ASYMPTOTIC ANALYSIS OF SYMMETRIC FUNCTIONS

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Dedicated to Professor Yuri Prokhorov on the occasion of his 85th birthday (15.12.1929–16.07.2013)

Abstract. In this paper we consider asymptotic expansions for a class of sequences of symmetric functions of many variables. Applications to classical and free probability theory are discussed.

1. Introduction

Most limit theorems such as the central limit theorem in finite dimensional and abstract spaces and the functional limit theorems admit refinements in terms of asymptotic expansions in powers of $n^{-1/2}$, where n denotes the number of observations. Results on asymptotic expansions of this type are summarized in many monographs, see for example [2].

These expansions are obtained by very different techniques such as expanding the characteristic function of the particular statistic, or in discrete cases starting even from a combinatorial formula for its distribution function. Alternatively one might use an expansion for an underlying empirical process and evaluate it on a domain defined by a functional or statistic of this process. In those cases one would need to make approximations by Gaussian processes in suitable function spaces.

The aim of this paper is to show that for most of these expansions one could safely ignore the underlying probability model and its ingredients (like e.g. proof of existence of limiting processes and its properties). In fact one can obtain expansions in a very similar way based on a simple general scheme reflecting the common nature of these models that is a universal collective behavior caused by many independent asymptotically negligible variables influencing the distribution of a functional.

The results of this paper may be considered as the extension of the results given by F. Götze in [5] where the following scheme of sequences of symmetric

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functions is studied. Let $h_n(\varepsilon, ..., \varepsilon_n), n \geq 1$ denote a sequence of real functions defined on $\mathbb{R}^n$ and suppose that the following conditions hold:

$$h_{n+1}(\varepsilon_1, ..., \varepsilon_j, 0, \varepsilon_{j+1}, ..., \varepsilon_n) = h_n(\varepsilon_1, ..., \varepsilon_j, \varepsilon_{j+1}, ..., \varepsilon_n); \tag{1.1}$$

$$\frac{\partial}{\partial \varepsilon_j} h_n(\varepsilon_1, ..., \varepsilon_j, ..., \varepsilon_n)\bigg|_{\varepsilon_j=0} = 0 \text{ for all } j = 1, ..., n; \tag{1.2}$$

$$h_n(\varepsilon_{\pi(1)}, ..., \varepsilon_{\pi(n)}) = h_n(\varepsilon_1, ..., \varepsilon_n) \text{ for all } \pi \in S_n, \tag{1.3}$$

where by S_n denotes the symmetric group.

This symmetry property follows e.g. from the independence and identical distribution of an underlying vector of random elements X_j (in an arbitrary space) with common distribution P, if $h_n = \mathbb{E}F(\varepsilon_1(\delta_{X_1} - P) + \ldots + \varepsilon_n(\delta_{X_n} - P))$ is the expected value of a functional F of a *weighted* process (based on the Dirac-measures in $X_1, \ldots, X_n$). Here h_n may be regarded as function of "influences" of the various random components X_j. In [5] we considered limits and expansions for functions h_n of equal weights $\varepsilon_j = n^{-1/2}, 1 \leq j \leq n$. In the following this scheme will be extended to the case of *non identical* weights ε_j, which occurs e.g. for expectations of functionals of weighted i.i.d. random X_j elements in Probability Theory.

Denote by ε the n-vector $\varepsilon_j, 1 \leq j \leq n$ and by $\varepsilon^d := \sum_{j=1}^n \varepsilon_j^d, d \geq 1$ the d-th power sum. In the following we shall show that (1.1)-(1.3) ensures the existence of a "limit" function $h_\infty(\varepsilon^2, \lambda_1, \ldots, \lambda_s)$ as a first order approximation of h_n together with "Edgeworth-type" asymptotic expansions. These are given in terms of polynomials of power sums $\varepsilon^d, d \geq 3$. The coefficients of these "Edgeworth"-Polynomials, defined in (2.8) below, are given by derivatives of the limit function h_∞ at $\lambda_1 = 0, \ldots, \lambda_s = 0$.

Remark (Algebraic Representations). In case that h_n is a *multivariate polynomial* of ε itself, satisfying (1.1)–(1.3), we may express it as polynomial in the algebraic base, $\varepsilon^d, d \geq 1$, of symmetric power sums of ε with constant coefficients. Note that

$$\frac{\partial}{\partial_{\varepsilon_j}} \varepsilon^d \bigg|_{\varepsilon_j=0} = \delta_{d,1},$$

where $\delta_{d,1} = 1$, if $d = 1$ and zero otherwise. Hence, (1.2) entails that h_n does not polynomially depend on ε^1. Now we may write

$$h_n(\varepsilon) = P_{\varepsilon^2}(\varepsilon^3, \ldots, \varepsilon^n),$$

where P_{ε^2} denotes a polynomial with coefficients in the polynomial ring $\mathbb{C}[\varepsilon^2]$ of the variable ε^2. Restricting ourselves to the sphere $\varepsilon^2 = 1$ for convenience, P_{ε^2} is the desired "Edgeworth" expansion, provided we introduce the following grading of monomials in the variables $\varepsilon^d, d \geq 3$ via $\deg(\varepsilon^\mathrm{d}) = \mathrm{d} - 2$ and expand the polynomial in monomials of increasing grade.

1.1. **Notations.** Throughout the paper we will use the following notations. We denote $\varepsilon^d := \sum_{i=1}^n \varepsilon_i^d$ and $|\varepsilon|^d := \sum_{i=1}^n |\varepsilon_i|^d$. We also introduce additional notation and denote by $(\varepsilon)_d$ and $|\varepsilon|_d$ the d-th root of ε^d and $|\varepsilon|^d$ respectively, i.e. $(\varepsilon)_d := (\varepsilon^d)^{1/d}$ and $|\varepsilon|_d := (|\varepsilon|^d)^{1/d}$. By c with induces or without we denote absolute constants, they can be different in different places. Let D^α, where α a nonnegative integral vector, denote partial derivatives $\frac{\partial^{\alpha_1}}{\partial\varepsilon_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial\varepsilon_m^{\alpha_m}}$, and $\alpha = \sum_{j=1}^m \alpha_j$.

2. Results

We denote by

$$h_\infty(\lambda_1, ..., \lambda_s, \lambda) := \lim_{k\to\infty} h_{k+s}\left(\lambda_1, ..., \lambda_s, \frac{\lambda}{\sqrt{k}}, ..., \frac{\lambda}{\sqrt{k}}\right). \tag{2.1}$$

The following theorem is an analogue of the Berry-Esseen type inequality for sums of non identically distributed independent random variables in probability theory.

Theorem 2.1. *Assume $h_n(\cdot), n \ge 1$, satisfies conditions* (1.1), (1.2) *and* (1.3) *together with*

$$|D^\alpha h_n(\varepsilon_1, ..., \varepsilon_n)| \le B, \tag{2.2}$$

for all $\varepsilon_1, ..., \varepsilon_n$, where B denotes some positive constant, $\alpha = (\alpha_1, ..., \alpha_r), r \le 3$, and

$$\alpha_j \ge 2, \quad j = 1, ..., r, \quad \sum_{j=1}^r (\alpha_j - 2) \le 1.$$

Then there exists function $h_\infty(|\varepsilon|_2)$ defined by (2.1) *and*

$$|h_n(\varepsilon_1, ..., \varepsilon_n) - h_\infty(|\varepsilon|_2)| \le c \cdot B \cdot \max(|\varepsilon|_2, |\varepsilon|_2^3)|\varepsilon|^3,$$

where c is an absolute constant.

In case that ε depends on n, this theorem shows that if

$$\lim_{n\to\infty} |\varepsilon|_3 = 0 \tag{2.3}$$

then $h_n(\varepsilon_1, ..., \varepsilon_n)$ converges to the limiting function $h_\infty(|\varepsilon|_2)$, which doesn't depend on $\varepsilon_1, ..., \varepsilon_n$ but on the l_2-norm $|\varepsilon|_2$. This means that the sequence of symmetric functions (invariant with respect to S_n) may be approximated by a rotationally invariant function (invariant with respect to the orthogonal group $\mathcal{O}_n$).

Note though that if (2.3) holds, Theorem 2.1 doesn't provide an explicit formula for the function $h_\infty(|\varepsilon|_2)$, but guarantees its existence.

Remark. It was shown in [10, Lemma 4.1] that (2.3) holds with high probability.

Proof of Theorem 2.1. We divide the proof into three steps. In the first step we substitute each argument ε_j by a block of the length k of equal variables

$\varepsilon_j/\sqrt{k}$. This procedure doesn't change the l_2-norm $|\varepsilon|_2$. After n steps we we arrive at a function which depend on $n \times k$ arguments

$$h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right). \tag{2.4}$$

We show that

$$\left| h_n(\varepsilon_1, ..., \varepsilon_n) - h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right) \right| \le c \cdot B \cdot |\varepsilon|^3. \tag{2.5}$$

This step corresponds to Lindeberg's scheme of replacing arbitrary by Gaussian random variables in the central limit theorem in probability theory. In the *second step*, still fixing n, we determine the limit of the sequence of functions (2.4), as k goes to infinity. We will show that in this case the limit depends $\varepsilon_1, ..., \varepsilon_n$, through the l_2-norm $|\varepsilon|_2$ only. It will be shown that

$$\begin{aligned} &\left| h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right) - h_k\left(\frac{|\varepsilon|_2}{\sqrt{k}}, ..., \frac{|\varepsilon|_2}{\sqrt{k}}\right) \right| \\ &\qquad \le c \cdot B \cdot k^{-1/2}. \end{aligned} \tag{2.6}$$

Finally, we may apply the result of [5][Proposition 2.1]. For the reader's convenience we repeat the proof of this proposition here in the *third step*. We show that

$$\left| h_k\left(\frac{|\varepsilon|_2}{\sqrt{k}}, ..., \frac{|\varepsilon|_2}{\sqrt{k}}\right) - h_\infty(|\varepsilon|_2) \right| \le c \cdot B \cdot k^{-1/2}. \tag{2.7}$$

From (2.5)– (2.7) it follows that

$$|h_n(\varepsilon_1, ..., \varepsilon_n) - h_\infty(|\varepsilon|_2)| \le C \cdot B \cdot (|\varepsilon|^3 + k^{-1/2}).$$

Taking the limit with respect to k we conclude the statement of Theorem.

In the following we give the details for the steps outlined above.

Proof of the first step. We denote $h_k(\delta_1, ..., \delta_k) := h_{n+k-1}(\delta_1, ..., \delta_k, \varepsilon_2, ..., \varepsilon_n)$. Set

$$\begin{aligned} \underline{\delta}_k &:= (\delta_1, ..., \delta_k) := \left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}\right) \\ \underline{\delta}_k^0 &:= (\delta_1^0, ..., \delta_k^0) := (\varepsilon_1, 0, ..., 0). \end{aligned}$$

Expanding by Taylor's formula we may write

$$\begin{aligned} h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) &= \sum_{j=1}^k \frac{\partial h_k(\underline{\delta}_k^0)}{\partial \delta_j}(\delta_j - \delta_j^0) \\ &\quad + \frac{1}{2}\sum_{j,l=1}^k \frac{\partial^2 h_k(\underline{\delta}_k^0)}{\partial \delta_j \partial \delta_l}(\delta_j - \delta_j^0)(\delta_l - \delta_l^0) + R_{31}, \end{aligned}$$

where R_{31} is the remainder term which will be considered later. In what follows we shall denote by R_{3i}, for some $i \in \mathbb{N}$, the remainder terms in Taylor's expansion. By (1.1) all summands in the first sum equals zero except for $j = 1$.

Consider the second sum. If $j \neq l$ and $j, l \neq 1$ then the corresponding summand equals zero. Condition (1.2) yields

$$\frac{\partial}{\partial \delta_j}\frac{\partial}{\partial \delta_l} h_k(\underline{\delta}_k^0) = 0$$

provided that $j \neq l$, $j, l \neq 1$ and

$$\frac{\partial}{\partial \delta_1}\frac{\partial}{\partial \delta_l} h_k(\underline{\delta}_k^0) = R_{32}.$$

for all $l = 2, ..., k$. Since

$$\frac{\partial}{\partial \delta_1} h_k(\underline{\delta}_k^0) = \frac{\partial^2}{\partial \delta_1^2} h_k(\underline{\delta}_k^0)\big|_{\delta_1^0=0}\delta_1^0 + R_{33},$$
$$\frac{\partial^2}{\partial \delta_j^2} h_k(\underline{\delta}_k^0) = \frac{\partial^2}{\partial \delta_1^2} h_k(\underline{\delta}_k^0)\big|_{\delta_1^0=0} + R_{34},$$

and applying condition (1.3) we may sum the coefficients in front of the second derivatives of h_k and get

$$\varepsilon_1\left(\frac{\varepsilon_1}{\sqrt{k}} - \varepsilon_1\right) + \frac{1}{2}\left(\frac{\varepsilon_1}{\sqrt{k}} - \varepsilon_1\right)^2 + \frac{k-1}{2}\left(\frac{\varepsilon_1}{\sqrt{k}}\right)^2 = 0.$$

We now investigate the terms $R_{3l}, l = 1, ..., 4$, and show that $|R_{3l}| \leq c \cdot B \cdot |\varepsilon_1|^3$. Let us consider R_{31}. First we note that R_{31} is the sum of the third derivatives of h_k at some middle point $\hat{\underline{\delta}}_k^0$:

$$\sum_{j,l,m=1}^{k} \frac{\partial^3}{\partial \delta_j \partial \delta_l \partial \delta_m} h_k(\hat{\underline{\delta}}_k^0)(\delta_j - \hat{\delta}_j^0)(\delta_l - \hat{\delta}_l^0)(\delta_m - \hat{\delta}_m^0).$$

If the partial derivative with respect to δ_j (or δ_l, δ_m) is of order one then we have to expand further with respect to this variable at zero using (1.2). Finally we come to the bound

$$\sum_{j,l,m=1}^{k} \frac{\partial^3}{\partial \delta_j \partial \delta_l \partial \delta_m} h_k(\hat{\underline{\delta}}_k^0)(\delta_j - \hat{\delta}_j^0)(\delta_l - \hat{\delta}_l^0)(\delta_m - \hat{\delta}_m^0)$$
$$\leq C \cdot B \cdot (|\varepsilon_1|^3 + |\varepsilon_1|^4 + |\varepsilon_1|^6).$$

The other terms $R_{3l}, l = 2, 3, 4$, may be studied in the similar way.

Repeating this procedure $k-1$ times we come to the function (2.4) and the bound (2.5).

Proof of the second step is similar to the previous step. In this case n is fixed and we may write down all bound with respect to k.

Proof of the third step. We consider the difference between h_m at the point

$$\underline{\varepsilon}_m = (|\varepsilon|_2 m^{-1/2}, ..., |\varepsilon|_2 m^{-1/2})$$

and h_{k+r} at the point

$$\underline{\varepsilon}_{m+r} = (|\varepsilon|_2 (m+r)^{-1/2}, ..., |\varepsilon|_2 (m+r)^{-1/2})$$

and show, similar to the previous steps or the proof of [5][Proposition 2.1], that

$$h_m(\underline{\varepsilon}_m) - h_{m+r}(\underline{\varepsilon}_{m+r}) = O(d_3(h,m)) \sum_{p=m}^{m+r-1} p^{-3/2}.$$

Therefore, $h_m(\underline{\varepsilon}_m)$ is a Cauchy sequence in m with a limit, say $h_\infty(|\varepsilon|_2)$. Taking $m = k$ we finish the proof of Theorem. □

To formulate the asymptotic expansion of the function $h_n(\cdot), n \geq 1$, we have to introduce further notations. We introduce the following differential operators by means of formal power series identities. Define cumulant differential operators $\kappa_p(D)$ by means of

$$\sum_{p=2}^{\infty} p!^{-1} \varepsilon^p \kappa_p(D) = \ln \left(1 + \sum_{p=2}^{\infty} p!^{-1} \varepsilon^p D^p\right)$$

in the formal variable ε. One may easily compute the first cumulants. For example, $\kappa_2 = D^2, \kappa_3 = D^3, \kappa_4 = D^4 - 3D^2D^2$. Define Edgeworth polynomials by means of the following formal series in κ_r, τ_r and the formal variable ε

$$\sum_{r=0}^{\infty} \varepsilon^r P_r(\tau_* \kappa_*) = \exp \left(\sum_{r=3}^{\infty} r!^{-1} \varepsilon^{r-2} \kappa_r \tau_r\right)$$

which yields

$$P_r(\tau_*\kappa_*) = \sum_{m=1}^{r} m!^{-1} \sum_{j_1,...,j_m} (j_1+2)!^{-1} \tau_{j_1+2} \kappa_{j_1+2} \times (j_2+2)!^{-1} \tau_{j_2+2} \kappa_{j_2+2} ... (j_m+2)!^{-1} \tau_{j_m+2} \kappa_{j_m+2}, \tag{2.8}$$

where the sum $\sum_{j_1,...,j_m}$ extends over all m-tuples of positive integers $(j_1, ..., j_m)$ satisfying $\sum_{q=1}^m j_q = r$ and $\kappa_* = (\kappa_3, ..., \kappa_{r+2}), \tau_* = (\tau_3, ..., \tau_{r+2})$. For example,

$$P_1(\tau_*\kappa_*) = \frac{1}{6}\tau_3^3\kappa_3 = \frac{1}{6}\tau_3^3 D^3,$$

$$P_2(\tau_*\kappa_*) = \frac{1}{24}\tau_4\kappa_4 + \frac{1}{72}\tau_3^2\kappa_3\kappa_3 = \frac{1}{24}\tau_4(D^4 - 3D^2D^2) + \frac{1}{72}\tau_3^2 D^3 D^3. \tag{2.9}$$

In the following theorem we will assume that ε is a vector on the unit sphere, i.e. $|\varepsilon|_2 = 1$. It is also possible to consider the general case $|\varepsilon|_2 = r, r > 1$, but then the remainder terms will have more difficult structure. In what follows we shall omit the argument $|\varepsilon|_2$ from the notations of h_∞.

Theorem 2.2. *Assume $h_n(\varepsilon_1, ..., \varepsilon_n), n \geq 1$, satisfies conditions* (1.1), (1.2) *and* (1.3) *together with $|\varepsilon|_2 = 1$. Suppose that*

$$|D^\alpha h_n(\varepsilon_1, ..., \varepsilon_n)| \leq B, \tag{2.10}$$

for all $\varepsilon_1, ..., \varepsilon_n$, *where* B *denotes some positive constant,* $\alpha = (\alpha_1, ..., \alpha_r), r \le s$, *and*

$$\alpha_j \ge 2, \quad j = 1, ..., r, \quad \sum_{j=1}^{r} (\alpha_j - 2) \le s - 2.$$

Then

$$h_n(\varepsilon_1, ..., \varepsilon_n) = h_\infty + \sum_{l=1}^{s-3} P_l(\varepsilon^* \kappa_*) h_\infty(\lambda_1, ..., \lambda_s)\big|_{\lambda_1 = ... = \lambda_s = 0} + R_s,$$

where

$$|R_s| \le c_s \cdot B \cdot |\varepsilon|^s.$$

where c_s *is an absolute constant.*

As an example consider the case $s = 5$. Then by (2.9)

$$\begin{aligned} h_n(\varepsilon_1, ..., \varepsilon_n) &= h_\infty + \frac{\varepsilon^3}{6} \frac{\partial^3}{\partial \lambda^3} h_\infty(\lambda)\big|_{\lambda=0} \\ &+ \left[\frac{\varepsilon^4}{24} \left(\frac{\partial^4}{\partial \lambda_1^4} - 3 \frac{\partial^2}{\partial \lambda_1^2} \frac{\partial^2}{\partial \lambda_2^2} \right) + \frac{\varepsilon^6}{72} \frac{\partial^3}{\partial \lambda_1^3} \frac{\partial^3}{\partial \lambda_2^3} \right] h_\infty(\lambda_1, \lambda_2)\big|_{\lambda_1 = 0, \lambda_2 = 0} \\ &+ \mathcal{O}(|\varepsilon|^5). \end{aligned} \tag{2.11}$$

Before we start to prove Theorem 2.2 we have to introduce one more notation. For any sequence $\tau_p, p \ge 1$, of formal variables define $\tilde{P}(\tau_* \kappa_*)$ as a polynomial in the cumulant operators κ_p multiplied by τ_p by the following formal power series in μ:

$$\sum_{j=0}^{\infty} \tilde{P}_j(\tau_* \kappa_*) \mu^j := \exp\left(\sum_{j=2}^{\infty} j!^{-1} \tau_j \kappa_j(D) \mu^j \right). \tag{2.12}$$

For example, $\tilde{P}_0 = 1, \tilde{P}_1 = 0, \tilde{P}_2 = \frac{1}{2} \tau_2 D^2, \tilde{P}_3 = \frac{1}{6} \tau_3 D^3$, and

$$\tilde{P}_4 = \frac{1}{24} \tau_4 (D^4 - 3D^2 D^2) + \frac{1}{8} \tau_2^2 D^2 D^2.$$

If $\tau_p = \tau^p, p \ge 1$ then

$$\tilde{P}_j = j!^{-1} \tau_j D^j. \tag{2.13}$$

One may also see that

$$\sum_{j+l=r} \tilde{P}_j(\tau_* \kappa_*) \tilde{P}_j(\tau'_* \kappa_*) = \tilde{P}_r((\tau_* + \tau'_*) \kappa_*). \tag{2.14}$$

There is a relation between Edgeworth polynomial $P_r(\cdot)$ and $\tilde{P}_r(\cdot)$ which is given by the following formula

$$\sum_{r=1}^{\infty} [P_r(\tau_* \kappa_*)]_l = \sum_{r=1}^{l} \tilde{P}_r(\tau_* \kappa_*), \tag{2.15}$$

where $[\cdot]_l$ means the sum of all monomials $\tau_1^{p_1} \cdots \tau_{r+2}^{p_r+2}$ in $P_r(\tau_* \kappa_*)$ such that $p_1 + 2p_2 + ... + (r+2)p_{r+2} \le l$. We will use (2.15) in the proof of Theorem 2.2. The following Lemma gives the possibility to rewrite the derivatives of $h_n(\varepsilon_1, ..., \varepsilon_n)$ via derivatives in the additional variables using definition of $\tilde{P}_r$.

Lemma 2.3. *Suppose that the conditions* (1.1), (1.2) *and* (1.3) *hold. Then*

$$\sum_{j=2}^{m} \frac{1}{j!} \frac{\partial}{\partial \varepsilon^j} h_n(\varepsilon, \varepsilon_2, ..., \varepsilon_n)\big|_{\varepsilon=0} (\eta^j - \varepsilon^j)$$
$$= \sum_{r=2}^{m} \tilde{P}_r((\eta^* - \varepsilon^*)\kappa_*(D)) h_{n+m}(\lambda_1, .., \lambda_k, \varepsilon, \varepsilon_2, ..., \varepsilon_n)\big|_{\lambda_1 = ... = \lambda_m = 0} + \mathcal{O}(\varepsilon^m).$$

Proof. The following simple calculations prove the statement of Lemma

$$\sum_{r=1}^{m} \tilde{P}_r((\eta^* - \varepsilon^*)\kappa_*(D)) h_{m+n}(\underbrace{0, ..., 0}_{m}, \varepsilon, \varepsilon_2, ..., \varepsilon_n)$$
$$\overset{(2.13)}{=} \sum_{j=1}^{m} \sum_{\substack{l+r=j \\ r \geq 1}} \tilde{P}_r((\eta^* - \varepsilon^*)\kappa_*(D)) \tilde{P}_r(\varepsilon^* \kappa_*(D))$$
$$\times h_{m+n}(\underbrace{0, ..., 0}_{m+1}, \varepsilon_2, ..., \varepsilon_n) + \mathcal{O}(\varepsilon^m).$$
$$\overset{(2.14)}{=} \sum_{j=1}^{m} (\tilde{P}_r(\eta^* \kappa_*(D)) - \tilde{P}_r(\varepsilon^* \kappa_*(D)) h_{m+k}(\underbrace{0, ..., 0}_{m+1}, \varepsilon_2, ..., \varepsilon_n) + \mathcal{O}(\varepsilon^m).$$
$$\overset{(2.13)}{=} \sum_{j=2}^{m} \frac{1}{j!} \frac{\partial}{\partial \varepsilon^j} h_n(\varepsilon, \varepsilon_2, ..., \varepsilon_n)\big|_{\varepsilon=0} (\eta^j - \varepsilon^j) + \mathcal{O}(\varepsilon^m).$$

□

Proof of Theorem 2.2. We prove this theorem by the induction on the length of the expansion. We consider the difference

$$h_n(\varepsilon_1, ..., \varepsilon_n) - h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right)$$

and similarly to the idea of the previous Theorem 2.1 divide it into the sum of n terms, each time changing ε_j by $(k^{-1/2}\varepsilon_j, ..., k^{-1/2}\varepsilon_j)$. We start from the case $j = 1$ and denote $h_k(\delta_1, ..., \delta_k) := h_{n+k-1}(\delta_1, ..., \delta_k, \varepsilon_2, ..., \varepsilon_n)$. Set

$$\underline{\delta}_k := (\delta_1, ..., \delta_k) := \left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}\right)$$
$$\underline{\delta}_k^0 := (\delta_1^0, ..., \delta_k^0) := (\varepsilon_1, 0, ..., 0).$$

Expanding by Taylor's formula we get

$$h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) = \sum_{0 < |\alpha| < s} \alpha!^{-1} D^\alpha h_k(\underline{\delta}_k^0)((\underline{\delta}_k - \underline{\delta}_k^0)^\alpha + R_{1,s},$$

where $R_{1,s}$ is the remainder term and $|R_{1,s}| \leq c_s \cdot B \cdot |\varepsilon_1|^s$. To simplify our notations we introduce the following agreement. We shall denote by $R_l, l \geq 1$ a remainder term which has the order $\mathcal{O}(|\varepsilon|^l)$ and omit the explicit dependence on B. By $R_{1,l}, l \geq 1$ we shall denote a remainder term which has the order $\mathcal{O}(|\varepsilon_1|^l)$.

In order to use the condition (1.2) we expand each derivative $D^\alpha h_k(\underline{\delta}_k^0)$, where $\alpha = (\alpha_{j_1}, ..., \alpha_{j_p}), 1 \leq j_1 \leq ... \leq j_p \leq k$, around $\delta_{j_r} = 0, r = 1, ..., p$. This yields

$$D^\alpha h_k(\underline{\delta}_k^0) = \sum_{0<|\alpha|+|\beta|<s} \beta!^{-1} D^{\alpha+\beta} h_k(\underline{\delta}_k^0)(\underline{\delta}_k^0)^\beta + R_{1,s}, \tag{2.16}$$

Using the binom formula one may see that

$$\sum_{j+k=r, j\geq 1} \frac{1}{j! \cdot k!} (\varepsilon - \eta)^j \eta^k = \frac{1}{r!} (\varepsilon^r - \eta^r),$$

and applying it to (2.16) we get

$$h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) = \sum_{0<|\gamma|<s} \gamma!^{-1} D^\gamma h_k(0, ..., 0) \prod_{i=1}^k [\delta_i^{\gamma_i} - (\delta_i^0)^{\gamma_i}] + R_{1,s}.$$

To use an induction assumption which will be formulated later we have to return to the function $h_k(\underline{\delta}_k)$ taking derivatives in the additional variables at zero. Applying Lemma 2.3 we get

$$\begin{aligned} h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) \\ = \sum \tilde{P}_{r_1}(\Delta_{j_1}^* \kappa_*) \cdots \tilde{P}_{r_m}(\Delta_{j_m}^* \kappa_*) h_{k+s}(\underline{\delta}_k, 0, ..., 0) + R_{1,s}, \end{aligned} \tag{2.17}$$

here the sum extends over all combination of $r_1, ..., r_m \geq 2, m = 1, 2, ...$, such that $r_1 + ... + r_m < s$ and all ordered m - tuples of positive induces $1 \leq j_r \leq m$ without repetition. Assume that is has been already proved for $l = 3, ..., s-1$ that

$$D^\alpha h(\varepsilon_1, ..., \varepsilon_n) = \sum_{j=0}^{l-3} P_j(\varepsilon^* \kappa_*) h_\infty + R_{1,l}. \tag{2.18}$$

For example, the case $l = 3$ follows from Theorem 2.1. Applying the induction assumption (2.18) to (2.17) we get

$$h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) = \sum \tilde{P}_{r_1}(\Delta_{j_1}^* \kappa_*) \cdots \tilde{P}_{r_k}(\Delta_{j_k}^* \kappa_*) P_{r_0}(\varepsilon^* \kappa_*) h_\infty + R_{1,s},$$

where the sum extends over all induces $r_1, ..., r_m \geq 1, r_0 \geq 0$, such that $r_0 + r_1 + ... + r_m \leq s$. We will rewrite this in the following way

$$\begin{aligned} h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) \\ = \sum_{r_0=0}^{s-4} P_{r_0}(\varepsilon^* \kappa_*) \sum_j \left[\prod_{l=1}^{s-r_0} \left[\sum_{v_l=1}^s \tilde{P}_{v_l}(\Delta_{j_l}^* \kappa_*) \right] \right]_{s-r_0} h_\infty + R_{1,s}. \end{aligned}$$

The last may be rewritten as

$$\sum_{r_0=0}^{s-4} P_{r_0}(\varepsilon^* \kappa_*) \left[\exp \left[\sum_{p=2}^\infty \left(\sum_{j=1}^k \Delta_j^p \right) p!^{-1} \kappa_p \right] - 1 \right]_{s-r_0} h_\infty + R_{1,s} \tag{2.19}$$

It is easy to see that

$$\sum_{j=1}^{k} \Delta_j^2 = \varepsilon_1^2 \left[\left(\frac{1}{k} - 1\right) + \frac{k-1}{k}\right] = 0$$

and

$$\sum_{j=1}^{k} \Delta_j^p = \varepsilon_1^p \left[\left(\frac{1}{k^{p/2}} - 1\right) + \frac{k-1}{k^{p/2}}\right] = -\varepsilon_1^p + \mathcal{O}\left(\frac{|\varepsilon_1|^p}{k^{p/2}}\right), \quad p > 2. \tag{2.20}$$

Using (2.15) we get

$$\begin{aligned}
&P_{r_0}(\varepsilon^* \kappa_*) \left[\exp\left[\sum_{p=2}^{\infty}\left(\sum_{j=1}^{k}\Delta_j^p\right) p!^{-1}\kappa_p\right] - 1\right]_{s-r_0} h_\infty \\
&= P_{r_0}(\varepsilon^* \kappa_*) \sum_{i=3}^{s-r_0-1} \tilde{P}_i \left(\left(\sum_{j=1}^{k}\Delta_j^*\right)\kappa_*\right) h_\infty \\
&= P_{r_0}(\varepsilon^* \kappa_*) \sum_{i=1}^{\infty} \left[P_i\left(\left(\sum_{j=1}^{k}\Delta_j^*\right)\kappa_*\right)\right]_{s-r_0-1} h_\infty
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
&P_{r_0}(\varepsilon^* \kappa_*) P_r\left(\sum_{j=1}^{k}\Delta_j^* \kappa_*\right) h_\infty \\
&\qquad = P_{r_0}(\varepsilon^* \kappa_*)\left[P_r\left(\sum_{j=1}^{k}\Delta_j^* \kappa_*\right)\right]_{s-r_0-1} h_\infty + R_{1,s}
\end{aligned} \tag{2.22}$$

By (2.21)–(2.22) we may rewrite (2.19) in the following way

$$h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) = \sum_{r_0=0}^{s-4} P_{r_0}(\varepsilon^*\kappa_*) \sum_{r=1}^{s-3-r_0} P_r\left(\sum_{j=1}^{k}\Delta_j^*\kappa_*\right) h_\infty + R_{1,s}.$$

Since

$$\sum_{r+q=k} P_r(\tau_*\kappa_*)P_q(\tau_*'\kappa_*) = P_k((\tau_* + \tau_*')\kappa_*), \quad q, r, k \geq 0.$$

we will have

$$\begin{aligned}
&h_k(\underline{\delta}_k) - h_k(\underline{\delta}_k^0) \\
&= \sum_{r=1}^{s-3} \left[P_r((\varepsilon^* - \varepsilon_1^*)\kappa_*) - P_r(\varepsilon^*\kappa_*)\right] h_\infty(\lambda_1, ..., \lambda_s)\big|_{\lambda_1 = ... = \lambda_s = 0} + R_{1,s},
\end{aligned}$$

where we have used (2.20).

Now we may change ε_j by $(k^{-1/2}\varepsilon_j, ..., k^{-1/2}\varepsilon_j)$ for $j \geq 2$. It is easy to see that if one replace the function $h(\varepsilon_1, ..., \varepsilon_n)$ by $h_k(\underline{\delta}_k)$ than (2.18) will be true replacing ε by $\varepsilon_{[2:n]}$, where $\varepsilon_{[2:n]} := (\varepsilon_2, ..., \varepsilon_n)$. But the function h_∞ will be the same since it depends only on the $|\varepsilon|_2$ and $|\underline{\delta}_k|^2 + |\varepsilon_{[2:n]}|^2 = |\varepsilon|_2$. The same is true for all $j > 2$. Repeating this procedure $n-1$ times we will arrive at

$$h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right) - h_k(\varepsilon_1, ..., \varepsilon_n)$$
$$= \sum_{j=1}^{n}\sum_{r=1}^{s-3}\Big[P_r((\varepsilon^*_{[j:n]} - \varepsilon^*_j)\kappa_*) - P_r(\varepsilon^*_{[j:n]}\kappa_*)\Big]\, h_\infty(\lambda_1, ..., \lambda_s)\big|_{\lambda_1=...=\lambda_s=0}$$
$$+ R_s,$$

where $\varepsilon_{[j:n]} := (\varepsilon_j, ..., \varepsilon_n)$.

To finish the proof we need to show that for fixed r we get

$$\sum_{j=1}^{n}\Big[P_r((\varepsilon^*_{[j:n]} - \varepsilon^*_j)\kappa_*) - P_r(\varepsilon^*_{[j:n]}\kappa_*)\Big] = -P_r(\varepsilon^*\kappa_*) \tag{2.23}$$

Validity of (2.23) follows from the following simple observation. Let $m \geq 1$ be a fixed integer and $(j_1, ..., j_m)$ be a vector of positive numbers such that $j_1 + ... + j_m = r$. Then

$$\sum_{i=1}^{n}(\varepsilon_{[i:n]}^{j_1+2} - \varepsilon_i^{j_1+2}) \cdots (\varepsilon_{[i:n]}{}^{j_m+2} - \varepsilon_i^{j_m+2}) - \sum_{i=1}^{n}\varepsilon_{[i:n]}^{j_1+2} \cdots \varepsilon_{[i:n]}{}^{j_m+2}$$
$$= -\varepsilon^{j_1+2} \cdots \varepsilon^{j_m+2}.$$

The proof of it is trivial, it is enough to see that for all $i \geq 1$

$$(\varepsilon_{[i:n]}^{j_1+2} - \varepsilon_i^{j_1+2}) \cdots (\varepsilon_{[i:n]}{}^{j_m+2} - \varepsilon_i^{j_m+2}) = \varepsilon_{[i+1:n]}{}^{j_1+2} \cdots \varepsilon_{[i+1:n]}{}^{j_m+2}.$$

Applying (2.23) we arrive at

$$h_n(\varepsilon_1, ..., \varepsilon_n) - h_{nk}\left(\frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_1}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}, ..., \frac{\varepsilon_n}{\sqrt{k}}\right)$$
$$= \sum_{r=1}^{s-3} P_r(\varepsilon^*\kappa_*) h_\infty(\lambda_1, ..., \lambda_s)\big|_{\lambda_1=...=\lambda_s=0} + R_s.$$

Now we may repeat the last two steps in the proof of the previous Theorem 2.1 and taking the limit with respect to $k \to \infty$ we get

$$h_n(\varepsilon_1, ..., \varepsilon_n) - h_\infty = \sum_{r=1}^{s-3} P_r(\varepsilon^*\kappa_*) h_\infty(\lambda_1, ..., \lambda_s)\big|_{\lambda_1=...=\lambda_s=0} + R_s.$$

This proves (2.18) for $l = s$ and $\alpha = 0$. Hence, the induction is completed and the Theorem is proved. □

3. Application of Theorem 2.2

In this section we illustrate on different examples how one may apply Theorem 2.2 to derive an asymptotic expansion of different functions in probability theory.

3.1. **Expansion in the Central Limit Theorem.** As the first example let us consider the sequence of independent random variables $X, X_j, j \in \mathbb{N}$, taking values in $\mathbb{R}$ with a common distribution function F. Suppose the $\mathbb{E}\, X = 0, \mathbb{E}\, X^2 = 1$. Consider the sum $S_\varepsilon = \varepsilon_1 X_1 + ... + \varepsilon_n X_n$. As h_n we may take the characteristic function of S_ε, i.e.,

$$h_n(\varepsilon_1, ..., \varepsilon_n) = \mathbb{E}\, e^{it(\varepsilon_1 X_1 + ... + \varepsilon_n X_n)}.$$

From Theorem (2.1) we know that $h_\infty(|\varepsilon|_2)$ exists provided that the condition (2.2) holds. In our setting this condition means that $\mathbb{E}\,|X|^3 < \infty$. It is well known that

$$h_\infty(|\varepsilon|_2) = \mathbb{E}\, e^{itG},$$

where $G \sim N(0, |\varepsilon|_2)$. The speed of convergence is given by $|\varepsilon|_3^3$. In what follows we shall assume that $|\varepsilon|_2 = 1$. If ε is well spread, for example, $\varepsilon_j = n^{-1/2}$ for all $1 \le j \le n$ then

$$|h_n(\varepsilon_1, ..., \varepsilon_n) - h_\infty(|\varepsilon|_2)| \le C \cdot \frac{E|X|^3}{n^{1/2}}. \tag{3.1}$$

This doesn't hold for all ε on the sphere. Consider a simple counter example. Let $X \sim \text{Uniform}([-\sqrt{3}, \sqrt{3}])$ and $\varepsilon = e_1$. Then $S_\varepsilon = X_1 \sim \text{Uniform}([-\sqrt{3}, \sqrt{3}])$, which is not Gaussian as $n \to \infty$.

It is interesting to mention here the following result of Klartag and Sodin [10] who showed that with high probability the right hand side of (3.1) has the order $\mathcal{O}(1/n)$ for the uniform distribution σ_{n-1} on S^{n-1}.

Let us construct an asymptotic expansion applying Theorem 2.2. We have

$$h_\infty(\lambda_1, ..., \lambda_s) = \mathbb{E}\, e^{it(\lambda_1 X_1 + ... + \lambda_s X_s + G)}.$$

We may take derivatives with respect to $\lambda_1, ..., \lambda_s$ at zero and get

$$\begin{aligned}
\frac{\partial^3}{\partial \lambda_1^3} h_\infty(\lambda_1)\Big|_{\lambda_1=0} &= (it)^3 e^{-t^2/2}\beta_3, \\
\frac{\partial^4}{\partial \lambda_1^4} h_\infty(\lambda_1)\Big|_{\lambda_1=0} &= (it)^4 e^{-t^2/2}\beta_4, \\
\frac{\partial^4}{\partial \lambda_1^2 \partial \lambda_2^2} h_\infty(\lambda_1, \lambda_2)\Big|_{\lambda_1=0,\lambda_2=0} &= (it)^4 e^{-t^2/2}\beta_2^2, \\
\frac{\partial^6}{\partial \lambda_1^3 \partial \lambda_2^3} h_\infty(\lambda_1, \lambda_2)\Big|_{\lambda_1=0,\lambda_2=0} &= (it)^6 e^{-t^2/2}\beta_3^2,
\end{aligned}$$

where $\mathbb{E}\, X^2 = \beta_2 = 1, \mathbb{E}\, X^3 = \beta_3$ and $\mathbb{E}\, X^4 = \beta_4$. Substituting these equations to (2.11) we get

$$\begin{aligned}
h_n(\varepsilon_1, ..., \varepsilon_n) = \mathbb{E}\, e^{itG} &+ \frac{\varepsilon^3}{6}(it)^3 e^{-t^2/2}\beta_3 \\
&+ \frac{\varepsilon^4}{24}[\beta_4 - 3](it)^4 e^{-t^2/2} + \frac{(\beta_3\varepsilon^3)^2}{72}(it)^6 e^{-t^2/2} + R_5.
\end{aligned}$$

This expansion coincides with the well known Edgeworth expansion for sums of random variables, see Petrov [12][Chapter 6, Paragraph 1].

Let us concentrate now on the so-called short asymptotic expansion

$$h_n(\varepsilon_1, ..., \varepsilon_n) = \mathbb{E}\, e^{itG} + \frac{\varepsilon^3}{6}(it)^3 e^{-t^2/2}\beta_3 + R_4, \tag{3.2}$$

where

$$R_4 \le B \cdot |t|^4 \cdot \sum_{k=1}^{n} \varepsilon_k^4.$$

Suppose that the following two conditions from [10] hold:

$$\left|\sum_{k=1}^{n} \varepsilon_k^3\right| \le \frac{C}{n} \text{ and } \sum_{k=1}^{n} \varepsilon_k^4 \le \frac{C}{n}, \tag{3.3}$$

where C is some constant independent of n. It follows from [10, Lemma 4.1] that these bound hold with high probability. Then it follows from (3.2) and (3.3) that

$$|h_n(\varepsilon_1, ..., \varepsilon_n) - \mathbb{E}\, e^{itG}| = \mathcal{O}\left(\frac{1}{n}\right)$$

Since conditions (3.3) hold with high probability with respect to the uniform distribution σ_{n-1} on S^{n-1} the previous estimate is valid for most choices of $\varepsilon_1, ..., \varepsilon_n$. This property may be generalized for arbitrary function $h_n(\varepsilon_1, ..., \varepsilon_n)$ which satisfies the conditions of Theorem 2.2.

It also possible to apply our result for asymptotic expansion in the central limit theorem for quadratic forms in sums of random elements with values in a Hilbert space including infinite dimensional case, see, e.g. [2], [7], [15], [14], [13] and [9].

3.2. **Expansion in Free Central Limit theorem.** It has been shown in the recent paper [8] that one may apply the results of Theorem 2.2 in the setting of Free Probability theory.

Denote by $\mathcal{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. Let $X_1, X_2, \dots$ be free self-adjoint identically distributed random variables with distribution $\mu \in \mathcal{M}$. We always assume that μ has zero mean and unit variance. Let μ_n be the distribution of the normalized sum $S_n := \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j$. In free probability a sequence of measures μ_n converges to the semicircle law ω. Moreover, μ_n is absolutely continuous with respect to the Lebesgue measure for sufficiently large n. We denote by p_{μ_n} the density of μ_n. Define the Cauchy transform of a measure μ:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x}, \quad z \in \mathbb{C}^+,$$

where $\mathbb{C}^+$ denotes the upper half plane.

In [3] Chistyakov and Götze obtained a formal power expansion for the Cauchy transform of μ_n and the Edgeworth type expansions for μ_n and p_{μ_n}. In [8] the general scheme from [5] was applied to derive a similar result.

3.3. **Expansion of quadratic von Mises Statistics.** Let $X, \overline{X}, X_1, ..., X_n$ be independent identically distributed random elements taking values in an arbitrary measurable space $(\mathcal{X}, \mathcal{B})$. Assume that $g : \mathcal{X} \to \mathbb{R}$ and $h : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be real-valued measurable functions. We additionally assume that h is symmetric. We consider the quadratic functional

$$w_n(\varepsilon_1, ..., \varepsilon_n) = \sum_{k=1}^{n} \varepsilon_j g(X_j) + \sum_{j,k=1}^{n} \varepsilon_j \varepsilon_k h(X_j, X_k),$$

assuming that

$$\mathbb{E}\, g(X) = 0, \quad \mathbb{E}(h(X, \overline{X})|X) = 0.$$

We derive an asymptotic expansion of $h_n(\varepsilon_1, ..., \varepsilon_n) := \mathbb{E} \exp(itw_n(\varepsilon_1, ..., \varepsilon_n))$.

Consider the measurable space $(\mathcal{X}, \mathcal{B}, \mu)$ with measure $\mu := \mathcal{L}(X)$. Let $L^2 := L^2(\mathcal{X}, \mathcal{B}, \mu)$ denote the real Hilbert space of square integrable real functions. The Hilbert-Schmidt operator $\mathbb{Q} : L^2 \to L^2$ is defined via

$$\mathbb{Q} f(x) = \int_{\mathcal{X}} h(x, y) f(y) \mu(dy) = \mathbb{E}\, h(x, X) f(X), \quad f \in L^2.$$

Let $\{e_j, j \geq 1$ denote an orthonormal complete system of eigenfunctions of $\mathbb{Q}$ ordered by decreasing absolute values of the corresponding eigenvalues $q_1, q_2, ...$, that is, $|q_1| \geq |q_2| \geq ....$ Then

$$\mathbb{E}\, h^2(X, \overline{X}) = \sum_{j=1}^{\infty} q_j^2 < \infty, \quad h(x, y) = \sum_{j=1}^{\infty} q_j e_j(x) e_j(y)$$

If the closed span $\langle \{e_j, j \geq 1\} \rangle \subset L^2$ is a proper subset, it might be necessary to choose functions e_{-1}, e_0 such that $\{e_j, j = -1, 0, 1, ...\}$ is an orthonormal system and

$$g(x) = \sum_{k=0}^{\infty} g_k e_k(x), \quad h(x, x) = \sum_{k=-1}^{\infty} h_k e_k(x).$$

It is easy to see that $\mathbb{E}\, e_j(X) = 0$ for all j. Therefore $\{e_j(X), j = -1, 0, 1, ...\}$ is an orthonormal system of mean zero random variables.

We derive an expression for the the derivatives of $h_\infty(\lambda_1, ..., \lambda_r)$. Since for every fixed k the sum $n^{-1/2}(e_k(X_1) + ... + e_k(X_n))$ weakly converges to the standard normal random variable we get that $w_{n+r}(\lambda_1, ..., \lambda_r, n^{-1/2}, ..., n^{-1/2})$

weakly converges to the random variable

$$w_\infty(\lambda_1,...,\lambda_r) := w_r(\lambda_1,...,\lambda_r) + \sum_{k=0}^{\infty} g_k Y_k + \sum_{k=1}^{\infty} q_k^2 (Y_k^2 - 1) + \mathbb{E}\, h(X,X) + 2\sum_{k=1}^{\infty} q_k \left(\sum_{l=1}^{r} \lambda_l e_k(X_l)\right) Y_k,$$

where $Y_k, k \geq 0$ are independent standard normal random variables. For every fixed T by complex integration we get

$$\mathbb{E}\exp\left[itq_k(Y_k^2-1) + 2itTY_k\right] = \frac{1}{\sqrt{1-2itq_k}} \exp(-itq_k) \exp\left[-\frac{2t^2T^2}{\sqrt{1-2itq_k}}\right].$$

This yields that

$$h_\infty(\lambda_1,...,\lambda_r) = \varphi(t)\,\mathbb{E}\exp[itw_r(\lambda_1,...,\lambda_r) + (it)^2 \sum_{k=1}^{\infty} q_k T_k(\lambda)(2q_k T_k(\lambda) + g_k)(1-2itq_k)^{-1}], \tag{3.4}$$

where $T_k(\lambda) = \sum_{l=1}^{r} \lambda_l e_k(X_l)$ and

$$\varphi(t) = \left[\prod_{k=1}^{\infty} \frac{1}{\sqrt{1-2itq_k}} \exp(-itq_k)\right] \cdot \exp\left[it\,\mathbb{E}\, h(X_1,X_1) - t^2 \sum_{k=0}^{\infty} g_k^2 (1-2itq_k)^{-1}/2\right].$$

Let us introduce the following functions of X and $\overline{X}$:

$$h_t(X,\overline{X}) := h(X,\overline{X}) + 2it\sum_{k=1}^{\infty} q_k^2 e_k(X) e_k(\overline{X})(1-2itq_k)^{-1},$$

$$g_t(X) := g(X) + it\,\mathbb{E}\, h_t(X,\overline{X}) g(\overline{X})|X).$$

Applying these notations we may rewrite (3.4) in the following way

$$h_\infty(\lambda_1,...,\lambda_r) = \varphi(t)\,\mathbb{E}\exp\left[it\sum_{j,k=1}^{n} h_t(X_j,X_k)\lambda_j\lambda_k + it\sum_{j=1}^{r} \lambda_j g_t(X_j)\right].$$

Taking the derivatives of h_∞ with respect with $\lambda_1,...,\lambda_r$ at zero we get

$$h_n(\varepsilon_1,...,\varepsilon_n) = \varphi(t)\sum_{r=0}^{s-3} a_r(t,h,g) + R_s,$$

where

$$a_r(t,h,g) := P_r(\varepsilon^*\kappa_*)\,\mathbb{E}\exp\left[it\sum_{j,k=1}^{n} h_t(X_j,X_k)\lambda_j\lambda_k + it\sum_{j=1}^{r} \lambda_j g_t(X_j)\right]\Bigg|_{\lambda_1=...=\lambda_r=0}.$$

One may also consider the higher order U statistics by similar arguments. See, for example, the result of [6] and [1].

3.4. **Expansion for weighted one sided Kolmogorov-Smirnov statistic.** Let $X_1, ...X_n$ be an independent identically distributed random variables with uniform distribution in $[0, 1]$. Consider the following statistic $D^+(\varepsilon_1, ..., \varepsilon_n, t) = \sum_{j=1}^n \varepsilon_j(\mathbb{I}(X_j \le t) - t)$. For example, if $\varepsilon_j = n^{-1/2}, j = 1, ..., n$ then we have $D^+(t) = n^{1/2}(F_n(t) - t)$, where $F_n(t)$ is a classical empirical distribution function of $X_1, ..., X_n$. We are interested in the asymptotic expansion of

$$\mathbb{P}(\sup_{0 \le t \le 1} D^+(\varepsilon_1, ..., \varepsilon_n, t) > a), \quad a > 0.$$

It is well known that $h_\infty(0) = \exp[-2a^2]$ and

$$h_\infty(\lambda) = \int_0^1 \mathbb{P}(x(t) + \lambda(\mathbb{I}(s < t) - t) > a, 0 \le t \le 1)\, ds =$$
$$\int_0^1 \mathbb{E}\, f_a(s, x(s), \lambda) f_a(1 - s, x(s), -\lambda)\, ds,$$

where $f_a(s, x, \lambda) = \mathbb{P}(x(t) > a + \lambda t, 0 \le t \le s | x(s) = x) = \exp(-2a(a + \lambda s - x)/s)$ and $x(t), 0 \le t \le 1$ is a Brownian bridge. See [5] for details. Then it follows from Theorem 2.2 that

$$\mathbb{P}(\sup_{0 \le t \le 1} D^+(\varepsilon_1, ..., \varepsilon_n, t) > a) = \left[1 + \frac{1}{6}\varepsilon^3 \frac{\partial}{\partial a} + \mathcal{O}(|\varepsilon|_4^4)\right] \exp(-2a^2).$$

Such expansions for equal weights were derived, for example, in [11], [4].

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