrm{d}\lambda}\eta(\lambda,\sigma=0). \tag{3.21}$$

Equation (3.20) demonstrates the principal difference between the solutions of (3.4) for even and odd spaces (if we consider n instead of m). The run-up displacement corresponding to an odd-dimension space can be simplified and presented as a high-order time-derivative of the initial wave shape. A similar result is given in Cally (2012) with an application to the solar atmosphere. So, for $m=2$ (which corresponds to the spatial dimension $n=3$), the wave run-up is the first temporal derivative of the initial wave profile (Didenkulova & Pelinovsky 2009)

$$\eta(\lambda,\sigma=0)=2g\alpha\tau\frac{\mathrm{d}\eta_{in}(\lambda/g\alpha-\tau)}{\mathrm{d}\lambda}-\frac{u^2}{2g}. \tag{3.22}$$

By straightforward computation, it follows from (3.20) that, for $m=2/3$ ($n=5$), the wave run-up is the order two time-derivative

$$\eta(\lambda,\sigma=0)=\frac{8g^2\alpha^2}{3}\left(\frac{\tau}{2}\right)^2\frac{\mathrm{d}\eta_{in}^2(\lambda/g\alpha-\tau)}{\mathrm{d}\lambda^2}-\frac{u^2}{2g}. \tag{3.23}$$

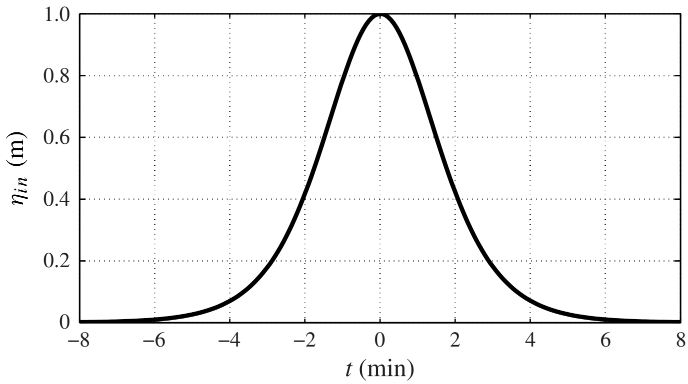

FIGURE 3. Incident solitary wave ($A = 1$ m, $T = 2$ min).

For $m = 2/5$ ($n = 7$), it is the order three time-derivative

$$\eta\,(\lambda, \sigma = 0) = \frac{16 g^3 \alpha^3}{15}\left(\frac{\tau}{2}\right)^3 \frac{\mathrm{d}\eta_{in}^3\,(\lambda/g\alpha - \tau)}{\mathrm{d}\lambda^3} - \frac{u^2}{2g}, \tag{3.24}$$

and so on. It also follows from (3.21)–(3.24) that a wave of finite duration will produce wave oscillations at the coast of finite duration.

As we discussed before, this simple representation of the wave run-up as temporal derivatives of the initial wave profile occurs only for odd-dimensional spaces. For spaces of even dimension the integral in (3.20) does not reduce to differentiation. For instance, for $m = 1$ ($n = 4$), we have

$$\begin{aligned}\eta\,(\lambda, \sigma = 0) \;=\; & 2\sqrt{\pi}\left(\frac{\tau}{2}\right)^{3/2} \int_{-\infty}^{+\infty} |\omega|^{3/2} A\,(\omega) \\ & \times \exp\left(\mathrm{i}\left[\omega\left(\frac{\lambda}{g\alpha} - \tau\right) - \frac{3\pi}{4}\mathrm{sign}\,(\omega)\right]\right) \mathrm{d}\omega - \frac{u^2}{2g}.\end{aligned} \tag{3.25}$$

The run-up of a soliton-type incident wave (i.e. the one-soliton solution of the Korteweg–de Vries equation, see figure 3)

$$\eta_{in}(t, L) = A\,\mathrm{sech}^2(t/T) \tag{3.26}$$

on the coast in channels with different m is shown in figure 4 for an incident soliton registered 3.7 km from the coast at a water depth of 100 m.

It can be seen that a smaller m leads to a greater wave run-up height, a larger number of waves hitting the coast, and a longer delay of the first wave arrival. Note that a larger run-up height means a greater probability of wave breaking. For example, in the case given in figure 4, the breaking parameter Br for a bay with $m = 100$ corresponds to an almost linear wave ($Br = 0.06$), while for a bay with $m = 2/3$ the wave is nearly at the breaking point ($Br = 0.99$), which is indicated in figure 4 by the sharp and narrow wedge at the run-down. This can be observed even better in figure 5 for the corresponding shoreline velocity. The wave breaking here corresponds to the vertical profile of the shoreline velocity (see the red dash-dotted line). Figure 5 also shows that the shoreline velocity is greater for narrower channels – decreasing parameter m (recall that the transversal profile is $z \sim |y|^m$).

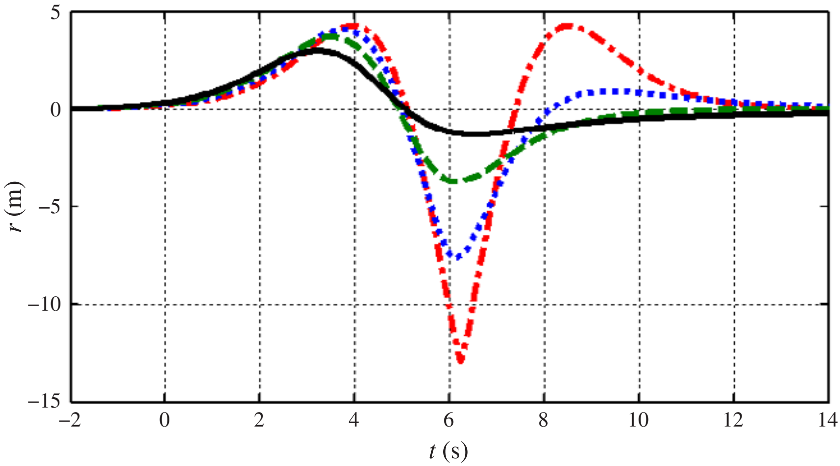

FIGURE 4. (Colour online) Run-up of the solitary wave in a bay with $m = 100$ (black solid), $m = 2$ (green dashed), $m = 1$ (blue dotted) and $m = 2/3$ (red dash-dotted).

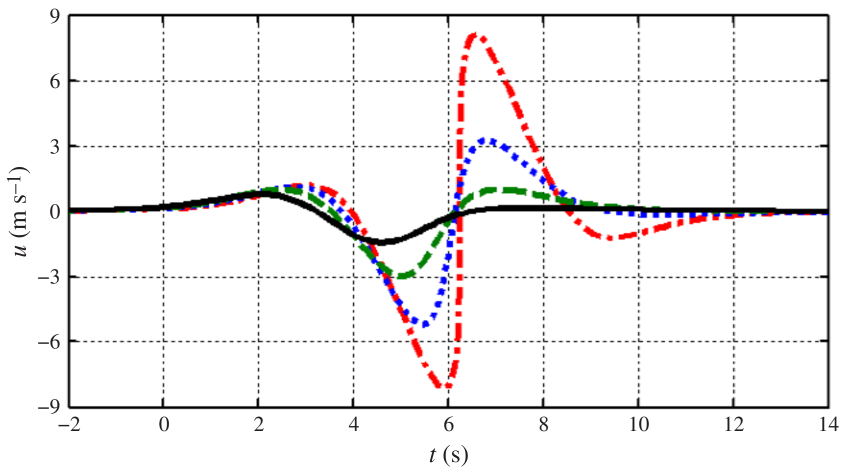

FIGURE 5. (Colour online) Shoreline velocity in a bay with $m = 100$ (black solid), $m = 2$ (green dashed), $m = 1$ (blue dotted) and $m = 2/3$ (red dash-dotted).

4. Conclusions

This paper is concerned with generalizing the Carrier–Greenspan approach to nonlinear wave dynamics originally developed for a plane beach to inclined bays of arbitrary cross-section. This allows exact solutions in the nonlinear shallow water theory. Our general approach easily recovers all previous results on the run-up problem for plane beaches and U-shaped bays. It also yields some new analytical solutions for waves in inclined bays of arbitrary cross-section.

For all inclined geometries (plane beach, inclined bays of any cross-sections) we prove that extreme wave run-up characteristics (maximum run-up/minimum run-down and maximum run-up/run-down velocities) coincide in both linear and nonlinear shallow water theories. This can directly be used to estimate run-up of the leading tsunami wave and the subsequent inundation without time consuming numerical computations. The latter is particularly important for tsunami broadcast in the coastal zone of Alaska, where the use of expensive full-scale numerical models is not feasible due to the low population density.

Our model, as many others, is no longer valid after the wave breaking point. However, our breaking criterion determining the range of validity of the shallow water theory can be written in terms of the linear problem. This effect was initially discovered for waves on a plane beach (Pelinovsky & Mazova 1992). We demonstrate that our breaking criterion is universal for all inclined bathymetries – that is, the same for every cross-section.

Acknowledgements

The research leading to this paper is supported in part by the United States NSF under grant DMS1009673, the European Union through support of the European Regional Development Fund to the Centre of Excellence in Nonlinear Studies and by grants ETF-8870, MK-1146.2014.5 and RFBR (14-02-00983, 14-05-00092, 13-05-97037). ID acknowledges support provided by the Alexander von Humboldt Foundation. EP acknowledges support from Volkswagen Stiftung and the Austrian Science Foundation (FWF) under project P246710.

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