A Selection Principle for Mappings of Bounded Variation

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Submitted by Hélene Frankowska

   E. Helly's selection principle states that *an infinite bounded family of real functions on the closed interval*, *which is bounded in variation*, *contains a pointwise convergent sequence whose limit is a function of bounded variation*. We extend this theorem to metric space valued mappings of bounded variation. Then we apply the extended Helly selection principle to obtain the existence of regular selections of (non-convex) set-valued mappings: *any set-valued mapping from an interval of the real line into nonempty compact subsets of a metric space*, *which is of bounded variation with respect to the Hausdorff metric*, *admits a selection of bounded variation*. Also, we show that a compact-valued set-valued mapping which is Lipschitzian, absolutely continuous, or of bounded Riesz Φ-variation admits a selection which is Lipschitzian, absolutely continuous, or of bounded Riesz Φ-variation, respectively.
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   Key Words: Helly's selection principle; bounded variation; metric space valued mappings; set-valued mappings; regular selections.

1. INTRODUCTION

   This paper is devoted to the question of existence of those selections of a given set-valued mapping which preserve certain nice (or regular) properties of the mapping. There exists a vast literature on the existence of regular selections for set-valued mappings with convex images (cf. Michael [16], Dommisch [10], Aubin and Frankowska [1], Dentcheva [9], and references therein). If the images are not convex, one cannot in general expect more than measurable selections or selections which are Baire mappings (see Kuratowski and Ryll-Nardzewski [15] and Čoban [7]); indeed, many

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examples exist to show that a continuous mapping from a closed interval into nonempty compact subsets of a ball in $\mathbb{R}^2$ or even a Lipschitz-continuous mapping from $\mathbb{R}^3$ into nonempty compact subsets of a ball in $\mathbb{R}^3$ need not admit a continuous selection (see, e.g., Hermes [13]).

We are going to show that the situation is different if the domain of the set-valued mapping F under consideration is an interval of the real line, the values of F lie in an arbitrary metric space, and the total variation of F with respect to the Hausdorff metric is finite. We are interested in finding those selections of bounded variation of F which pass through a given point of the graph of F and whose total variation does not exceed the total variation of F. If X is a (finite- or infinite-dimensional) Banach space, the existence of regular selections of bounded variation in different senses was proved in [2−6] under the assumption that the graph of the set-valued mapping is compact (for more details see Section 2). In this paper we generalize certain theorems on the existence of regular selections for set-valued mappings of bounded Jordan variation with images that are arbitrary, nonempty, compact subsets of a metric space.

First we establish the following Helly type selection principle: *an infinite pointwise precompact family of metric space valued mappings on the closed interval of the real line*, *which is bounded in variation*, *contains a pointwise convergent sequence whose limit is a mapping of bounded variation* (Theorem 1). Then we apply the Helly selection principle to obtain the existence of regular selections: *any set-valued mapping from an interval of the real line into nonempty compact subsets of a metric space*, *which is of bounded variation with respect to the Hausdorff metric*, *admits a selection of bounded variation* (Theorem 2). Finally, we show that the more regular the set-valued mapping is under consideration (i.e., Lipschitzian, absolutely continuous, or of bounded generalized Riesz Φ-variation) the more regular selection it admits (Theorem 3).

2. PRELIMINARIES AND PRINCIPAL RESULTS

We begin with reviewing some definitions and facts needed for our results.

Let (X, d) be a metric space, and let $E \subset \mathbb{R}$ be a nonempty set. Denote by X^E the set of all mappings $f: E \to X$ from E into X. Given $f \in X^E$, the (total) *Jordan variation* of f is the quantity

$$V(f, E) = \sup_T \sum_{i=1}^m d(f(t_i), f(t_{i-1})), \qquad (1)$$

where the supremum is taken over all partitions $T = \{t_i\}_{i=0}^m$ of E, i.e., $m \in \mathbb{N}$, $\{t_0, t_1, \ldots, t_m\} \subset E$, and $t_{i-1} < t_i$, $i = 1, \ldots, m$. If $V(f, E) < \infty$, the

mapping f is said to be *of bounded* (*Jordan*) *variation* on E. In particular, if $E = [a, b]$ is a closed interval, then $V(f, [a, b])$ is equal to the right hand side of (1) with the supremum taken over partitions $T = \{t_i\}_{i=0}^m$ of $[a, b]$ of the form $m \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$.

A family of mappings $\mathscr{F} \subset X^E$ is said to be (a) *of uniformly bounded variation* or *bounded in variation* if there exists a constant $C \geq 0$ such that $V(f, E) \leq C$ for all $f \in \mathscr{F}$; (b) *pointwise precompact* if the set $\mathscr{F}(t) = \{f(t) \mid f \in \mathscr{F}\}$ is precompact (i.e., its closure is compact) in X for all $t \in E$; (c) *bounded* in the case $X = \mathbb{R}$ if there is a constant $C \geq 0$ such that $|f(t)| \leq C$ for all $t \in E$ and $f \in \mathscr{F}$. A sequence of mappings $\{f_n\}_{n=1}^\infty \subset X^E$ is said to be *pointwise convergent* on E to a mapping $f \in X^E$ if $\lim_{n \to \infty} d(f_n(t), f(t)) = 0$ for all $t \in E$.

The classical Helly selection principle [12; 18, Chap. 8, Sect. 4, Helly's Theorem] asserts that *an infinite bounded family of functions $\mathscr{F} \subset \mathbb{R}^{[a, b]}$ of uniformly bounded variation contains a sequence which converges on $[a, b]$ to a function of bounded variation*. This theorem is essentially based on the Jordan decomposition theorem (i.e., a function $f \in \mathbb{R}^{[a, b]}$ is of bounded variation if and only if it is the difference of two bounded nondecreasing functions) and the following theorem which was originally due to Helly [12] and is sometimes called a selection principle for monotone functions (its proof can also be found, e.g., in the book of Natanson [18, Chap. 8, Sect. 4, Lemma 2]):

THEOREM A. *An infinite bounded family $\mathscr{F} \subset \mathbb{R}^{[a, b]}$ of nondecreasing functions contains a sequence which converges pointwise on $[a, b]$ to a bounded non-decreasing function.*

The interest in the Helly selection principle(s) is natural due to its numerous applications in analysis. Let us mention the recent work of Fuchino and Plewik [11] where Theorem A is generalized onto monotone mappings between linearly ordered sets. In this paper we present another generalization of Helly's selection principle for mappings of bounded variation with values in metric spaces and its application to the selection problem for set-valued mappings of bounded variation. In order to establish the existence of regular selections of multifunctions (= set-valued mappings) of bounded Jordan variation with respect to the Hausdorff metric, the second author proved the following extension of the Helly selection principle [2, 3]:

THEOREM B. *If K is a compact subset of the metric space X and $\mathscr{F} \subset K^{[a, b]}$ is an infinite family of continuous mappings of uniformly bounded variation, then $\mathscr{F}$ contains a sequence which converges pointwise on $[a, b]$ to a mapping $f \in X^{[a, b]}$ of bounded variation. Moreover, if X is a Banach space* (*over the field $\mathbb{R}$ or $\mathbb{C}$*), *the continuity condition on the family $\mathscr{F}$ is redundant.*

Theorem A was extensively used in the proof of Theorem B, but for metric space valued mappings the Jordan decomposition theorem is inapplicable, and, hence, the following *structural theorem* was used in [3]; see Theorem C below. In order to formulate it, recall that a mapping g: $E \to X$ is said to be *Lipschitzian* if the following quantity, called the (minimal) *Lipschitz constant* of g, is finite:

$$\mathrm{Lip}(g) = \sup\{d(g(t), g(s)) / |t - s| \, | \, t, s \in E, t \neq s\}. \qquad (2)$$

Also, the *composition $g \circ \varphi$*: $J \to X$ of two mappings g: $E \to X$ and φ: $J \to E$ is defined as usual by $(g \circ \varphi)(t) = g(\varphi(t))$ for all $t \in J$.

THEOREM C. *A mapping $f \in X^E$ is of bounded variation if and only if $f = g \circ \varphi$ on E, where $\varphi \in \mathbb{R}^E$ is bounded and nondecreasing and g maps the image $\varphi(E) = \{\varphi(t) \, | \, t \in E\}$ of φ into X and is Lipschitzian with $\mathrm{Lip}(g) \leq 1$. In the necessity part one can define the function φ by $\varphi(t) = V(f, E \cap (-\infty, t])$, $t \in E$.*

Roughly speaking, to prove Theorem B (see [2, 3]), one should write each $f \in \mathscr{F}$ in the form $f = g_f \circ \varphi_f$ according to Theorem C, apply Theorem A to the family $\{\varphi_f \, | \, f \in \mathscr{F}\}$ to extract a pointwise convergent sequence $\{\varphi_{f_n}\}_{n=1}^{\infty}$, and then apply Arzelà-Ascoli's theorem to the sequence $\{g_{f_n}\}_{n=1}^{\infty}$ in order to get a uniformly convergent subsequence $\{g_{f_{n_k}}\}_{k=1}^{\infty}$. Then the sequence $\{f_{n_k}\}_{k=1}^{\infty}$ converges pointwise on $[a, b]$ to a mapping of bounded variation.

However (cf. [3, Remark (1) after Theorem 5.1]), it was not clear if the condition "$\mathscr{F} \subset K^{[a, b]}$ where $K \subset X$ is compact" in Theorem B could be replaced by a weaker condition: "for every $t \in [a, b]$ the set $\mathscr{F}(t) = \{f(t) \, | \, f \in \mathscr{F}\}$ is precompact in X" (recall the Arzelà-Ascoli theorem!). In the present paper we give the affirmative answer to this question. Moreover, we will show that the continuity condition on the family $\mathscr{F}$ as well as the completeness of X are redundant even for metric space valued mappings. The first main result of this paper, which will be proved in Section 3, is the following

THEOREM 1 (Helly type selection principle). *Suppose that (X, d) is an arbitrary metric space. An infinite pointwise precompact family of mappings $\mathscr{F} \subset X^{[a, b]}$ of uniformly bounded variation contains a sequence which converges pointwise on $[a, b]$ to a mapping $f \in X^{[a, b]}$ of bounded variation.*

To treat the selection problem for set-valued mappings of bounded variation, we recall the definitions of the Hausdorff metric and set-valued mappings. If (X, d) is a metric space and $A, B \subset X$ are nonempty subsets, the *Hausdorff distance $D = D_d$ between A and B* is defined by

$$D(A, B) = \max\{e(A, B), e(B, A)\},$$

where

$$e(A, B) = \sup_{x \in A} \text{dist}(x, B) \quad \text{and} \quad \text{dist}(x, B) = \inf_{y \in B} d(x, y).$$

It is well known that the mapping $D(\cdot, \cdot)$ is a *metric* on the set of all nonempty closed bounded subsets of X and, in particular, on the set of all nonempty compact subsets of X.

Given two nonempty sets E and X, any mapping F associating to each point $t \in E$ a set $F(t) \subset X$, the image of t under F, is called a *set-valued mapping* from E into X (in symbols, $F: E \rightrightarrows X$). If all the images $F(t)$ of F are nonempty, we write $F: E \rightrightarrows \dot{X}$. Given a metric space (X, d), the set-valued mapping $F: E \rightrightarrows \dot{X}$ is said to be (a) *compact-valued* if the image $F(t)$ is a compact subset of X for each $t \in E$; (b) *of bounded variation* (with respect to $D = D_d$) on $E \subset \mathbb{R}$ if

$$V(F, E) \equiv V_D(F, E) = \sup_T \sum_{i=1}^{m} D\big(F(t_i), F(t_{i-1})\big) < \infty,$$

where the supremum is taken over all partitions $T = \{t_i\}_{i=0}^m$ of the set E. The mapping $F: E \rightrightarrows \dot{X}$ is said to *admit a selection* if there exists a (single-valued) mapping $f \in X^E$ such that $f(t) \in F(t)$ for all $t \in E$.

In [2] the following theorem on the existence of selections was proved (note that no convexity of images $F(t)$ of F are assumed):

THEOREM D. *Suppose that X is a Banach space* (*over $\mathbb{R}$ or $\mathbb{C}$*), $F:$ $[a, b] \rightrightarrows \dot{X}$ *is a set-valued mapping with compact graph* $\text{Gr}(F) = \{(t, x) \in [a, b] \times X \mid x \in F(t)\}$, $t_0 \in [a, b]$ *and* $x_0 \in F(t_0)$. *If F is of bounded variation* (*respectively, of bounded variation and continuous*) *on* $[a, b]$, *then it admits a selection* $f \in X^{[a, b]}$ *of bounded variation* (*respectively, which is of bounded variation and continuous*) *such that* $f(t_0) = x_0$ *and* $V(f, [a, b]) \leq V(F, [a, b])$.

Moreover, under conditions of Theorem D (one may assume that F is compact-valued if X is finite-dimensional) Lipschitzian F admits a Lipschitzian selection (Hermes [14] if dim $X < \infty$, Mordukhovich [17, Theorem D1.8] if dim $X \leq \infty$); absolutely continuous F admits an absolutely continuous selection (Zhu Qiji [19] if dim $X < \infty$, Chistyakov [3] if dim $X \leq \infty$); and F of bounded generalized Riesz Φ-variation admits a selection of bounded generalized Riesz Φ-variation (Chistyakov [4−6] if dim $X \leq \infty$ and F is compact-valued). Thus, a set-valued mapping F having a property $\mathcal{P}$ (where $\mathcal{P}$ is either Lipschitzian, absolutely continuous, of bounded variation, continuous and of bounded variation, or of bounded generalized Riesz Φ-variation) with respect to the Hausdorff metric D, generated by the norm in X, admits a selection with the property $\mathcal{P}$ with respect to the

norm in X. If a selection of F inherits the property $\mathscr{P}$ of F, it is called a *regular selection* of F.

Having Helly's type selection principle (Theorem 1) at hand, we are able to remove the conditions of linearity and completeness of X and the compactness of the graph of F from Theorem D and to obtain the second main result of this paper, which will be proved in Section 4:

THEOREM 2 (existence of regular selections).   *If (X, d) is a metric space, $F: [a, b] \rightrightarrows \dot{X}$ is a compact-valued set-valued mapping of bounded variation, $t_0 \in [a, b]$ and $x_0 \in F(t_0)$, then F admits a regular selection $f \in X^{[a, b]}$ of bounded variation such that $f(t_0) = x_0$ and $V(f, [a, b]) \le V(F, [a, b])$.*

In Section 5 we show that if the compact-valued set-valued mapping F from $[a, b]$ into X is Lipschitzian (respectively, continuous and of bounded variation, absolutely continuous, of bounded Riesz Φ-variation) with respect to the Hausdorff metric $D = D_d$, then it admits a regular selection $f \in X^{[a, b]}$ which is Lipschitzian (respectively, continuous and of bounded variation, absolutely continuous, of bounded Riesz Φ-variation) with respect to the metric d on X.

3. PROOF OF THEOREM 1

In order to prove Theorem 1 we need a lemma.

LEMMA 1.   *Given a metric space (X, d), if $\mathscr{F} \subset X^E$ is an infinite pointwise precompact family of mappings, then for any countable set $J \subset E$ there exists a sequence in $\mathscr{F}$ which converges pointwise on J.*

Proof.   We use the standard Cantor diagonal process. Let $J = \{t_k\}_{k=1}^{\infty}$. Since the family $\mathscr{F}(t_1) = \{f(t_1) \,|\, f \in \mathscr{F}\}$ is precompact in X, it contains a sequence denoted by $\{f_n^1(t_1)\}_{n=1}^{\infty}$, which is convergent in X. Similarly, let $\{f_n^2(t_2)\}_{n=1}^{\infty}$ be a convergent subsequence of $\{f_n^1(t_2)\}_{n=1}^{\infty}$, and inductively, given $k \in \mathbb{N}$, $k \ge 2$, denote by $\{f_n^k(t_k)\}_{n=1}^{\infty}$ a convergent subsequence of $\{f_n^{k-1}(t_k)\}_{n=1}^{\infty}$. Then the diagonal sequence $\{f_n^n\}_{n=1}^{\infty} \subset \mathscr{F}$ is pointwise convergent on the set J. ∎

Proof of Theorem 1.   For the sake of clarity we divide the proof into six steps.

Step 1 (*common auxiliary part*).   Since the family $\mathscr{F}$ is of uniformly bounded variation, there exists a constant $C \ge 0$ such that $V(f, [a, b]) \le C$ for all $f \in \mathscr{F}$. According to Theorem C, any $f \in \mathscr{F}$ can be written in the form $f = g_f \circ \varphi_f$ on $[a, b]$, where $\varphi_f(t) = V(f, [a, t])$, $t \in [a, b]$, and $g_f: E_f = \varphi_f([a, b]) \to X$ is Lipschitzian with $\mathrm{Lip}(g_f) \le 1$. Observe that any φ_f

is nondecreasing, nonnegative, and $\varphi_f(a) = 0$. Moreover, the family $\{\varphi_f \,|\, f \in \mathscr{F}\}$ is infinite and bounded (since $\varphi_f(t) \le \varphi_f(b) = V(f, [a, b]) \le C$ for all $t \in [a, b]$ and $f \in \mathscr{F}$), and so, by Theorem A, it contains a sequence $\{\varphi_n\}_{n=1}^{\infty}$ corresponding to the decomposition $f_n = g_n \circ \varphi_n$ (i.e., $\varphi_n = \varphi_{f_n}$ and $g_n = g_{f_n}$) for all $n \in \mathbb{N}$, which converges pointwise on $[a, b]$ to a nondecreasing and bounded (and nonnegative) function $\varphi \in \mathbb{R}^{[a, b]}$ as $n \to \infty$. Setting $\ell_n = V(\varphi_n, [a, b])$ and $\ell = V(\varphi, [a, b])$, we have $\ell_n = \varphi_n(b) \to \varphi(b) = \ell$ as $n \to \infty$.

In Steps 2–4 the following hypothesis will be used:

$$\text{functions } \{\varphi_n\}_{n=1}^{\infty} \text{ and } \varphi \text{ are continuous on } [a, b]. \qquad (3)$$

Step 2. Under the hypothesis (3) the domain of g_n is the closed interval $E_{f_n} = \varphi_n([a, b]) = [0, \ell_n]$, $n \in \mathbb{N}$. Set $L = \sup_{n \in \mathbb{N}} \ell_n$, so that $0 \le L < \infty$ and $\ell = \lim_{n \to \infty} \ell_n \le L$. We extend each mapping g_n to the interval $[\ell_n, L]$ by setting $g_n(\tau) = g_n(\ell_n)$ if $\tau \in [\ell_n, L]$. Clearly, the new g_n is Lipschitzian on the interval $[0, L]$ with $\mathrm{Lip}(g_n) \le 1$.

Let us prove that the set $\{g_n\}_{n=1}^{\infty}$ is pointwise precompact on $[0, L]$. Given $\tau \in [0, L]$, for each $n \in \mathbb{N}$ there exists $t_n \in [a, b]$ such that $\varphi_n(t_n) = \tau$. Using the compactness of $[a, b]$ and choosing a suitable subsequence of $\{t_n\}_{n=1}^{\infty}$, we may suppose that $t_n \to t \in [a, b]$ as $n \to \infty$. By the assumption, the sequence $\{f_n(t) = g_n(\varphi_n(t))\}_{n=1}^{\infty} \subset \mathscr{F}(t)$ is precompact in X, and so it contains a subsequence (denoted as the whole sequence) such that $d(f_n(t), x) \to 0$ as $n \to \infty$ for some $x \in X$. We will show that $d(g_n(\tau), x) \to 0$ as $n \to \infty$.

For all $n \in \mathbb{N}$ we have

$$d(g_n(\tau), x) = d(g_n(\varphi_n(t_n)), x) \le d(g_n(\varphi_n(t_n)), g_n(\varphi_n(t))) + d(f_n(t), x)$$

$$\le |\varphi_n(t_n) - \varphi_n(t)| + d(f_n(t), x).$$

By Step 1, $\varphi_n(t)$ tends to $\varphi(t)$ as $n \to \infty$, and so it suffices to show that $\varphi_n(t_n) \to \varphi(t)$ as $n \to \infty$. Suppose that $a < t < b$. Given $\varepsilon > 0$, by the continuity of φ there is a $\delta = \delta(\varepsilon) > 0$, $\delta \le \min\{t - a, b - t\}$, such that $|\varphi(s) - \varphi(t)| \le \varepsilon/2$ for all $s \in [a, b]$ with $|s - t| \le \delta$. As $t_n \to t$ and $\varphi_n \to \varphi$ pointwise on $[a, b]$ as $n \to \infty$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n \ge N(\varepsilon)$ we have $t - \delta \le t_n \le t + \delta$, $|\varphi_n(t - \delta) - \varphi(t - \delta)| \le \varepsilon/2$ and $|\varphi_n(t + \delta) - \varphi(t + \delta)| \le \varepsilon/2$. Since φ_n is nondecreasing, it follows that

$$\varphi_n(t_n) \le \varphi_n(t + \delta) \le \varphi(t + \delta) + \varepsilon/2 \le \varphi(t) + \varepsilon,$$

$$\varphi_n(t_n) \ge \varphi_n(t - \delta) \ge \varphi(t - \delta) - \varepsilon/2 \ge \varphi(t) - \varepsilon,$$

or $|\varphi_n(t_n) - \varphi(t)| \le \varepsilon$ for all $n \ge N(\varepsilon)$. Now the cases $t = a$ and $t = b$ are treated with obvious modifications.

Step 3. Using the hypothesis (3) let us prove that a subsequence of $\{g_n\}_{n=1}^{\infty}$ converges uniformly on the whole interval $[0, L]$ as $n \to \infty$. Let $\{\tau_k\}_{k=1}^{\infty}$ be a dense sequence in $[0, L]$. Since $\{g_n\}_{n=1}^{\infty}$ is pointwise precompact on $[0, L]$ by Step 2, by virtue of Lemma 1 (taking a suitable subsequence) we may suppose that $\{g_n\}_{n=1}^{\infty}$ converges at each point of the set $\{\tau_k\}_{k=1}^{\infty}$. We are going to show that, actually, the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly on $[0, L]$. Given $\varepsilon > 0$, choose a number $k_0(\varepsilon) \in \mathbb{N}$ with the following property: if $\tau \in [0, L]$, there exists a $k \in \{1, \ldots, k_0(\varepsilon)\}$ such that $|\tau - \tau_k| \leq \varepsilon$. Since, as $n \to \infty$, the sequence $\{g_n(\tau_k)\}_{n=1}^{\infty}$ converges in X as $n \to \infty$ (and, hence, is Cauchy) for all $k \in \{1, \ldots, k_0(\varepsilon)\}$, there exists $N_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq N_0(\varepsilon)$ we have

$$d\big(g_n(\tau_k), g_m(\tau_k)\big) \leq \varepsilon, \qquad k = 1, \ldots, k_0(\varepsilon).$$

Now, given $\tau \in [0, L]$, there exists $k \in \{1, \ldots, k_0(\varepsilon)\}$ with $|\tau - \tau_k| \leq \varepsilon$, so that the inequality $\mathrm{Lip}(g_n) \leq 1$ ($n \in \mathbb{N}$) yields

$$
\begin{aligned}
d\big(g_n(\tau), g_m(\tau)\big) &\leq d\big(g_n(\tau), g_n(\tau_k)\big) + \big(g_n(\tau_k), g_m(\tau_k)\big) \\
&\quad + d\big(g_m(\tau_k), g_m(\tau)\big) \\
&\leq 3\varepsilon
\end{aligned}
\tag{4}
$$

for all $n, m \geq N_0(\varepsilon)$. It follows that the sequence $\{g_n(\tau)\}_{n=1}^{\infty}$ is Cauchy in X and, since it is precompact in X by Step 2, we infer that it converges in X as $n \to \infty$ for all $\tau \in [0, L]$. Passing to the limit as $m \to \infty$ in (4) we get the uniform convergence of $\{g_n\}_{n=1}^{\infty}$ on $[0, L]$. If $g: [0, L] \to X$ is the uniform limit of $\{g_n\}_{n=1}^{\infty}$, then, clearly, g is Lipschitzian with $\mathrm{Lip}(g) \leq 1$.

Step 4. Now we establish Theorem 1 provided (3) holds. Since the mapping $g: [0, L] \to X$ from the end of Step 3 is Lipschitzian with $\mathrm{Lip}(g) \leq 1$, $\varphi \in \mathbb{R}^{[a, b]}$ is bounded and nondecreasing, and $\varphi([a, b]) = [0, \ell] \subset [0, L]$, the composite mapping $f = g \circ \varphi: [a, b] \to X$ is of bounded variation by Theorem C. For all $t \in [a, b]$ we have

$$
\begin{aligned}
d\big(f_n(t), f(t)\big) &= d\big((g_n \circ \varphi_n)(t), (g \circ \varphi)(t)\big) \\
&\leq d\big(g_n(\varphi_n(t)), g_n(\varphi(t))\big) + d\big(g_n(\varphi(t)), g(\varphi(t))\big) \\
&\leq |\varphi_n(t) - \varphi(t)| + \sup_{\tau \in [0, L]} d\big(g_n(\tau), g(\tau)\big)
\end{aligned}
$$

with the right hand side tending to zero as $n \to \infty$. This proves the pointwise convergence of f_n to f on the interval $[a, b]$.

Step 5. Suppose now that functions $\{\varphi_n\}_{n=1}^{\infty}$ and φ from Step 1 are continuous on an open interval $(\alpha, \beta) \subset [a, b]$. We will prove that a

subsequence of $\{f_n\}_{n=1}^{\infty}$ converges pointwise on (α, β). Let $\{[\alpha_k, \beta_k]\}_{k=1}^{\infty}$ be an exhausting sequence of closed intervals for (α, β), i.e., $\alpha_{k+1} < \alpha_k < \beta_k < \beta_{k+1}$, $k \in \mathbb{N}$, with $\alpha_k \to \alpha$ and $\beta_k \to \beta$ as $k \to \infty$. For each $n \in \mathbb{N}$ denote by $\tilde{f}_n$ the restriction of f_n to $[\alpha_1, \beta_1]$, so that we have the decomposition $\tilde{f}_n = \tilde{g}_n \circ \tilde{\varphi}_n$ where

$$\tilde{\varphi}_n(t) = V(\tilde{f}_n, [\alpha_1, t]) = V(f_n, [\alpha_1, t]) = V(f_n, [a, t]) - V(f_n, [a, \alpha_1])$$

$$= \varphi_n(t) - \varphi_n(\alpha_1), \qquad t \in [\alpha_1, \beta_1],$$

and $\tilde{g}_n : \tilde{\varphi}_n([\alpha_1, \beta_1]) \to X$ is Lipschitzian with $\mathrm{Lip}(\tilde{g}_n) \leq 1$. Moreover, as $n \to \infty$, we have $\tilde{\varphi}_n(t) \to \tilde{\varphi}(t) = \varphi(t) - \varphi(\alpha_1)$ for all $t \in [\alpha_1, \beta_1]$. Since φ_n and φ are continuous on (α, β), it follows that $\tilde{\varphi}_n$ and $\tilde{\varphi}$ are continuous on $[\alpha_1, \beta_1]$, $n \in \mathbb{N}$. Applying the result of Steps 2–4 to $\{\tilde{f}_n\}_{n=1}^{\infty}$ on $[\alpha_1, \beta_1]$, we can choose a subsequence $\{f_n^1\}_{n=1}^{\infty}$ of $\{\tilde{f}_n\}_{n=1}^{\infty}$, which is in fact a subsequence of $\{f_n\}_{n=1}^{\infty}$, such that $\{f_n^1\}_{n=1}^{\infty}$ converges pointwise on the interval $[\alpha_1, \beta_1]$. In a similar manner (denoting by $\tilde{f}_n^1$ the restriction of f_n^1 to $[\alpha_2, \beta_2]$, and so on), choose a subsequence $\{f_n^2\}_{n=1}^{\infty}$ of $\{f_n^1\}_{n=1}^{\infty}$ which is convergent on the interval $[\alpha_2, \beta_2]$, and inductively, given $k \in \mathbb{N}$, $k \geq 2$, choose a subsequence $\{f_n^k\}_{n=1}^{\infty}$ of $\{f_n^{k-1}\}_{n=1}^{\infty}$ which is convergent on $[\alpha_k, \beta_k]$. Then the diagonal sequence $\{f_n^n\}_{n=1}^{\infty}$ converges pointwise on the interval $(\alpha, \beta) = \bigcup_{k=1}^{\infty} [\alpha_k, \beta_k]$ as $n \to \infty$.

Step 6 (*general case*).   Denote by E the set consisting of the discontinuity points of functions $\{\varphi_n\}_{n=1}^{\infty}$ and φ and points a and b. Since φ_n and φ are nondecreasing on $[a, b]$, the set $E \subset [a, b]$ is at most countable. By the assumption, the sequence $\{f_n\}_{n=1}^{\infty}$ is pointwise precompact on $[a, b]$, and so, by Lemma 1, it contains a subsequence (denoted by the same symbol) which converges pointwise on E. The difference $[a, b] \setminus E$ is at most a countable union of open intervals (a_k, b_k), $k \in \mathbb{N}$, and functions $\{\varphi_n\}_{n=1}^{\infty}$ and φ are continuous on each interval (a_k, b_k). Applying Step 5, choose a subsequence $\{f_n^1\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ which is convergent pointwise on (a_1, b_1); then choose a subsequence $\{f_n^2\}_{n=1}^{\infty}$ of $\{f_n^1\}_{n=1}^{\infty}$ which is convergent pointwise on (a_2, b_2), and so on. As a result, we get the diagonal subsequence $\{f_n^n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ which is convergent pointwise on $\bigcup_{k=1}^{\infty} (a_k, b_k)$ and E, i.e., on the whole interval $[a, b]$. The pointwise limit $f : [a, b] \to X$ of $\{f_n^n\}_{n=1}^{\infty} \subset \mathscr{F}$ is a mapping of bounded variation by virtue of the lower semi-continuity of the functional $V(\cdot, [a, b])$:

$$V(f, [a, b]) \leq \liminf_{n \to \infty} V(f_n^n, [a, b]) \leq C.$$

This completes the proof of Theorem 1.   ∎

EXAMPLE 1.   (a) If X is a finite-dimensional normed linear space with the norm $\|\cdot\|$, then, by virtue of the inequality $\|f(t)\| \leq \|f(t_0)\| + V(f, [a, b])$, one may replace the condition "pointwise precompact family $\mathscr{F} \subset X^{[a,b]}$" in Theorem 1 by "family $\mathscr{F} \subset X^{[a,b]}$, for which $\mathscr{F}(t)$ is closed and bounded at $t = t_0 \in [a, b]$." However, for an infinite-dimensional Banach space X one cannot weaken the conditions of Theorem 1 by assuming that the family $\mathscr{F}(t) = \{f(t) \,|\, f \in \mathscr{F}\}$ is precompact in X only at a given point $t = t_0 \in [a, b]$. To see this, let $[a, b] = [0, 1]$ and $X = \ell^1(\mathbb{N})$ be the Banach space of all absolutely summable sequences $x = \{x_i\}_{i=1}^\infty \in \mathbb{R}^\mathbb{N}$ with the norm $\|x\| = \sum_{i=1}^\infty |x_i|$. For each $n \in \mathbb{N}$ define $f_n : [0, 1] \to \ell^1(\mathbb{N})$ as follows: $f_n(t) = t e_n$, $t \in [0, 1]$, where $e_n = \{x_i\}_{i=1}^\infty$ with $x_i = 0$ if $i \neq n$ and $x_n = 1$. If $\mathscr{F} = \{f_n\}_{n=1}^\infty$, we have that $\mathscr{F}(t) = t\{e_n\}_{n=1}^\infty$ is precompact in $\ell^1(\mathbb{N})$ if and only if $t = 0$; $V(f_n, [0, 1]) = 1$ for all $n \in \mathbb{N}$; no subsequence of $\{f_n(t)\}_{n=1}^\infty = \{t e_n\}_{n=1}^\infty$ converges in $\ell^1(\mathbb{N})$ if $0 < t \leq 1$. This example shows also that the precompactness of the families $\mathscr{F}(t)$, $t \in [a, b]$, cannot be replaced by their closedness and boundedness.

(b)   On the other hand, Theorem 1 is wrong even in the classical situation $X = \mathbb{R}$ if we drop the assumption that the family $\mathscr{F}$ is of uniformly bounded variation: in fact, if functions are defined by $f_n(t) = \sin(2\pi n t)$, $t \in [0, 1]$, then $V(f_n, [0, 1]) = 4n$, $n \in \mathbb{N}$, and no subsequence of $\mathscr{F} = \{f_n\}_{n=1}^\infty$ converges at all points of the interval $[0, 1]$.

COROLLARY 1.   *Theorem* 1 *remains valid if we replace the closed interval* $[a, b]$ *there by an arbitrary* (*open*, *half-closed*, *bounded*, *or unbounded*) *interval* $I \subset \mathbb{R}$.

Proof.   We apply the standard diagonal method. Let $[a_n, b_n]$, $n \in \mathbb{N}$, be an increasing sequence of exhausting closed intervals for I. Since

$$V(f, [a_n, b_n]) \leq V(f, I) \leq C < \infty, \qquad f \in \mathscr{F}, \qquad n \in \mathbb{N},$$

by Theorem 1, the family $\mathscr{F}$ contains a sequence $\{f_n^1\}_{n=1}^\infty$ which is pointwise convergent on the interval $[a_1, b_1]$. Similarly, $\{f_n^1\}_{n=1}^\infty$ contains a subsequence $\{f_n^2\}_{n=1}^\infty$ which is pointwise convergent on $[a_2, b_2] \supset [a_1, b_1]$, and so forth. The diagonal sequence $\{f_n^n\}_{n=1}^\infty \subset \mathscr{F}$ converges pointwise on I to a mapping from X^I of bounded variation.   ∎

4. PROOF OF THEOREM 2

Proof of Theorem 2.   Let $E \subset [a, b]$ be the set of all discontinuity points of F. If $E = \{t_j\}_{j=1}^m$ is finite, we set $t_j = t_m$ for $j \in \mathbb{N}$, $j > m$, and since F is of bounded variation, E is at most countable, and we may write

$E = \{t_j\}_{j=1}^{\infty}$. For each $n \in \mathbb{N}$ let $T_n = \{t_i^{2n}\}_{i=0}^{2n}$ be a partition of the interval $[a, b]$, that is, $a = t_0^{2n} < t_1^{2n} < \cdots < t_{2n-1}^{2n} < t_{2n}^{2n} = b$, with the following two properties:

$$t_j \in T_n \left(\text{i.e., } t_j = t_{k_j(n)}^{2n} \text{ for some } k_j(n) \in \{0, 1, \ldots, 2n\}\right),$$

$$j = 0, 1, \ldots, n; \quad (5)$$

$$\lim_{n \to \infty} \max_{1 \le i \le 2n} \left(t_i^{2n} - t_{i-1}^{2n}\right) = 0. \quad (6)$$

We define elements $x_i^{2n} \in F(t_i^{2n})$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, 2n$, inductively as follows. Let $n \in \mathbb{N}$. Assume first that $a < t_0 < b$, so that $k_0(n) \in \{1, \ldots, 2n - 1\}$.

(a) Set $x_{k_0(n)}^{2n} = x_0$.

(b) If $i \in \{1, \ldots, k_0(n)\}$ and the element $x_i^{2n} \in F(t_i^{2n})$ is already chosen, pick $x_{i-1}^{2n} \in F(t_{i-1}^{2n})$ such that $d(x_i^{2n}, x_{i-1}^{2n}) = \text{dist}(x_i^{2n}, F(t_{i-1}^{2n}))$.

(c) If $i \in \{k_0(n) + 1, \ldots, 2n\}$ and the element $x_{i-1}^{2n} \in F(t_{i-1}^{2n})$ is already chosen, pick $x_i^{2n} \in F(t_i^{2n})$ such that $d(x_{i-1}^{2n}, x_i^{2n}) = \text{dist}(x_{i-1}^{2n}, F(t_i^{2n}))$.

Now, if $t_0 = a$, i.e., $k_0(n) = 0$, we define $x_i^{2n} \in F(t_i^{2n})$ following (a) and (c), and if $t_0 = b$, i.e., $k_0(n) = 2n$, we define $x_i^{2n} \in F(t_i^{2n})$ in accordance with (a) and (b).

Also, for each $n \in \mathbb{N}$ we define a mapping $f_n \in X^{[a, b]}$ as follows:

$$f_n(t) = \begin{cases} x_i^{2n} & \text{if } t = t_i^{2n}, & i = 0, 1, \ldots, 2n, \\ x_{i-1}^{2n} & \text{if } t \in \left(t_{i-1}^{2n}, t_i^{2n}\right), & i = 1, 2, \ldots, 2n. \end{cases} \quad (7)$$

We have $f_n(t_0) = f_n(t_{k_0(n)}^{2n}) = x_{k_0(n)}^{2n} = x_0$ and, by virtue of (b) and (c),

$$V(f_n, [a, b]) = \sum_{i=1}^{2n} V\left(f_n, \left[t_{i-1}^{2n}, t_i^{2n}\right]\right) = \sum_{i=1}^{2n} d\left(x_i^{2n}, x_{i-1}^{2n}\right)$$

$$\le \sum_{i=1}^{2n} D\left(F\left(t_i^{2n}\right), F\left(t_{i-1}^{2n}\right)\right) \le V(F, [a, b]), \quad n \in \mathbb{N}. \quad (8)$$

In order to apply the Helly type selection principle, we have to verify that the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is precompact in X for all $t \in [a, b]$. If $t \in E \cup \{a, t_0, b\}$, by (5) there exists a number $n_0(t) \in \mathbb{N}$ (depending on t) such that $t \in T_n$ for all $n \ge n_0(t)$, and so

$$f_n(t) \in F(t), \quad n \ge n_0(t), \quad (9)$$

thanks to (7), (a), (b), and (c), and it suffices to take into account the compactness of $F(t)$. Now, if $t \in (a, b) \setminus E$ and $t \ne t_0$, F is continuous at t

with respect to $D = D_d$, and for every $n \in \mathbb{N}$ there exists $i(n) \in \{0, 1, \dots,$ $2n - 1\}$ such that $t^{2n}_{i(n)} \le t < t^{2n}_{i(n)+1}$. It follows from (7) that $f_n(t) = x^{2n}_{i(n)} \in$ $F(t^{2n}_{i(n)})$, $n \in \mathbb{N}$, and (6) implies $t^{2n}_{i(n)} \to t$ as $n \to \infty$. Choosing, for each $n \in \mathbb{N}$, an element $x^n_t \in F(t)$ such that $d(x^{2n}_{i(n)}, x^n_t) = \text{dist}(x^{2n}_{i(n)}, F(t))$, by the continuity of F and the definition of the Hausdorff metric we have

$$d(f_n(t), x^n_t) \le D\big(F\big(t^{2n}_{i(n)}\big), F(t)\big) \to 0 \qquad \text{as } n \to \infty.$$

Since $F(t)$ is compact and $\{x^n_t\}^\infty_{n=1} \subset F(t)$, there exists a subsequence of $\{x^n_t\}^\infty_{n=1}$ (denoted by the same symbol) which converges to an element $x_t \in F(t)$ as $n \to \infty$. Therefore,

$$d(f_n(t), x_t) \le d(f_n(t), x^n_t) + d(x^n_t, x_t) \to 0 \qquad \text{as } n \to \infty, \quad (10)$$

and so the sequence $\{f_n(t)\}^\infty_{n=1}$ is precompact in X.

By Theorem 1, the family $\mathscr{F} = \{f_n\}^\infty_{n=1}$ contains a subsequence (for which we use the notation of the whole sequence) which converges pointwise on $[a, b]$ to a mapping $f \in X^{[a,b]}$ of bounded variation. Clearly, $f(t_0) = x_0$. The inclusion $f(t) \in F(t)$ for all $t \in [a, b]$ follows from (9) and (10). It remains to observe that the lower semicontinuity of $V(\cdot, [a, b])$ and (8) yield

$$V(f, [a, b]) \le \liminf_{n \to \infty} V(f_n, [a, b]) \le V(F, [a, b]).$$

This completes the proof of Theorem 2. ∎

EXAMPLE 2.   The inequality in Theorem 2 may be wrong if we drop the assumption that F is compact-valued. To see this, let $\ell^1(\mathbb{N})$ be as in Example 1(a), set $A = \{(1 + 1/n)e_n\}^\infty_{n=2}$ and $B = \{e_1\} \cup A$ (so that A and B are only closed and bounded in $\ell^1(\mathbb{N})$), and define $F: [0, 1] \rightrightarrows \ell^1(\mathbb{N})$ by $F(t) = A$ if $0 \le t < 1$ and $F(1) = B$. It follows that if $f: [0, 1] \to \ell^1(\mathbb{N})$ is any selection of F such that $f(1) = e_1$, we have $V(f, [0, 1]) > 2 = D(A, B)$ $= V(F, [0, 1])$.

Set-valued mappings of bounded variation with noncompact images (as $\mathscr{F}(t)$ in Example 1(a) and $F(t)$ in Example 2) may admit regular selections as can be seen from the following observation. Suppose that F satisfies the conditions of Theorem 2 except that the values of F are not necessarily compact, but assume also that for any $t \in [a, b]$ there exists a compact subset $F_0(t) \subset F(t)$ such that $D(F_0(t), F_0(s)) \le D(F(t), F(s))$ for all $t, s \in [a, b]$. Then $F_0: [a, b] \rightrightarrows \dot{X}$ is of bounded variation and, by Theorem 2, F_0 admits a selection, which is at the same time a selection of F.

COROLLARY 2.   *Theorem* 2 *remains valid if we replace the closed interval* $[a, b]$ *there by an arbitrary* (*open, half-closed, bounded, or unbounded*) *interval* $I \subset \mathbb{R}$.

Proof. Let $I \subset \mathbb{R}$ be an open interval, and let $\{r_k\}_{k \in \mathbb{Z}} \subset I$ be an increasing sequence such that $r_0 \leq t_0 \leq r_1$, $\lim_{k \to \infty} r_k = \sup I$ and $\lim_{k \to \infty} r_{-k} = \inf I$. Setting $I_k = [r_k, r_{k+1}]$, $k \in \mathbb{Z}$, we have $I = \bigcup_{k \in \mathbb{Z}} I_k$. Applying Theorem 2 on the interval I_0 we find a selection $f_0 \in X^{I_0}$ of F (more precisely, of the restriction $F|_{I_0}$ of F to I_0) of bounded variation such that $f_0(t_0) = x_0$ and $V(f_0, I_0) \leq V(F, I_0)$. We define inductively $f_k \in X^{I_k}$ to be a selection of F on I_k such that $f_k(r_k) = f_{k-1}(r_k)$ and $V(f_k, I_k) \leq V(F, I_k)$, $k \in \mathbb{Z}$. Given $t \in I$, so that $t \in I_k$ for some $k \in \mathbb{Z}$, we set $f(t) = f_k(t)$. Clearly, $f \in X^I$ is a selection of F on I such that $f(t_0) = x_0$. The properties of the functional $V(\cdot, \cdot)$ yield (cf. [3, 2.1])

$$V(f, I) = \lim_{n \to \infty} V(f, [r_{-n}, r_n]) = \lim_{n \to \infty} \sum_{k=-n}^{n-1} V(f_k, I_k)$$

$$\leq \lim_{n \to \infty} \sum_{k=-n}^{n-1} V(F, I_k) = \lim_{n \to \infty} V(F, [r_{-n}, r_n]) = V(F, I).$$

The case when $I \subset \mathbb{R}$ is a half-closed interval is treated similarly. $\blacksquare$

5. MORE REGULAR SELECTIONS

In this section we will show that the more regular the set-valued mapping of bounded variation is under consideration, the more regular selection it admits.

Let (X, d) be a metric space, and let $E \subset \mathbb{R}$ be a nonempty set. Recall that a mapping $f \in X^E$ is said to be: (a) *absolutely continuous on E* if there exists a function $\delta : (0, \infty) \to (0, \infty)$, depending on f, such that if $\varepsilon > 0$, $\{a_i, b_i\}_{i=1}^n \subset E$ (with arbitrary $n \in \mathbb{N}$), $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and $\sum_{i=1}^n (b_i - a_i) \leq \delta(\varepsilon)$, then $\sum_{i=1}^n d(f(b_i), f(a_i)) \leq \varepsilon$; more precisely, we say that f is $\delta(\cdot)$-*absolutely continuous* and write $\delta(\cdot) = \delta_f(\cdot)$; (b) *of bounded* (*generalized*) *Riesz Φ-variation* provided the following quantity is finite:

$$V_\Phi(f, E) \equiv V_{\Phi, d}(f, E) = \sup_T \sum_{i=1}^m \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{t_i - t_{i-1}} \right)(t_i - t_{i-1}),$$

where the supremum is taken over all partitions $T = \{t_i\}_{i=0}^m$ of the set E and $\Phi : [0, \infty) \to [0, \infty)$ is a convex continuous function vanishing at zero only and such that $\lim_{\rho \to \infty} \Phi(\rho)/\rho = \infty$.

Given $f \in X^E$, it is known that (e.g., [6, 8]): (i) if f is Lipschitzian or of bounded Riesz Φ-variation, then f is absolutely continuous; (ii) if E is

bounded and f is Lipschitzian, then f is of bounded Riesz Φ-variation; (iii) if E is compact and f is absolutely continuous or of bounded Riesz Φ-variation, then f is of bounded (Jordan) variation. Moreover [4, 6], $f \in X^{[a, b]}$ is absolutely continuous if and only if there exists a function Φ (with the properties as above) such that f is of bounded Riesz Φ-variation on $[a, b]$.

In what follows we need the counterpart of Theorem C (see [3, 6]):

THEOREM E.   *Let E be a compact set. A mapping $f \in X^E$ is absolutely continuous* (*respectively, of bounded Riesz Φ-variation*) *if and only if $f = g \circ \varphi$ on E, where $\varphi \in \mathbb{R}^E$ is absolutely continuous* (*respectively, of bounded Riesz Φ-variation*), *bounded, and nondecreasing and g maps the image $\varphi(E) = \{\varphi(t) \mid t \in E\}$ of φ into X and is Lipschitzian with $\mathrm{Lip}(g) \leq 1$. In the necessity part one can define the function φ by $\varphi(t) = V(f, E \cap (-\infty, t])$, $t \in E$, in which case we have $\delta_\varphi(\cdot) = \delta_f(\cdot)$ if f is absolutely continuous, and $V_\Phi(\varphi, E) = V_\Phi(f, E)$ if f is of bounded Riesz Φ-variation.*

Given a set-valued mapping $F: E \rightrightarrows \dot{X}$ with compact images, we say that it is *absolutely continuous* (*on E*) or *of bounded Riesz Φ-variation* provided (a) or (b) above holds, respectively, with d there replaced by the Hausdorff metric $D = D_d$ and f by F. A similar definition applies to *Lipschitzian F* (cf. (2)).

The main result of this section is the following.

THEOREM 3 (existence of more regular selections).   *Suppose that (X, d) is a metric space, $F: [a, b] \rightrightarrows \dot{X}$ is a compact-valued set-valued mapping, $t_0 \in [a, b]$ and $x_0 \in F(t_0)$. We have*: (a) *if F is Lipschitzian, then it admits a Lipschitzian selection f such that $\mathrm{Lip}(f) \leq \mathrm{Lip}(F)$*; (b) *if F is continuous and of bounded variation, then it admits a continuous selection f of bounded variation*; (c) *if F is $\delta(\cdot)$-absolutely continuous, then it admits a $\delta(\cdot)$-absolutely continuous selection f*; (d) *if F is of bounded Riesz Φ-variation, then it admits a selection f of bounded Riesz Φ-variation such that $V_\Phi(f, [a, b]) \leq V_\Phi(F, [a, b])$. Moreover, in the cases* (a)–(d) *the selection f of F can be additionally chosen in such a way that $f(t_0) = x_0$ and $V(f, [a, b]) \leq V(F, [a, b])$.*

Proof. (a) Since F is Lipschitzian, it is of bounded variation, so let $f \in X^{[a, b]}$ be a selection of F constructed in the proof of Theorem 2. With no loss of generality, we may assume that the sequence (7) converges to f pointwise on $[a, b]$ as $n \to \infty$. To prove that f is Lipschitzian, let $a \leq t < s < b$. Then for any $n \in \mathbb{N}$ there exist $i(n), j(n) \in \{0, 1, \ldots, 2n - 1\}$ such that $t_{i(n)}^{2n} \leq t < t_{i(n)+1}^{2n}$ and $t_{j(n)}^{2n} \leq s < t_{j(n)+1}^{2n}$, and so (7) implies $f_n(t) = x_{i(n)}^{2n} \in F(t_{i(n)}^{2n})$ and $f_n(s) = x_{j(n)}^{2n} \in F(t_{j(n)}^{2n})$. Properties (b) and (c) in the proof of Theorem 2 yield

$$d\bigl(x_i^{2n}, x_{i-1}^{2n}\bigr) \leq D\bigl(F\bigl(t_i^{2n}\bigr), F\bigl(t_{i-1}^{2n}\bigr)\bigr) \leq \mathrm{Lip}(F) \cdot \bigl(t_i^{2n} - t_{i-1}^{2n}\bigr).$$

By (6), $i(n) < j(n)$ for n large enough, and it follows that

$$d\big(f_n(t), f_n(s)\big) = d\big(x_{i(n)}^{2n}, x_{j(n)}^{2n}\big) \leq \sum_{i=i(n)+1}^{j(n)} d\big(x_i^{2n}, x_{i-1}^{2n}\big)$$

$$\leq \sum_{i=i(n)+1}^{j(n)} \mathrm{Lip}(F) \cdot \big(t_i^{2n} - t_{i-1}^{2n}\big) = \mathrm{Lip}(F) \cdot \big(t_{i(n)}^{2n} - t_{j(n)}^{2n}\big).$$

Since $f_n(t) \to f(t)$, $f_n(s) \to f(s)$, $t_{i(n)}^{2n} \to t$, and $t_{j(n)}^{2n} \to s$ as $n \to \infty$, we have $d(f(t), f(s)) \leq \mathrm{Lip}(F) \cdot |t - s|$. Now, if $a \leq t < b$ and $s = b$, the argument above applies with $j(n) = 2n$, i.e., $s = t_{j(n)}^{2n}$. Thus, $\mathrm{Lip}(f) \leq \mathrm{Lip}(F)$.

(b), (c), and (d). Suppose that F satisfies conditions (b), (c), or (d). Then F is continuous, and so the function $\varphi(t) = V(F, [a, t])$, $t \in [a, b]$, is continuous as well. By virtue of Theorems C and E, we have the decomposition $F = G \circ \varphi$ on $[a, b]$, where φ is in addition continuous, $\delta(\cdot)$-absolutely continuous, or of bounded Riessz Φ-variation such that $V_\Phi(\varphi, [a, b]) = V_\Phi(F, [a, b])$ in accordance with (b), (c), or (d), and G: $[0, \ell] \rightrightarrows \dot{X}$ is a Lipschitzian compact-valued set-valued mapping for which $\ell = V(F, [a, b]) = \varphi(b)$ and $\mathrm{Lip}(G) \leq 1$. If $\tau_0 = \varphi(t_0)$, then $x_0 \in G(\tau_0)$, and so, by Theorem 3(a), there exists a Lipschitzian mapping $g \in X^{[0, \ell]}$ such that $g(\tau) \in G(\tau)$ for all $\tau \in [0, \ell]$, $g(\tau_0) = x_0$ and $\mathrm{Lip}(g) \leq \mathrm{Lip}(G) \leq 1$.

We claim that the composite mapping $f = g \circ \varphi$ is the desired continuous selection of F on $[a, b]$: in fact,

$$f(t) = g(\varphi(t)) \in G(\varphi(t)) = F(t) \qquad \text{for all } t \in [a, b]$$

and $f(t_0) = g(\varphi(t_0)) = g(\tau_0) = x_0$. Moreover, since $\mathrm{Lip}(g) \leq 1$, applying Theorems C and E one more time we have: in case (b), f is of bounded variation and

$$V(f, [a, b]) = V(g, [0, \ell]) \leq \ell \cdot \mathrm{Lip}(g) \leq \ell = V(F, [a, b]); \quad (11)$$

in case (c), f is $\delta(\cdot)$-absolutely continuous for which (11) holds as well; and in case (d), f is of bounded Riesz Φ-variation for which (11) holds and such that

$$V_\Phi(f, [a, b]) = V_\Phi(g \circ \varphi, [a, b]) \leq V_\Phi(\varphi, [a, b]) = V_\Phi(F, [a, b]).$$

This completes the proof of Theorem 3. ∎

Finally, making use of the idea in the proof of Corollary 2 one can replace the closed interval $[a, b]$ in Theorem 3 by an arbitrary open, half-closed, bounded, or unbounded interval $I \subset \mathbb{R}$.

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