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Unobstructed symplectic packing for tori and hyper-Kähler manifolds

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Let M be a closed symplectic manifold of volume V. We say that the symplectic packings of M by balls are unobstructed if any collection of disjoint symplectic balls (of possibly different radii) of total volume less than V admits a symplectic embedding to M. In 1994, McDuff and Polterovich proved that symplectic packings of Kähler manifolds by balls can be characterized in terms of the Kähler cones of their blow-ups. When M is a Kähler manifold which is not a union of its proper subvarieties (such a manifold is called Campana simple), these Kähler cones can be described explicitly using the Demailly and Paun structure theorem. We prove that for any Campana simple Kähler manifold, as well as for any manifold which is a limit of Campana simple manifolds in a smooth deformation, the symplectic packings by balls are unobstructed. This is used to show that the symplectic packings by balls of all even-dimensional tori equipped with Kähler symplectic forms and of all hyper-Kähler manifolds of maximal holonomy are unobstructed. This generalizes a previous result by Latschev–McDuff–Schlenk. We also consider symplectic packings by other shapes and show, using Ratner's orbit closure theorem, that any even-dimensional torus equipped with a Kähler form whose cohomology class is not proportional to a rational one admits a full symplectic packing by any number of equal polydisks (and, in particular, by any number of equal cubes).

Keywords: Symplectic manifold; symplectic packing; Kähler manifold; hyper-Kähler manifold; Campana simple complex structure; Kähler cone; symplectic cone.

1. Introduction

The symplectic packing problem is one of the major problems of symplectic topology that was introduced, along with the first results on it, in the famous foundational

paper by Gromov [19]. The most extensively studied version of the problem is the question about symplectic packings of symplectic manifolds by balls. In [27] McDuff and Polterovich reduced the question about such packings of a symplectic manifold (M, ω) to a question about the structure of the symplectic cone in the cohomology of a blow-up of M. In the same paper they showed that symplectic packings of Kähler manifolds by balls are deeply related to algebraic geometry that allows sometimes to describe the shape of the Kähler cone in the cohomology of a Kähler manifold.

In this paper we use several strong results from complex geometry in order to prove the flexibility of symplectic packings by balls for all even-dimensional tori equipped with Kähler symplectic forms as well as for certain hyper-Kähler manifolds. Namely, we show that such a packing is possible as long as the natural volume constraint is satisfied. In this case we will say that the symplectic packings of the symplectic manifold by balls are *unobstructed*. (We use the term "unobstructed" when we consider packings by *all* collections of balls of *arbitrary relative sizes* and the term "a full symplectic packing" when we talk about packings by a *specific* number of balls, or other shapes, of *specific* relative sizes — e.g. "a full symplectic packing by k equal balls").

Our unobstructed symplectic packing theorem extends a previous result of Latschev–McDuff–Schlenk [26].

Our strategy is the following. Let (M, I, ω) be a closed connected Kähler manifold, where I is the complex structure and ω is the symplectic form forming the Kähler structure. If the complex structure I is *Campana simple*, meaning that the union of the positive-dimensional proper complex submanifolds of (M, I) is of measure zero, then the Demailly–Paun theorem [10] allows to give a complete description of the Kähler cones of the relevant blow-ups of M. Together with McDuff–Polterovich theorem mentioned above this implies that the symplectic packings of (M, ω) by balls are unobstructed. Then, *and this is the main novelty of our approach, compared to* [26], we use the Kodaira–Spencer stability theorem [24] to show that even if the complex structure I on M is not Campana simple but can only be approximated in a smooth deformation by Campana simple complex structures, the result about the Kähler cones in the Campana simple case gives enough information about the *symplectic* cones of the blow-ups of M to yield that the symplectic packings of (M, ω) by balls are unobstructed in this case as well (see Sec. 4.3 for a more detailed outline of this argument and Sec. 8.3 for a complete proof). We then use methods of complex geometry to show that if M, $\dim_{\mathbb{R}} M \geq 4$, is a torus (respectively, a hyper-Kähler manifold of a certain type) and ω is a Kähler (respectively, a hyper-Kähler) form on M, then I, appearing with ω in a Kähler (respectively, a hyper-Kähler) structure, on M, can indeed be approximated in a smooth deformation by Campana simple complex structures.

In this paper we also study symplectic packings of the tori and certain hyper-Kähler manifolds by arbitrary shapes (with the vanishing second real cohomology). For such a manifold M we prove the following theorem. Let ω_1, ω_2 be two Kähler (respectively, hyper-Kähler) forms on M whose cohomology classes

are not proportional to rational ones. Assume that $\int_M \omega_1^n = \int_M \omega_2^n > 0$, where $2n = \dim_\mathbb{R} M$. In the hyper-Kähler case assume also that ω_1 and ω_2 lie in the same deformation class of hyper-Kähler forms. Then the maximal fraction of the total volume that can be filled by packing copies of a given shape of an arbitrary relative size into the symplectic manifolds (M, ω_1), (M, ω_2) is the same. The proof is based on the ideas from [44, 45]. Its key step, involving an application of Ratner's orbit closure theorem, is to show the ergodicity of the action of the group of orientation-preserving diffeomorphisms of M on the space of Kähler forms on M with a fixed positive total volume.

Combining the latter result with the fact that the symplectic packings of tori by balls are unobstructed, we show that T^{2n}, equipped with any Kähler form ω whose cohomology class is not proportional to a rational one, admits a full symplectic packing by any number of equal $2n$-dimensional polydisks (that is, $2n$-dimensional direct products of arbitrary symplectic balls), and, in particular, by any number of equal $2n$-dimensional cubes.

Let us now recall a few preliminaries and present exact statements of our results.

2. Preliminaries

Symplectic and complex structures. We will view complex structures as tensors, that is, as integrable almost complex structures.

We say that an almost complex structure J and a differential 2-form ω on a smooth manifold M are *compatible* with each other if $\omega(\cdot, J\cdot)$ is a J-invariant Riemannian metric on M.

Any closed differential 2-form compatible with an almost complex structure is automatically symplectic.[a]

The compatibility between a *complex* structure J and a symplectic form ω means exactly that $\omega(\cdot, J\cdot) + i\omega(\cdot, \cdot)$ is a Kähler metric on M.

We will call a symplectic form *Kähler*, if it is compatible with *some* complex structure.

We will say that a complex structure is of *Kähler type* if it is compatible with *some* symplectic form.

Symplectic forms on tori. Consider a torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n} = T^{2n}$ be the natural projection.

A differential form (respectively, a complex structure) on $\mathbb{R}^{2n}$ is called *linear* if it has constant coefficients with respect to the standard coordinates on $\mathbb{R}^{2n}$, or, in other words, if it defines a linear exterior form (respectively, a linear complex structure) on the vector space $\mathbb{R}^{2n}$. A linear differential form (respectively, a linear complex structure) on $\mathbb{R}^{2n}$ descends under π to a differential form (respectively, a

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[a]Every symplectic form admits compatible almost complex structures [19] but does not necessarily admit compatible complex structures. At the same time an almost complex structure may not be compatible with any symplectic form.

complex structure) on T^{2n}. We will call a differential form (respectively, a complex structure) on T^{2n} *linear* if it can be obtained in this way.

Any linear symplectic form on T^{2n} is compatible with a linear complex structure (and thus is Kähler). Likewise, any linear complex structure on T^{2n} is compatible with a linear symplectic form (and thus is a complex structure of Kähler type), since the same holds for linear symplectic forms and linear complex structures on $\mathbb{R}^{2n}$. In fact, any Kähler form on T^{2n} can be mapped by a symplectomorphism to a linear symplectic form — see Proposition 6.1.

Hyper-Kähler manifolds. There are several equivalent definitions of a hyper-Kähler manifold. Since we study hyper-Kähler manifolds from the symplectic viewpoint, here is a definition which is close in spirit to symplectic geometry: A *hyper-Kähler manifold* is a manifold equipped with three complex structures I_1, I_2, I_3 satisfying the quaternionic relations and three symplectic forms $\omega_1, \omega_2, \omega_3$ compatible, respectively, with I_1, I_2, I_3, so that the three Riemannian metrics $\omega_i(\cdot, I_i \cdot)$, $i = 1, 2, 3$, coincide. Such a collection of complex structures and symplectic forms on a manifold is called a *hyper-Kähler structure* and will be denoted by $\mathfrak{h} = \{I_1, I_2, I_3, \omega_1, \omega_2, \omega_3\}$.

We will say that a symplectic form is *hyper-Kähler* and a complex structure is of *hyper-Kähler type*, if each of them appears in *some* hyper-Kähler structure. In particular, any hyper-Kähler symplectic form is Kähler and any complex structure of hyper-Kähler type is also of Kähler type.

We say that two hyper-Kähler forms are *hyper-Kähler deformation equivalent* if they can be connected by a smooth path of hyper-Kähler forms.

The real dimension of a manifold admitting a hyper-Kähler structure has to be divisible by 4 — this follows readily from the fact that the complex structures I_1, I_2, I_3 appearing in a hyper-Kähler structure on M induce an action of the quaternions on TM.

Here is the complete list of currently known closed manifolds admitting a hyper-Kähler structure: T^{4n}, K3-surfaces, the deformations of Hilbert schemes of points of K3-surfaces, deformations of generalized Kummer manifolds,[b] two more "sporadic" manifolds due to O'Grady [31, 32], and finite quotients of direct products of the examples above.

It follows from the famous Calabi–Yau theorem [6, 48] that any Kähler form on a hyper-Kähler manifold is cohomologous to a unique hyper-Kähler form [21, Theorem 23.5]. It was conjectured that any symplectic form on a hyper-Kähler manifold is hyper-Kähler but even for K3-surfaces this is unknown [11].

A hyper-Kähler manifold $(M, \mathfrak{h})$ is called *irreducible holomorphically symplectic* (*IHS*) if $\pi_1(M) = 0$ and $\dim_{\mathbb{C}} H_I^{2,0}(M, \mathbb{C}) = 1$, where I is any of the three complex structures appearing in $\mathfrak{h}$ and $H_I^{2,0}(M, \mathbb{C})$ is the $(2,0)$-part in the Hodge decomposition of $H^2(M, \mathbb{C})$ defined by I (see Sec. 5.1; for all three complex structures

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[b]A generalized Kummer manifold is a fiber of the Albanese map from Hilbert scheme of a torus T^4 to T^4.

in $\mathfrak{h}$ the space $H^{2,0} = (M, \mathbb{C})$ has the same dimension). K3-surfaces, as well as the Hilbert schemes of points for T^4 and for K3-surfaces, are IHS. Any closed hyper-Kähler manifold admits a finite covering which is the product of a torus and several IHS hyper-Kähler manifolds [4]. The IHS hyper-Kähler manifolds are also called *hyper-Kähler manifolds of maximal holonomy*, because the holonomy group of a hyper-Kähler manifold is $\mathrm{Sp}(n)$ (the group of invertible quaternionic $n \times n$-matrices) — and not its proper subgroup — if and only if it is IHS [3].

3. Main Results

3.1. *Unobstructed symplectic packings by balls*

By Vol we will always denote the symplectic volume of a symplectic manifold.

Let (M, ω), $\dim_{\mathbb{R}} M = 2n$, be a closed connected symplectic manifold. We say that the symplectic packings of (M, ω) by balls are *unobstructed*, if any finite collection of pairwise disjoint closed round balls in the standard symplectic $\mathbb{R}^{2n}$ of total volume less than $\mathrm{Vol}(M, \omega)$ has an open neighborhood that can be symplectically embedded into (M, ω).

Theorem 3.1. *Let M be a torus T^{2n} with a Kähler form ω, or an IHS hyper-Kähler manifold with a hyper-Kähler symplectic form ω. Then the symplectic packings of (M, ω) by balls are unobstructed.*

This extends a previous result of Latschev–McDuff–Schlenk [26]. For the proof see Sec. 4.2.

3.2. *Symplectic packing by arbitrary shapes*

Let (U, η), $\dim_{\mathbb{R}} U = 2n$, be an open, possibly disconnected, symplectic manifold, and let $V \subset U$, $\dim_{\mathbb{R}} V = 2n$, be a compact, possibly disconnected, submanifold of U with piecewise smooth boundary.

By a *symplectic embedding of (V, η) in (M, ω)* we mean a symplectic embedding of an open neighborhood of V in (U, η) to (M, ω).

Set

$$\nu(M, \omega, V) := \frac{\sup_\alpha \mathrm{Vol}(V, \alpha\eta)}{\mathrm{Vol}(M, \omega)},$$

where the supremum is taken over all α such that $(V, \alpha\eta)$ admits a symplectic embedding into (M, ω). If there is no such α, set $\nu(M, \omega, V) := -\infty$.

We say that (M, ω) *can be fully packed by k equal copies of (V, η)* if

$$\nu(M, \omega, W) = 1,$$

where W is a disjoint union of k equal copies of (V, η), or, in other words, if a disjoint union of k copies of $(V, \alpha\eta)$, $\alpha > 0$, admits a symplectic embedding into (M, ω) if and only if $k \, \mathrm{Vol}(V, \alpha\eta) < \mathrm{Vol}(M, \omega)$.

Theorem 3.2. *With $V \subset (U, \eta)$ as above, assume that $H^2(V, \mathbb{R}) = 0$. Let M, $\dim_{\mathbb{R}} M =: 2n \geq 4$, be either an oriented torus T^{2n} or, respectively, a closed connected oriented manifold admitting IHS hyper-Kähler structures (compatible with the orientation). Let ω_1, ω_2 be either Kähler forms on T^{2n} or, respectively, hyper-Kähler forms on M. Assume that $\int_M \omega_1^n = \int_M \omega_2^n > 0$ and that the cohomology classes $[\omega_1]$, $[\omega_2]$ are not proportional to rational ones. In the hyper-Kähler case assume also that ω_1, ω_2 are hyper-Kähler deformation equivalent.*

Then $\nu(M, \omega_1, V) = \nu(M, \omega_2, V)$.

Equivalently, the claim of the theorem can be stated as follows: (V, η) can be symplectically embedded in (M, ω_1) if and only if it can be symplectically embedded in (M, ω_2).

The theorem also immediately implies that (M, ω_1) can be fully packed by k equal copies of (V, η) if and only if so can (M, ω_2).

For the proof of the theorem see Sec. 9.

The strategy of the proof is as follows. Without loss of generality, we may assume that $\mathrm{Vol}(M, \omega) = 1$. The group Diff^+ of orientation-preserving diffeomorphisms of M acts on the space $\mathcal{F}$ of Kähler (respectively, in the hyper-Kähler case, hyper-Kähler) forms on M of total volume 1. The function $\omega \mapsto \nu(M, \omega, V)$ is clearly invariant under the action. We will show that this function is lower semicontinuous (with respect to the C^∞-topology on $\mathcal{F}$) and that the orbit of ω under action of Diff^+ is dense in $\mathcal{F}$ for $M = T^{2n}$ (respectively, in a connected component $\mathcal{F}^0$ of $\mathcal{F}$ containing ω in the hyper-Kähler case) as long as the cohomology class $[\omega]$ is not proportional to a rational one. Then, since the orbits of both ω_1 and ω_2 are dense in $\mathcal{F}$ (respectively, in $\mathcal{F}^0$), we get, by the lower semicontinuity, that $\nu(M, \omega_1, V) \leq \nu(M, \omega_2, V)$ and $\nu(M, \omega_1, V) \geq \nu(M, \omega_2, V)$, which means that $\nu(M, \omega_1, V) = \nu(M, \omega_2, V)$.

Clearly, ν can be replaced in the theorem by any symplectic invariant of a symplectic manifold which depends (lower or upper) semicontinuously on the symplectic form — for instance, by any symplectic capacity defined by means of symplectic embeddings (like the Gromov width). *It would be interesting to find other kinds of semicontinuous symplectic invariants.*

As an application of Theorem 3.2, consider the case where V is the union of k disjoint translated copies of a polydisk

$$B^{2n_1}(R_1) \times \cdots \times B^{2n_l}(R_l) \subset \mathbb{R}^{2n}$$

for some $n_1 + \cdots + n_l = n$ and $R_1, \ldots, R_l > 0$. (Here $B^{2n_i}(R_i)$, $R_i > 0$, is a closed round ball in $\mathbb{R}^{2n_i}$ equipped with the standard symplectic form Ω_{2n_i}, $i = 1, \ldots, l$, and $\Omega_{2n_1} \oplus \cdots \oplus \Omega_{2n_l} = dp \wedge dq$. Accordingly U can be taken to be the standard symplectic $\mathbb{R}^{2n}$). Theorem 3.2 allows to prove the following corollary (the proof also uses Theorem 3.1).

Corollary 3.3. *Let ω be a Kähler form on T^{2n}, $n \geq 2$, and assume that the cohomology class $[\omega]$ is not proportional to a rational one. Then for any $k \in \mathbb{N}$ the*

symplectic manifold (T^{2n}, ω) *can be fully packed by* k *equal polydisks* $B^{2n_1}(R_1) \times \cdots \times B^{2n_l}(R_l)$ *(for any* $n_1, \ldots, n_l$, $n_1 + \cdots + n_l = n$, *and any* $R_1, \ldots, R_l > 0$*).*

For the proof see Sec. 9.

Question 3.4. Does Corollary 3.3 hold for all Kähler forms on T^{2n}?

Remark 3.5. In dimension 2, volume (that is, symplectic area) is the only constraint for symplectic packing of any closed 2-dimensional symplectic manifold by any shapes. This easily follows from Moser's theorem [30] stating that two symplectic forms on a closed surface are symplectomorphic if and only if their integrals over the surface are equal. (This explains why we left out the 2-dimensional case in Theorem 3.2 and Corollary 3.3.)

By the above-mentioned Moser's theorem, the closed disk $B^2(1/\sqrt{\pi}) \subset (\mathbb{R}^2, \Omega_2)$ and the square $[0,1]^2 \subset (\mathbb{R}^2, \Omega_2)$ (that both have area 1) have arbitrarily small symplectomorphic open neighborhoods and therefore so do the $2n$-dimensional polydisk $\mathcal{P}oly := B^2(1/\sqrt{\pi}) \times \cdots \times B^2(1/\sqrt{\pi}) \subset (\mathbb{R}^{2n}, dp \wedge dq)$ and the $2n$-dimensional cube $\mathcal{C}ube := [0,1]^2 \subset (\mathbb{R}^{2n}, dp \wedge dq)$. Therefore

$$\nu(M, \omega, \mathcal{P}oly) = \nu(M, \omega, \mathcal{C}ube)$$

for all (M, ω). Thus, Corollary 3.3 yields the following corollary.

Corollary 3.6. *Let* ω *be a Kähler form on* T^{2n}, $n \geq 2$, *and assume that the cohomology class* $[\omega]$ *is not proportional to a rational one. Then for any* $k \in \mathbb{N}$ *the symplectic manifold* (T^{2n}, ω) *can be fully packed by* k *equal cubes.* $\qquad\square$

4. Campana Simple Kähler Manifolds

Definition 4.1. A complex structure on a (closed, connected) manifold M, $\dim_{\mathbb{C}} M > 1$, is called *Campana simple*, if the union $\mathfrak{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} M$ has measure[c] zero.

The points of $M \setminus \mathfrak{U}$ are called *Campana-generic*.

If a complex structure J on M is Campana simple, we will also say that the complex manifold (M, J) is *Campana simple*.

Remark 4.2. If J is a complex structure of Kähler type on a closed connected manifold M, then the union $\mathfrak{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim_{\mathbb{C}} Z < \dim_{\mathbb{C}} M$ either has measure zero or is the whole M.

Indeed, the proper positive-dimensional complex subvarieties of (M, J) are parametrized by points of the so-called Douady space $\mathcal{D}$ of (M, J) which is itself a complex variety, and moreover, there exists a complex variety $\mathcal{X}$ and proper

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[c]The measure is defined by means of a volume form on M. One can easily see that if a set is of measure zero with respect to some volume form, then it is of measure zero with respect to any other volume form.

holomorphic projections $\pi_1 : \mathcal{X} \to \mathcal{D}$ and $\pi_2 : \mathcal{X} \to M$ so that for each $a \in \mathcal{D}$ the set $\pi_2(\pi_1^{-1}(x))$ is exactly the complex subvariety of M parametrized by a [12]. By a theorem of Fujiki [15], each connected component $\mathcal{D}'$ of $\mathcal{D}$ is compact. Thus $\pi_1^{-1}(\mathcal{D}') \subset \mathcal{X}$ is a compact complex variety. By Remmert's proper holomorphic mapping theorem [36], since π_2 is holomorphic and proper, $\pi_2(\pi_1^{-1}(\mathcal{D}'))$ is a complex subvariety of (M, J). Note that $\pi_2(\pi_1^{-1}(\mathcal{D}'))$ is the union of all the complex subvarieties of (M, J) parametrized by the points of $\mathcal{D}'$. By another theorem of Fujiki [16], the space $\mathcal{D}$ can have at most countably many connected components. Thus, $\mathfrak{U}$ is a union of at most countably many complex subvarieties of (M, J). If all of these subvarieties are proper, $\mathfrak{U}$ has measure 0, otherwise $\mathfrak{U} = M$.

Remark 4.3. Campana simple manifolds are non-algebraic. Indeed, a manifold which admits a globally defined meromorphic function f is a union of zero divisors of the functions $f - a$, for all $a \in \mathbb{C}$, and the zero divisor of f^{-1}. Hence, unlike algebraic manifolds, Campana simple manifolds admit no globally defined meromorphic functions.

The following conjecture is due to F. Campana.

Conjecture 4.4. ([7, Question 1.4; 8, Conjecture 1.1]) *Let (M, J) be a Campana simple Kähler manifold. Then (M, J) is bimeromorphic to a hyper-Kähler orbifold or a finite quotient of a torus.*

4.1. *Deformations of complex structures*

Let M, $\dim_{\mathbb{R}} M = 2n$, be a closed connected manifold and let J be a complex structure on M.

Assume $\mathcal{X}$, $\mathcal{B}$ are connected — not necessarily Hausdorff! — complex manifolds, $t_0 \in \mathcal{B}$ is a marked point and $\mathcal{X} \to \mathcal{B}$ is a proper holomorphic submersion whose fiber over $t_0 \in \mathcal{B}$ is M. By Ehresmann's lemma, the fibration $\mathcal{X} \to \mathcal{B}$ admits *smooth local trivializations*. The fibers of $\mathcal{X} \to \mathcal{B}$ are closed complex submanifolds of $\mathcal{X}$ that are diffeomorphic to M. Denote by M_t the fiber over $t \in \mathcal{B}$ and denote by J_t the complex structure on M_t induced by the complex structure on $\mathcal{X}$. Assume that $J_{t_0} = J$. In such a case we say that $\mathcal{X} \to (\mathcal{B}, t_0)$ is *a smooth deformation of* (M, J).

A smooth local trivialization of $\mathcal{X} \to \mathcal{B}$ over a small neighborhood of t_0 in $\mathcal{B}$, identified with a ball $B^{2m} \subset \mathbb{C}^m$ (so that t_0 is identified with $0 \in B^{2m}$), allows to view all J_t, $t \in B^{2m}$, as complex structures on M. Such a family $\{J_t\}$, $t \in B^{2m}$, $J_0 = J$, will be called *a smooth local deformation of* J. (Note that we work only with deformations with a smooth base).

We say that J *can be approximated by Campana-simple complex structures in a smooth deformation* if there exists a smooth local deformation $\{J_t\}$, $t \in B^{2m}$, $J_0 = J$, of J and a sequence $\{t_i\} \to 0$ in B^{2m} so that each J_{t_i} is Campana simple.

4.2. *Proof of Theorem 3.1*

The proof of Theorem 3.1 is based on the following two claims.

Theorem 4.5. (*A*) *Any complex structure of Kähler type on T^{2n} can be approximated in a smooth deformation by complex structures J such that (T^{2n}, J) does not admit any proper complex subvarieties of positive dimension and, in particular, is Campana simple.*
(*B*) *Let $(M, \mathfrak{h})$, $\dim_{\mathbb{R}} M \geq 4$, be a closed connected IHS hyper-Kähler manifold and let I be a complex structure appearing in $\mathfrak{h}$. Then I can be approximated in a smooth deformation by Campana simple complex structures.*

For the proofs of (A) and (B) see Secs. 6 and 7.

Theorem 4.6. *Let (M, I, ω) be a Kähler manifold and assume that I can be approximated in a smooth deformation by Campana simple complex structures. Then the symplectic packings of (M, ω) by balls are unobstructed.*

Below we give a sketch of the proof — for the complete proof (in fact, of a somewhat stronger result) see Sec. 8 and, in particular, Theorem 8.7.

Proof of Theorem 3.1. In the case $M = T^2$ the theorem is obvious — see Remark 3.5. In all the other cases the proof follows right away from Theorems 4.5 and 4.6. $\square$

4.3. *A sketch of the proof of Theorem 4.6*

The basic notions used in this sketch will be recalled in further sections.

Let (M, ω) be as in Theorem 4.6. Assume we want to show that the symplectic packings of (M, ω) by k balls are unobstructed. Let $\widetilde{M}$ be a complex blow-up of M at k Campana-generic points $x_1, \ldots, x_k$. More precisely, we think of $\widetilde{M}$ as a fixed smooth manifold — the connected sum of M with k copies of $\overline{\mathbb{C}P^n}$ — so that each complex structure I on M induces a complex structure $\widetilde{I}$ on $\widetilde{M}$ and a projection $\Pi_I : \widetilde{M} \to M$ whose fibers over $x_1, \ldots, x_k$ are the exceptional divisors $E_1(I), \ldots, E_k(I)$ that are complex submanifolds of $(\widetilde{M}, \widetilde{I})$. The exceptional divisors vary as the complex structure I varies but their homology classes remain the same. Denote by $[E_1], \ldots, [E_k] \in H^2(\widetilde{M}, \mathbb{Z})$ the corresponding Poincaré-dual cohomology classes. Similarly, the projection Π_I varies with I but the induced map $H^*(M, \mathbb{C}) \to H^*(\widetilde{M}, \mathbb{C})$, that will be denoted by Π^*, remains the same.

We are going to use the following theorem of McDuff and Polterovich.

Theorem 4.7. (McDuff–Polterovich, [27, Proposition 2.1.B and Corollary 2.1.D]) *Let M, $\dim_{\mathbb{R}} M = 2n$, be a closed connected manifold equipped with a Kähler form ω. Let $r_1, \ldots, r_k$ be a collection of positive numbers. Assume there exists a complex structure I of Kähler type on M tamed by ω and a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ taming $\widetilde{I}$ so that $[\widetilde{\omega}] = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$. Then (M, ω) admits a symplectic embedding of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$.*

For sufficiently small $r_1, \ldots, r_k > 0$ the cohomology class $[\widetilde{\omega}] = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$ is Kähler.

We are also going to use the Demailly–Paun theorem [10] that says the following: Let N be a closed connected Kähler manifold and let $\hat{K}(N)$ be the subset of $H^{1,1}(N, \mathbb{R})$ consisting of all classes η such that $\int_Z \eta^{\dim Z} > 0$ for any closed complex subvariety $Z \subset N$. The Demailly–Paun theorem says that the Kähler cone of N is one of the connected components of $\hat{K}(N)$.

At the first stage of the proof assume that (M, I, ω) is a Kähler manifold and I is Campana simple. There are three kinds of irreducible complex subvarieties of $\widetilde{M}$:

(a) The preimages under Π_I of proper complex subvarieties $Z \subset M$ not containing the points x_i, $i = 1, \ldots, k$.
(b) The subvarieties of $E_i(I)$, $i = 1, \ldots, k$.
(c) The manifold $\widetilde{M}$ itself.

Any real $(1,1)$-cohomology class η on $\widetilde{M}$ can be written as

$$\eta = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$$

for some $r_1, \ldots, r_k > 0$. To see when η lies in the Demailly–Paun set $\hat{K}(\widetilde{M})$ we consider separately the three types of subvarieties defined above and using the Demailly–Paun theorem deduce (see Sec. 8 for the precise argument) that the Kähler cone of $\widetilde{M}$ is the set of all $\eta := \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$ that satisfy the following conditions:

(A) $r_i > 0$ for all $i = 1, \ldots, k$,

and

(B) $\int_M \eta^n = \int_M [\omega]^n - \pi^n \sum_{i=1}^{k} r_i^{2n} > 0$.

Combining Theorem 4.7 with this description of the Kähler cone of $\widetilde{M}$ we immediately obtain that the symplectic packings of (M, ω) by balls are unobstructed.

Till this point the argument has been similar to the one in [26]. Let us now consider the general case when I is only a limit of Campana simple complex structures in a smooth deformation — this part of the argument is new, compared to [26], and yields a stronger result already when $\dim M = 4$: for instance, it implies that for a product symplectic form $\omega = \omega' \oplus \omega'$ on $T^4 = T^2 \times T^2$ the symplectic packings of (T^4, ω) by *any* number of balls are unobstructed (in [26] this was proved only for the packings by one ball).

Namely, assume we want to embed $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$, $\mathrm{Vol}(\bigsqcup_{i=1}^{k} B^{2n}(r_i)) < \mathrm{Vol}(M, \omega)$, symplectically into (M, ω). Let J be a Campana simple complex structure of Kähler type close to I in a smooth deformation (in particular, we may

assume that ω tames J) and let $\widetilde{J}$ be the corresponding complex structure $\widetilde{J}$ on $\widetilde{M}$. We use the Kodaira–Spencer stability theorem [24] to show that the (1,1)-part $[\omega]_J^{1,1}$ of the cohomology class $[\omega]$ with respect to J can be represented by a Kähler form ω' compatible with J and close to ω so that $\mathrm{Vol}(\bigsqcup_{i=1}^k B^{2n}(r_i)) < \mathrm{Vol}(M, \omega')$ (see Sec. 5.3 for a rigorous proof). Since the symplectic packings by balls in the Campana simple case are unobstructed, together with Theorem 4.7, this yields that the class $\alpha := \Pi^*[\omega]_J^{1,1} - \pi \sum_{i=1}^k r_i^2 [E_i] \in H^2(\widetilde{M}, \mathbb{R})$ is Kähler. The next step is crucial: we observe that the cohomology class $\eta = \Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i]$ can be written as $\eta = \alpha + \beta$, where $\beta \in H^2(\widetilde{M}, \mathbb{R})$ is a (2,0)+(0,2) class *with respect to $\widetilde{J}$* (see Sec. 8 for the precise argument). This implies that η is a symplectic class. Moreover, it is not hard to show (see Sec. 8 for details) that η can be represented by a symplectic form taming $\widetilde{J}$. Applying Theorem 4.7 we obtain that (M, ω) admits a symplectic embedding of $\bigsqcup_{i=1}^k B^{2n}(r_i)$, as required.

5. Background from Complex Geometry

5.1. *Hodge decomposition*

We recall a few basic facts about Hodge structures — for details see e.g. [46].

Let M, $\dim_\mathbb{R} M = 2n$, be a closed connected manifold admitting Kähler structures.

An almost complex structure J on M acts on the space of $\mathbb{R}$-linear $\mathbb{C}$-valued differential forms on M (the action is induced by the pointwise action of J on the tangent spaces of M). This action induces a well-known (p, q)-decomposition of the space of such forms. The (p, q)-component of a differential form ω with respect to this decomposition will be denoted by $\omega_J^{p,q}$; sometimes, in order to emphasize the dependence of the decomposition on J, we will also say that a form is a (p, q)-*form with respect to J*. The action of J on a (p, q)-form is simply the multiplication of the form by i^{p-q}. In particular, all the (p, p)-forms are preserved by the action.

For the remainder of this section let J be a complex structure of Kähler type on M. Then, by the famous theorem of Hodge, saying that any (p, q)-form is cohomologous to a unique harmonic (p, q)-form, the (p, q)-decomposition on the space of differential forms induces a (p, q)-decomposition on $H^*(M, \mathbb{C})$, called *the Hodge decomposition*. Although the proof of the existence of the Hodge decomposition involves the whole Kähler structure of which J is a part, one can show (see e.g. [46, Vol. 1, Proposition 6.11]) that, in fact, the Hodge decomposition depends only on the complex structure J. We will denote it by

$$H^*(M, \mathbb{C}) = \bigoplus_{p,q} H_J^{p,q}(M, \mathbb{C}).$$

We will also say that a (nonzero) cohomology class in $H_J^{p,q}(M, \mathbb{C})$ is a (p, q)-*class* (*with respect to J*).

Set

$$H_J^{p,q}(M, \mathbb{Q}) := H_J^{p,q}(M, \mathbb{C}) \cap H^{p+q}(M, \mathbb{Q}),$$

$$H_J^{p,q}(M, \mathbb{R}) := H_J^{p,q}(M, \mathbb{C}) \cap H^{p+q}(M, \mathbb{R}).$$

The following proposition is also well-known — see e.g. [46, Vol. 1, Sec. 11].

Proposition 5.1. *Let J be a complex structure of Kähler type on a closed connected manifold M, $\dim_{\mathbb{R}} M = 2n$. If $L \subset (M, J)$, $\dim_{\mathbb{R}} L = 2l$, is a closed complex subvariety, then L represents a well-defined fundamental integral homology class of degree $2l$; denote its Poincaré-dual cohomology class by $[L] \in H^{2n-2l}(M, \mathbb{Q})$. Then $[L] \in H_J^{n-l,n-l}(M, \mathbb{Q})$ and $[L] \neq 0$.* $\square$

Let $\mathcal{X} \to (\mathcal{B}, t_0)$ be a smooth deformation of (M, J) over a connected base $\mathcal{B}$ which is trivial as a smooth fibration. Assume that for each $t \in \mathcal{B}$ the complex structure J_t on the fiber M_t of $\mathcal{X} \to (\mathcal{B}, t_0)$ is of Kähler type. Note that there is a canonical (that is, independent of trivializations) identification of the homology/cohomology of each fiber M_t with the homology/cohomology of M. Thus for each $t \in \mathcal{B}$ we have a well-defined Hodge decomposition of $H^*(M, \mathbb{C})$ defined by J_t.

Definition 5.2. Let $a \in H^{2p}(M, \mathbb{Q})$, $a \neq 0$. The *Hodge locus of a for the smooth local deformation* $\mathcal{X} \to (\mathcal{B}, t_0)$ is the set of $t \in \mathcal{B}$ such that $a \in H_{J_t}^{p,p}(M, \mathbb{Q})$.

The following proposition can be found e.g. in [46, Vol. 2, Lem. 5.13].

Proposition 5.3. *The Hodge locus of $a \in H^{2p}(M, \mathbb{Q})$, $a \neq 0$, is a complex subvariety of $\mathcal{B}$.* $\square$

5.2. *Complex structures tamed by symplectic forms*

Definition 5.4. We say that a differential 2-form ω *tames* an almost complex structure J if $\omega(v, Jv) > 0$ for any non-zero tangent vector v.

Clearly, a closed differential 2-form ω taming an almost complex structure J is automatically symplectic.

Proposition 5.5. (A) *A differential 2-form ω tames an almost complex structure J if and only if so does $\omega_J^{1,1}$.*

(B) *A differential 2-form ω is compatible with an almost complex structure J if and only if ω tames J and $\omega = \omega_J^{1,1}$.*

(C) *Assume ω is a symplectic form on a manifold M taming an almost complex structure J. Let η be a closed (real-valued) $(2,0)+(0,2)$-form with respect to J. Then $\omega + \eta$ is also a symplectic form.*

Proof. The $(2,0) + (0,2)$-forms (with respect to J) are exactly the 2-forms anti-invariant under the action of J, while the $(1,1)$-forms are exactly the invariant ones. In particular, if η is a $(2,0) + (0,2)$-form, then $\eta(v, Jv) = 0$ for all v. The claims (A), (B) and (C) now follow easily. $\square$

5.3. *Symplectic and Kähler cones*

Let M be a closed connected smooth manifold.

A cohomology class $a \in H^2(M, \mathbb{R})$ is called *symplectic* if it can be represented by a symplectic form. The set of all symplectic classes $a \in H^2(M, \mathbb{R})$ is called *the symplectic cone* of M.

Assume J is a complex structure on M. A cohomology class $a \in H^2(M, \mathbb{R})$ is called *Kähler*, or *Kähler with respect to J*, if it can be represented by a Kähler form compatible with J. The set of all Kähler classes in $H^2(M, \mathbb{R})$ is a convex cone $\mathsf{Kah}(M, J)$, called the *Kähler cone* of (M, J). Clearly, it is a subset of the symplectic cone of M.

Theorem 5.6. (A version of Kodaira–Spencer stability theorem) *Let (M, J, ω) be a closed Kähler manifold, and let $\{J_t\}$, $t \in B$, $J_0 = J$, be a smooth local deformation of J. Then there exists a neighborhood of $U \subset B$ of zero in B such that the complex structure J_t on M is of Kähler type and $[\omega]_{J_t}^{1,1} \in \mathsf{Kah}(M, J_t)$ for all $t \in U$. Moreover, the class $[\omega]_{J_t}^{1,1}$, $t \in U$, depends smoothly on t.*

Proof. The Kodaira–Spencer stability theorem [24] states that there exists an open neighborhood $U \subset B$ of zero such that J_t is of Kähler type for any $t \in U$ and ω can be extended to a smooth family $\{\omega_t\}$, $t \in U$, $\omega_0 = \omega$, of Kähler forms on the complex manifolds (M, J_t).

For $t \in U$ denote by Ω_t the $(1,1)$-component of the unique harmonic 2-form on the closed Kähler manifold (M, J_t, ω_t) representing the cohomology class $[\omega] \in H^2(M, \mathbb{R}) \subset H^2(M, \mathbb{C})$. Then Ω_t is a real closed 2-form on M of type $(1,1)$ (with respect to J_t) and $[\Omega_t] = [\omega]_{J_t}^{1,1}$ for all $t \in U$.

Note that Ω_0 and ω are cohomologous closed real-valued $(1,1)$-forms on (M, J). Therefore, by the $\partial\overline{\partial}$-lemma (see e.g. [46, Vol. 1, Proposition 6.17]), $\omega = \Omega_0 + \partial_J \overline{\partial}_J F$ for a smooth function $F : M \to \mathbb{R}$.

Consider now the forms $\alpha_t := \Omega_t + \partial_{J_t} \overline{\partial}_{J_t} F$, $t \in U$. The form α_t is a closed $(1,1)$-form with respect to J_t which is cohomologous to Ω_t. Thus $[\alpha_t] = [\Omega_t] = [\omega]_{J_t}^{1,1}$.

Since Ω_t depends smoothly on t, so does α_t and therefore so does its cohomology class $[\alpha_t]$.

Since the condition on a complex structure to be tamed by a symplectic form is open and since $\omega = \alpha_0$ tames $J = J_0$, we can assume without loss of generality that U is sufficiently small so that the form α_t tames J_t for all $t \in U$. By Proposition 5.5, part B, this means that α_t is Kähler on (M, J_t). Hence, $[\alpha_t] = [\omega]_{J_t}^{1,1}$ lies in $\mathsf{Kah}(M, J_t)$ and depends smoothly on t. $\qquad\square$

6. Campana Simple Complex Structures on a Torus

Let $M = T^{2n}$, $n \geq 2$. As above, let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n} = T^{2n}$ be the standard projection.

Proposition 6.1. *Any complex structure of Kähler type on T^{2n} is biholomorphic to a linear complex structure. Any Kähler form on T^{2n} is symplectomorphic to a linear symplectic form.*

Proof. Let (J, ω) be a Kähler structure on T^{2n}. The Albanese map (see e.g. [46, Vol. 1, Definition 12.10 and Theorem 12.15] defines a biholomorphism $f : (T^{2n}, J) \to (\mathbb{C}^n/\Gamma, I)$, where I is the standard complex structure on $\mathbb{C}^n/\Gamma$ and $\Gamma \subset \mathbb{C}^n$ is a lattice.

Denote by $\pi_\Gamma : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ the natural projection and let $F : \mathbb{R}^{2n} \to \mathbb{C}^n$ be an $\mathbb{R}$-linear isomorphism of vector spaces mapping $\mathbb{Z}^{2n}$ to Γ. Then F covers a diffeomorphism $\overline{f} : T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{C}^n/\Gamma$, that is, $\overline{f} \circ \pi = \pi_\Gamma \circ F$.

Clearly, $\overline{f}^* I$ is a linear complex structure on $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Therefore $\overline{f}^{-1} \circ f$ is a diffeomorphism mapping J to a linear complex structure on T^{2n}. This proves the first claim of the proposition.

Let us prove the second claim. The symplectic form $\zeta := (f^{-1})^* \omega$ on the torus $\mathbb{C}^n/\Gamma$ is compatible with I and is cohomologous to a linear 2-form η (i.e. $\pi_\Gamma^* \eta$ is a linear symplectic form on $\mathbb{C}^n$). By Proposition 5.5, $\zeta = \zeta_I^{1,1}$, i.e. ζ is a $(1,1)$-form. Since η is linear, it is harmonic, and since it is cohomologous to ζ, it is also a $(1,1)$-form. Therefore, by Proposition 5.5, η is symplectic.

Now note that ζ and η are two cohomologous symplectic forms on $\mathbb{C}^n/\Gamma$ that are compatible with the same complex structure I. Therefore ζ and η can be connected by a straight path $t\zeta + (1-t)\eta$, $t \in [0,1]$, of closed cohomologous 2-forms; all these forms tame I and hence are symplectic. By Moser's theorem [30], it implies that the symplectic forms ζ and η on $\mathbb{C}^n/\Gamma$ are symplectomorphic. Hence the symplectic forms $\omega = f^* \zeta$ and $\overline{f}^* \eta$ on $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ are also symplectomorphic. Clearly, $\overline{f}^* \eta$ is a linear symplectic form because $\pi^* \overline{f}^* \eta = (\pi_\Gamma \circ F)^* \eta$. Thus, ω is symplectomorphic to a linear symplectic form on $T^{2n} = \mathbb{R}^{2n}\mathbb{Z}^{2n}$. This finishes the proof of the proposition. $\qquad\square$

Proposition 6.2. *The space of linear orientation-preserving complex structures on $\mathbb{T}^{2n}$ (which we identify with the space of orientation-preserving linear complex structures on $\mathbb{R}^{2n}$) is a connected manifold Comp_l that can be equipped with a complex structure.*

Proof. It is well-known (see e.g. [28], Proposition 2.48) that the space of orientation-preserving linear complex structures on $\mathbb{R}^{2n}$ can be identified with

$$\mathsf{Comp}_l := \mathrm{GL}^+(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$$

which is a connected smooth manifold. Here the group $\mathrm{GL}(n, \mathbb{C})$ is embedded in $\mathrm{GL}(2n, \mathbb{R})$ by the map

$$C \mapsto \begin{pmatrix} \mathrm{Re}\ C & -\mathrm{Im}\ C \\ \mathrm{Im}\ C & \mathrm{Re}\ C \end{pmatrix}.$$

The tangent space $T_J \mathsf{Comp}_l$ to Comp_l at a point $J \in \mathsf{Comp}_l$ is given by all $R \in \mathrm{GL}^+(2n, \mathbb{R})$ satisfying $RJ + JR = 0$. The complex structure on $T_J \mathsf{Comp}_l$ is given by $R \mapsto JR$. Since Comp_l is identified with the symmetric space $\mathrm{GL}^+(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$, the almost complex structure on Comp_l is integrable [3]. $\qquad\square$

Consider the projection $T^{2n} \times \mathsf{Comp}_l \to \mathsf{Comp}_l$. Denote by I the complex structure on Comp_l. Equip $T^{2n} \times \mathsf{Comp}_l$ with an almost complex structure which is defined at a point $(x, J) \in T^{2n} \times \mathsf{Comp}_l$ as $J \oplus I$ with respect to the obvious splitting of $T_{(x,J)}(T^{2n} \times \mathsf{Comp}_l)$. One easily checks that this almost complex structure is integrable and thus $T^{2n} \times \mathsf{Comp}_l \to \mathsf{Comp}_l$ is a smooth deformation of (T^{2n}, J) for any $J \in \mathsf{Comp}_l$.

Given $A \in H^{2p}(T^{2n}, \mathbb{Q})$, $A \neq 0$, $0 < p < n$, define $\mathcal{T}_A \subset \mathsf{Comp}_l$ as the Hodge locus of A with respect to the smooth deformation $T^{2n} \times \mathsf{Comp}_l \to \mathsf{Comp}_l$.

Proposition 6.3. *The set $\mathcal{T}_A$ is either empty or a proper complex subvariety of (Comp_l, I).*

Proof. Given a complex structure J of Kähler type on T^{2n}, denote by $\overline{J}$ the linear complex structure on $\mathbb{R}^{2n}$ which is the lift of J under $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n} = T^{2n}$ (see Proposition 6.1).

For any cohomology class $A \in H^*(T^{2n}, \mathbb{R})$ there is a unique closed linear form ω_A on T^{2n} that represents A. Let $\overline{\omega}_A := \pi^* \omega_A$ be the corresponding linear form on $\mathbb{R}^{2n}$.

Now let $A \in H^{2p}(T^{2n}, \mathbb{R})$, $A \neq 0$, $0 < p < n$. It follows from Proposition 5.3 that $\mathcal{T}_A$ is a complex subvariety of Comp_l. Let us prove that the complex subvariety $\mathcal{T}_A$ is proper.

Indeed, assume by contradiction that it is not proper. Then, since Comp_l is connected, $\mathcal{T}_A$ has to coincide with Comp_l — in other words, $A \in H_J^{p,p}(T^{2n}, \mathbb{R})$ for any $J \in \mathsf{Comp}_l$. Hence, the linear form $\overline{\omega}_A$ is $\overline{J}$-invariant for any orientation-preserving linear complex structure $\overline{J}$ on $\mathbb{R}^{2n}$ and thus it is preserved by the subgroup $G \subset \mathrm{SL}(2n, \mathbb{R})$ generated by such $\overline{J}$ (each such $\overline{J}$ lies in $\mathrm{SL}(2n, \mathbb{R})$). Since the set of orientation-preserving linear complex structures is conjugacy-invariant in $\mathrm{SL}(2n, \mathbb{R})$, G is a normal subgroup of $\mathrm{SL}(2n, \mathbb{R})$. But $\mathrm{SL}(2n, \mathbb{R})$ is a simple Lie group and therefore its normal subgroup is either contained in its center (equal to $\{\mathrm{Id}, -\mathrm{Id}\}$) or coincides with the whole $\mathrm{SL}(2n, \mathbb{R})$ (see e.g. [34]). Clearly, the first option does not hold for G and therefore $G = \mathrm{SL}(2n, \mathbb{R})$. Therefore the exterior form $\overline{\omega}_A$ is $\mathrm{SL}(2n, \mathbb{R})$-invariant. But the only $\mathrm{SL}(2n, \mathbb{R})$-invariant exterior forms on $\mathbb{R}^{2n}$ are scalar multiples of a volume form[d] or forms of degree 0 (that is, constants),

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[d]This can be easily seen by considering the action of diagonal matrices of the form

$$\mathrm{diag}\,\{1, \ldots, 1, \lambda, 1, \ldots, 1, 1/\lambda, 1, \ldots, 1\}$$

lying in $\mathrm{SL}(2n, \mathbb{R})$ on exterior forms.

while $\overline{\varpi}_A$ is of degree $0 < 2p < 2n$. Thus we have obtained a contradiction. Hence, $\mathcal{T}_A$ is a proper subvariety of Comp_l and the proposition is proved. $\qquad\square$

The following theorem is a precise formulation of Theorem 4.5 in the case of a torus.

Theorem 6.4. *Let Ξ be the set of linear complex structures $J \in \mathsf{Comp}_l$ such that (T^{2n}, J) admits no proper complex subvarieties of positive dimension. Then Ξ is dense in Comp_l.*

In fact, as it follows from the proof below, Ξ is not only dense in Comp_l but of "full measure". Proposition 6.3 and Theorem 6.4, which is easily deduced from Proposition 6.3, seem to be well known but we have not been able to find them in the literature.

Proof of Theorem 6.4. Let $\mathcal{C} \subset \mathsf{Comp}_l$ be the set of linear complex structures J for which $H_J^{p,p}(T^{2n}, \mathbb{Q}) = 0$ for all $0 < p < n$. By Proposition 5.1, each proper positive-dimensional compact complex subvariety of a Kähler manifold carries a nonzero integral fundamental class whose Poincaré-dual cohomology class is of Hodge type (p,p) for some $0 < p < n$. Therefore $\mathcal{C} \subset \Xi$ and it is enough to show that $\mathcal{C}$ is dense in Comp_l. Note that $\mathcal{C}$ is the complement of $\bigcup_A \mathcal{T}_A$ in Comp_l, where the union is taken over all $A \in H^{2p}(T^{2n}, \mathbb{Q})$, $A \neq 0$, for all $0 < p < n$. But the latter union is a countable union of proper complex subvarieties and therefore (by Baire's theorem) its complement in Comp_l — that is, $\mathcal{C}$ — is dense. This finishes the proof. $\qquad\square$

Proof of Theorem 4.5 in the torus case. By Proposition 6.1, it suffices to prove the claim of Theorem 4.5 for an arbitrary linear complex structure J on T^{2n}. Since $T^{2n} \times \mathsf{Comp}_l \to \mathsf{Comp}_l$ yields a smooth deformation of J, the claim follows immediately from Theorem 6.4. $\qquad\square$

7. Campana Simple Complex Structures on an IHS Hyper-Kähler Manifold

First, let us recall a few relevant results about topology and deformation theory of hyper-Kähler manifolds — for more details see [21, 43], cf. [9, 13, 23].

Let $(M, \mathfrak{h})$, $\dim_{\mathbb{R}} M = 4n$, be a closed connected IHS hyper-Kähler manifold, $\mathfrak{h} = \{I_1, I_2, I_3, \omega_1, \omega_2, \omega_3\}$.

7.1. *Bogomolov–Beauville–Fujiki form*

The content of this section will be used only in Sec. 9.2. We put it here as it belongs to the general theory of hyper-Kähler manifolds.

Theorem 7.1. (Fujiki formula, [17]) *Let M be an IHS hyper-Kähler manifold as above.*

Then there exists a primitive integral quadratic form q on $H^2(M, \mathbb{R})$, and a positive rational number κ so that $\int_M \eta^{2n} = \kappa q(\eta, \eta)^n$ for each $\eta \in H^2(M, \mathbb{R})$.

Moreover, q has signature $(3, b_2 - 3)$, where $b_2 = \dim_{\mathbb{R}} H^2(M, \mathbb{R})$. In particular, for an IHS closed connected hyper-Kähler manifold $b_2 \geq 3$.

Definition 7.2. The constant κ is called *Fujiki constant*, and the quadratic form q is called *Bogomolov–Beauville–Fujiki form*. It is defined by the Fujiki formula uniquely, up to a sign.

To fix the sign, note that the complex-valued form $\Omega = \omega_2 + \sqrt{-1}\omega_3$ is a closed, non-degenerate, holomorphic $(2, 0)$-form with respect to I_1. Let $\overline{\Omega}$ be the complex conjugate 2-form with respect to I_1.

The sign of q is determined from the following formula (Bogomolov, Beauville, [2]):

$$cq(\eta, \eta) = (n/2) \int_M \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1}$$
$$- (1 - n) \left(\int_M \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_M \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right),$$

where $c > 0$ is a positive constant.

7.2. *Teichmüller space for hyper-Kähler manifolds*

Denote by Comp_h the set of complex structures of hyper-Kähler type on M. (Recall that we view complex structures as tensors on M, that is, as integrable almost complex structures).

The set Comp_h can be equipped with C^∞-topology in a standard way (see e.g. [20]) so that a sequence of complex structures converges in the topology if it converges uniformly with all derivatives.

Let $\mathrm{Diff}_0 \subset \mathrm{Diff}^+$ be the group of smooth isotopies of M. Note that Diff_0 acts on Comp_h.

Definition 7.3. The quotient topological space $\mathsf{Teich} := \mathsf{Comp}_h / \mathrm{Diff}_0$ is called *the Teichmüller space of M*. For $J \in \mathsf{Comp}_h$ we denote by $[J]$ the corresponding point in Teich.

For any $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, the tensor $aI_1 + bI_2 + cI_3$ is a complex structure of hyper-Kähler type on M. We will call any such $aI_1 + bI_2 + cI_3$ *an induced complex structure on M* or *a complex structure induced by* $\mathfrak{h}$. Clearly, all induced complex structures lie in the same connected component of Comp_h whose image under $\mathsf{Comp}_h \to \mathsf{Teich} = \mathsf{Comp}_h / \mathrm{Diff}_0$ will be denoted by Teich_0.

By [5], Teich_0 admits the structure of a smooth (possibly, non-Hausdorff) complex manifold so that the fibration $M \times \mathsf{Teich}_0 \to \mathsf{Teich}_0$, with appropriate complex structures on $M \times \mathsf{Teich}_0$ and Teich_0, defines a smooth deformation of J_0 for any J_0 such that $[J_0] \in \mathsf{Teich}_0$.

More precisely, a combination of the fundamental theorems of Kuranishi [25] and Bogomolov–Tian–Todorov [5, 39, 40] shows that the Kuranishi space of (M, J_0) (the base of a certain universal smooth local deformation of a complex structure defined in [25]) is a smooth complex manifold as long as the canonical bundle of (M, J_0) is trivial, which, of course, is true in the hyper-Kähler case. Moreover (see e.g. [9]), there exists a homeomorphism between a neighborhood U of $[J_0]$ in Teich_0 and an open subset of the Kuranishi space of (M, J_0). The homeomorphism maps each $[J] \in \mathsf{Teich}_0$ to the point of the Kuranishi space corresponding to a complex structure on M representing $[J]$. The smooth and the complex structures on U are the pullbacks of the corresponding structures on the Kuranishi space and the pullback of the universal smooth local deformation over the Kuranishi space under the homeomorphism induces a smooth deformation $M \times U \to U$.

Note that the universal deformation over the Kuranishi space is defined uniquely up to appropriate bi-holomorphic reparametrizations of the base and the total space and therefore in the construction above the complex structures $M \times \mathsf{Teich}_0$ and Teich_0 are not defined uniquely. Nevertheless, the restriction of the complex structure on $M \times \mathsf{Teich}_0$ to a fiber over $x \in \mathsf{Teich}_0$ is always a complex structure J representing $x = [J]$. Since Diff_0 acts trivially on the homology of M, different complex structures representing the same point in Teich_0 define the same Hodge structure on $H^*(M, \mathbb{C})$. Thus, the Hodge decomposition on the cohomology of a fiber of $M \times \mathsf{Teich}_0 \to \mathsf{Teich}_0$ does not depend on any of the choices.

By Proposition 5.3, the Hodge locus of $A \in H^{2p}(M, \mathbb{Q})$, $A \neq 0$, with respect to the smooth deformation $M \times \mathsf{Teich}_0 \to \mathsf{Teich}_0$ is a complex subvariety of Teich_0. Denote it by $\hat{\mathcal{T}}_A$.

7.3. *Trianalytic subvarieties*

Definition 7.4. ([41]) A closed subset $X \subset M$ of the hyper-Kähler manifold $(M, \mathfrak{h})$ is called *trianalytic* (with respect to $\mathfrak{h}$), if X is a complex subvariety of (M, I) for each complex structure I induced by $\mathfrak{h}$.

Theorem 7.5. *Let $A \in H^*(M, \mathbb{Q})$, $A \neq 0$. Then $A = [N]$ for a trianalytic $N \subset M$ if and only if A is invariant with respect to the action of any induced complex structure on $H^*(M, \mathbb{Q})$.*

Proof. This is Theorem 4.1 of [41]. $\qquad\square$

Definition 7.6. Let J be a complex structure of Kähler type on M. We say that J is *of general type* with respect to the hyper-Kähler structure $\mathfrak{h}$ on M, if all the elements of the group

$$\bigoplus_p H_J^{p,p}(M, \mathbb{Q}) \subset H^*(M, \mathbb{Q})$$

are invariant under the action of any complex structure induced by $\mathfrak{h}$.

Theorem 7.5 has the following immediate corollary:

Corollary 7.7. *Assume that a complex structure J of Kähler type on M is of general type with respect to $\mathfrak{h}$. Let $L \subset (M, J)$ be a closed complex subvariety. Then L is trianalytic with respect to $\mathfrak{h}$.* □

Deformations of trianalytic subvarieties were studied in [42] where the following theorem was proved.

Theorem 7.8. ([42, Theorems 9.1 and 1.2]) *Let $Z \subset (M, \mathfrak{h})$ be a compact trianalytic subvariety. Then all deformations of Z are isometric[e] to Z, and their union is locally isometric to a product $Z \times W$, where W is a hyper-Kähler variety.*

This yields the following corollary, used further on in this paper.

Corollary 7.9. *Assume that the hyper-Kähler manifold $(M, \mathfrak{h})$ is IHS and J is a complex structure on M which is of general type with respect to $\mathfrak{h}$. Then the union of all proper positive-dimensional complex subvarieties of (M, J) (by Corollary 7.7 all such subvarieties are trianalytic) has measure 0.*

Proof. Let $\mathfrak{J}$ be the set of all deformation classes of proper positive-dimensional subvarieties of (M, J). For each $\alpha \in \mathfrak{J}$ the union of all subvarieties belonging to α is either a proper complex analytic subvariety of M [15] or the whole M. However, by Theorem 7.8, in the latter case the hype-Kähler manifold M is (locally) a direct product of two hyper-Kähler manifolds and therefore is not IHS [3], contrary to our assumptions. Thus the former case holds and therefore the union of all subvarieties belonging to each $\alpha \in \mathfrak{J}$ is of measure zero. Since $\mathfrak{J}$ is countable [16], the union of all proper positive-dimensional complex subvarieties of (M, J) has measure 0. □

7.4. *Density of Campana simple complex structures on a hyper-Kähler manifold*

Note that the action of Diff_0 on Comp_h maps Campana simple complex structures to Campana simple complex structures.

Theorem 7.10. *Assume that the hyper-Kähler manifold M is IHS. Then the set*

$$\mathrm{Camp} := \{[J] \in \mathsf{Teich}_0 \mid J \text{ is Campana simple}\}$$

is dense in Teich_0. Consequently, the set of Campana simple complex structures of hyper-Kähler type on M is dense in Comp_h.

---

[e]It means that Z and any subvariety Z' of M obtained by a deformation of Z in the class of complex subvarieties of M are isomorphic as complex varieties and the isomorphism respects the Riemannian metrics obtained by the restriction of the Riemannian metric induced by $\mathfrak{h}$ to Z and Z'.

Proof of Theorem 7.10. Define $\hat{\mathcal{C}} \subset \mathsf{Teich}_0$ as $\hat{\mathcal{C}} := \bigcup_A \hat{\mathcal{T}}_A$, where the union is taken over all nonzero $A \in H^*(M, \mathbb{Q})$. Then $\hat{\mathcal{C}}$ is a countable union of proper complex subvarieties and therefore (by Baire's theorem) its complement in Teich_0 is dense. Let us show that $\mathsf{Teich}_0 \backslash \hat{\mathcal{C}} \subset \mathrm{Camp}$ — this would prove the theorem.

Indeed, let J be a complex structure of Kähler type on M such that $[J] \in \mathsf{Teich}_0 \backslash \hat{\mathcal{C}}$. Let us prove that J is Campana simple.

Consider the set of $A \in H^*(M, \mathbb{Q})$, $A \neq 0$, $0 < \deg A < 2n = \dim_{\mathbb{R}} M/2$, such that $J \in \hat{\mathcal{T}}_A$.

If this set is empty, then Proposition 5.1 implies that J is Campana simple.

If the set is not empty, pick an arbitrary $A \in H^{p,p}_J(M, \mathbb{Q})$, $A \neq 0$, $0 < p < 2n$. Recall that $\hat{\mathcal{T}}_A$ is a complex subvariety of Teich_0. By the choice of J, this complex subvariety is non-proper, which means that $\hat{\mathcal{T}}_A = \mathsf{Teich}_0$, since Teich_0 is connected. In other words, $A \in H^{p,p}_{J'}(M, \mathbb{Q})$ for all complex structures J' such that $[J'] \in \mathsf{Teich}_0$ and, in particular, $A \in H^{p,p}_{J'}(M, \mathbb{Q})$ for all induced complex structures J'. Since it holds for arbitrary $A \in H^{p,p}_J(M, \mathbb{Q})$, $A \neq 0$, $0 < p < 2n$, we get that J is of general type with respect to $\mathfrak{h}$. By Corollary 7.7, any complex subvariety of (M, J) is trianalytic. By 7.9, the union of all such subvarieties is of measure zero, which means that J is Campana simple. This finishes the proof. $\square$

Proof of Theorem 4.5 in the hyper-Kähler case. As we saw above, $M \times \mathsf{Teich}_0 \to \mathsf{Teich}_0$ is a smooth deformation of any complex structure appearing in $\mathfrak{h}$. Therefore the claim of Theorem 4.5 in the hyper-Kähler case follows immediately from Theorem 7.10. $\square$

8. Symplectic Packing for Campana Simple Kähler Manifolds

8.1. *Blow-ups and McDuff–Polterovich theorem*

The blow-up operation can be performed both in complex and symplectic categories.

Since we are going to compare blow-ups of the same smooth manifold with different complex structures, let us take the following point of view on the (simultaneous) complex blow-ups of a complex manifold M at k points. From this point on we fix $k \in \mathbb{N}$.

We first define a smooth manifold $\widetilde{M}$ as a connected sum of M^{2n} with k copies of $\overline{\mathbb{C}P^n}$. Any complex structure I on M defines uniquely, up to a smooth isotopy, a complex structure $\widetilde{I}$ on $\widetilde{M}$ and complex submanifolds $E_1(I), \ldots, E_k(I) \subset (\widetilde{M}, \widetilde{I})$ so that there exists a diffeomorphism between $\widetilde{M}$ and the complex blow-up $\widetilde{M}'$ of (M, I) at k points identifying $\widetilde{I}$ with the canonical complex structure on $\widetilde{M}'$ and $E_1(I), \ldots, E_k(I)$ with the k exceptional divisors in $\widetilde{M}'$. We will call $E_1(I), \ldots, E_k(I)$ *the exceptional divisors defined by I.* The canonical projection $\widetilde{M}' \to M$ is then identified with a projection $\Pi_I : \widetilde{M} \to M$. If J is another complex structure on M, we get another complex structure $\widetilde{J}$ on $\widetilde{M}$ with another projection $\Pi_J : \widetilde{M} \to M$ which is smoothly isotopic to Π_I and therefore induces the same map on cohomology which is independent of the complex structure and will be denoted by Π^*.

The exceptional divisors $E_1(J), \ldots, E_k(J)$ defined by J might be different from $E_1(I), \ldots, E_k(I)$ but lie in the same homology classes which are independent of the complex structure. We will denote the cohomology classes that are Poincaré-dual to these homology classes by $[E_1], \ldots, [E_k] \in H^2(\widetilde{M}, \mathbb{Z})$.

Now let us briefly recall the notion of a (simultaneous) symplectic blow-up at k points — for details see [27, 28]. Denote by $B^{2n}(r)$ a round closed ball of radius r in the standard symplectic $\mathbb{R}^{2n}$. Given a symplectic manifold (M^{2n}, ω) and a symplectic embedding $\iota : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$, one can construct a new manifold, diffeomorphic to $\widetilde{M}$, by removing $\iota(\bigsqcup_{i=1}^{k} B^{2n}(r_i))$ from M and contracting the boundary of the resulting manifold along the fibers of the fibration induced by ι from the Hopf fibration on the boundary of $B^{2n}(r)$. The form ω is then extended in a certain way to a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ and the resulting symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ is independent of the extension choices, up to a symplectic isotopy (though it does depend on $r_1, \ldots, r_k$ and ι), and is called *the symplectic blow-up of (M, ω) along ι*. Alternatively, one can describe the construction of $(\widetilde{M}, \widetilde{\omega})$ as follows: extend ι to a symplectic embedding of the union of a slightly larger closed balls, remove the interior of the image of the union of the larger balls from M and glue in symplectically the disjoint union of appropriate neighborhoods of $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$ with the standard symplectic form on them (normalized so that its integral over the projective line is equal to π).

The cohomology class of $\widetilde{\omega}$ is given by

$$[\widetilde{\omega}] = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i].$$

McDuff and Polterovich discovered in [27] that, in fact, the existence of a symplectic form on $\widetilde{M}$ in the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$ satisfying certain additional conditions is sufficient for the existence of a symplectic embedding $\iota : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$. We will state their result in the case when the symplectic manifold M is Kähler (since we are going to work only with such manifolds).

Theorem 8.1. (McDuff–Polterovich, [27, Corollary 2.1.D and Remark 2.1.E]) *Let M, $\dim_{\mathbb{R}} M = 2n$, be a closed connected complex manifold equipped with a Kähler form ω. Let $k \in \mathbb{N}$ and let $\widetilde{M}$, $\Pi^* : H^2(M, \mathbb{R}) \to H^2(\widetilde{M}, \mathbb{R})$ and $[E_i] \in H^2(\widetilde{M}, \mathbb{Z})$, $i = 1, \ldots, k$, be defined as above. Let $r_1, \ldots, r_k$ be a collection of positive numbers. Assume there exists a complex structure I of Kähler type on M tamed by ω and a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ taming $\widetilde{I}$ so that $[\widetilde{\omega}] = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i]$. Let $\Gamma \subset M$ be a (possibly empty) closed complex submanifold of (M, I) that does not intersect the image of $\bigcup_{i=1}^{k} E_i(I)$ under Π_I. Then $(M \backslash \Gamma, \omega)$ admits a symplectic embedding of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$.*

Remark 8.2. The proof of Theorem 8.1 in [27] actually shows that for any sufficiently small $c_1, \ldots, c_k > 0$ the cohomology class $\Pi^*[\omega] - \sum_{i=1}^{k} \pi c_i [E_i]$ is Kähler (with respect to $\widetilde{I}$). Thus $\mathsf{Kah}(\widetilde{M}, \widetilde{I})$ is non-empty for any complex structure I of

Kähler type on M or, in other words, if I is of Kähler type on M, then $\widetilde{I}$ is of Kähler type on $\widetilde{M}$.

Theorem 8.3. *Let* (M, I, ω), $\dim_{\mathbb{R}} M = 2n$, *be a closed connected Kähler manifold. Let* $k \in \mathbb{N}$ *and let* $\widetilde{M}$, $\Pi^* : H^2(M, \mathbb{R}) \to H^2(\widetilde{M}, \mathbb{R})$, $[E_1], \ldots, [E_k] \in H^2(\widetilde{M}, \mathbb{Z})$, $r_1, \ldots, r_k > 0$ *be as above. Let* $\Gamma \subset M$ *be a (possibly empty) closed complex submanifold of* (M, I) *that does not intersect the image of* $\bigcup_{i=1}^k E_i(I)$ *under* Π_I. *Assume that there exists a complex structure* J *of Kähler type on* M *which is tamed by* ω *so that* $\Pi^*[\omega]_J^{1,1} - \pi \sum_{i=1}^k r_i^2 [E_i] \in \mathsf{Kah}(\widetilde{M}, \widetilde{J})$.

Then $(M \backslash \Gamma, \omega)$ *admits a symplectic embedding of* $\bigsqcup_{i=1}^k B^{2n}(r_i)$.

Proof. First, let us remark that any complex structure on M defines a complex structure of Kähler type on $\widetilde{M}$ (see Remark 8.2) that defines a Hodge decomposition on $H^2(\widetilde{M}, \mathbb{R})$. Under the identification $H^2(\widetilde{M}, \mathbb{R}) = H^2(M, \mathbb{R}) \oplus \mathrm{Span}_{\mathbb{R}}\{[E_1], \ldots, [E_k]\}$ the homomorphism Π^* (which is independent of the complex structure on M) acts as an identification of $H^2(M, \mathbb{R})$ with the first summand. The classes $[E_1], \ldots, [E_k]$ are all of type $(1, 1)$. This identification preserves the Hodge types (with respect to the complex structures on M and $\widetilde{M}$).

Since, by our assumption, the cohomology class $\Pi^*[\omega]_J^{1,1} - \pi \sum_{i=1}^k r_i^2 [E_i]$ lies in $\mathsf{Kah}(\widetilde{M}, \widetilde{J})$, it can be represented by a Kähler form $\widetilde{\alpha}$ on $(\widetilde{M}, \widetilde{J})$.

Note that $\Pi^*[\omega]_J^{1,1} \in H^2(\widetilde{M}, \mathbb{R})$ is of type $(1, 1)$ with respect to $\widetilde{J}$. Hence, the class $\Pi^*[\omega] - \Pi^*[\omega]_J^{1,1} \in H^2(\widetilde{M}, \mathbb{R})$ is of type $(2, 0) + (0, 2)$ with respect to $\widetilde{J}$ and can be represented as $\Pi^* b$ for a $(2, 0) + (0, 2)$-class $b \in H^2(M, \mathbb{R})$ with respect to J. Represent b by a closed real-valued form β on M of type $(2, 0) + (0, 2)$ with respect to J. Then the class $\Pi^*[\omega] - \Pi^*[\omega]_J^{1,1}$ is represented by a closed real-valued form $\Pi_J^* \beta$ on $\widetilde{M}$ of type $(2, 0) + (0, 2)$ with respect to $\widetilde{J}$.

Set $\widetilde{\omega} := \widetilde{\alpha} + \Pi_J^* \beta$. By Proposition 5.5, parts (A) and (C), the form $\widetilde{\omega}$ is symplectic and tames $\widetilde{J}$. The cohomology class of $\widetilde{\omega}$ can be written as

$$[\widetilde{\omega}] = [\widetilde{\alpha}] + [\Pi_J^* \beta] = \Pi^*[\omega]_J^{1,1} - \pi \sum_{i=1}^k r_i^2 [E_i] + \Pi^*[\omega] - \Pi^*[\omega]_J^{1,1}$$

$$= \Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i].$$

Now we can apply Theorem 8.1 *with* J *instead of* I, which yields the needed claim. $\square$

A necessary condition for $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i]$ to be Kähler is

$$\left\langle \left(\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i] \right)^n, [\widetilde{M}] \right\rangle > 0.$$

The following proposition shows that in terms of symplectic packings the latter inequality means the following simple fact: if a finite disjoint union of closed balls

is symplectically embedded in (M, ω), then its total volume is less than the volume of (M, ω).

Proposition 8.4. *With the notation as in Theorem 8.1,*

$$\left\langle \left(\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 [E_i] \right)^n, [\widetilde{M}] \right\rangle = \int_M \omega^n - \pi^n \sum_{i=1}^{k} r_i^{2n}$$

$$= \int_M \omega^n - \mathrm{Vol}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i) \right).$$

Proof. Note that for all $i = 1, \ldots, k$ we have $\Pi^*[\omega] \cup [E_i] = 0$, as well as $[E_i]^n = -1$, if n is even, and $[E_i]^n = 1$, if n is odd. Also note that $[E_i] \cup [E_j] = 0$ for all $i \neq j$. Finally, recall that the symplectic volume of $B^{2n}(r)$ equals $\pi^n r^{2n}$. The claim follows directly from these observations. □

8.2. *Demailly–Paun theorem and the Kähler cone*

Our results depend crucially on the following deep result by Demailly and Paun.

Theorem 8.5. (Demailly–Paun, [10]) *Let N be a closed connected Kähler manifold. Let $\hat{K}(N) \subset H^{1,1}(N, \mathbb{R})$ be a subset consisting of all $(1,1)$-classes η which satisfy $\langle \eta^m, [Z] \rangle > 0$ for any homology class $[Z]$ realized by a complex subvariety $Z \subset N$ of complex dimension m. Then the Kähler cone of N is one of the connected components of $\hat{K}(N)$.*

For Campana simple manifolds this theorem can be used to study the Kähler cone of a blow-up.

Theorem 8.6. *Let (M, I), $\dim_{\mathbb{C}} M = n$, be a Campana simple closed connected Kähler manifold. Consider a complex blow-up $\widetilde{M}$ of (M, I) at k Campana-generic points $x_1, \ldots, x_k$. Define $\Pi_I : \widetilde{M} \to M$, $E_i := \Pi_I^{-1}(x_i)$ and $[E_i] \in H^2(\widetilde{M}, \mathbb{Z})$, $i = 1, \ldots, k$, as above.*

Assume that η is a Kähler class in $H^2(M, \mathbb{R})$. Then, given $c_1, \ldots, c_k \in \mathbb{R}$, the following claims are equivalent:

(A) *The cohomology class $\widetilde{\eta} := \Pi^*\eta - \sum_{i=1}^{k} c_i[E_i] \in H^2(\widetilde{M}, \mathbb{R})$ is Kähler.*
(B) *The conditions* (B1) *and* (B2) *below are satisfied:*

(B1) *All c_i are positive.*
(B2) $\langle \widetilde{\eta}^n, [\widetilde{M}] \rangle > 0.$

Proof of (A) ⇒ (B). The implication (A) ⇒ (B2) is obvious. To prove (A) ⇒ (B1) note that, since $\widetilde{\eta}$ is Kähler, for each $i = 1, \ldots, k$ we have

$$0 < \int_{E_i(I)} \widetilde{\eta}^{n-1} = \int_{E_i(I)} (-c_i[E_i])^{n-1},$$

and since the restriction of $-[E_i]$ to $E_i(I)$ is a positive multiple of the Fubini–Study form on the exceptional divisor (see the discussion on the symplectic blow-up in Sec. 8.1) and the integral of the exterior power of the latter form over $E_i(I)$ is positive, we readily get that $c_i > 0$. □

Proof of (B) $\Rightarrow$ (A). Assume (B1) and (B2) are satisfied.

Since x_i are Campana-generic, any connected proper complex subvariety of $(\widetilde{M}, \widetilde{I})$ is either contained in an exceptional divisor E_i, or does not intersect any exceptional divisor.

Since $\eta \in H^2(M, \mathbb{R})$ is a Kähler cohomology class with respect to I, we have $\langle \widetilde{\eta}^m, Z \rangle = \langle \eta^m, \Pi_I(Z) \rangle > 0$ for any complex subvariety $Z \subset (\widetilde{M}, \widetilde{I})$, $\dim_{\mathbb{C}} Z = m$, that does not intersect the exceptional divisors.

On the other hand, note that for each $i = 1, \ldots, k$ the restriction of the cohomology class $[E_i]$ to the submanifold $E_i(I)$ is a positive multiple of $-[\omega_{E_i(I)}]$, where $\omega_{E_i(I)}$ is the Fubini–Study form, and therefore (B1) yields that $\langle \widetilde{\eta}^m, [Z] \rangle > 0$ for any complex variety $Z \subset (\widetilde{M}, \widetilde{I})$, $\dim_{\mathbb{C}} Z = m$, lying in $E_i(I)$.

Thus $\langle \widetilde{\eta}^m, Z \rangle > 0$ for any complex subvariety $Z \subsetneq (\widetilde{M}, \widetilde{I})$, $\dim_{\mathbb{C}} Z = m$. This shows that for any positive $c_1, \ldots, c_k$ the class $\widetilde{\eta} = \eta - \sum_{i=1}^{k} c_i[E_i]$ lies in $\hat{K}(\widetilde{M})$ as long as it satisfies (B2).

By Theorem 8.5, in order to show that $\widetilde{\eta}$ is Kähler, it remains to check that there exists a Kähler form in the connected component of $\hat{K}(\widetilde{M})$ containing $\widetilde{\eta}$.

Indeed, similarly to Proposition 8.4, one gets that (B2) is equivalent to the condition

$$\sum_{i=1}^{k} c_i^n < \langle \widetilde{\eta}^n, [\widetilde{M}] \rangle.$$

If this condition holds for $c_1, \ldots, c_k > 0$, it also holds for $\varepsilon c_1, \ldots, \varepsilon c_k$ for any $\varepsilon \in (0, 1]$. The numbers $\varepsilon c_1, \ldots, \varepsilon c_k$ are still positive and therefore, by the argument above, for any $\varepsilon \in (0, 1]$ the class $\widetilde{\eta}_\varepsilon := \Pi^*\eta - \varepsilon \sum_{i=1}^{k} c_i[E_i]$ also lies in $\hat{K}(\widetilde{M})$. But, as it is explained in Remark 8.2, for any sufficiently small positive ε the class $\widetilde{\eta}_\varepsilon$ is Kähler. Thus, $\widetilde{\eta}$ lies in the same connected component of $\hat{K}(\widetilde{M})$ as a Kähler class $\widetilde{\eta}_\varepsilon$. Therefore, by Theorem 8.5, $\widetilde{\eta}$ is Kähler. □

8.3. *Unobstructed symplectic packings by balls for Campana simple complex manifolds and their limits*

The following result is a slightly stronger version of Theorem 4.6.

Theorem 8.7. *Let (M, I, ω) be a closed connected Kähler manifold. Assume I admits a smooth local deformation $\{I_t\}$, $t \in U$, $I = I_{t_0}$, and there is a sequence $\{t_l\}_{l \in \mathbb{N}} \to t_0$ in U such that I_{t_l} is Campana simple for all $l \in \mathbb{N}$. Let $\Gamma \subset (M, I)$ be a (possibly empty) closed complex submanifold.*

Then the symplectic packings of $(M \backslash \Gamma, \omega)$ by balls are unobstructed.

Proof of Theorem 8.7. Consider a disjoint union $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$ whose total symplectic volume is less than the symplectic volume of M, that is,

$$\pi^n \sum_{i=1}^{k} r_i^{2n} < \mathrm{Vol}(M) = \langle [\omega]^n, [M] \rangle. \tag{8.1}$$

We need to show that it admits a symplectic embedding into $(M \backslash \Gamma, \omega)$.

It follows from Theorem 5.6 and the assumptions of the theorem that for a sufficiently large $l \in \mathbb{N}$ and the corresponding Campana simple complex structure $J := I_{t_l}$ the cohomology class $[\omega]_J^{1,1}$ is Kähler (with respect to J). Note that J can be chosen arbitrarily C^∞-close to I. In particular, we can assume that J is tamed by ω (since I is tamed by ω) and that

$$\pi^n \sum_{i=1}^{k} r_i^{2n} < \langle ([\omega]_J^{1,1})^n, [M] \rangle, \tag{8.2}$$

because of (8.1) and Theorem 5.6, combined with the fact that $[\omega] = [\omega]_J^{1,1}$.

Choose k Campana-generic points $x_1, \ldots, x_k \in (M, J)$ lying in $M \backslash \Gamma$ and consider the complex blow-up $(\widetilde{M}, \widetilde{J})$ of (M, J) at those points. By Theorem 8.6 applied to the Kähler class $[\omega]_J^{1,1}$, the cohomology class $\Pi^*[\omega]_J^{1,1} - \pi \sum_{i=1}^{k} r_i^2 [E_i]$ is Kähler with respect to $\widetilde{J}$ (note that, by Proposition 8.4, the condition (B2) in Theorem 8.6 is equivalent to (8.2)). Therefore, by Theorem 8.3, $(M \backslash \Gamma, \omega)$ admits a symplectic embedding of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$. $\qquad\square$

9. Symplectic Packing by Arbitrary Shapes

Let M be either an oriented torus T^{2n}, $n \geq 2$, or, respectively, a closed connected oriented manifold admitting IHS hyper-Kähler structures compatible with the orientation. Without loss of generality we are going to prove the results for symplectic forms on M of total volume 1.

9.1. *Semicontinuity of symplectic packing constants in families of symplectic forms*

Let $\mathcal{F}$ denote the space of Kähler forms on T^{2n} (respectively, hyper-Kähler forms on M in the hyper-Kähler case) of total volume 1. Equip $\mathcal{F}$ with the C^∞-topology. The group Diff^+ of orientation-preserving diffeomorphisms of M acts on $\mathcal{F}$ and the function $\omega \mapsto \nu(M, \omega, V)$, defined in Sec. 3.2, is clearly invariant under the action.

Proposition 9.1. *The function $\omega \mapsto \nu(M, \omega, V)$ on $\mathcal{F}$ is lower semicontinuous.*

Theorem 9.2. *Let $\omega \in \mathcal{F}$ be a symplectic form such that the cohomology class $[\omega]$ is not proportional to a rational one. Then the orbit of ω under action of Diff^+ is dense in $\mathcal{F}$ in the toric case and in the connected component $\mathcal{F}^0$ of $\mathcal{F}$ containing ω in the hyper-Kähler case.*

We will prove Theorem 9.2 in Sec. 9.2.

Proof of Theorem 3.2. Since the orbit of ω_1 is dense in $\mathcal{F}$ and $\nu(M, \cdot, V)$ is constant on it, we get, by the lower semicontinuity of $\nu(M, \cdot, V)$, that

$$\nu(M, \omega_2, V) \leq \nu(M, \omega_1, V).$$

Switching ω_1 and ω_2 and applying the same argument we get

$$\nu(M, \omega_1, V) \geq \nu(M, \omega_2, V).$$

Thus

$$\nu(M, \omega_1, V) = \nu(M, \omega_2, V). \qquad \square$$

Proof of Proposition 9.1. Let us start with a number of preparations.

Fix Riemannian metrics on U and on M. These Riemannian metrics induce C^0-norms on the spaces of vector fields and differential forms defined on an open neighborhood of V in U and on M. Abusing the notation, we will denote all these norms by the same symbol $\|\cdot\|$:

$$\|v\| := \max_x |v(x)|$$

for a vector field v and

$$\|\Omega\| := \max_{\|v_1\|, \ldots, \|v_l\| \leq 1} |\Omega(v_1, \ldots, v_l)|$$

for a differential l-form Ω (for vector fields and forms on a neighborhood of V we take the maximums only over V — recall that V is compact). $\qquad \square$

Lemma 9.3. *Let (U, η), $\dim_{\mathbb{R}} U = 2n$, be an open, possibly disconnected symplectic manifold, and let $V \subset U$, $\dim_{\mathbb{R}} V = 2n$, be a compact submanifold with a piecewise smooth boundary. Given an exact 2-form Ω on a neighborhood of V, one can choose a 1-form σ on the same neighborhood so that $d\sigma = \Omega$ and $\|\sigma\| \leq C_1 \|\Omega\|$ for some constant $C_1 > 0$ depending only on V.*

Proof. We present only an outline of the proof leaving the technical details to the reader.

We triangulate V and proceed by induction on the number of $2n$-dimensional simplices. In the case of one simplex the claim follows from an explicit formula for the primitive of an exact form on a star-shaped domain in a Euclidean space appearing in the proof of the classical Poincare lemma (see e.g. [38]).

Assume now that the result holds for any manifold whose triangulation consists of k simplices and consider a manifold which is the union of $k+1$ simplices. Apply the induction assumption to the union A of the first k simplices and, separately, to the $(k+1)$th simplex B.

If $A \cap B = \emptyset$, the claim is obvious. Therefore we may assume that $A \cap B \neq \emptyset$. We get two small 1-forms σ_1 and σ_2 defined on open neighborhoods Z and W of A and B so that $d\sigma_1 = \Omega$ and $d\sigma_2 = \Omega$. Thus on $Z \cap W$ the 1-form $\sigma_1 - \sigma_2$ is exact and small. Without loss of generality, we may assume that W is a ball and

$Z \cap W$ is, topologically, either a ball or a spherical shell. In either case it is not hard to see that $\sigma_1 - \sigma_2$ can be written on $Z \cap W$ as $\sigma_1 - \sigma_2 = dh$ for a C^1-small function h. Extend the function h from $Z \cap W$ to W keeping it C^1-small (this is not hard to do, since $Z \cap W$ is a ball or a spherical shell inside the ball W). This allows to extend $\sigma_1 - \sigma_2$ to a small exact 1-form on W. Thus σ_1 (which is equal to $\sigma_1 = \sigma_2 - (\sigma_2 - \sigma_1)$ on $Z \cap W$) can be extended to a small 1-form on the open set $Z \cup W$ (which is a neighborhood of our original manifold V) so that $d\sigma_1 = \Omega$ everywhere. The loss of "smallness" of the differential forms at each step above is by a factor that depends only on the geometry of V and not on the differential forms. Setting $\sigma := \sigma_1$ finishes the proof. $\qquad\square$

Having fixed V and η, denote for brevity $\nu(\omega) := \nu(M, \omega, V)$. Consider an arbitrary $\omega_0 \in \mathcal{F}$. Let us prove that ν is lower semicontinuous at ω_0. If $\nu(\omega_0) = -\infty$, the claim is obvious, so we can assume without loss of generality that $\nu(\omega_0) \neq -\infty$, meaning that there exist symplectic embeddings $(V, \alpha\eta) \to (M, \omega_0)$ for some $\alpha > 0$. To prove the lower semicontinuity of ν at ω_0 it suffices to prove the following claim:

For any symplectic embedding $f : (\mathcal{U}, \alpha\eta) \to (M, \omega_0)$, where $\mathcal{U} \subset U$ is an open neighborhood of V in U, and any sufficiently small $\varepsilon > 0$ there exists $\delta = \delta(f, \varepsilon) > 0$ such that for any $\omega \in \mathcal{F}$, $\|\omega - \omega_0\| \leq \delta$, the following two conditions are satisfied:

(A) $\dfrac{\mathrm{Vol}(V, \alpha\eta)}{\mathrm{Vol}(M, \omega)} > \dfrac{\mathrm{Vol}(V, \alpha\eta)}{\mathrm{Vol}(M, \omega_0)} - \varepsilon$.

(B) There exists a symplectic embedding $g : (\mathcal{U}', \alpha\eta) \to (M, \omega)$, where $\mathcal{U}' \subset \mathcal{U}$ is a possibly smaller neighborhood of V.

In order to prove the claim let us fix $\varepsilon > 0$ and choose $\delta > 0$ so that (A) is satisfied — this is, of course, not a problem. Let us now show (B): namely, consider a form $\omega \in \mathcal{F}$, $\|\omega - \omega_0\| \leq \delta$ and construct g by the classical Moser method [30] as follows. If δ is sufficiently small (depending on ω_0 and f), then, since $f^*\omega_0 = \alpha\eta$ is symplectic, the straight path $\theta_t := f^*\omega_0 + tf^*(\omega - \omega_0)$, $0 \leq t \leq 1$, connecting $f^*\omega_0$ and $f^*\omega$ is formed by symplectic forms on V. Since $H^2(V, \mathbb{R}) = 0$, the form $f^*(\omega - \omega_0)$ is an exact. By Lemma 9.3, one can choose a 1-form σ on $\mathcal{U}$ so that $f^*(\omega - \omega_0) = d\sigma$ and $\|\sigma\| \leq C_2\|\omega - \omega_0\|$ for some constant $C_2 > 0$ depending only on V, ω_0 and f. Let v_t be the vector field on $\mathcal{U}$ defined by $\theta_t(v_t, \cdot) = \sigma(\cdot)$. Then $\max_{0 \leq t \leq 1} \|v_t\| \leq C_3\|\omega - \omega_0\| \leq C_3\delta$ for some constant $C_3 > 0$ depending only on V, ω_0 and f. Therefore if δ is sufficiently small, the time-$[0, 1]$ flow of v_t yields a well-defined map $\psi : \mathcal{U}' \to \mathcal{U}$ for some smaller neighborhood $\mathcal{U}' \subset \mathcal{U}$ of V. Moser's argument [30] shows that $\psi^*(f^*\omega) = f^*\omega_0$. This implies that $g := f \circ \psi : (\mathcal{U}', \alpha\eta) \to (M, \omega)$ is a symplectic embedding: indeed,

$$g^*\omega = (f \circ \psi)^*\omega = \psi^*(f^*\omega) = f^*\omega_0 = \alpha\eta.$$

Proof of Corollary 3.3. Let (V, η) be the disjoint union of k copies of $(B^{2n_1}(R_1) \times \cdots \times B^{2n_l}(R_l), dp \wedge dq)$, where $dp \wedge dq$ is the standard symplectic form on $\mathbb{R}^{2n}$. For brevity set $\nu(\omega) := \nu(T^{2n}, \omega, V)$. We need to show that $\nu(\omega) = 1$.

For each $i = 1, \ldots, l$ set $v_i := \mathrm{Vol}(B^{2n_i}(R_i), \Omega_{n_i})$, where Ω_{n_i} is the standard symplectic form on $\mathbb{R}^{2n_i}$. Note that

$$v_i = \mathrm{Vol}(B^{2n_i}(R_i), \Omega_{n_i}) = \pi^{n_i} n_i! R_i^{2n_i},$$

$$dp \wedge dq = \Omega_{n_1} \oplus \cdots \oplus \Omega_{n_l}$$

and

$$\mathrm{Vol}(V, \eta) = k \, \mathrm{Vol}(B^{2n_1}(R_1) \times \cdots \times B^{2n_l}(R_l), dp \wedge dq) = kN \prod_{i=1}^{l} v_i,$$

where

$$N := \frac{n!}{n_1! \cdot \ldots \cdot n_l!}.$$

Assume without loss of generality that

$$\mathrm{Vol}(T^{2n}, \omega) = k \, \mathrm{Vol}(B^{2n_1}(R_1) \times \cdots \times B^{2n_l}(R_l), dp \wedge dq) = kN v_1 \cdot \ldots \cdot v_l = 1.$$

Given $m \in \mathbb{N}$ and $w_1, \ldots, w_m > 0$, set $\overline{w} := (w_1, \ldots, w_m) \in \mathbb{R}^m$, $v_{\overline{w}} := m! w_1 \cdot \ldots \cdot w_m$, and denote by $\omega_{\overline{w}}$ the symplectic form $\omega_{\overline{w}} = \sum_{i=1}^{m} w_i dp_i \wedge dq_i$ on the torus $T^{2m} = \mathbb{R}^{2m}/\mathbb{Z}^{2m}$. Note that the form $\omega_{\overline{w}}$ is Kähler (since it is linear) and $\mathrm{Vol}(T^{2m}, \omega_{\overline{w}}) = v_{\overline{w}}$.

Set $k_1 := k$, $k_2 = \cdots = k_l := 1$, so that

$$k = k_1 \cdot \ldots \cdot k_l.$$

Given $0 < \alpha \leq 1$, one can choose $\overline{w}_i \in \mathbb{R}^{2n_i}$, $i = 1, \ldots, l$, depending on α (for brevity we suppress this dependence in the notation below) so that the following conditions hold:

(A) $\prod_{i=1}^{l} v_{\overline{w}_i} = \prod_{i=1}^{l} k_i v_i = 1/N$.
(B) $v_{\overline{w}_i} > \alpha k_i v_i$ for all $i = 1, \ldots, l$.
(C) The vector $\overline{\mathbf{w}} := (\overline{w}_1, \ldots, \overline{w}_k) \in \mathbb{R}^{2n}$ is not proportional to a vector with rational coordinates.

Condition (C) can be achieved since the set of vectors that are not proportional to a vector with rational coordinates is dense in the set

$$\{(w_1, \ldots, w_{2n}) \in \mathbb{R}^{2n} \mid w_1, \ldots, w_{2n} > 0, w_1 \cdot \ldots \cdot w_{2n} = C\}$$

for any $C > 0$.

Consider the symplectic form $\omega_{\overline{\mathbf{w}}}$ on $\mathbb{R}^{2n}$ — it is Kähler (since it is a linear symplectic form) and its cohomology class is not proportional to a rational one.

Note that, by condition (A),

$$\int_{T^{2n}} \omega_{\mathbf{w}}^n = N \prod_{i=1}^{l} v_{\overline{w}_i} = \int_{T^{2n}} \omega^n = 1.$$

Thus to prove that $\nu(\omega) = 1$ it is enough to show that $\nu(\omega_{\overline{\mathbf{w}}}) \to 1$ as $\alpha \to 1$. Indeed, by Theorem 3.2 and in view of condition (C), $\nu(\omega_{\overline{\mathbf{w}}})$ is constant for all $\alpha > 1$ (recall that $\overline{\mathbf{w}}$ depends on α) and equal to $\nu(\omega)$.

Now note that, by condition (B), for any $\alpha > 1$ for all i

$$\mathrm{Vol}(T^{2n_i}, \omega_{\overline{w}_i}) = v_{\overline{w}_i} > \alpha k_i v_i = \mathrm{Vol}\left(\bigsqcup_{k_i} (B^{2n_i}(R_i), \alpha\Omega_{n_i}) \right),$$

where $\bigsqcup_{k_i}$ denotes the disjoint union of k_i copies of the ball. Therefore, by Theorem 3.1, there exists a symplectic embedding

$$f_i : \bigsqcup_{k_i} \left(B^{2n_i}(R_i), \alpha\Omega_{n_i} \right) \to (T^{2n_i}, \omega_{\overline{w}_i}).$$

Accordingly, the direct product of all such embeddings f_i, $i = 1, \ldots, l$, is a symplectic embedding

$$f : (V, \alpha) \to (T^{2n}, \omega_{\overline{\mathbf{w}}})$$

of $k_1 \cdot \ldots \cdot k_l = k$ disjoint equal copies of the polydisk $(B^{2n_1}(R_1) \times \cdots \times \times B^{2n_l}(R_l), dp \wedge dq)$ into $(T^{2n}, \omega_{\overline{\mathbf{w}}})$. The fraction of the volume of $(T^{2n}, \omega_{\overline{\mathbf{w}}})$ filled by the image of f tends to 1 as $\alpha \to 1$. In other words, $\nu(\omega_{\overline{\mathbf{w}}})$ converges to 1 as $\alpha \to 1$ which yields the needed result. $\qquad\square$

9.2. *Ergodic action on the symplectic Teichmüller space — the proof of Theorem 9.2*

The proof of Theorem 9.2 follows the same lines as the proof of the ergodicity theorem in [44] and [45].

First, let us make a few preparations.

Definition 9.4. Let G be a real Lie group. We say that $g \in G$ is *unipotent*, if $g = e^h$ for a nilpotent element h the Lie algebra of G. A group G is said to be *generated by unipotents*, if it is multiplicatively generated by unipotent elements.

Our proof will be based on the following fundamental theorem of Ratner (see e.g. [29] for a friendly introduction to the subject).

Theorem 9.5. (Ratner's orbit closure theorem, [35]) *Let G be a connected Lie group, $H \subset G$ its subgroup generated by unipotents and $\Gamma \subset G$ a lattice (that is, a discrete subgroup of finite covolume). Then for any $g \in G$ one has*

$$\overline{\Gamma g H} = \Gamma g S$$

for some closed Lie subgroup S, $H \subset S \subset G$.

In particular, if H is a closed Lie subgroup, then the closure of the orbit $\Gamma \cdot gH$ of gH in G/H is $\Gamma(gSg^{-1}) \cdot gH$.

A combination of Ratner's Theorem 9.5 with a result of Shah [37, Proposition 3.2]) yields a more precise description of the group S from Theorem 9.5 in the case where G is a linear algebraic group and $\Gamma \subset G$ is an arithmetic lattice — see [22, Proposition 3.3.7]. We will state the result for $g = e$, since this is exactly what we are going to use in our proof.

Claim 9.6. (Ratner's theorem for arithmetic lattices; [22, Proposition 3.3.7] or [37, Proposition 3.2]) *Let $G \subset SL(m, \mathbb{R})$, $m \in \mathbb{N}$, be a linear algebraic real $\mathbb{Q}$-group (that is, G is a Lie subgroup of $SL(m, \mathbb{R})$ defined by algebraic equations with rational coefficients on the entries of a real $m \times m$-matrix). Assume G has no non-trivial $\mathbb{Q}$-characters (that is, characters defined by algebraic equations with rational coefficients). Let $\Gamma \subset G$ be an arithmetic lattice (that is, a lattice lying in $G \cap SL(m, \mathbb{Z})$). Let $H \subset G$ be a closed Lie subgroup generated by unipotents. Let $x := eH \in G/H$, where $e \in G$ is the identity of G.*

Then the closure of the orbit $\Gamma \cdot x$ in G/H is $\Gamma S \cdot x$, where $S \subset G$ is the smallest real algebraic $\mathbb{Q}$-subgroup of G containing H.

Let $\mathcal{F}$ be the space of Kähler (respectively, hyper-Kähler) forms on M defined as in Sec. 9.1 and viewed as an infinite-dimensional Fréchet manifold. Let $\mathcal{F}^0$ be the connected component of $\mathcal{F}$ containing the form ω as in Theorem 9.2. The quotient topological space $\mathsf{Teich}_s := \mathcal{F}/\mathrm{Diff}_0$ is called *the Teichmüller space of symplectic structures*. Set $\mathsf{Teich}_s^0 := \mathcal{F}^0/\mathrm{Diff}_0$ — it is a connected component of Teich_s (in fact, in the torus case it coincides with Teich_s).

Let $\mathsf{Per} : \mathsf{Teich}_s \to H^2(M, \mathbb{R})$ be *the period map* associating to a symplectic structure its cohomology class. Using Moser's stability theorem for symplectic structures, it is not hard to obtain that Teich_s is a finite-dimensional manifold and Per is locally a diffeomorphism [14].

In the case when M is a torus it is easy to see that Per is an embedding whose image is the set $\Theta_t \subset H^2(T^{2n}, \mathbb{R})$ of cohomology classes η such that $\int_{T^{2n}} \eta^n = 1$. Indeed, any such η is the cohomology class of a linear symplectic form of total volume 1). It is also not hard to show that Θ_t is connected (since, by the linear Darboux theorem, the space of all linear symplectic forms on a vector space compatible with a fixed orientation is connected).

In the hyper-Kähler case we will use the following theorem proved in [1].

Theorem 9.7. *Let M, $\dim_\mathbb{R} M = 2n$, be a closed connected oriented manifold admitting IHS hyper-Kähler structures (compatible with the orientation). Then Per is an open embedding on each connected component of Teich_s and its image is the set $\Theta_h := \{\eta \in H^2(M, \mathbb{R}) \,|\, q(\eta, \eta) > 0, \ \int_M \eta^n = 1\}$, where q is the Bogomolov–Beauville–Fujiki form defined in Sec. 7.1. Moreover, Teich_s has finitely many connected components.*

Let us return to the setup where M, $\dim_{\mathbb{R}} M = 2n \geq 4$, is either an oriented torus or a closed connected oriented manifold admitting IHS hyper-Kähler structures (compatible with the orientation). Let $P \subset H^2(M, \mathbb{R})$ be the image of Per, that is, $P = \Theta_t$ in the torus case and $P = \Theta_h$ in the hyper-Kähler case. The group $\mathrm{Diff}^+ / \mathrm{Diff}_0$ acts in an obvious way on Teich_s and on $H^2(M, \mathbb{R})$. The period map $\mathsf{Per} : \mathsf{Teich}_s \to H^2(M, \mathbb{R})$ respects the actions. In particular, the action of $\mathrm{Diff}^+ / \mathrm{Diff}_0$ on $H^2(M, \mathbb{R})$ preserves P. Let $\overline{\Gamma} \subset \mathrm{Diff}^+ / \mathrm{Diff}_0$ be a finite index subgroup such that $\overline{\Gamma} \cdot \mathsf{Teich}_s^0 = \mathsf{Teich}_s^0$ (recall that Teich_s is connected in the torus case and has finitely many connected components in the hyper-Kähler case).

Theorem 9.8. *For any $\eta \in P$ the orbit $\overline{\Gamma} \cdot \eta$ is dense in P if and only if the cohomology class η is not proportional to a rational one.*

Before proving Theorem 9.8 let us see how it implies Theorem 9.2.

Proof of Theorem 9.2. The orbit $\mathrm{Diff}^+ \cdot \omega$ is dense in $\mathcal{F}$ (in the torus case) or, respectively, in $\mathcal{F}^0$ (in the hyper-Kähler case) if and only if the orbit of the image of ω in Teich_s under the action of $\mathrm{Diff}^+/\mathrm{Diff}_0$ is dense in Teich_s (in the torus case) or, respectively, in Teich_s^0 (in the hyper-Kähler case). The latter condition holds if and only if the orbit $\overline{\Gamma} \cdot [\omega]$ is dense in P, since $\mathsf{Per} : \mathsf{Teich}_s^0 \to P$ is a diffeomorphism and $\overline{\Gamma} \subset \mathrm{Diff}^+/\mathrm{Diff}_0$ is a subgroup preserving Teich_s^0. By Theorem 9.8, $\overline{\Gamma} \cdot [\omega]$ is dense in P if and only if the cohomology class $[\omega]$ is not proportional to a rational one. We conclude that the orbit $\mathrm{Diff}^+ \cdot \omega$ is dense in $\mathcal{F}$ (in the torus case) or, respectively, in $\mathcal{F}^0$ (in the hyper-Kähler case) if and only if the cohomology class $[\omega]$ is not proportional to a rational one. This finishes the proof. $\square$

Proof of Theorem 9.8. Both in the torus and in the hyper-Kähler case set $k := b_2 := \dim H^2(M, \mathbb{R})$.

In the case of the torus we identify $H^2(T^{2n}, \mathbb{R})$ with the space $\left(\bigwedge^2 \mathbb{R}^{2n} \right)^*$ of linear 2-forms on $\mathbb{R}^{2n}$ and the action of $\overline{\Gamma} = \mathrm{Diff}^+/\mathrm{Diff}_0$ on $H^2(T^{2n}, \mathbb{R})$ with the action of $\Gamma := \mathrm{SL}(2n, \mathbb{Z})$ on $\left(\bigwedge^2 \mathbb{R}^{2n} \right)^*$. The latter action extends to the natural $\mathrm{SL}(2n, \mathbb{R})$-action on $\left(\bigwedge^2 \mathbb{R}^{2n} \right)^*$. We also fix a linear identification of $H^2(T^{2n}, \mathbb{R}) = \left(\bigwedge^2 \mathbb{R}^{2n} \right)^*$ with $\mathbb{R}^k$, so that $H^2(T^{2n}, \mathbb{Q}) \subset H^2(T^{2n}, \mathbb{R})$ is identified with $\mathbb{Q}^k \subset \mathbb{R}^k$.

The stabilizer of a point $\eta \in P \subset H^2(T^{2n}, \mathbb{R})$ under the $\mathrm{SL}(2n, \mathbb{R})$-action is a closed Lie subgroup $H_\eta \subset \mathrm{SL}(2n, \mathbb{R})$ isomorphic to $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$ by an inner automorphism of $\mathrm{SL}(2n, \mathbb{R})$. (Indeed, Per identifies elements of P with linear symplectic forms on $\mathbb{R}^{2n}$ defining the same volume form and the $\mathrm{SL}(2n, \mathbb{R})$-action on $\left(\bigwedge^2 \mathbb{R}^{2n} \right)^*$ clearly preserves the set of such forms).

In the hyper-Kähler case we identify $H^2(M, \mathbb{R})$ with $\mathbb{R}^k$ so that the Bogomolov–Beauville–Fujiki form q defined in Sec. 7.1 is identified with the standard $(3, k-3)$-quadratic form on $\mathbb{R}^k$ (and $H^2(M, \mathbb{Q}) \subset H^2(M, \mathbb{R})$ is identified with $\mathbb{Q}^k \subset \mathbb{R}^k$). The group of linear automorphisms of $H^2(M, \mathbb{R})$ preserving

$$P = \Theta_h = \left\{ \eta \in H^2(M, \mathbb{R}) \mid q(\eta, \eta) > 0, \ \int_M \eta^n = 1 \right\}$$

is then identified with

$$\text{SO}(3, k-3) \cong O(H^2(M, \mathbb{R}), q) \cap \text{SL}(H^2(M, \mathbb{R}), \mathbb{R}).$$

Since $\overline{\Gamma} \cdot P = P$, there is a natural homomorphism $\overline{\Gamma} \to \text{SO}(3, k-3)$. As it was shown in [43], the image of this homomorphism is an arithmetic lattice $\Gamma \subset \text{SO}(3, k-3)$. The stabilizer of a point $\eta \in P$ under the action of $\text{SO}(3, k-3)$ is a closed Lie subgroup $H_\eta \subset \text{SO}(3, k-3)$ isomorphic to $\text{SO}(2, k-3) \subset \text{SO}(3, k-3)$ by an inner automorphism of $\text{SO}(3, k-3)$.

We are going to apply Claim 9.6 to $G = \text{SL}(2n, \mathbb{R}) \supset H := H_\eta \cong \text{Sp}(2n, \mathbb{R})$, $\Gamma = \text{SL}(2n, \mathbb{Z}) \subset G$ in the torus case and to $G = \text{SO}(3, k-3) \supset H := H_\eta \cong \text{SO}(2, k-3)$ and the arithmetic lattice $\Gamma \subset G$ in the hyper-Kähler case. Indeed, in both cases G is a connected linear algebraic group that does not admit nontrivial $\mathbb{Q}$-characters (since it has no normal subgroups except for the center which is discrete — see e.g. [34]); Γ is arithmetic; H_η is a closed Lie subgroup generated by unipotents (since $\text{Sp}(2n, \mathbb{R})$ and $\text{SO}(3, k-3)$ have the latter property — see e.g. [18, Theorem 11.2.11]). Hence Claim 9.6 can be applied.

In both cases $G \cdot P = P$ and it is not hard to show that G acts transitively on P. Thus the orbit $G \cdot \eta = P$ is identified with the homogeneous space G/H_η. Let $x_\eta := eH_\eta \in G/H_\eta = P$, where $e \in G$ is the identity element. The orbit $\Gamma \cdot x_\eta$ in $G/H_\eta = P$ is exactly the orbit $\Gamma \cdot x \subset P$. Thus, in order to prove Theorem 9.2 we need to show that $\overline{\Gamma \cdot x_\eta} = G/H_\eta$ if and only if the cohomology class η is not proportional to a rational one.

We will need the following lemmas.

Lemma 9.9. *There are no intermediate Lie subgroups* $\text{Sp}(2n, \mathbb{R}) \subsetneq S \subsetneq \text{SL}(2n, \mathbb{R})$ *and* $\text{SO}(p-1, q) \subsetneq S \subsetneq \text{SO}(p, q)$ *(for any $p, q \in \mathbb{N}$).*

Lemma 9.10. *The group H_η a $\mathbb{Q}$-subgroup (that is, it can be defined by algebraic equations with rational coefficients) if and only if $\eta \in H^2(M, \mathbb{R})$ is proportional to a rational cohomology class.*

Postponing the proof of the lemmas let us finish the proof of the theorem.

It follows from Claim 9.6 that

$$\overline{\Gamma \cdot x_\eta} = \Gamma S_\eta \cdot x_\eta = \Gamma S_\eta e H_\eta = \Gamma S_\eta H_\eta,$$

where S_η is the smallest connected $\mathbb{Q}$-subgroup of G containing H_η.

It follows Lemma 9.9 that for any $\eta \in P$ there are exactly two possibilities: either $S_\eta = H_\eta$ or $S_\eta = G$.

In the first case we have

$$\overline{\Gamma \cdot x_\eta} = \Gamma S_\eta H_\eta = \Gamma H_\eta \subsetneq G.$$

Indeed, Γ is a discrete (hence countable) subgroup of G and $H_\eta \subset G$ is a proper closed Lie subgroup (hence a set of measure 0) hence ΓH_η is a proper subset of G.

In the second case, when $S_\eta = G$, we clearly have

$$\overline{\Gamma \cdot x_\eta} = \Gamma S_\eta H_\eta = \Gamma G H_\eta = G.$$

Thus, it suffices to show that $S_\eta \neq H_\eta$ (and thus $S_\eta = G$) if and only if the cohomology class η is not proportional to a rational one.

Let us now note that $S_\eta = H_\eta$ if and only if the stabilizer H_η itself is a $\mathbb{Q}$-subgroup. By Lemma 9.10, the latter condition holds if and only if η is proportional to a rational cohomology class. Thus $S_\eta = G$ if and only if the cohomology class η is not proportional to a rational one.

This finishes the proof of Theorem 9.2. □

Proof[f] of Lemma 9.9. Let us denote by $\mathfrak{g}$ the Lie algebra of $\mathrm{SL}(2n, \mathbb{R})$ (respectively, $\mathrm{SO}(p, q)$) and by $\mathfrak{h}$ the Lie algebra of $\mathrm{Sp}(2n, \mathbb{R})$ (respectively, $\mathrm{SO}(p-1, q)$). In both cases $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and it suffices to show that there is no intermediate Lie subalgebra $\mathfrak{s}$ such that $\mathfrak{h} \subsetneq \mathfrak{s} \subsetneq \mathfrak{g}$.

Indeed, assume $\mathfrak{h} \subset \mathfrak{s} \subset \mathfrak{g}$ and let us show that either $\mathfrak{s} = \mathfrak{g}$ or $\mathfrak{s} = \mathfrak{h}$. Denote by $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes \mathbb{C}$, $\mathfrak{h}_\mathbb{C} := \mathfrak{h} \otimes \mathbb{C}$, $\mathfrak{s}_\mathbb{C} := \mathfrak{s} \otimes \mathbb{C}$ the corresponding complex Lie algebras. It is enough to show that either $\mathfrak{s}_\mathbb{C} = \mathfrak{g}_\mathbb{C}$ or $\mathfrak{s}_\mathbb{C} = \mathfrak{h}_\mathbb{C}$.

View $\mathfrak{g}_\mathbb{C}$, $\mathfrak{h}_\mathbb{C}$ and $\mathfrak{s}_\mathbb{C}$ as modules over $\mathfrak{h}_\mathbb{C}$ with respect to the adjoint action. Then $\mathfrak{s}_\mathbb{C}/\mathfrak{h}_\mathbb{C} \subset \mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ is an inclusion of $\mathfrak{h}_\mathbb{C}$-modules. Since $\mathfrak{h}_\mathbb{C}$ is semi-simple (both in the toric and the hyper-Kähler cases), it is enough to show that $\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ is an irreducible $\mathfrak{h}_\mathbb{C}$-module.

The case of $\mathfrak{g} = \mathfrak{so}(p, q)$. The complexification of $\mathfrak{g} = \mathfrak{so}(p, q)$ is $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(p+q, \mathbb{C})$.

The algebra $\mathfrak{so}(n+1, \mathbb{C})$, $n \in \mathbb{N}$, is the algebra of complex skew-symmetric $(n+1) \times (n+1)$-matrices. A skew-symmetric $(n+1) \times (n+1)$-matrix is of the form

$$T_{a,v} := \left(\begin{array}{c|c} a & v \\ \hline -v^t & 0 \end{array} \right),$$

where a is a skew-symmetric $n \times n$-matrix and v is a $1 \times n$-column.

The embedding $\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{so}(n+1, \mathbb{C})$ corresponds to $v = 0$: $\mathfrak{so}(n, \mathbb{C}) = \{T_{a,0}\}$. Clearly, there is an isomorphism of vector spaces:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus E, \quad \text{where } E := \{T_{0,v}\} \subset \mathfrak{g}_\mathbb{C}.$$

Let us show that $E \cong \mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$ is an irreducible $\mathfrak{so}(n, \mathbb{C})$-module. Indeed, any element of $\mathfrak{so}(n, \mathbb{C})$ is $T_{a,0}$ and one readily sees that

$$[T_{a,0}, T_{0,v}] = T_{0,av}.$$

In particular, $(ad\, g)e \in E$ for each $g \in \mathfrak{so}(n, \mathbb{C}), e \in E$, so E is an $\mathfrak{so}(n, \mathbb{C})$-submodule of $\mathfrak{g}_\mathbb{C}$. Moreover, E is isomorphic to the standard module (that is, the n-dimensional complex vector space with the natural $\mathfrak{so}(n, \mathbb{C})$-action: $a \cdot v := av$

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[f]This proof is due to M. Gorelik — we thank her for communicating it to us.

for $a \in \mathfrak{so}(n, \mathbb{C})$). One easily checks that the standard module is irreducible and therefore $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ is irreducible as required.

The case of $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$. First, let us recall the following basic facts concerning representations of Lie algebras. Let $\mathfrak{k}$ be a complex Lie algebra and let V, W be $\mathfrak{k}$-modules.

We define the adjoint action of $\mathfrak{k}$ on $Hom(V, W)$ as follows:

$$((ad\, g)(\psi))(v) := g(\psi(v)) - \psi(gv) \quad \text{where } g \in \mathfrak{k}, \psi \in \mathrm{Hom}(V, W), v \in V.$$

(This action is called adjoint since $(ad\, g)\psi := [g, \psi] = g\psi - \psi g$.)

We define the $\mathfrak{k}$-module structure on $V^* := \mathrm{End}(V, \mathbb{C})$ by viewing the base field $\mathbb{C}$ as the trivial $\mathfrak{k}$-module (that is, $gv = 0$ for all $g \in \mathfrak{k}$, $v \in \mathbb{C}$). Then the $\mathfrak{k}$-module structure is given by

$$(gf)(v) = -f(gv), \quad \text{where } g \in \mathfrak{k}, f \in V^*, v \in V.$$

If V is finite-dimensional, then the natural isomorphism

$$V^* \otimes V \cong \mathrm{End}(V)$$

of vector spaces, given by $(f \otimes v)(v') := f(v')v$, is an isomorphism of $\mathfrak{k}$-modules.

If V is finite-dimensional and admits a non-degenerate $\mathfrak{k}$-invariant bilinear form $(\cdot, \cdot)$ (that is, such that $(gv, v') + (v, gv') = 0$ for all $g \in \mathfrak{k}$, $v, v' \in V$), then the canonical isomorphism $\iota : V \cong V^*$, given by $\iota(V)(v') = (v, v')$, is a $\mathfrak{k}$-isomorphism.

Let us now return to the case $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{R})$. The complex Lie algebra $\mathfrak{h}_{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$ is of type C_n. Let R be the natural representation of $\mathfrak{sp}(2n, \mathbb{C})$: it is a $2n$-dimensional complex vector space with the natural action of $\mathfrak{sp}(2n, \mathbb{C})$. One has $R^* \cong R$, since R admits a non-degenerate invariant bilinear form (the symplectic form).

Observe that $\mathfrak{g}_{\mathbb{C}} := \mathfrak{sl}(2n, \mathbb{C})$ is a $\mathfrak{h}_{\mathbb{C}}$-submodule in $\mathrm{End}(R) = \mathfrak{gl}(2n, \mathbb{C}) = \mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial $\mathfrak{h}_{\mathbb{C}}$-module. Thus, in order to show that $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ is irreducible, it is enough to verify that $\mathrm{End}(R)$ is the sum of three irreducible representations (they are automatically $\mathfrak{h}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ and $\mathbb{C}$).

Indeed, we have

$$\mathrm{End}(R) = R \otimes R = S^2(R) \oplus \Lambda^2(R),$$

where $S^2(R), \Lambda^2(R)$ are, respectively, the symmetric and the exterior squares of R. Using [33, Table 5], we obtain

$$S^2(R) = R(2\pi_1), \quad \Lambda^2(R) = R(\pi_2) \oplus R(\pi_0),$$

where $R(2\pi_1), R(\pi_2), R(\pi_0)$ are some irreducible representations of $\mathfrak{sp}(2n, \mathbb{C})$ (in fact, $R(\pi_1) = R$, $R(\pi_0) = \mathbb{C}$ and $R(2\pi_1) = \mathfrak{h}_{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$ is the adjoint representation of $\mathfrak{sp}(2n, \mathbb{C})$).

Thus, $\mathrm{End}(R) = \mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}$ is a sum of three irreducible $\mathfrak{h}_{\mathbb{C}}$-modules and therefore $\mathfrak{g}_{\mathbb{C}}$ is a sum of two irreducible $\mathfrak{h}_{\mathbb{C}}$-modules one of which is $\mathfrak{h}_{\mathbb{C}}$. Therefore $\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}}$ is an irreducible $\mathfrak{h}_{\mathbb{C}}$-module as required. This finishes the proof of Lemma 9.9. $\qed$

Proof of Lemma 9.10. The "if" part is obvious.

To prove the "only if" part let us assume by contradiction that the cohomology class η is not proportional to a rational one but at the same time H_η is a $\mathbb{Q}$-subgroup. Define $G^\mathbb{C} := \mathrm{SL}(2n, \mathbb{C}) \supset \mathrm{SL}(2n, \mathbb{R}) = G$ in the torus case and $G^\mathbb{C} := \mathrm{SO}(k, \mathbb{C}) \supset \mathrm{SO}(3, k - 3) = G$ in the hyper-Kähler case. We view η as a vector in $\mathbb{R}^k \subset \mathbb{C}^k$. The complex points of the real algebraic variety H_η defined over $\mathbb{Q}$ form a complex algebraic group $H_\eta^\mathbb{C} \subset G^\mathbb{C}$, also defined over $\mathbb{Q}$. The group $H_\eta^\mathbb{C}$ is the stabilizer of η under the action of $G^\mathbb{C}$ on $\mathbb{C}^k$.

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ of $\mathbb{C}$ over $\mathbb{Q}$ (that is, the group of field automorphisms of $\mathbb{C}$) acts coordinate-wise on $G^\mathbb{C}$ and on $\mathbb{C}^k$. Since $H_\eta^\mathbb{C}$ is defined over $\mathbb{Q}$, the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $G^\mathbb{C}$ maps $H_\eta^\mathbb{C}$ to itself. Therefore, since $H_\eta^\mathbb{C}$ is the stabilizer of η in $G^\mathbb{C}$, the action of any $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathbb{C}^k$ maps η to a vector $g(\eta) \in \mathbb{C}^k$ also stabilized by $H_\eta^\mathbb{C}$. Note that the space of vectors in $\mathbb{C}^k$ stabilized by $H_\eta^\mathbb{C}$ is the $\mathbb{C}$-span of η: in the hyper-Kähler case this is obvious ($H_\eta^\mathbb{C}$ acts transitively on the orthogonal complement of η with respect to the non-degenerate complex-valued inner product on $\mathbb{C}^k$ preserved by the action of $G^\mathbb{C}$), and in the torus case it follows from the fact that the space of complex linear 2-forms invariant under the action of $\mathrm{Sp}(2n, \mathbb{C}) \cong H_\eta^\mathbb{C}$ on $\mathbb{C}^{2n}$ equipped with the standard complex symplectic form Ω is the $\mathbb{C}$-span of Ω (see e.g. [18, Sec. 5.3.2]). Thus for all $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ the vectors $\eta = (\eta_1, \ldots, \eta_k)$ and $g(\eta) := (g(\eta_1), \ldots, g(\eta_k))$ are proportional: $g(\eta_i) = \lambda(g)\eta_i$ for all $i = 1, \ldots, k$ and some $\lambda(g) \in \mathbb{C}$ depending only on g.

Recall that $\eta \neq \overline{0}$ and thus $\eta' \neq \overline{0}$, $\lambda(g) \neq 0$. Without loss of generality, assume that the coordinate η_1 of η is nonzero. Thus for any $i = 1, \ldots, k$ and any $g \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we have

$$g\left(\frac{\eta_i}{\eta_1}\right) = \frac{g(\eta_i)}{g(\eta_1)} = \frac{\lambda(g)\eta_i}{\lambda(g)\eta_1} = \frac{\eta_i}{\eta_1}.$$

In other words, for each i the complex number η_i/η_1 is preserved by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. But the complex numbers with this property are exactly the rationals (a proof of this well-known fact can be easily extracted e.g. from [47]). Hence η_i/η_1 is rational for each $i = 1, \ldots, k$. This immediately implies that $\eta = (\eta_1, \ldots, \eta_k)$ is proportional to a vector with rational coordinates, in contradiction to our assumption. This finishes the proof of the lemma. $\square$

Acknowledgments

References

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