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On the Dynamical Coherence of Structurally Stable 3-diffeomorphisms

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Abstract—We prove that each structurally stable diffeomorphism f on a closed 3-manifold M^3 with a two-dimensional surface nonwandering set is topologically conjugated to some model dynamically coherent diffeomorphism.

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1. INTRODUCTION AND STATEMENT OF RESULTS

We consider a diffeomorphism f on a closed 3-manifold M^3 which satisfies Smale's axiom A (A-*diffeomorphism*). According to Smale's spectral theorem [14], the nonwandering set $NW(f)$ of f can be represented as a finite union of pairwise disjoint closed invariant sets, called *basic sets*, each of which contains a dense trajectory.

It is known that the existence of a basic set of codimension 0 or 1 implies strong constraints on the dynamics of f and topology of M^3. Indeed, if $NW(f)$ contains a basic set $\mathcal{B}$ with $dim\ \mathcal{B} = 3$, then f is an Anosov diffeomorphism and the manifold M^3 is the torus $\mathbb{T}^3$. The topological classification of Anosov diffeomorphisms on $\mathbb{T}^3$ was obtained by J. Franks in [3].

If dim $\mathcal{B} = 2$, then, due to [12], $\mathcal{B}$ is either an attractor or a repeller. It follows from [1] that any two-dimensional attractor (repeller) of an A-diffeomorphism $f : M^3 \to M^3$ is either an expanding attractor (contracting repeller) or a surface attractor (surface repeller). Recall that, due to [15], an attractor $\mathcal{B}$ of f is said to be *expanding* if the topological dimension of $\mathcal{B}$ is equal to the dimension of $W^u_x, x \in \mathcal{B}$. We say that $\mathcal{B}$ is a *contracting repeller* of f if it is an expanding attractor for f^{-1}. According to [6], a basic set $\mathcal{B}$ of a diffeomorphism $f : M^3 \to M^3$ is called a *surface basic set* if it is contained in an f-invariant closed surface $M^2_{\mathcal{B}}$ (not necessarily connected) topologically embedded in M^3. We say that *the nonwandering set $NW(f)$ is a two-dimensional surface set* if each of its basic sets is two-dimensional and surface.

It follows from [7] and [11] that any manifold M^3 which admits a structurally stable diffeomorphism $f : M^3 \to M^3$ with a two-dimensional expanding attractor (contracting repeller) is diffeomorphic to the torus $\mathbb{T}^3$, and moreover, f is topologically conjugated with the diffeomorphism obtained from Anosov diffeomorphism by the surgery operation (see Fig. 1).

In this paper we consider a class G of A-diffeomorphisms $f : M^3 \to M^3$ with two-dimensional surface nonwandering sets. Due to [4, 5], a manifold M^3 which admits a diffeomorphism $f \in G$ is

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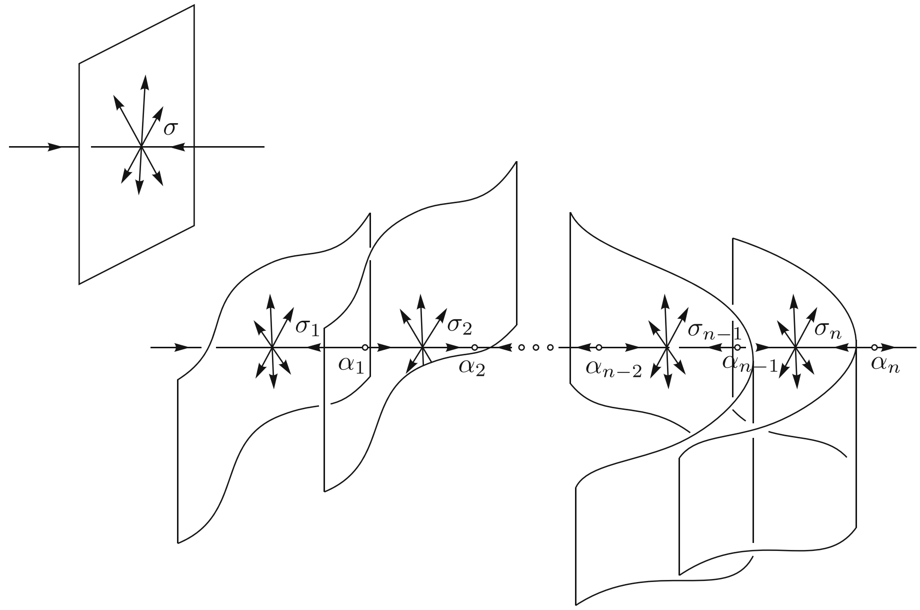

Fig. 1. Structurally stable diffeomorphism on $\mathbb{T}^3$ with a two-dimensional expanding attractor obtained from Anosov diffeomorphism by the surgery operation.

diffeomorphic to the mapping torus[1)] $M_{\widehat{J}}$, where $\widehat{J}$ is an algebraic automorphism[2)] of the torus defined by the matrix J which is either hyperbolic or equals the identity matrix $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or the matrix $-I = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Denote by $\mathcal{C}$ the set of the hyperbolic matrices from $GL(2,\mathbb{Z})$ and set $\mathcal{J} = \mathcal{C} \cup Id \cup (-Id)$.

Let us recall (see, for example, [2, 9]) that a diffeomorphism g on M^3 is *partially hyperbolic* if the following holds:

(i) there exists a continuous splitting of the tangent bundle $T_{M^3} = E^s \oplus E^c \oplus E^u$ invariant under the derivative Dg, where $\dim\ E^s = \dim\ E^c = \dim\ E^u = 1$;

(ii) the strong expansion of the unstable bundle E^u and the strong contraction of the stable bundle E^s dominate any expansion or contraction on the center E^c.

A partially hyperbolic diffeomorphism g is called *dynamically coherent* if there are g-invariant foliations tangent to both $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ and hence, there is a g-invariant foliation tangent to E^c.

In the next sections we construct models of dynamically coherent diffeomorphisms on $M_{\widehat{J}}$ and prove the following result.

Theorem 1. *Each structurally stable diffeomorphism $f \in G$ is topologically conjugated to a dynamically coherent model diffeomorphism.*

2. MODELS OF DYNAMICALLY COHERENT DIFFEOMORPHISMS

For the description of the dynamics of diffeomorphism from G let us recall the topological classification of structural stable diffeomorphisms on the circle $\mathbb{S}^1$.

Let $MS(\mathbb{S}^1)$ be a class of structural stable transformations of the circle, which is coincides, due to Mayer's results [10], with the class of Morse–Smale diffeomorphisms on $\mathbb{S}^1$. Divide $MS(\mathbb{S}^1)$

[1)] *The mapping torus* is the factor space M_τ obtained from $\mathbb{T}^2 \times [0,1]$ by identification of points $(z,1)$ and $(\tau(z),0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism.

[2)] *An algebraic automorphism* $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a map defined by the matrix $C = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ which belongs to the set $GL(2,\mathbb{Z})$ of integer matrices with determinant ± 1, i.e., $\widehat{C}(x,y) = (ax+by, cx+dy) \pmod 1$. $\widehat{C}$ is called *hyperbolic* if the absolute value of each eigenvalue is different from one. In the latter case the matrix C is also called *hyperbolic*.

into two subclasses $MS_+(\mathbb{S}^1)$ and $MS_-(\mathbb{S}^1)$ consisting of orientation-preserving and orientation-reversing diffeomorphisms, respectively. Below we formulate the results of Mayer on the topological classification of structural stable transformations.

Statement 1.

1. *For each diffeomorphism $\varphi \in MS_+(\mathbb{S}^1)$ the set $NW(\varphi)$ consists of $2n, n \in \mathbb{N}$ periodic orbits, each of them has period k.*
2. *For each diffeomorphism $\varphi \in MS_-(\mathbb{S}^1)$ the set $NW(\varphi)$ consists of $2q, q \in \mathbb{N}$ periodic points, two of them are fixed, the others have period 2.*

Let $\varphi \in MS_+(\mathbb{S}^1)$. Enumerate the periodic points from $NW(\varphi)$: $p_0, p_1, \ldots, p_{2nk-1}, p_{2nk} = p_0$ starting from arbitrary periodic point p_0 clockwise, then $\varphi(p_0) = p_{2nl}$, where l is an integer such that for $k = 1$, $l = 0$, while for $k > 1$, $l \in \{1, \ldots, k-1\}$ and (k, l) are coprime[3)]. Notice that the number l does not depend on the choice of the point p_0.

Statement 2.

1. *Two diffeomorphisms $\varphi; \varphi' \in MS_+(\mathbb{S}^1)$ with parameters $n, k, l; n', k', l'$ are topologically conjugated if and only if $n = n', k = k'$ and one of the following assertions holds:*
   - *$l = l'$ (hence, if $l \neq 0$, then the conjugating homeomorphism is orientation-preserving),*
   - *$l = k' - l'$ (hence, the conjugating homeomorphism is orientation-reversing).*
2. *Two diffeomorphisms $\varphi; \varphi' \in MS_-(\mathbb{S}^1)$ with parameters $q; q'$ are topologically conjugated if and only if $q = q'$.*

For $n, k \in \mathbb{N}$ and integer l such that for $k = 1$, $l = 0$, while for $k > 1$, $l \in \{1, \ldots, k-1\}$, let us construct a standard representative φ_+ in $MS_+(\mathbb{S}^1)$ with parameters n, k, l. For $q \in \mathbb{N}$ let us construct a standard representative φ_- in $MS_-(\mathbb{S}^1)$ with parameter q.

Let us represent $\mathbb{S}^1$ as $\mathbb{S}^1 = \{e^{i2\pi r} = (\cos 2\pi r, \sin 2\pi r) \in \mathbb{R}^2 : r \in \mathbb{R}\}$. Denote by $\pi : \mathbb{R} \to \mathbb{S}^1$ the projection given by the formula $\pi(r) = e^{i2\pi r}$. Let us introduce the following maps:
$\psi_m : \mathbb{R} \to \mathbb{R}$ is the time-one map of the flow generated by $\dot{r} = ln\ \mu \cdot \sin(2\pi m r)$ for $m \in \mathbb{N}$, $\mu > 1$;
$\chi_{k,l} : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given by the formula $\chi_{k,l}(r) = r - \frac{l}{k}$;
$\chi : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given by the formula $\chi(r) = -r$;
$\tilde{\varphi}_+ = \psi_{n \cdot k} \chi_{k,l} : \mathbb{R} \to \mathbb{R}$ and $\tilde{\varphi}_- = \psi_q \chi : \mathbb{R} \to \mathbb{R}$.
Let $\sigma \in \{+, -\}$, then it can be directly verified that $\tilde{\varphi}_\sigma(r + \nu) = \tilde{\varphi}_\sigma(r)$ for $\nu \in \mathbb{Z}$. Hence the following diffeomorphisms are well defined: $\varphi_\sigma = \pi \tilde{\varphi}_\sigma \pi^{-1} : \mathbb{S}^1 \to \mathbb{S}^1$. Here $\pi^{-1}(s)$ is a complete preimage of point $s \in \mathbb{S}^1$.

Using φ_σ and a hyperbolic matrix C, let us construct the model diffeomorphism ϕ_σ on $M_{\widehat{J}}$ for $J \in \mathcal{J}$ from the class G.

Denote by $\tilde{\phi}_\sigma : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ the product of the diffeomorphism $\tilde{\varphi}_\sigma$ and the automorphism $\widehat{C}$, $C \in \mathcal{C}$, that is, $\tilde{\phi}_\sigma(z, r) = (\widehat{C}(z), \tilde{\varphi}_\sigma(r))$. As $M_{\widehat{J}} = (\mathbb{T}^2 \times \mathbb{R})/\Gamma$ and Γ is a cyclic group with generator $\gamma(z, r) = (\widehat{J}(z), r - 1)$, then either $\tilde{\phi}_\sigma \gamma = \gamma \tilde{\phi}_\sigma$ or $\tilde{\phi}_\sigma \gamma^{-1} = \gamma \tilde{\phi}_\sigma$ is a necessary and sufficient condition to well define the diffeomorphism $\phi_\sigma : M_{\widehat{J}} \to M_{\widehat{J}}$ by the formula $\phi_\sigma = p_{_J} \tilde{\phi}_\sigma p_{_J}^{-1}$ (see, for example, [8]). Hence it follows that $CJ = JC$ for $\sigma = +$ and $CJ^{-1} = JC$ for $\sigma = -$. The condition $CJ^{-1} = JC$ implies $C^2 J = JC^2$. Then $\widehat{J}$ belongs to the centralizer of $\widehat{C}^2$ and, hence, due to [13], $J \in \{Id, -Id\}$ for $\sigma = -$. Thus we get such a description of the models.

Let $J_+ \in \mathcal{J}$ and $C_+ \in \mathcal{C}$ such that $C_+ J_+ = J_+ C_+$. Let $J_- \in \{Id, -Id\}$ and $C_- \in \mathcal{C}$. Set $\tilde{\phi}_\sigma(z, r) = (\widehat{C}_\sigma(z), \tilde{\varphi}_\sigma(r))$. It is immediately verified that $\tilde{\phi}_\sigma \gamma_\sigma = \gamma_\sigma \tilde{\phi}_\sigma$, where $\gamma_\sigma(z, r) = (J_\sigma(z), r - 1)$ is the generator of the group $\Gamma_\sigma = \{\gamma_\sigma^i, i \in \mathbb{Z}\}$. Then the following concept is well defined.

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3) Indeed, instead the number l A. G. Mayer used the number r_1, which he called *ordering number*, such that $l \cdot r_1 \equiv 1 \pmod{k}$.

Definition 1. *We say that the diffeomorphism $\phi_\sigma : M_{\widehat{J}_\sigma} \to M_{\widehat{J}_\sigma}, \sigma \in \{+,-\}$ is a locally direct product of $\widehat{C}_\sigma$ and φ_σ, if $\phi_\sigma = p_{J_\sigma} \tilde{\phi}_\sigma p^{-1}_{J_\sigma}$ and write $\phi_\sigma = \widehat{C}_\sigma \otimes \varphi_\sigma$.*

Let us denote by Φ_+ (Φ_-) the set of all locally direct products ϕ_+ (ϕ_-). Set $\Phi = \Phi_+ \cup \Phi_-$. By the construction each diffeomorphism ϕ_σ from Φ_σ has a two-dimensional surface nonwandering set and, moreover, ϕ_σ is dynamically coherent if $\mu < |\lambda|$, where $|\lambda|$, $\frac{1}{|\lambda|}$ are absolute values of eigenvalues of C_σ.

Now let $f \in G$. Denote by $\mathcal{A}$ and $\mathcal{R}$ the attractor and the repeller of f, respectively, such that $NW(f) = \mathcal{A} \cup \mathcal{R}$.

Definition 2. *We say that $f \in G$ is topologically coherent if the following properties hold:*

(i) *if the intersection $W^s(x) \cap W^u(y)$ is nonempty for some points $x \in \mathcal{A}$, $y \in \mathcal{R}$, then each of its connected components is an open arc which has exactly two boundary points, one of them being from $\mathcal{A}$ and the other from $\mathcal{R}$ (see Fig. 2);*

(ii) *There is an f-invariant one-dimensional continuous foliation on M^3, each of its leaves being the union of the closure of arcs determined in (i).*

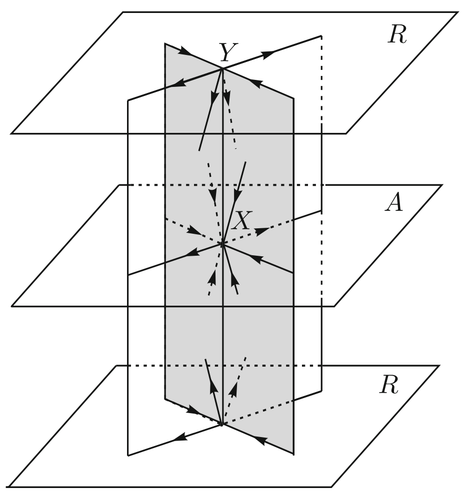

Fig. 2. Topologically coherent diffeomorphism.

The following facts were proved in [5].

Statement 3.

(I) *each $f \in G$ is Ω-conjugated[4)] with a dynamically coherent diffeomorphism ϕ from Φ;*

(II) *if $f \in G$ is topologically coherent, then f is topologically conjugated with a dynamically coherent diffeomorphism ϕ from Φ.*

In the next section we will prove that an arbitrary structurally stable diffeomorphism f from G is topologically coherent, which implies that Theorem 1 holds.

[4)] Two diffeomorphisms $f : M^3 \to M^3$, $f' : M'^3 \to M'^3$ are said to be Ω-*cojugated* if there exists a homeomorphism $h : M^3 \to M'^3$ such that $h(NW(f)) = NW(f')$ and $hf|_{NW(f)} = f'h|_{NW(f)}$.

3. THE EXISTENCE OF A ONE-DIMENSIONAL FOLIATION

This section is devoted to the proof of the following fact.

Lemma 1. *Each structurally stable diffeomorphism f from G is topologically coherent.*

Proof. Let us divide the proof into three steps.

Step 1. By Statement 3, f is Ω-conjugated with some diffeomorphism $\phi : M_{\widehat{J}} \to M_{\widehat{J}}$ from the class Φ by means a homeomorphism $h : M^3 \to M_{\widehat{J}}, J \in \mathcal{J}$. To prove the lemma, it is enough to assume that the diffeomorphism ϕ belongs to Φ_+ and is defined by parameters $C \in \mathcal{C}$, $n \in \mathbb{N}$, $k = 1, l = 0$ (in the opposite case we can take some power of f). Set $\psi = hfh^{-1} : M_{\widehat{J}} \to M_{\widehat{J}}$. By the construction ψ is topologically conjugated with f by means of h. Then we will apply to the homeomorphism ψ the notions and denotations of the nonwandering set and also stable and unstable manifolds of nonwandering points, understanding this as a preimage under h of similar objects for the diffeomorphism f. Thus ψ coincides with ϕ on their common nonwandering set and its lift $\tilde{\psi} : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ coincides with $\tilde{\phi}$ on the set $\mathbb{T}^2 \times (\bigcup\limits_{i\in\mathbb{Z}} \frac{i}{2n})$.

Denote by $p : \mathbb{R}^2 \to \mathbb{T}^2$ a universal cover such that $p(x, y) = (x \pmod 1, y \pmod 1)$. Let the map $\widehat{C} : \mathbb{T}^2 \to \mathbb{T}^2$ given by the matrix $C = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\check{C} : \mathbb{R}^2 \to \mathbb{R}^2$ diffeomorphism given by the formula $\check{C}(x, y) = (ax + by, cx + dy)$. Let $(\hat{x}_0, \hat{y}_0)$ be some fixed point of $\widehat{C}$ and $x_0, y_0 \in [0, 1)$ such that $p(x_0, y_0) = (\hat{x}_0, \hat{y}_0)$. Then there are integers α, β such that $\check{C}(x_0, y_0) = (x_0 + \alpha, y_0 + \beta)$. Hence $\check{\psi}_0(x, y) = (ax + by - \alpha, cx + dy - \beta)$ is the lift of $\widehat{C}$ with respect to p with the unique fixed point (x_0, y_0). Let $\eta : \mathbb{R}^3 \to \mathbb{T}^2 \times \mathbb{R}$ be a cover given by the formula $\eta(x, y, z) = (p(x, y), z)$. Denote by $\check{\psi} : \mathbb{R}^3 \to \mathbb{R}^3$ the lift of $\tilde{\psi}$ with respect to η coinciding with $\check{\psi}_0$ on $\Pi_0 = \mathbb{R}^2 \times \{0\}$.

As $\check{\psi}$ is a lift of $\tilde{\psi}$ and $\tilde{\psi}_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix C then $\check{\psi} b = b'\check{\psi}$, where $b : \mathbb{R}^3 \to \mathbb{R}^3$ given by formula $b(x, y, z) = (x + \nu_1, y + \nu_2, z)$, $(\nu_1, \nu_2) \in \mathbb{Z}^2$ and $b'(x, y, z) = (x + a\nu_1 + b\nu_2, y + c\nu_1 + d\nu_2, z)$. Thus the diffeomorphism $\check{\psi}(x, y, z) = (\check{\psi}_1(x, y, z), \check{\psi}_2(x, y, z), \check{\psi}_3(x, y, z))$ has the form

$$\begin{aligned} \check{\psi}_1(x, y, z) &= ax + by + h_1(x, y, z), \\ \check{\psi}_2(x, y, z) &= cx + dy + h_2(x, y, z), \\ \check{\psi}_3(x, y, z) &= h_3(x, y, z), \end{aligned} \qquad (*)$$

where $h_j(x + \nu_1, y + \nu_2, z) = h_j(x, y, z), j = 1, 2, 3$ for each $\nu_1, \nu_2 \in \mathbb{Z}$.

Step 2. For the stable (unstable) manifold $W^s(x)$ $(W^u(x))$ of the nonwandering point $x \in NW(\psi)$ of homeomorphism ψ we will denote by $w^s(\check{x})$ $(w^u(\check{x}))$ the connected component of the set $\eta^{-1}(W^s(x))$ $\eta^{-1}(W^u(x))$ passing through the point $\check{x}$. The homeomorphism $\check{\psi}$ has exactly one fixed saddle point P_i belonging to the plane $\Pi_i = \mathbb{R}^2 \times \{\frac{i}{2n}\}$ for each $i \in \mathbb{Z}$. Let us show that the intersection $(cl\ w^u(P_0)) \cap \Pi_1 = w^u(P_1)$.

For this purpose let us introduce the notion of (x, y)-*asymptotic direction* for a curve $\gamma(t) = (x(t), y(t), z(t)), t \in \mathbb{R}$ in $\mathbb{R}^3$ as $\lim\limits_{t\to\pm\infty} \frac{y(t)}{x(t)}$.

Notice that the homeomorphism $\check{\psi}$ possesses two transversal one-dimensional $\check{\psi}$-invariant foliations F_0^s, F_0^u (F_1^s, F_1^u) on Π_0 (Π_1) consisting of parallel straight lines with different (x, y)-asymptotic directions μ_s and μ_u which are irrational numbers. Let $L_0^u(P_0)$ $(L_1^s(P_1))$ be a leaf of foliation F_0^u (F_1^s) passing through the point P_0 (P_1). Further it is useful to look at Fig. 3.

Choose a closed box B_u (B_s) bounded by the planes $\Pi_0, \Pi_1, Q_1^u = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b_1^u\}$, $Q_2^u = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_u x + b_2^u\}$ $(\Pi_0, \Pi_1, Q_1^s = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b_1^s\}$, $Q_2^s = \{(x, y, z) \in \mathbb{R}^3 : y = \mu_s x + b_2^s\})$ and containing $L_0^u(P_0)$ $(L_1^s(P_1))$. Then there is a homeomorphism $h_u : \mathbb{R} \times [0, 1] \to W^u(P_0)$ $(h_s : \mathbb{R} \times [0, 1] \to W^s(P_1))$ such that $h_u(\mathbb{R} \times \{0\}) = L_0^u(P_0)$ $(h_s(\mathbb{R} \times \{0\}) = L_1^s(P_1))$ and the collar $N(L_0^u(P_0)) = h_u(\mathbb{R} \times [0, 1])$ $(N(L_1^s(P_1)) = h_s(\mathbb{R} \times [0, 1]))$ is situated in B_u (B_s). Thus the curve $l_u = h_u(\mathbb{R} \times \{1\})$ $(l_s = h_s(\mathbb{R} \times \{1\}))$ has the (x, y)-asymptotic direction μ_u (μ_s).

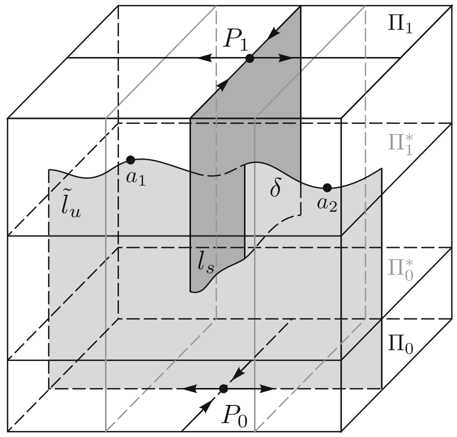

Fig. 3. Box $\tilde{B}_u$.

Let us choose $z_0^*, z_1^* \in (0, \frac{1}{2n})$ such that the planes $\Pi_0^* = \{(x,y,z)\in\mathbb{R}^3 : z = z_0^*\}, \Pi_1^* = \{(x,y,z)\in \mathbb{R}^3 : z = z_1^*\}$ divide the domain $\mathbb{R}^2 \times (0, \frac{1}{2n})$ into three open sets V_0, V, V_1 with the properties $(cl\ V_0) \setminus V_0 = \Pi_0 \cup \Pi_0^*, (cl\ V_1) \setminus V_1 = \Pi_1 \cup \Pi_1^*$ and $l_u, l_s \subset V$. Let us show that there is a number $k_* \in \mathbb{N}$ such that the curve $\tilde{l}_u = \check{\psi}^{k_*}(l_u)$ is a subset of V_1.

Indeed, $\eta(cl\ V) = \mathbb{T}^2 \times [z_0^*, z_1^*]$ and each point $t \in \eta(cl\ V)$ is wandering for $\tilde{\psi}$ and its positive iterations go to the attractor $T_1 = \eta(\Pi_1)$. Then there is a neighborhood $U_t \subset \mathbb{T}^2 \times (0, \frac{1}{2n})$ of the point t and a natural number $k(t)$ such that $\tilde{\psi}^k(U_t) \subset \eta(V_1)$ for $k \geqslant k(t)$. As the set $\eta(cl\ V)$ is compact, there is a finite subcover for the cover $\{U_t, t \in \eta(cl\ V)\}$. Thus there is a natural number k_* such that $\tilde{\psi}^k(\eta(cl\ V)) \subset \eta(V_1)$ for $k \geqslant k_*$. Hence, $\check{\psi}^k(cl\ V) \subset V_1$ for $k \geqslant k_*$ and also $\tilde{l}_u \subset V_1$.

Due to the form (*), the collar $\tilde{N}(L_0^u(P_0)) = \check{\psi}^{k_*}(N(L_0^u(P_0)))$ belongs to the box $\tilde{B}_u$ bounded by the planes $\Pi_0, \Pi_1, \tilde{Q}_1^u = \{(x,y,z) \in \mathbb{R}^3 : y = \mu_u x + \tilde{b}_1^u\}, \tilde{Q}_2^u = \{(x,y,z) \in \mathbb{R}^3 : y = \mu_u x + \tilde{b}_2^u\}$ for some numbers $\tilde{b}_1^u, \tilde{b}_2^u$. Hence the curve $\tilde{l}_u$ has the (x,y)-asymptotic direction μ_u. By the construction $\tilde{l}_u$ is situated between Π_1^* and Π_1. Then $\tilde{l}_u \cap B_s \neq \varnothing$ and, moreover, there is an arc $\delta \subset (\tilde{l}_u \cap B_s)$ which has end points $a_1 \in Q_1^s, a_2 \in Q_2^s$. Since $l_s \subset V$, we have $\delta \cap N(L_1^s(P_1)) \neq \varnothing$.

Thus $w^u(P_0) \cap w^s(P_1) \neq \varnothing$. As the original diffeomorphism f is structurally stable, the intersection $w^u(P_0) \cap w^s(P_1)$ is topologically transversal. So δ is topologically transversal to $w^s(P_1)$. Since in a neighborhood of P_1 the homeomorphism $\check{\psi}$ is topologically conjugated with the hyperbolic saddle point, by the λ-lemma we have $w^u(P_1) \subset cl\ (\bigcup\limits_{n\geqslant 1} \check{\psi}^n(\delta))$. Hence $w^u(P_1) \subset (cl\ w^u(P_0))$. On the other hand, $N(L_0^u(P_0)) \subset B_u$ and the set $K_u = (\tilde{N}(L_0^u(P_0)) \setminus N(L_0^u(P_0)))$ belongs to a part $\tilde{B}_u^*$ of $\tilde{B}_u$ bounded by Π_0^* and Π_1.

Set $w_+^u(P_0) = w^u(P_0) \cap (\mathbb{R}^2 \times [0, \frac{1}{2n}))$. Let us represent the set $w_+^u(P_0)$ as the union $w_+^u(P_0) = N(L_0^u(P_0)) \cup \bigcup\limits_{i\in\mathbb{N}} f^i(K_u)$. Hence, $N(L_0^u(P_0)) \subset B_u$ and $\bigcup\limits_{i\in\mathbb{N}} \check{\psi}^i(K_u) \subset \bigcup\limits_{i\in\mathbb{N}} \check{\psi}^i(\tilde{B}_u^*))$. Since Π_1 is an attractor for $\check{\psi}$ on $\mathbb{R}^2 \times (-\frac{1}{2n}, \frac{1}{2n})$ and due to (*) we have $\check{\psi}^i(\tilde{B}_u^*) \subset \tilde{B}_u^*$ for some $i \in \mathbb{N}$. Without loss of generality we can assume that $i = 1$. Thus $\bigcup\limits_{i\in\mathbb{N}} \check{\psi}^i(K_u) \subset \tilde{B}_u^*$ and hence $(cl\ w_+^u(P_0) \setminus w_+^u(P_0)) \subset \tilde{B}_u$. Since $cl\ w_+^u(P_0) \setminus w_+^u(P_0)$ is $\check{\psi}$-invariant, $(cl\ w_+^u(P_0) \setminus w_+^u(P_0)) \subset \bigcap\limits_{i\in\mathbb{N}} \check{\psi}^i(\tilde{B}_u^*)$. As $w^u(P_1)$ is an attractor for $\check{\psi}|_{\Pi_1}$ and $w^u(P_1)$ is a unique $\check{\psi}$-invariant set in Π_1, we have $(cl\ w^u(P_0)) \cap \Pi_1 = w^u(P_1)$.

Step 3. On the set $\mathbb{R}^2 \times [0, \frac{1}{2n})$ there is a $\check{\psi}$-invariant two-dimensional foliation R_0 each leaf of which is homeomorphic to the semiplane and coincides with $w^u(x) \cap (\mathbb{R}^2 \times [0, \frac{1}{2n}))$ for some point $x \in \Pi_0$. Since $\psi|_{\eta(\Pi_0)}$ is a hyperbolic automorphism, the projection (by virtue of η) of an arbitrary point with the rational coordinates on Π_0 is a periodic point of $\psi|_{\eta(\Pi_0)}$. As each periodic point of ψ is the fixed point for some power of ψ, for an arbitrary leaf G_0 of foliation R_0 passing through a point with the rational coordinates on Π_0 we have $cl\ (G_0) \cap \Pi_1 = \{(x,y,z) \in \Pi_1 : y = \mu_u x + b_{G_0}\}$ for some $b_{G_0} \in \mathbb{R}$. By the continuity we have the same property for an arbitrary leaf G_0 of foliation R_0.

Analogously, on the set $\mathbb{R}^2 \times (0, \frac{1}{2n}]$ there is a $\check{\psi}$-invariant two-dimensional foliation R_1 each leaf of which is homeomorphic to the semiplane and coincides with $w^s(x) \cap (\mathbb{R}^2 \times [0, \frac{1}{2n}))$ for some point $x \in \mathbb{T}^2 \times \{\frac{1}{2n}\}$. Similarly, for an arbitrary leaf G_1 of foliation R_1 we have $cl\ (G_1) \cap \Pi_0 = \{(x,y,z) \in \Pi_0 : y = \mu_s x + b_{G_1}\}$ for some $b_{G_1} \in \mathbb{R}$.

Thus the intersection $Y = G_0 \cap G_1$ is not empty for each leaf $G_0 \in R_0, G_1 \in R_1$ and $cl\ (Y) \setminus Y$ consists of two points $P_{G_0} \in \Pi_0, P_{G_1} \in \Pi_1$. Moreover, $cl\ Y$ is compact as G_0 and G_1 have different (x,y)-asymptotic directions. As f is structurally stable and it is topologically conjugated with ψ, the set $cl(Y)$ consists of curves. If we suppose that the number of curves in $cl(Y)$ is greater than one, then there appears an open 2-disc $D \subset cl(G_0)$, bounded by curves from $cl(Y)$ foliated by closed curves, which are an intersection of the leaves of foliation R_1 with D. This is in contradiction with the fact that an open 2-disc does not admit a foliation without singularities that consists of simple closed curves. □

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