arXiv:1209.3960v1 [math.AG] 18 Sep 2012

DESINGULARIZATION OF QUIVER GRASSMANNIANS FOR DYNKIN QUIVERS

G. CERULLI IRELLI, E. FEIGIN, M. REINEKE

Abstract. A desingularization of arbitrary quiver Grassmannians for representations of Dynkin quivers is constructed in terms of quiver Grassmannians for an algebra derived equivalent to the Auslander algebra of the quiver.

1. Introduction

Quiver Grassmannians are varieties parametrizing subrepresentations of a quiver representation. They first appeared in [7, 16] in relation to questions on generic properties of quiver representations; later it was observed (see [3]) that these varieties play an important role in cluster algebra theory [11] since cluster variables can be described in terms of the Euler characteristic of quiver Grassmannians. A rather well-studied specific class of quiver Grassmannians are the varieties of subrepresentations of exceptional quiver representations since they are smooth projective varieties, see for example [4].

In [5, 6] the authors of the present paper initiated a systematic study of a class of singular quiver Grassmannians, starting from the observation that the type A degenerate flag varieties of [8, 9, 10] are quiver Grassmannians. Namely, Grassmannians of subrepresentations of the direct sum $P \oplus I$ of a projective representation P and an injective representation I of a Dynkin quiver Q, of the same dimension vector as P, are considered. They are shown in [5] to be reduced irreducible normal local complete intersection varieties, admitting a group action with finitely many orbits, as well as a cell decomposition. Moreover, a detailed description of the singular locus of degenerate flag varieties is given in [6].

In pursuing the analysis of singular quiver Grassmannians, it is thus desirable to have an explicit desingularization at our disposal; this is done in [10] for the type A degenerate flag varieties.

The main result (see Section 7) of the present work is that an appropriate representation theoretic re-interpretation of the construction of [10] generalizes, and provides desingularizations of arbitrary quiver Grassmannians over Dynkin quivers. In fact, the desingularization is itself a quiver Grassmannian over a certain quiver $\widehat{Q}$, and can be described explicitly once the irreducible components of the quiver Grassmannian to be desingularized are known. Moreover, every fibre of the desingularization map is a quiver Grassmannian over $\widehat{Q}$ itself.

At the heart of the construction of the desingularization lies the definition of a certain algebra B_Q of global dimension at most two (see Section 4), together with a fully faithful functor Λ from the category of representations of Q to the one of B_Q with special homological properties. Namely, the essential image of Λ consists of representations of B_Q without self-extensions and of projective and injective dimension at most 1 (see Section 5).

The algebra B_Q arises as the endomorphism ring of an additive generator of a certain category $\mathcal{H}_Q$ of embeddings of projective representations of Q (see Section 3);

it is derived equivalent (but not Morita equivalent) to the Auslander algebra of Q (see Section 6).

The algebra B_Q and the functor Λ should be of independent interest. The authors believe that the associated representation varieties are related to graded Nakajima quiver varieties in the spirit of [14, 15], and that they admit applications to the geometry of orbit closures in representation varieties of Q. These topics will be discussed elsewhere.

Although (or precisely because?) the construction of the desingularization is a very general and conceptual one, it is still nontrivial to analyse, say, the dimension of the singular fibres; no general formulas or estimates are known at the moment. But a detailed analysis of all cases in type A_2 is given, suggesting that good geometric properties of the desingularization (like, for example, being one to one precisely over the smooth locus) can only be expected precisely in the case considered above, namely the quiver Grassmannians generalizing the type A degenerate flag varieties (see Section 8).

The paper is organized as follows. In Section 2 we collect some standard material on categories of functors and Auslander-Reiten theory. In Section 3 the category $\mathcal{H}_Q$ is introduced and in Section 4 the homological properties of the category $\mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$ are studied. In Section 5 we define the key functor $\Lambda : \mathrm{mod}\,kQ \to \mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$. Section 6 contains the computation of the ordinary quiver of the algebra B_Q and several examples. In Section 7 we use the results of the previous sections to construct the desingularizations of quiver Grassmannians. Finally, in Section 8 we close with examples of desingularizations.

2. Reminder on categories of functors and Auslander-Reiten theory

We collect some standard material (see e.g. [1, IV.6., A.2.], [12, 3.]) on the functorial approach to the representation theory of algebras and to Auslander-Reiten theory in particular. Throughout the paper, we will make free use of the basic concepts of Auslander-Reiten theory [1, IV.], like almost split maps, Auslander-Reiten sequences, the Auslander-Reiten translation and its relation to the Nakayama functor, and the structure of the Auslander-Reiten quiver of a Dynkin quiver.

Let k be a field. All categories in the following are assumed to be k-linear, finite length (that is, all objects admit finite composition series) Krull-Schmidt (that is, the Krull-Schmidt theorem holds) categories with finite dimensional morphism spaces.

For a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ its opposite category, and by $\mathrm{mod}\,\mathcal{C}^{\mathrm{op}}$ the category of k-linear additive, contravariant functors from $\mathcal{C}$ to $\mathrm{mod}\,k$, the category of finite dimensional k-vector spaces. For an object M of $\mathcal{C}$, the functor $\mathrm{Hom}(\_, M)$ is an object of $\mathrm{mod}\,\mathcal{C}^{\mathrm{op}}$. By Yoneda's lemma, we have

$$\mathrm{Hom}_{\mathrm{mod}\ \mathcal{C}^{\mathrm{op}}}(\mathrm{Hom}(\_, M), F) \simeq F(M)$$

for every $M \in \mathcal{C}$, $F \in \mathrm{mod}\,\mathcal{C}^{\mathrm{op}}$. From this we can conclude that $\mathrm{Hom}(\_, M)$ is a projective object of $\mathrm{mod}\,\mathcal{C}^{\mathrm{op}}$; in fact, every projective object is of this form. Dually, the injective objects in $\mathrm{mod}\ \mathcal{C}^{\mathrm{op}}$ are the functors $\mathrm{Hom}(M, \_)^*$ for $M \in \mathcal{C}$, where V^* denotes the linear dual of a k-vector space V. Moreover, the simple objects in $\mathrm{mod}\ \mathcal{C}^{\mathrm{op}}$ are parametrized by the isomorphism classes of indecomposable objects in $\mathcal{C}$: for such an object U, there exists a unique simple functor S_U which is a quotient of $\mathrm{Hom}(\_, U)$ and embeds into $\mathrm{Hom}(U, \_)^*$.

For a finite dimensional k-algebra A, let $\mathrm{mod}\,A$ be the category of finite dimensional (left) A-modules. We can consider the subcategory $\mathrm{proj}\,A$ of $\mathrm{mod}\,A$, which is the category of finite dimensional projective A-modules. Then mod $(\operatorname{proj} A)^{\mathrm{op}}$ is equivalent to mod A by associating to a module M the functor $\operatorname{Hom}(\_, M)$.

Somewhat conversely, assume that $\mathcal{C}$ admits only finitely many indecomposables; let $U_1, \ldots, U_N$ be a system of representatives. Then $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$ is equivalent to $\operatorname{mod} B(\mathcal{C})$, where $B(\mathcal{C}) := \operatorname{End}_{\mathcal{C}}(\bigoplus_i U_i)^{\mathrm{op}}$; namely, by the above, $\operatorname{Hom}(\_, \bigoplus_i U_i)$ is a projective generator of $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$. In particular, we have $B(\operatorname{proj} A) \simeq A$. If A admits only finitely many indecomposables, the algebra $B(\operatorname{mod} A)$ is called the Auslander algebra of A.

The structure of $\operatorname{mod}(\operatorname{mod} A)^{\mathrm{op}}$ is related to Auslander-Reiten theory: the simple functors are precisely those of the form S_U for an indecomposable U in $\operatorname{mod} A$, where S_U (being additive) is determined on indecomposables by $S_U(U) = k$ and $S_U(V) = 0$ for all $V \not\simeq U$. If $0 \to \tau U \to B \to U \to 0$ is the Auslander-Reiten sequence ending in U, then

$$0 \to \operatorname{Hom}(\_, \tau U) \to \operatorname{Hom}(\_, B) \to \operatorname{Hom}(\_, U) \to S_U \to 0$$

is a projective resolution of S_U in $\operatorname{mod}(\operatorname{mod} A)^{\mathrm{op}}$. In particular, this category has global dimension at most two.

For the category $\mathcal{C}$, we can consider the category $\operatorname{Hom}\mathcal{C}$ with objects being morphisms $f : M \to N$ in $\mathcal{C}$, and with morphisms from $f : M \to N$ to $f' : M' \to N'$ being pairs $(\phi : M \to M', \psi : N \to N')$ of morphisms such that $\psi f = f'\phi$; composition is defined naturally. We also consider the full subcategories $\operatorname{Hom}^{\mathrm{mono}}\mathcal{C}$ (resp. $\operatorname{Hom}^{\mathrm{iso}}\mathcal{C}$) with objects the monomorphisms (resp. isomorphisms) between objects of $\mathcal{C}$.

We sometimes denote by $\operatorname{ind} A$ the set of indecomposable (finite dimensional, left) A–modules.

3. The category $\mathcal{H}_Q$

From now on, let Q be a Dynkin quiver with set of vertices Q_0, and let kQ be its path algebra. We denote by S_i the simple left module corresponding to a vertex i of Q, and by P_i (resp. I_i) its projective cover (resp. injective hull). Then every object of $\operatorname{proj} kQ$ is isomorphic to a finite direct sum of the P_i.

We consider the category $\operatorname{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)$; its objects are thus injective maps between projective representations of Q. We note a few obvious formulas for morphisms in this category whose verification is immediate:

Lemma 3.1. *For all projectives P, Q and R, we have in $\operatorname{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)$:*

- (i) $\operatorname{Hom}(P \stackrel{\mathrm{id}}{\to} P, Q \to R) \simeq \operatorname{Hom}(P, Q)$,
- (ii) $\operatorname{Hom}(P \to Q, R \stackrel{\mathrm{id}}{\to} R) \simeq \operatorname{Hom}(Q, R)$,
- (iii) $\operatorname{Hom}(0 \to P, Q \to R) \simeq \operatorname{Hom}(P, R)$,
- (iv) *Every non-zero morphism in* $\operatorname{Hom}(P \to Q, 0 \to R)$ *is split.*

Taking the cokernel of an injective map between projectives induces a functor $\operatorname{Coker} : \operatorname{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ) \to \operatorname{mod} kQ$; on morphisms it is defined by mapping a pair (φ, ψ) to the unique f making the following diagram commutative:

$$\begin{array}{ccccccc} 0 & \to & P & \stackrel{\iota}{\to} & Q & \to & \operatorname{Coker}\iota & \to 0 \\ & & \varphi\downarrow & & \psi\downarrow & & f\downarrow & \\ 0 & \to & P' & \stackrel{\iota'}{\to} & Q' & \to & \operatorname{Coker}\iota' & \to 0 \end{array}$$

Lemma 3.2. *The functor* Coker *is full and dense. We have*

$$\operatorname{Hom}((P \stackrel{\iota}{\to} Q), (P' \stackrel{\iota'}{\to} Q'))/\operatorname{Hom}(Q, P') \simeq \operatorname{Hom}_{\operatorname{mod} kQ}(\operatorname{Coker}\iota, \operatorname{Coker}\iota').$$

Proof. Given $M \in \operatorname{mod} kQ$, there exists a projective resolution $0 \to P \xrightarrow{\iota} Q \to M \to 0$, thus $\iota : P \to Q$ is an object of $\mathrm{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)$ mapping to M under Coker. This proves density. Every map $f : M \to N$ between representations lifts over projective resolutions as in the above diagram, which proves that Coker is full. Again in the above diagram, the induced morphism f on cokernels is 0 if and only if ψ factors over ι', which proves the claimed isomorphism. ☐

Corollary 3.3. *The functor* Coker *induces an equivalence between the quotient category* $\mathrm{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)/\mathrm{Hom}^{\mathrm{iso}}(\operatorname{proj} kQ)$ *and* $\operatorname{mod} kQ$*.*

Proof. If a morphism $\mathrm{Hom}((P \xrightarrow{\iota} Q), (P' \xrightarrow{\iota'} Q'))$ is of the form $(h\iota, \iota' h)$ for some $h \in \mathrm{Hom}(Q, P')$, it admits a factorization

$$(P \xrightarrow{\iota} Q) \xrightarrow{(\iota,\mathrm{id})} (Q \xrightarrow{\mathrm{id}} Q) \xrightarrow{(h,\iota' h)} (P' \xrightarrow{\iota'} Q').$$

Conversely, a factorization

$$(P \xrightarrow{\iota} Q) \xrightarrow{(\varphi\iota,\varphi)} (R \xrightarrow{\mathrm{id}} R) \xrightarrow{(\psi,\iota'\psi)} (P' \xrightarrow{\iota'} Q')$$

for $\varphi : P \to R$ and $\psi : R \to Q'$ yields a map $h = \psi\varphi$ as above. Combining this with the previous lemma, the statement follows. ☐

Proposition 3.4. *Up to isomorphism, the indecomposable objects of the category* $\mathrm{Hom}^{\mathrm{mono}}\operatorname{proj} kQ$ *are the following:*

(i) $P_U \xrightarrow{\iota_U} Q_U$ *for* $0 \to P_U \xrightarrow{\iota_U} Q_U \to U \to 0$ *a minimal projective resolution of a non-projective indecomposable* U *in* $\operatorname{mod} kQ$*,*
(ii) $0 \to P_i$ *for* $i \in Q_0$*,*
(iii) $P_i \xrightarrow{\mathrm{id}} P_i$ *for* $i \in Q_0$*.*

Proof. Indecomposability of the objects in (ii) and (iii) is clear from indecomposability of P_i. Indecomposability of the objects in (i) follows from minimality of the resolution. Conversely, assume that $P \xrightarrow{\iota} Q$ is an indecomposable object, not of the form in (ii) or (iii). Then ι is not an isomorphism, thus $U = Q/P \neq 0$. Since $\mathrm{End}(P \xrightarrow{\iota} Q)$ is local, so is $\mathrm{End}(U)$ by Lemma 3.2, and thus U is indecomposable. It is also non-projective, since ι is non-split. But then $P \xrightarrow{\iota} Q$ admits $P_U \xrightarrow{\iota_U} Q_U$ as a direct summand, proving that they are isomorphic. ☐

Definition 3.5. Let $\mathcal{H}_Q$ be the full subcategory of $\mathrm{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)$ of objects without direct summands of the form $0 \to P$. Let B_Q be the algebra $B(\mathcal{H}_Q)$.

Equivalently, $\mathcal{H}_Q$ is the full subcategory of embeddings $P \subset Q$ whose image is not contained in a proper direct summand. Moreover, $\mathcal{H}_Q$ is equivalent to the quotient category of $\mathrm{Hom}^{\mathrm{mono}}(\operatorname{proj} kQ)$ by morphisms factoring through an object $0 \to P$ by Lemma 3.1. The cokernel functor thus induces an equivalence between $\mathcal{H}_Q/\mathrm{Hom}^{\mathrm{iso}}(\operatorname{proj} kQ)$ and $\underline{\operatorname{mod}}\, kQ$, the quotient category of $\operatorname{mod} kQ$ by $\operatorname{proj} kQ$.

4. The category $\operatorname{mod} \mathcal{H}_Q^{\mathrm{op}}$ and its homological properties

Now we consider the category $\operatorname{mod} \mathcal{H}_Q^{\mathrm{op}}$, whose objects thus are contravariant functors from embeddings $P \subset Q$ without direct summands $0 \subset R$ to vector spaces.

Note again that the projective objects of this functor category are the objects of the form $\mathrm{Hom}(\_, (P \subset Q))$; more precisely, the projective indecomposable objects are the $\mathrm{Hom}(\_, (P_U \subset Q_U))$ for non-projective indecomposables U in $\operatorname{mod} kQ$, and the $\mathrm{Hom}(\_(, P_i = P_i))$ for $i \in Q_0$. Dually, the injective objects on $\operatorname{mod} \mathcal{H}_Q^{\mathrm{op}}$ are of the form $\mathrm{Hom}(P \subset Q, \_)^*$.

The cokernel functor Coker : $\mathcal{H}_Q \to \operatorname{mod} kQ$ induces an exact functor mod (mod

$kQ)^{\mathrm{op}} \to \mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$. We will now use this functor to construct projective resolutions in $\mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$ using those in $\mathrm{mod}\,(\mathrm{mod}\,kQ)^{\mathrm{op}}$ coming from the Auslander-Reiten theory. We first need a lemma relating homomorphism spaces in $\mathcal{H}_Q$ and in $\mathrm{mod}\,kQ$ via the cokernel functor.

Lemma 4.1. *For all $(P \subset Q)$ in $\mathcal{H}_Q$, we have an exact sequence of functors*

$$0 \to \mathrm{Hom}_{\mathcal{H}_Q}(\_, (P = P)) \to \mathrm{Hom}_{\mathcal{H}_Q}(\_, (P \subset Q)) \to \mathrm{Hom}_{kQ}(\mathrm{Coker}\,\_, Q/P) \to 0.$$

Proof. We prove exactness of the above sequence by evaluating on an arbitrary object $(R \subset S)$ of $\mathcal{H}_Q$. We have an induced commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}(S/R, P) & \to & \mathrm{Hom}(S/R, Q) & \to & \mathrm{Hom}(S/R, Q/P) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}(S, P) & \to & \mathrm{Hom}(S, Q) & \to & \mathrm{Hom}(S, Q/P) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}(R, P) & \to & \mathrm{Hom}(R, Q) & \to & \mathrm{Hom}(R, Q/P) & \to & 0 \end{array}$$

A standard diagram chase yields an exact sequence

$$0 \to \mathrm{Hom}(S, P) \to Y \to \mathrm{Hom}(S/R, Q/P) \to 0,$$

where Y denotes the subspace of $\mathrm{Hom}(R, P) \oplus \mathrm{Hom}(S, Q)$ of pairs mapping to the same element of $\mathrm{Hom}(R, Q)$. But this sequence immediately identifies with the evaluation of the claimed exact sequence at $(R \subset S)$. □

Theorem 4.2. *The category $\mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$ has global dimension at most two.*

Proof. We exhibit projective resolutions of the simple objects in $\mathrm{mod}\,\mathcal{H}_Q^{\mathrm{op}}$. These simple objects are the $S_{P_U \subset Q_U}$ for U a non-projective indecomposable, and the $S_{P_i = P_i}$ for $i \in Q_0$.

Recall the projective resolution

$$0 \to \mathrm{Hom}(\_, \tau U) \to \mathrm{Hom}(\_, B) \to \mathrm{Hom}(\_, U) \to S_U \to 0$$

in $\mathrm{mod}\,(\mathrm{mod}\,A)^{\mathrm{op}}$, where $0 \to \tau U \to B \to U \to 0$ is the Auslander-Reiten sequence ending in U. The above minimal projective resolutions of U and τU induce a (not necessarily minimal) projective resolution

$$0 \to \underbrace{P_U \oplus P_{\tau U}}_{=P_B} \to \underbrace{Q_U \oplus Q_{\tau U}}_{=Q_B} \to B \to 0$$

of the middle term B. Together with the exact sequences of the previous lemma, we can thus consider the following commutative diagram (in which $\mathrm{Hom}(X, Y)$ is abbreviated to (X, Y)):

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & (\_, (P_{\tau U} = P_{\tau U})) & \rightarrow & (\_, (P_B = P_B)) & \rightarrow & (\_, (P_U = P_U)) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (\_, (P_{\tau U} \subset Q_{\tau U})) & \rightarrow & (\_, (P_B \subset Q_B)) & \rightarrow & (\_, (P_U \subset Q_U)) & \rightarrow & S_{P_U \subset Q_U} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Coker}\_, \tau U) & \longrightarrow & (\mathrm{Coker}\_, B) & \longrightarrow & (\mathrm{Coker}\_, U) & \rightarrow & S_U(\mathrm{Coker}\_) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

All columns being exact and the top and bottom row being exact, a double application of the 3×3-lemma yields exactness of the middle row. This sequence provides the desired projective resolution of $S_{P_U \subset Q_U}$ as long as τU is non-projective. In case it is, we note that the restriction of the functor $\mathrm{Hom}(\_, (0 \subset \tau U))$ to $\mathcal{H}_Q$ is zero, thus the middle row provides an even shorter projective resolution. It remains to exhibit a projective resolution of $S_{P_i = P_i}$, which is provided by

$$0 \to \mathrm{Hom}(\_, (\bigoplus_{i \to j} P_j \subset P_i)) \to \mathrm{Hom}(\_, (P_i = P_i)) \to S_{P_i = P_i} \to 0.$$

This can be verified by evaluating on indecomposable objects, using that $\bigoplus_{i \to j} P_j \simeq \mathrm{rad}\, P_i$, and that the inclusion $\mathrm{rad} P_i \subset P_i$ is almost split. The theorem is proved. □

5. The functor Λ

We have an embedding $\mathrm{proj}\, kQ \to \mathcal{H}_Q$ associating to P the object $(P = P)$. This induces a restriction functor $\mathrm{res} : \mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}} \to \mathrm{mod}\, (\mathrm{proj}\, kQ)^{\mathrm{op}} \simeq \mathrm{mod}\, kQ$.

Central to the following is the definition of a functor $\Lambda : \mathrm{mod}\, kQ \to \mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$ (see Section 6 for concrete examples):

Definition 5.1. For M in $\mathrm{mod}\, kQ$, define an object $\widehat{M}$ in $\mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$ as follows:

$$\widehat{M}(\iota : P \to Q) = \mathrm{Im}(\mathrm{Hom}(\iota, M) : \mathrm{Hom}(Q, M) \to \mathrm{Hom}(P, M)),$$

with the natural definition on morphisms. This defines a functor $\Lambda : \mathrm{mod}\, kQ \to \mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$.

Lemma 5.2. *We have* $\mathrm{res}\, \widehat{M} \simeq M$ *naturally.*

Proof. We have

$$(\mathrm{res}\, \widehat{M})(P) = \widehat{M}(P = P) = \mathrm{Im}(\mathrm{Hom}(P, M) \stackrel{\mathrm{id}}{\to} \mathrm{Hom}(P, M)) = \mathrm{Hom}(P, M);$$

using the equivalence between $\mathrm{mod} kQ$ and $\mathrm{mod}(\mathrm{proj}\, kQ)^{\mathrm{op}}$, the statement follows. □

We note the following weak adjunction properties:

Lemma 5.3. *For all M in* $\mathrm{mod}\, kQ$ *and all F in* $\mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$, *the natural maps*

$$\mathrm{Hom}(\widehat{M}, F) \to \mathrm{Hom}(\mathrm{res}\, \widehat{M}, \mathrm{res}\, F) \simeq \mathrm{Hom}(M, \mathrm{res}\, F)$$

and

$$\mathrm{Hom}(F, \widehat{M}) \to \mathrm{Hom}(\mathrm{res}\, F, \mathrm{res}\, \widehat{M}) \simeq \mathrm{Hom}(\mathrm{res}\, F, M)$$

are injective.

Proof. For every object $(P \subset Q)$ in $\mathcal{H}_Q$, we have a natural chain of morphisms

$$(P = P) \to (P \subset Q) \to (Q = Q)$$

in $\mathcal{H}_Q$. Applying the functor $\widehat{M}$, this induces a chain

$$\mathrm{Hom}(Q, M) \to \mathrm{Im}(\mathrm{Hom}(Q, M) \to \mathrm{Hom}(P, M)) \to \mathrm{Hom}(P, M),$$

so that the first map is surjective, and the second map is injective. Suppose that $\varphi : \widehat{M} \to F$ maps to 0 under res. This yields a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(Q, M) & \to & \mathrm{Im}(\mathrm{Hom}(Q, M) \to \mathrm{Hom}(P, M)) & \to & \mathrm{Hom}(P, M) \\ 0 \downarrow & & \downarrow & & \downarrow 0 \\ F(Q = Q) & \to & F(P \subset Q) & \to & F(P = P). \end{array}$$

The outer vertical maps being zero and the first map in the upper row being surjective, we see that the middle vertical map is zero. This being true for an arbitrary embedding $P \subset Q$, the map φ is already zero. The second statement is proved dually. □

Corollary 5.4. *The functor Λ is fully faithful.*

Proof. For all M and N in $\mathrm{mod}\, kQ$, we have a chain of maps

$$\mathrm{Hom}(M, N) \xrightarrow{\Lambda} \mathrm{Hom}(\widehat{M}, \widehat{N}) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\mathrm{res}\, \widehat{M}, \mathrm{res}\, \widehat{N}) \simeq \mathrm{Hom}(M, N)$$

whose composition is the identity, thus the first map is injective. The second map being injective by the previous lemma, the claim follows. □

Remark 5.5. The functor Λ is neither left nor right exact in general; from it being fully faithful we can at least conclude that injective and surjective maps are preserved.

Now we come to the central result on the functor Λ:

Theorem 5.6. *The following holds for all M in $\mathrm{mod}\, kQ$:*

(i) *Both the projective and the injective dimension of $\widehat{M}$ are at most one.*
(ii) *We have $\mathrm{Ext}^1(\widehat{M}, \widehat{M}) = 0$.*

Proof. We claim that if $0 \to P \xrightarrow{\iota} Q \to M \to 0$ is a projective resolution of M, we have a projective resolution

$$0 \to \mathrm{Hom}(\_, (P \xrightarrow{\iota} Q)) \to \mathrm{Hom}(\_, (Q = Q)) \to \widehat{M} \to 0$$

of $\widehat{M}$. First note that $\widehat{Q} \simeq \mathrm{Hom}(\_, (Q = Q))$ for Q projective, thus the projection $Q \to M$ induces a projection $\widehat{Q} \to \widehat{M}$, thus it suffices to verify that $\mathrm{Hom}(\_, (P \subset Q))$ is indeed the kernel. Evaluating the above sequence on an embedding $R \subset S$, we get the sequence

$$0 \rightarrowtail \mathrm{Hom}((R \subset S), (P \subset Q)) \rightarrowtail \mathrm{Hom}(S, Q) \rightarrowtail \mathrm{Im}(\mathrm{Hom}(S, M) \to \mathrm{Hom}(R, M)) \rightarrowtail 0$$

whose exactness follows from the inspection of the following diagram, noting that $\mathrm{Hom}((R \subset S), (P \subset Q))$ equals the space of pairs in $\mathrm{Hom}(R, P) \oplus \mathrm{Hom}(S, Q)$ mapping to the same element of $\mathrm{Hom}(R, Q)$:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(S, P) & \to & \mathrm{Hom}(S, Q) & \to & \mathrm{Hom}(S, M) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Hom}(R, P) & \to & \mathrm{Hom}(R, Q) & \to & \mathrm{Hom}(R, M) & \to & 0. \end{array}$$

To construct an injective coresolution of $\widehat{M}$, we use the inverse Nakayama functor $\nu^- = \mathrm{Hom}((kQ)^*, \_)$ which induces an equivalence between the full subcategory of $\mathrm{mod}\, kQ$ of injective modules and $\mathrm{proj}\, kQ$, namely $\mathrm{Hom}(\_, I) \simeq \mathrm{Hom}(\nu^- I, \_)^*$. If

M itself is an injective I, then $\widehat{I} \simeq \mathrm{Hom}((\nu^- I = \nu^- I), \_)^*$ is an injective object of $\mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$, since

$$\widehat{I}(P \subset Q) = \mathrm{Im}(\mathrm{Hom}(Q, I) \to \mathrm{Hom}(P, I)) = \mathrm{Hom}(P, I) \simeq$$
$$\simeq \mathrm{Hom}(\nu^- I, P)^* \simeq \mathrm{Hom}((\nu^- I = \nu^- I), (P \subset Q))^*.$$

Otherwise we can assume M to be without injective direct summands and choose an injective coresolution $0 \to M \to I \to J \to 0$. By definition, we have $\nu^- M = 0$, yielding an embedding $\nu^- I \subset \nu^- J$. Similar to the above case of a projective resolution, and making use of the Nakayama functor, we can verify that

$$0 \to \widehat{M} \to \mathrm{Hom}((\nu^- I = \nu^- I), \_)^* \to \mathrm{Hom}((\nu^- I \subset \nu^- J), \_)^* \to 0$$

is an injective coresolution of $\widehat{M}$.

To prove the second part of the theorem, we apply $\mathrm{Hom}(\_, \widehat{M})$ to the above projective resolution of $\widehat{M}$ and get

$$0 \to \mathrm{Hom}(\widehat{M}, \widehat{M}) \to \mathrm{Hom}(\mathrm{Hom}(\_, (Q = Q)), \widehat{M}) \to$$
$$\to \mathrm{Hom}(\mathrm{Hom}(\_, (P \to Q)), \widehat{M}) \to \mathrm{Ext}^1(\widehat{M}, \widehat{M}) \to 0.$$

The first term equals $\mathrm{Hom}(M, M)$, and the second and third term can be computed using Yoneda's lemma, yielding the sequence

$$0 \to \mathrm{Hom}(M, M) \to \widehat{M}(Q = Q) \to \widehat{M}(P \subset Q) \to \mathrm{Ext}^1(\widehat{M}, \widehat{M}) \to 0.$$

By the definition of $\widehat{M}$, this reads

$$0 \to \mathrm{Hom}(M, M) \to \mathrm{Hom}(Q, M) \xrightarrow{\alpha} \mathrm{Im}(\mathrm{Hom}(Q, M) \xrightarrow{\alpha} \mathrm{Hom}(P, M)) \to$$
$$\to \mathrm{Ext}^1(\widehat{M}, \widehat{M}) \to 0.$$

We see that the second map is tautologically surjective, thus the desired vanishing follows. The theorem is proved. □

6. The algebra B_Q

By the results of the previous section, the utility of the algebra $B_Q = B(\mathcal{H}_Q)$ is the following: it is an algebra of global dimension at most two, such that the original module category $\mathrm{mod}\, kQ$ embeds into the subcategory of $\mathrm{mod}\, B_Q$ of objects of projective and injective dimension of most one, in such a way that all non-trivial extensions in $\mathrm{mod}\, kQ$ vanish after the embedding. In contrast, the natural embedding $M \mapsto \mathrm{Hom}(\_, M)$ of $\mathrm{mod}\, kQ$ into $\mathrm{mod}\, (\mathrm{mod}\, kQ)^{\mathrm{op}}$ in general yields projective functors of injective dimension two. We will see in Proposition 7.1 why all these properties of Λ are essential for the construction of desingularizations of quiver Grassmannians.

In this section, we first determine the quiver of the algebra B_Q and compute some concrete examples of B_Q and of the functor Λ. We explain the relation of B_Q to the Auslander algebra of kQ and give a characterization of the essential image of Λ.

6.1. **Quiver of B_Q.** We are now able to compute the (ordinary) quiver of the algebra B_Q:

Proposition 6.1. *The quiver $\widehat{Q}$ of the algebra B_Q is given as follows: it has vertices $[U]$ parametrized by isomorphism classes of non-projective indecomposables in $\mathrm{mod}\, kQ$, together with vertices $[i]$ for $i \in Q_0$. There is an arrow $[U] \to [V]$ for every irreducible map $V \to U$ between non-projective indecomposables. Moreover, there are arrows $[i] \to [S_i]$, resp. $[\tau^{-1} S_i] \to [i]$, for every vertex $i \in Q_0$, as long as S_i is non-projective, resp. non-injective.*

Proof. Using the above projective resolutions of the simple functors, we can compute the Ext-quiver of the algebra B_Q. Namely, we can compute $\mathrm{Ext}^1(S_{P_U\subset Q_U}, F)$ as the first homology of the complex with terms $\mathrm{Hom}(\mathrm{Hom}(\_, (P_X \subset Q_X)), F)$ with X being τU or B or U, respectively, which using Yoneda simplifies to the complex

$$F(P_U \subset Q_U) \to F(P_B \subset Q_B) \to F(P_{\tau U} \subset Q_{\tau U}).$$

Now suppose that $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_V\subset Q_V})$ is non-zero. Then $S_{P_V\subset Q_V}(P_B \subset Q_B)$ is non-zero, thus V is a direct summand of B. But then V fulfills the following: it admits an irreducible map to U in mod kQ, it occurs as a direct summand of B with multiplicity one, and it is not a direct summand of U or of τU. This in turn implies that $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_V\subset Q_V})$ is one-dimensional. We have thus proved that $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_V\subset Q_V}) \neq 0$ if and only if V admits an irreducible map to U, in which case $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_V\subset Q_V})$ is one-dimensional.

Similarly we compute $\mathrm{Ext}^1(S_{P_i=P_i}, F)$ as the first homology of the complex

$$0 \to F(P_i = P_i) \to F(\bigoplus_{i\to j} P_j \subset P_i),$$

which for $F = S_{P_U\subset Q_U}$ is obviously non-zero (and one-dimensional in this case) if and only if $U = S_i$.

It also follows that $\mathrm{Ext}^1(S_{P_i=P_i}, S_{P_j=P_j}) = 0$ for all $i, j \in Q_0$.

Finally, to compute $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_i=P_i})$, we use an injective coresolution of $S_{P_i=P_i}$ analogous to the projective resolution exhibited in the proof of Theorem 4.2. We use the injective coresolution

$$0 \to S_i \to I_i \to I_i/\mathrm{soc} I_i \simeq \bigoplus_{j\to i} I_j \to 0,$$

which yields a projective resolution

$$0 \to P_i \to \bigoplus_{j\to i} P_j \to \tau^{-1} S_i \to 0$$

using the inverse Nakayama functor. From this, we can easily derive the injective coresolution

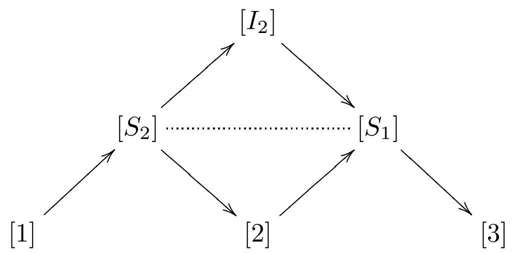
$$0 \to S_{P_i=P_i} \to \mathrm{Hom}((P_i = P_i), \_)^* \to \mathrm{Hom}((P_i \subset \bigoplus_{j\to i} P_j), \_)^* \to 0.$$

Similarly to the above, we see that $\mathrm{Ext}^1(S_{P_U\subset Q_U}, S_{P_i=P_i})$ is non-zero (and one-dimensional in this case) if and only if $U \simeq \tau^{-1} S_i$. The theorem is proved. □

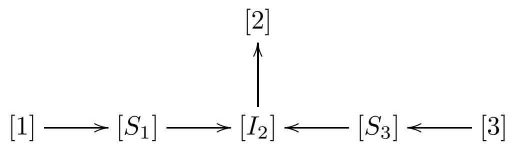
6.2. **Examples.** We now give some examples of the quivers $\widehat{Q}$ and of their representations $\widehat{M}$.

Example 6.2. Let Q be the equioriented quiver of type A_n. Then the quiver $\widehat{Q}$ of B_Q is the Auslander-Reiten quiver of kQ, and the algebra B_Q is given by imposing all commutativity relations, but no zero relations (see subsection 8.1 for more details). We want to stress that in general the quiver $\widehat{Q}$ **does not** coincide with the AR quiver of Q and the algebra B_Q is **not** isomorphic to the Auslander algebra of Q.

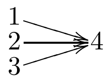
Example 6.3. Let Q be $1 \longrightarrow 2 \longrightarrow 3$, the equioriented quiver of type A_3. The algebra B_Q is given by the following quiver with one commutativity relation



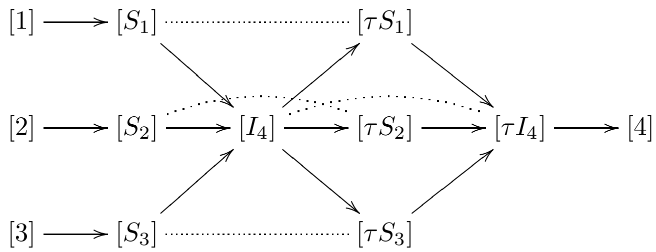
Example 6.4. Let Q be a quiver $1 \longrightarrow 2 \longleftarrow 3$ of type A_3. The algebra B_Q is given by the following quiver of type E_6



Example 6.5. Let Q be

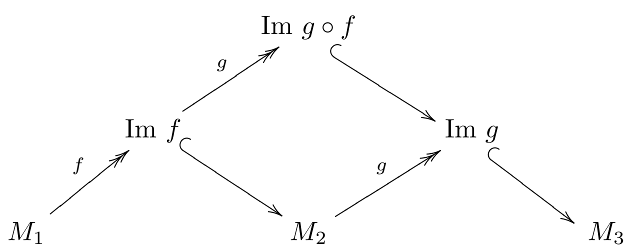


which is the "three subspaces" quiver of type D_4. The algebra B_Q is given by the following quiver with four mesh relations



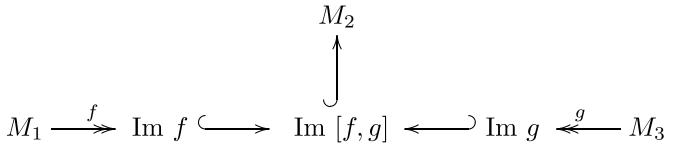
We now give examples of the functor $\Lambda : \mathrm{mod}kQ \to \mathrm{mod}B_Q : M \mapsto \widehat{M}$.

Example 6.6. Let $M := M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be a (finite dimensional) representation of the equioriented quiver of type A_3. The algebra B_Q is shown in Example 6.3. The B_Q-module $\widehat{M}$ is the following



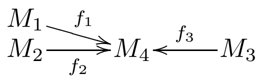
Example 6.7. Let $M := M_1 \xrightarrow{f} M_2 \xleftarrow{g} M_3$ be a (finite dimensional) representation of the quiver $Q := 1 \longrightarrow 2 \longleftarrow 3$. The algebra B_Q is shown in Example

6.4. The B_Q–module $\widehat{M}$ is the following

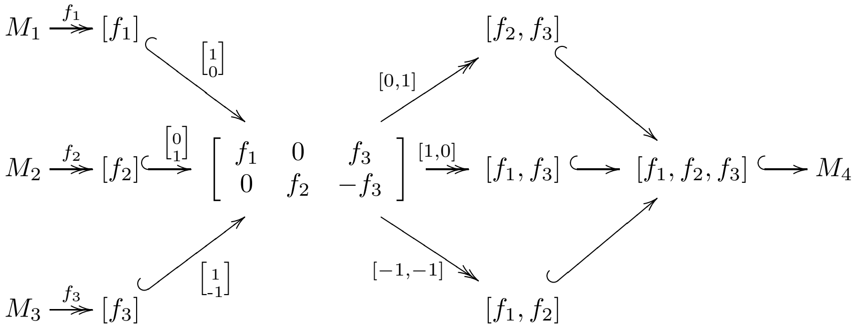


where $[f,g] : M_1 \oplus M_3 \to M_2 : (v,w) \mapsto f(v) + g(w)$.

Example 6.8. Let $M :=$



be a finite dimensional representation of the quiver of Example 6.5. The B_Q–module $\widehat{M}$ is given by



In the picture above we use the following convention: for a linear map $f : V \to W$ we denote by $[f] :=$ Im $f \subset W$. For example $\begin{bmatrix} f_1 & 0 & f_3 \\ 0 & f_2 & -f_3 \end{bmatrix}$ denotes the image of the linear map $\begin{bmatrix} f_1 & 0 & f_3 \\ 0 & f_2 & -f_3 \end{bmatrix} : M_1 \oplus M_2 \oplus M_3 \to M_4 \oplus M_4$.

6.3. **Comparison between B_Q and A_Q.** We discuss the relation between the algebra B_Q and the Auslander algebra $A_Q = B(\mathrm{mod}\, kQ)$ of $\mathrm{mod}\, kQ$. The equivalence $\mathcal{H}_Q/\mathrm{Hom}^{\mathrm{iso}}(\mathrm{proj}\, kQ) \simeq \underline{\mathrm{mod}}\, kQ$ immediately identifies a quotient of B_Q with the subalgebra $\mathrm{End}(\bigoplus_U U)^{\mathrm{op}}$ of A_Q, where the sum runs over all non-projective indecomposables of $\mathrm{mod}\, kQ$. Moreover, A_Q arises via tilting (see [1, VI.]) from B_Q:

Proposition 6.9. *The direct sum $T = \bigoplus_U \widehat{U}$ over indecomposable modules U in $\mathrm{mod}\, kQ$ is a tilting object in $\mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$, such that $\mathrm{End}(T)^{\mathrm{op}}$ is isomorphic to the Auslander algebra A_Q of kQ.*

Proof. Theorem 5.6 shows that $\mathrm{Ext}^1(T,T) = 0$, and that T has projective dimension at most 1. Rewriting the projective resolution of $\widehat{M}$ of the proof of Theorem 5.6 as

$$0 \to \mathrm{Hom}(\_, (P \subset Q)) \to \widehat{Q} \to \widehat{M} \to 0$$

for a projective resolution $0 \to P \to Q \to M \to 0$, we see that all indecomposable projective objects in $\mathrm{mod}\, \mathcal{H}_Q^{\mathrm{op}}$ admit a short coresolution by sums of direct summands of T, proving that T is a tilting object. The functor Λ being fully faithful, we see that

$$\mathrm{End}(T)^{\mathrm{op}} \simeq \mathrm{End}(\bigoplus_U U)^{\mathrm{op}} = A_Q.$$

□

The tilting B_Q–module T induces a derived equivalence $F := RHom_{B_Q}(T, \_) : \mathcal{D}^b(\text{mod–}B_Q) \to \mathcal{D}^b(\text{mod–}A_Q)$ [13, Theorem 1.6]. The image $F(B_Q)$ of B_Q under this functor is a tilting complex T' whose endomorphism ring is the algebra B_Q. The next proposition shows this tilting complex explicitly. In order to state the result we need a little preparation. Under the isomorphism $\text{End}_{B_Q}(T)^{\text{op}} \simeq A_Q$, every direct summand $\widehat{M}$ for $M \in \text{ind } kQ$ of T corresponds to an indecomposable projective A_Q–module which we denote by A_M.

Proposition 6.10. *The tilting complex T' in $\mathcal{D}^b(\text{mod–}A_Q)$ is given as follows*

$$T' = \bigoplus_{i\in Q_0} A_{P_i} \oplus \bigoplus_{U\in ind\ kQ\setminus proj\ kQ} (A_{Q_U} \to A_U)$$

where $0 \to P_U \to Q_U \to U \to 0$ is the minimal projective resolution of U, the complex A_{P_i} is concentrated in degree 0, and the complex $(A_{Q_U} \to A_U)$ is concentrated in degrees 0 and 1.

Proof. We write $B_Q = \bigoplus_i B_i$ as the sum of all the indecomposable projective B_Q–modules. Recall that these modules correspond to the functors $\text{Hom}(\_, (P_i = P_i))$, $i \in Q_0$ and $\text{Hom}(\_, (P_U \to Q_U))$, for $U \in \text{ind } kQ \setminus \text{proj } kQ$. Since $\text{Hom}(\_, (P_i = P_i)) \simeq \widehat{P}_i$, the corresponding B_i is a summand of T and hence $F(B_i) = A_{P_i}$ is an indecomposable projective A_Q–module. It remains to find the image of the remaining direct summands $B_j = \text{Hom}(\_, (P_U \to Q_U))$ of B, for $U \in \text{ind } kQ \setminus \text{proj } kQ$. Every such projective B_j arises in the projective resolution of B_Q–modules

$$0 \to B_j \to \widehat{Q_U} \to \widehat{U} \to 0.$$

This induces a triangle

$$B_j \to \widehat{Q_U} \to \widehat{U} \to B_j[1]$$

in $\mathcal{D}^b(B_Q)$. We apply the triangle functor F to this triangle and we get a triangle

$$F(B_j) \to A_{Q_U} \to A_U \to F(B_j)[1]$$

in $\mathcal{D}^b(A_Q)$. From this triangle we conclude that $F(B_j)$ is isomorphic to the complex $(A_{Q_U} \to A_U)$ (in degrees 0 and 1), as desired. □

6.4. **Essential image of Λ.** In Examples 6.6, 6.7 and 6.8 we see that all the linear maps of the $\widehat{Q}$–representation $\widehat{M}$ are either injective or surjective. The next proposition shows that such properties hold in general, encoded in the vanishing of certain homomorphism spaces. In fact, we can give a characterization of the essential image of the functor Λ as follows:

Proposition 6.11. *A functor $F \in \text{mod}\,\mathcal{H}_Q^{\text{op}}$ is isomorphic to $\widehat{M}$ for some $M \in \text{mod}\,kQ$ if and only if $\text{Hom}(S_{P_U\subset Q_U}, F) = 0 = \text{Hom}(F, S_{P_U\subset Q_U})$ for all non-projective indecomposables $U \in \text{mod}\,kQ$.*

Proof. We have $\text{res}\, S_{P_U\subset Q_U} = 0$ by definition. Lemma 5.3 implies

$$\text{Hom}(\widehat{M}, S_{P_U\subset Q_U}) \subset \text{Hom}(M, \text{res}\, S_{P_U\subset Q_U}) = 0,$$

$$\text{Hom}(S_{P_U\subset Q_U}, \widehat{M}) \subset \text{Hom}(\text{res}\, S_{P_U\subset Q_U}, M) = 0.$$

To prove the converse, assume that $\text{Hom}(S_{P_U\subset Q_U}, F) = 0 = \text{Hom}(F, S_{P_U\subset Q_U})$ for all non-projective indecomposables $U \in \text{mod}\,kQ$ for a functor F. We define $M = \text{res}\, F$ and have to prove that $F \simeq \widehat{M}$. By definition, this amounts to proving the following: given an object $P \subset Q$ in $\mathcal{H}_Q$, we have canonical maps $(P = P) \to (P \subset Q) \to (Q = Q)$ in $\mathcal{H}_Q$ inducing a sequence

$$F(Q = Q) \to F(P \subset Q) \to F(P = P).$$

Then we have to prove that the first map is surjective and the second map is injective. We prove injectivity of the second map; surjectivity of the first map is proved dually. First we can restrict to the case of $P \subset Q$ being indecomposable, thus $(P \subset Q) = (P_U \subset Q_U)$ for a non-projective indecomposable U. Assume that U is such that there exists an element $0 \neq x \in F(P_U \subset Q_U)$ mapping to zero in $F(P_U = P_U)$. Without loss of generality, we can assume U to be minimal with this property with respect to the ordering induced by irreducible maps. Using the description

$$S_{P_U \subset Q_U} \simeq \mathrm{Hom}(\_, (P_U \subset Q_U))/\mathrm{rad}\,\mathrm{Hom}(\_, (P_U \subset Q_U)),$$

we can rewrite $\mathrm{Hom}(S_{P_U \subset Q_U}, F)$ as the intersection of the kernels of the maps $F(f)$ for f ranging over the non-split maps $f : (P_V \subset Q_V) \to (P_U \subset Q_U)$ in $\mathcal{H}_Q$. Since this intersection is zero by assumption, there exists an indecomposable object $(P_V \subset Q_V)$ and a non-split map $f : (P_V \subset Q_V) \to (P_U \subset Q_U)$ such that $F(f)(x) \neq 0$. We have a natural square

$$\begin{array}{ccc} (P_V = P_V) & \to & (P_V \subset Q_V) \\ \downarrow & & \downarrow \\ (P_U = P_U) & \to & (P_U \subset Q_U) \end{array}$$

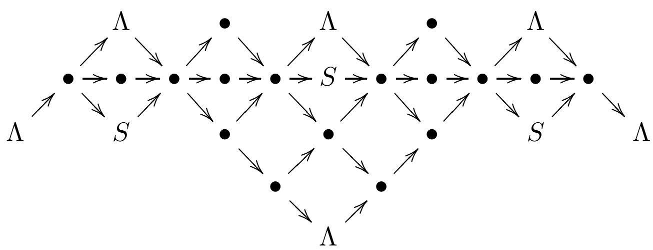
inducing the square

$$\begin{array}{ccc} F(P_V = P_V) & \leftarrow & F(P_V \subset Q_V) \\ \uparrow & & \uparrow \\ F(P_U = P_U) & \leftarrow & F(P_U \subset Q_U). \end{array}$$

The element x mapping to zero under the lower horizontal map, we see that $F(f)(x) \neq 0$ maps to zero under the upper horizontal map, a contradiction to the minimality of U. The proposition is proved. ☐

The following examples show the AR–quiver of some algebras B_Q of finite representation type. These pictures also illustrate the statement of the previous proposition.

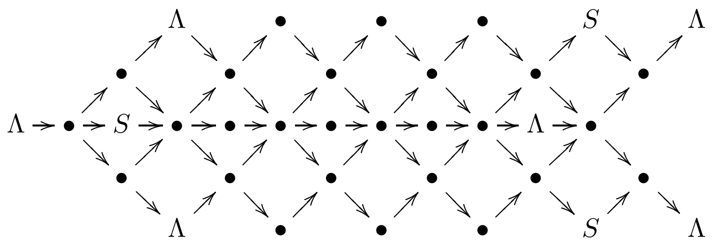
Example 6.12. Let $Q := 1 \longrightarrow 2 \longrightarrow 3$ be the quiver of type A_3 already considered in Examples 6.3 and 6.6. From the description of B_Q given in Example 6.3, it follows that B_Q is of finite representation type. The following quiver is the AR–quiver of B_Q.



In the picture above we denote by Λ the vertices corresponding to the B_Q–modules $\widehat{U}$, for $U \in \mathrm{ind}\ kQ \setminus \mathrm{proj}\ kQ$. We denote by S the vertices corresponding to the simple B_Q–modules $S_{P_U \subset Q_U}$, $U \in \mathrm{ind}\ kQ \setminus \mathrm{proj}\ kQ$.

Example 6.13. Let $Q := 1 \longrightarrow 2 \longleftarrow 3$ be the quiver of type A_3 already considered in Examples 6.4 and 6.7. From the description of B_Q given in Example 6.4, it follows that B_Q is of finite representation type. The following quiver is the

AR–quiver of B_Q (which is of type E_6).



We use the notation Λ and S in the same way as in the previous example.

7. Construction of the desingularization

Now we assume k to be algebraically closed.

Proposition 7.1. *Let Q be a quiver (not necessarily of Dynkin type), let M be a representation of kQ, and let $\mathbf{e}$ be a dimension vector for Q. Assume that M is a representation for a quotient algebra $A = kQ/I$, that is, the two-sided ideal I annihilates M, such that the following holds: A has global dimension at most two, both the injective and the projective dimension of M over A are at most one, and $\mathrm{Ext}^1_A(M, M) = 0$. Then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is smooth (and reduced).*

Proof. The dimension of the tangent space $T_N(\mathrm{Gr}_{\mathbf{e}}(M))$ being $\dim \mathrm{Hom}(N, M/N)$ (see [5]), we have to prove that the latter dimension only depends on the dimension vector $\mathbf{e}$ (and the fact that N is a subrepresentation of M of this dimension vector). Since M is a representation of A, so are N and M/N. Thus we have to prove that $\mathrm{Hom}_A(N, M/N)$ is constant. Since A has global dimension at most two, we know that

$$\dim \mathrm{Hom}_A(N, M/N) - \dim \mathrm{Ext}^1_A(N, M/N) + \dim \mathrm{Ext}^2_A(N, M/N)$$

only depends on the dimension vectors of N and M/N (see [1, III.3.]). Thus we are finished once we can prove that $\mathrm{Ext}^1_A(N, M/N) = 0 = \mathrm{Ext}^2_A(N, M/N)$. Applying $\mathrm{Hom}_A(\_, M)$ to the exact sequence $0 \to N \to M \to M/N \to 0$ and working out the resulting long exact sequence, we see that

$$0 = \mathrm{Ext}^2_A(M, M) \to \mathrm{Ext}^2_A(N, M) \to \mathrm{Ext}^3_A(M/N, M) = 0,$$

thus $\mathrm{Ext}^2_A(N, M) = 0$. Working out various other long exact cohomology sequences, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
\mathrm{Ext}^1_A(M, M) & \to & \mathrm{Ext}^1_A(M, M/N) & \to & \mathrm{Ext}^2_A(M, N) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_A(N, M) & \to & \mathrm{Ext}^1_A(N, M/N) & \to & \mathrm{Ext}^2_A(N, N) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^2_A(M/N, M) & \to & \mathrm{Ext}^2_A(M/N, M/N) & \to & \mathrm{Ext}^3_A(M/N, N)
\end{array}$$

By assumption, all four corners of this square are zero, thus the central term is zero, proving $\mathrm{Ext}^1_A(N, M/N) = 0$. □

For a given representation M, a dimension vector $\mathbf{e}$ and an isomorphism class $[N]$ for a Dynkin quiver Q, it is proved in [5, Section 2.3] that the subset $\mathcal{S}_{[N]}$ of $\mathrm{Gr}_{\mathbf{e}}(M)$, consisting of the subrepresentations which are isomorphic to N, is locally closed.

Proposition 7.2. *Suppose that $\emptyset \neq \mathcal{S}_{[U]}$ is contained in $\overline{\mathcal{S}_{[N]}}$. Then $\mathbf{dim}\widehat{U} \leq \mathbf{dim}\widehat{N}$ componentwise as dimension vectors of representations of B_Q.*

Proof. First we briefly recall the geometric definition of the above strata: we consider the variety $R_{\mathbf{e}}(Q)$ of representations of Q of dimension vector $\mathbf{e}$, with its standard base change action of the group $G_{\mathbf{e}}$, such that the orbits $\mathcal{O}_{[N]}$ correspond bijectively to the isomorphism classes $[N]$ of representations of Q of dimension vector $\mathbf{e}$. There exists a locally trivial $G_{\mathbf{e}}$-principal bundle $\pi : X_{\mathbf{e}}(M) \to \mathrm{Gr}_{\mathbf{e}}(M)$ which admits a $G_{\mathbf{e}}$-equivariant map $p : X_{\mathbf{e}}(M) \to R_{\mathbf{e}}(Q)$. The stratum $\mathcal{S}_{[N]}$ is then defined by $\pi(p^{-1}(\mathcal{O}_{[N]}))$.

Now suppose that $\emptyset \neq \mathcal{S}_{[U]}$ is contained in $\overline{\mathcal{S}_{[N]}}$. From the above definition, this means

$$\pi(p^{-1}(\mathcal{O}_{[U]})) \subset \overline{\pi(p^{-1}(\mathcal{O}_{[N]}))} \subset \pi(\overline{p^{-1}(\mathcal{O}_{[N]})}).$$

Thus $p^{-1}(\mathcal{O}_{[U]}) \subset \overline{p^{-1}(\mathcal{O}_{[N]})}$, which implies

$$\mathcal{O}_{[U]} = p(p^{-1}(\mathcal{O}_{[U]})) \subset p(\overline{p^{-1}(\mathcal{O}_{[N]})}) \subset p(p^{-1}(\overline{\mathcal{O}_{[N]}})) \subset \overline{\mathcal{O}_{[N]}},$$

where we used the fact that π is a principal bundle. The adherence relation $\mathcal{O}_{[U]} \subset \overline{\mathcal{O}_{[N]}}$ now implies that $\dim \mathrm{Hom}(P, U) = \dim \mathrm{Hom}(P, N)$ for all projective representations P, and $\dim \mathrm{Hom}(X, U) \geq \dim \mathrm{Hom}(X, N)$ for all non-projectives X (see [2]). Now consider the dimension vector of $\widehat{U}$, resp. of $\widehat{N}$, as a representation of B_Q, thus $(\mathbf{dim}\widehat{U})_{[X]} = \dim \widehat{U}(P_X \subset Q_X)$ and $(\mathbf{dim}\widehat{U})_{[i]} = \dim \widehat{U}(P_i = P_i) = (\mathbf{dim}U)_i$ as above. Using the exact sequence

$$0 \to \mathrm{Hom}(X, U) \to \mathrm{Hom}(Q_X, U) \to \mathrm{Hom}(P_X, U),$$

we can calculate

$$(\mathbf{dim}\widehat{U})_{[X]} = \dim \widehat{U}(P_X \subset Q_X) = \dim \mathrm{Im}(\mathrm{Hom}(Q_X, U) \to \mathrm{Hom}(P_X, U)) =$$
$$= \dim \mathrm{Hom}(Q_X, U) - \dim \mathrm{Hom}(X, U) \leq \dim \mathrm{Hom}(Q_X, N) - \dim \mathrm{Hom}(X, N),$$

which in turn equals $(\mathbf{dim}\widehat{N})_{[X]}$. This proves $\mathbf{dim}\widehat{U} \leq \mathbf{dim}\widehat{N}$ componentwise. □

Definition 7.3. We call $[N]$ a generic subrepresentation type of M of dimension vector $\mathbf{e}$ if the stratum $\mathcal{S}_{[N]}$ of $\mathrm{Gr}_{\mathbf{e}}(M)$ is open. Denote by $\mathrm{gsub}_{\mathbf{e}}(M)$ the set of all generic subrepresentation types.

In case $[N] \in \mathrm{gsub}_{\mathbf{e}}(M)$, the closure $\overline{\mathcal{S}_{[N]}}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{e}}(M)$, and every irreducible component arises in this way.

For representations M and N of Q, we now consider quiver Grassmannians for the quiver $\widehat{Q}$ of the form $\mathrm{Gr}_{\mathbf{dim}\widehat{N}}(\widehat{M})$. Here $\mathbf{dim}$ denotes the dimension vector of $\widehat{N}$ as a representation of $\widehat{Q}$, that is, $(\mathbf{dim}\widehat{N})_{[U]} = \dim \mathrm{Im}(\mathrm{Hom}(Q_U, N) \to \mathrm{Hom}(P_U, N))$ for all non-projective indecomposables U, and $(\mathbf{dim}\widehat{N})_{[i]} = (\mathbf{dim}N)_i$ for all $i \in Q_0$. We apply Proposition 7.1 to the quiver $\widehat{Q}$, the factor algebra B_Q of $k\widehat{Q}$ and the representation $\widehat{M}$ of $\widehat{Q}$, resp. of B_Q. Then the homological vanishing properties Theorem 4.2 and Theorem 5.6 imply that the assumptions of Proposition 7.1 hold, thus $\mathrm{Gr}_{\mathbf{dim}\widehat{N}}(\widehat{M})$ is smooth.

Theorem 7.4. *For arbitrary M and e as above, the map*

$$\pi : \bigsqcup_{[N] \in \mathrm{gsub}_{\mathbf{e}}(M)} \mathrm{Gr}_{\mathbf{dim}\widehat{N}}(\widehat{M}) \to \mathrm{Gr}_{\mathbf{e}}(M)$$

given by $(F \subset \widehat{M}) \mapsto (\mathrm{res}\, F \subset M)$ is a desingularization.

Proof. Smoothness is already proven. The map π is proper since the left hand side is projective. Given a generic embedding $N_0 \subset M$, we also have $\widehat{N_0} \subset \widehat{M}$ since Λ is fully faithful, thus the fibre over $N_0 \subset M$ is non-empty. This argument working for all generic subrepresentation types, we see that π is dominant, thus surjective, its image being closed since it is proper. That the generic fibre reduces to a single point is a special case of the following theorem describing all fibres of π as suitable quiver Grassmannians. □

Theorem 7.5. *We have an isomorphism*

$$\pi^{-1}(U \subset M) \simeq \bigsqcup_{\substack{[N]\in \mathrm{gsub}_{\mathbf{e}}(M)\\ \mathcal{S}_{[U]}\subset \overline{\mathcal{S}_{[N]}}}} \mathrm{Gr}_{\mathbf{dim}\widehat{N}-\mathbf{dim}\widehat{U}}(\widehat{M}/\widehat{U}).$$

Proof. More precisely, we prove that

$$\pi^{-1}(U \subset M) \simeq \bigsqcup_{\substack{[N]\in \mathrm{gsub}_{\mathbf{e}}(M)\\ \mathcal{S}_{[U]}\subset \overline{\mathcal{S}_{[N]}}}} \{F \subset \widehat{M} \,:\, \mathbf{dim}F = \mathbf{dim}\widehat{N},\, \widehat{U} \subset F\}.$$

By definition of the desingularization π, this immediately reduces to the following statement:
Suppose we are given a generic subrepresentation type N, a subrepresentation $U \in \overline{\mathcal{S}_{[N]}}$, and a subobject $F \subset \widehat{M}$ such that $\mathbf{dim}F = \mathbf{dim}\widehat{N}$. Then we have $\mathrm{res}F = U$ if and only if $\widehat{U} \subset F$.
So suppose $\mathbf{dim}F = \mathbf{dim}\widehat{N}$ and $\widehat{U} \subset F$. Then $U = \mathrm{res}\,\widehat{U} \subset \mathrm{res}\,F$ and

$$(\mathbf{dim}U)_i = \dim \mathrm{Hom}(P_i, U) = \dim \mathrm{Hom}(P_i, N) =$$
$$= \dim \widehat{N}(P_i = P_i) = \dim F(P_i = P_i) = (\mathbf{dim}\,\mathrm{res}\,F)_i,$$

and thus $U = \mathrm{res}\,F$.

Conversely, suppose that $\mathrm{res}\,F = U$ and $F \subset \widehat{M}$. For an object $(P \subset Q)$ of $\mathcal{H}_Q$, the canonical chain of maps $(P = P) \to (P \subset Q) \to (Q = Q)$ induces a diagram

$$\begin{array}{ccccc}
F(Q=Q) & \stackrel{\alpha}{\to} & F(P\subset Q) & \stackrel{\beta}{\to} & F(P=P)\\
\downarrow & & \downarrow & & \downarrow\\
\widehat{M}(Q=Q) & \to & \widehat{M}(P\subset Q) & \stackrel{\gamma}{\to} & \widehat{M}(P=P)\\
|| & & || & & ||\\
\mathrm{Hom}(Q,M) & \to & \mathrm{Im}(\mathrm{Hom}(Q,M)\to \mathrm{Hom}(P,M)) & \stackrel{\gamma}{\to} & \mathrm{Hom}(P,M).
\end{array}$$

The upper vertical maps being embeddings, and the map γ being an embedding, we see that β is an embedding. On the other hand, we have $\dim \mathrm{Hom}(Q, U) = \dim \mathrm{Hom}(Q, N) = \dim F(Q = Q)$ and similarly $\dim \mathrm{Hom}(P, U) = \dim F(P = P)$, which yields a diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(Q,U) & \to & \mathrm{Im}(\mathrm{Hom}(Q,U)\to \mathrm{Hom}(P,U)) & \to & \mathrm{Hom}(P,U)\\
|| & & & & ||\\
F(Q=Q) & \stackrel{\alpha}{\to} & F(P\subset Q) & \stackrel{\beta}{\to} & F(P=P)
\end{array}.$$

The upper middle term thus identifies with $\mathrm{Im}(\beta\alpha)$, whereas the lower middle term identifies with $\mathrm{Im}(\beta)$ since β is an embedding. But then $\mathrm{Im}(\beta\alpha)$ naturally embeds into $\mathrm{Im}(\beta)$, thus we have compatible embeddings $\widehat{U}(P \subset Q) \subset F(P \subset Q)$, thus an embedding of functors $\widehat{U} \subset F$ as desired. □

8. Examples

8.1. **Equioriented A_n case.** As the first example, we consider the equioriented type A_n quiver Q given by $1 \to 2 \to \ldots \to n$. A representation M is then given by a chain of linear maps

$$V_1 \stackrel{f_1}{\to} V_2 \stackrel{f_2}{\to} \ldots \stackrel{f_{n-1}}{\to} V_n;$$

a dimension vector $\mathbf{e}$ is given by a tuple $(e_1, \ldots, e_n)$.

The indecomposable representations of Q are the $U_{i,j}$ for $1 \le i \le j \le n$ of dimension vector $\mathbf{dim}U_{i,j} = (\overbrace{0,\ldots,0}^{i-1},\overbrace{1,\ldots,1}^{j-i+1},0,\ldots,0)$. In particular, we have $P_i = U_{i,n}$, $I_i = U_{1,i}$ and $S_i = U_{i,i}$. The quiver $\widehat{Q}$ thus has vertices $[i,j]$ for $1 \leq i \leq j < n$ and $[i]$ for $1 \leq i \leq n$ and the following arrows:

- $[i,j] \to [i,j+1]$ for $1 \leq i \leq j < n-1$,
- $[i,j] \to [i+1,j]$ for $1 \leq i < j < n$,
- $[i] \to [i,i] \to [i+1]$ for $1 \leq i < n$.

(see Example 6.2 and Example 6.3 for the case $n = 3$.) We have minimal projective resolutions

$$0 \to P_{j+1} \to P_i \to U_{i,j} \to 0$$

for all $1 \leq i \leq j < n$. Using the fact that all non-zero maps between the indecomposable projectives are scalar multiples of the natural embeddings induced by the chain $P_n \subset \ldots \subset P_1$, we can easily verify that the algebra B_Q is given as the path algebra of $\widehat{Q}$ modulo all commutativity relations. The representation $\widehat{M}$ of $\widehat{Q}$ is given by $M_{[i]} = V_i$ for all $1 \leq i \leq n$ and $M_{[i,j]} = \mathrm{Im}(f_j \circ \ldots \circ f_i : V_i \to V_{j+1})$ for $1 \leq i \leq j < n$. The maps representing the arrows of $\widehat{Q}$ are either natural inclusions or induced by the maps f_i.

To explicitly write down the desingularization map, it is thus necessary to determine the generic subrepresentation types; no general formula is known for these (see however the case A_2 below). We restrict to a special case where $\mathrm{Gr}_{\mathbf{e}}(M)$ is known to be irreducible, namely the type A_n degenerate flag variety of [5]. We define $V_1 = \ldots = V_n = k^{n+1}$, in which we choose a basis $w_1, \ldots, w_{n+1}$, and define f_i as the projection along w_{i+1}, that is, $f_i(w_{i+1}) = 0$ for $i = 1, \ldots, n-1$ and $f_i(w_j) = w_j$ for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, n+1$ such that $j \neq i+1$. Then $M \simeq kQ \oplus (kQ)^*$. We also define $e_i = i$ for $i = 1, \ldots, n$. Then $\mathrm{Gr}_{\mathbf{e}}(M)$ is irreducible with only generic subrepresentation type being $N = kQ$. It follows immediately from the above description of $\widehat{M}$ that the desingularization coincides with the one defined in [10], where this variety is proved to be a isomorphic to a tower of $\mathbf{P}^1$-bundles.

8.2. **Smooth locus.** In the second example, we show that, in general, our desingularization does not reduce to an isomorphism over the smooth locus, i.e. its fibres can be nontrivial even over smooth points. Namely, consider the quiver Q given by $1 \stackrel{\alpha}{\to} 2$ and the representation M given by $M_1 = \langle v_1, v_2, v_3 \rangle$, $M_2 = \langle w_1, w_2 \rangle$, $M_\alpha(v_1) = w_1$, $M_\alpha(v_2) = w_2$, $M_\alpha(v_3) = 0$, which is injective, hence exceptional. For $\mathbf{e} = (1,2)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is isomorphic to the projective plane, hence smooth and irreducible. The only generic subrepresentation type N is a generic representation of dimension vector $\mathbf{e}$. Calculating $\widehat{M}$ and $\widehat{N}$ as above, we see that $\widehat{M}$ is given by $M_1 \stackrel{M_\alpha}{\to} M_2 \stackrel{\mathrm{id}}{\to} M_2$, and $\mathbf{dim}\widehat{N} = (1,1,2)$. Now $\mathrm{Gr}_{\mathbf{dim}\widehat{N}}(\widehat{M})$ is easily seen to be isomorphic to the blowup of the projective plane in a single point, corresponding to a non-generic subrepresentation of M. Note, however, that the desingularization is an isomorphism over the smooth locus in the case of the degenerate flag variety discussed above, as is proved in [6].

8.3. A_2 **case.** Now we give a complete analysis of the A_2 case. We start with a general remark on how to approach the description of the $\mathrm{Aut}(M)$-orbits in $\mathrm{Gr}_{\mathbf{e}}(M)$ in small cases. Consider the quiver $Q \times A_2$ with vertices i and i' for all $i \in Q_0$ and with arrows $\alpha : i \to j$, $\alpha' : i' \to j'$ for all $\alpha : i \to j$ in Q and $\iota_i : i' \to i$ for all $i \in Q_0$. We consider the algebra $kQ \otimes kA_2$ which is the quotient of the path algebra $k(Q \times A_2)$ modulo the ideal generated by all commutativity relations $\alpha\iota_i = \iota_k\alpha'$ for all $\alpha : i \to j$ in Q. Given M and $\mathbf{e}$ as before, we consider the dimension vector $\mathbf{f}$ for $Q \times A_2$ given by $f_i = d_i$ and $f_{i'} = e_i$. The variety $R_{\mathbf{f}}(Q \times A_2)$ of representations of $Q \times A_2$ of dimension vector $\mathbf{f}$ admits a projection map to $R_{\mathbf{d}}(Q)$ by restricting to the vertices i. Inside $R_{\mathbf{f}}(Q \times A_2)$, we consider the locally closed subset Y consisting of representations of $kQ \otimes kA_2$ such that all arrows ι_i are represented by injections. By the definitions, the induced projection $p : Y \to R_{\mathbf{d}}(Q)$ is isomorphic to the universal quiver Grassmannian $\mathrm{Gr}^Q_{\mathbf{e}}(\mathbf{d}) \to R_{\mathbf{d}}(Q)$ of [5], thus $p^{-1}(\mathcal{O}_M)$ is isomorphic to the variety $X_{\mathbf{e}}(M)$ of the proof of Proposition 7.2, that is, it is a $G_{\mathbf{e}}$-principal bundle over $\mathrm{Gr}_{\mathbf{e}}(M)$. This immediately yields a correspondence between $\mathrm{Aut}(M)$-orbits in $\mathrm{Gr}_{\mathbf{e}}(M)$ and $G_{\mathbf{f}}$-orbits in $p^{-1}(\mathcal{O}_M)$, respecting orbit closure relations, types of singularities, etc.. Furthermore, the latter orbits are in natural bijection to the isomorphism classes of representations V of $kQ \otimes kA_2$ of dimension vector $\mathbf{f}$ such that V identifies with M under restriction to the vertices i, and such that all V_{ι_i} are represented by injections.

This approach to the study of $\mathrm{Gr}_{\mathbf{e}}(M)$ is only efficient once the class of representations V above is well-understood, but the algebra $kQ \otimes kA_2$ is wild in general.

Here we only consider the case of the quiver Q given by $1 \stackrel{\alpha}{\to} 2$. We fix a representation M of Q of dimension vector $\mathbf{d} = (d_1, d_2)$, which is thus determined by the rank $r \leq \min(d_1, d_2)$ of the map representing the single arrow, and a dimension vector $\mathbf{e} = (e_1, e_2)$ such that $e_1 \leq d_1$ and $e_2 \leq d_2$. The quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is thus given as the variety of pairs of subspaces $(U_1, U_2) \in \mathrm{Gr}_{e_1}(M_1) \times \mathrm{Gr}_{e_2}(M_2)$ such that $M_\alpha(U_1) \subset U_2$.

Proposition 8.1. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{e}}(M)$ *for type* A_2 *has the following geometric properties:*

- (i) *It is non-empty if and only if* $r \leq d_1 - e_1 + e_2$.
- (ii) *It is reduced and connected.*
- (iii) *The* $\mathrm{Aut}(M)$*-orbits* $\mathcal{O}(r', r'')$ *in* $\mathrm{Gr}_{\mathbf{e}}(M)$ *are uniquely determined by the ranks* r' *resp.* r'' *of the induced maps* $M_\alpha : U_1 \to U_2$ *and* $M_\alpha : M_1/U_1 \to M_2/U_2$.
- (iv) *If* $r \geq e_1 + e_2 - d_2$, *it is irreducible of dimension* $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$.
- (v) *If* $r < e_1 + e_2 - d_2$, *the irreducible components* $I(a)$ *of* $\mathrm{Gr}_{\mathbf{e}}(M)$ *are parameterized by the* a *such that* $\max(0, r+e_1-d_1, r-d_2+e_2) \leq a \leq \min(e_1, e_2, r)$, *namely,* $I(a)$ *consists of all pairs* (U_1, U_2) *such that the ranks* r', r'' *fulfill* $r' \leq a$ *and* $r'' \leq r - a$.
- (vi) *If* $e_1 = 0$ *or* $e_2 = d_2$ *or* $r = \min(d_1, d_2)$ *the variety* $\mathrm{Gr}_{\mathbf{e}}(M)$ *is smooth.*
- (vii) *If* $r \geq e_1 + e_2 - d_2$, *the smooth locus consists of all* (U_1, U_2) *such that* $r' = e_1$ *or* $r'' = d_2 - e_2$.
- (viii) *If* $r < e_1 + e_2 - d_2$, *and the conditions of* (vi) *are not fulfilled, the smooth locus consists of all* (U_1, U_2) *such that* $r' = a$ *and* $r'' = r - a$ *for one of the integers* a *as in* (v).

Proof. The algebra $kA_2 \otimes kA_2$ is of finite representation type; the Auslander-Reiten quiver of the subcategory $\mathcal{D}$ of representations with the arrows ι_i being represented by injections is of the form

$$\begin{array}{ccccccccc}
 & & \begin{smallmatrix}1&1\\0&0\end{smallmatrix} & & & & \begin{smallmatrix}1&1\\1&1\end{smallmatrix} & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & & \begin{smallmatrix}1&1\\0&1\end{smallmatrix} & & & & \begin{smallmatrix}1&0\\1&0\end{smallmatrix} \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & \begin{smallmatrix}0&1\\0&1\end{smallmatrix} & & & & \begin{smallmatrix}1&0\\0&0\end{smallmatrix} & &
\end{array}$$

A representation in $\mathcal{D}$ is thus completely determined up to isomorphism by the system

$$\begin{matrix} & p_2 & & p_5 & \\ p_1 & & p_4 & & p_7 \\ & p_3 & & p_6 & \end{matrix}$$

of multiplicities of the above indecomposables. A short calculation shows that the representations $V(r',r'')$ in $\mathcal{D}$ of dimension vector $\begin{smallmatrix} d_1 & d_2 \\ e_1 & e_2 \end{smallmatrix}$ restricting to M are given by the multiplicities

$$\begin{aligned} p_1 &= d_2 - e_2 - r'' \\ p_2 &= r'' \\ p_3 &= e_2 - r + r'' \\ p_4 &= r - r' - r'' \\ p_5 &= r' \\ p_6 &= d_1 - e_1 - r + r' \\ p_7 &= e_1 - r'', \end{aligned}$$

in terms of parameters r', r'', which thus have to fulfill the inequalities

$$\begin{aligned} \max(0, r + e_1 - d_1) &\leq r' \leq e_1, \\ \max(r - e_2) &\leq r' \leq d_2 - e_2, \\ r' + r'' &\leq r; \end{aligned}$$

we denote by R the subset of $\mathbf{N}^2$ of pairs (r', r'') fulfilling these inequalities.

Thus the $\mathrm{Aut}(M)$-orbits $\mathcal{O}(r', r'')$ in $\mathrm{Gr}_{\mathbf{e}}(M)$ are naturally indexed by these parameters. Moreover, the parameters r', r'' are chosen in such a way that a subrepresentation $U \in \mathcal{O}(r', r'')$ is a representation of dimension vector $\mathbf{e}$, with the map representing the unique arrow of Q being of rank r', and the corresponding factor representation M/U is of dimension vector $\mathbf{d} - \mathbf{e}$, with the map representing the unique arrow of Q being of rank r''. This proves claim (iii). Moreover, working out the condition for non-emptyness of R, we arrive at claim (i).

We can also work out the orbit closure relation using the description of degenerations of representations of $kQ \otimes kA_2$ (which is a representation directed algebra) in terms of the so-called Hom-ordering [2]. A straightforward calculation yields the following criterion:

We have $\mathcal{O}(r_1', r_1'') \subset \overline{\mathcal{O}(r_2', r_2'')}$ if and only if $r_1' \leq r_2'$ and $r_1'' \leq r_2''$.

With the aid of this criterion, we can determine the irreducible components of $\mathrm{Gr}_{\mathbf{e}}(M)$ as the closures of the maximal (with respect to orbit closure inclusion) orbits, yielding claim (v) and the first half of claim (iv). Since any two different irreducible components $I(a)$, $I(a')$ intersect, namely in the closure of the orbit $\mathcal{O}(a, r - a')$, we have proved the second half of claim (ii).

By computing the dimension of the endomorphism ring of the representation $V(r', r'')$, we can determine the dimension of the orbit $\mathcal{O}(r', r'')$ as

$$e_1(d_1-e_1)+e_2(d_2-e_2)-(d_2-e_2+e_1)r+(e_1+r)r'+(d_2-e_2+r)r''-r'^2-r'r''-r''^2.$$

This yields the second half of claim (iv).

The dimension of the tangent space to a point $U \in \mathcal{O}(r', r'')$ can be computed, using the formula $\dim T_U(\mathrm{Gr}_{\mathbf{e}}(M)) = \dim \mathrm{Hom}(U, M/U)$, as

$$e_1(d_1 - e_1) + e_2(d_2 - e_2) - (d_2 - e_2)r' - e_1 r'' + r'r''.$$

This yields claim (vii), as well as claim (viii) using that all all non-maximal orbits belong to the intersection of at least two irreducible components in this case. Finally, the first half of claim (ii) follows. □

Specializing the general properties of the desingularization in the present case, we arrive at:

Corollary 8.2. *The following properties of the desingularization of* $\mathrm{Gr}_{\mathbf{e}}(M)$ *hold:*

(i) *If* $r \geq e_1+e_2-d_2$*, the fibre of the desingularization over a point of* $\mathcal{O}(r', r'')$ *is isomorphic to the Grassmannian* $\mathrm{Gr}_{e_1-r'}(k^{r-r'-r''})$.

(ii) *If* $r \geq e_1+e_2-d_2$*, the desingularization is one to one over the smooth locus if and only if* $r = e_1+d_2-e_2$.

(iii) *In this case, it is even small.*

(iv) *If* $r < e_1+e_2-d_2$*, the fibre of the desingularization over a point of* $\mathcal{O}(r', r'')$ *is isomorphic to the disjoint union of the Grassmannians* $\mathrm{Gr}_{a-r'}(k^{r-r'-r''})$ *for* $\max(0, r+e_1-d_1, r-d_2+e_2, r') \leq a \leq \min(e_1, e_2, r, r-r'')$.

(v) *In this case, unless* $e_1 = 0$ *or* $e_2 = d_2$ *or* $r = \min(d_1, d_2)$*, the desingularization is one to one over the smooth locus.*

Concluding the discussion of the A_2 case, we remark that the case $r = e_1+e_2-d_2$ is precisely the case of quiver Grassmannians of the form $\mathrm{Gr}_{\mathbf{dim}P}(P \oplus I)$ for P a projective and I and injective representation studied in [5]. An open question is whether the desingularization is one to one over the smooth locus in this case for arbitrary Dynkin quivers.

8.4. **Del Pezzo surface.** Now we consider the quiver Q given by $1 \to 2 \leftarrow 3$ and the quiver Grassmannian $X = \mathrm{Gr}_{\mathbf{dim}kQ}(kQ \oplus kQ^*)$, which is thus a generalized degenerate flag variety in the sense of [5]. Choosing appropriate basis, the representation $kQ \oplus kQ^*$ can be written as

$$k^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0\\0&0&0 \end{pmatrix}} k^4 \xleftarrow{\begin{pmatrix} 0&0&0\\1&0&0\\0&1&0\\0&0&0 \end{pmatrix}} k^3$$

The dimension vector $\mathbf{dim}kQ$ equals $(1,3,1)$, thus, identifying $\mathrm{Gr}_3(k^4)$ with $\mathbf{P}^3$, the quiver Grassmannian X can be realized as

$$\{((a:b:c),(d:e:f),(n:p:q:r)) \in \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^3 \ : \ an+bp=0,\ dp+eq=0\},$$

which is a singular projective variety of dimension five. We work out the desingularization Y in this specific case. The quiver $\widehat{Q}$ is of type E_6 and it is shown in Example 6.4. The representation $\widehat{kQ \oplus kQ^*}$ of $\widehat{Q}$ admits the following explicit form:

$$\begin{array}{c} k^4 \\ \uparrow{\scriptstyle\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0 \end{pmatrix}} \\ k^3 \xrightarrow{\begin{pmatrix} 1&0&0\\0&1&0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 1&0\\0&1\\0&0 \end{pmatrix}} k^3 \xleftarrow{\begin{pmatrix} 0&0\\1&0\\0&1 \end{pmatrix}} k^2 \xleftarrow{\begin{pmatrix} 1&0&0\\0&1&0 \end{pmatrix}} k^3 \end{array}$$

The only generic subrepresentation type being $N = kQ$, we thus have to consider subrepresentations of dimension vector

$$\begin{array}{ccccc} & & 3 & & \\ 1 & 1 & 2 & 1 & 1 \end{array}$$

of this representation. Again identifying $\mathrm{Gr}_2(k^3)$ with $\mathbf{P}^2$, we arrive at the following realization of Y:

$$\{((a:b:c),(d:e:f),(g:h),(i:j),(k:l:m),(n:p:q:r))$$

$$\in \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^3 \ :$$
$$ah = bg,\ dj = ei,\ kp = ln,\ kq = mn,\ lq = mp,\ gk + hl = 0,\ il + jm = 0\},$$

with the desingularization map being the projection to the first, second and sixth component. Defining Z as

$$\{((g : h), (i : j), (k : l : m)) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2 \ : \ gk + hl = 0,\ il + jm = 0\},$$

we can view Y as a closed subvariety of $X \times Z$, with the desingularization map being the first projection.

The structure of Z is easily analysed by considering the projection to $\mathbf{P}^2$; namely, this proves that Z is isomorphic to a Del Pezzo surface, namely $\mathbf{P}^2$ blown up in two distinct points. By a straightforward analysis of the projection from Y to Z, we can see that Y is a three-fold tower of $\mathbf{P}^1$-fibrations over Z. Thus, the Poincaré polynomial of Y (in l-adic cohomology for an arbitrary algebraically closed field k) equals $(1 + 3t^2 + t^4)(1 + t^2)^3$.

The only two-dimensional fibre of the desingularization map is the one over the point $((0 : 0 : 1), (0 : 0 : 1), (0 : 0 : 0 : 1))$, namely, it is isomorphic to the Del Pezzo surface Z. If $(a, b) \neq 0$ or $(d, e) \neq 0$ or $(n, p, q) \neq 0$, the fibre is trivial, thus the locus of points of X with positive dimensional fibre is of codimension at least three (compare this to the general result of [6] that X is regular in codimension two), proving smallness of the desingularization map. Consequently, we also know the Poincaré polynomial of the (l-adic) intersection cohomology of X.

Acknowlegments

The authors would like to thank Claire Amiot, Klaus Bongartz, Aslak Buan, Lutz Hille, Bernhard Keller, Ragnar-Olaf Buchweitz and Michel Van den Bergh for interesting remarks on this work.

The work of Evgeny Feigin was partially supported by the Russian President Grant MK-3312.2012.1, by the Dynasty Foundation, by the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023, by the RFBR grants 12-01-00070, 12-01-00944 and by the Russian Ministry of Education and Science under the grant 2012-1.1-12-000-1011-016. This study comprises research fundings from the 'Representation Theory in Geometry and in Mathematical Physics' carried out within The National Research University Higher School of Economics' Academic Fund Program in 2012, grant No 12-05-0014. This study was carried out within the National Research University Higher School of Economics Academic Fund Program in 2012-2013, research grant No. 11-01-0017.

References

1. I. Assem, D. Simson, A. Skowronski, *Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory.* London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.
2. K. Bongartz, *On Degenerations and Extensions of Finite Dimensional Modules*, Adv. Math. **121** (1996), 245–287.
3. P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. **81** (2006), no. 3, 595–616. MR MR2250855 (2008b:16015)
4. P. Caldero and M. Reineke, *On the quiver Grassmannian in the acyclic case*, J. Pure Appl. Algebra **212** (2008), no. 11, 2369–2380. MR MR2440252 (2009f:14102)
5. G. Cerulli Irelli, E Feigin, M. Reineke, *Quiver Grassmannians and degenerate flag varieties*, Algebra Number Theory **6** (2012), 1, 165–194.
6. G. Cerulli Irelli, E. Feigin, M. Reineke, *Degenerate flag varieties: moment graphs and Schröder numbers*, Preprint 2012, to appear in J. Algebraic Combin., arXiv:1206.4178.
7. W. Crawley-Boevey, *Maps between representations of zero-relation algebras*, J. Algebra **126** (2001), no. 2, 259–263, 1989 .

8. E. Feigin, $\mathbb{G}_a^M$ *degeneration of flag varieties*, Selecta Mathematica: Volume 18, Issue 3 (2012), Page 513–537.
9. E. Feigin, *Degenerate flag varieties and the median Genocchi numbers*, Mathematical Research Letters, 18 (2011), no. 6, pp. 1–16.
10. E. Feigin, M. Finkelberg, *Degenerate flag varieties of type A: Frobenius splitting and BWB theorem*, arXiv:1103.1491.
11. S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.
12. P. Gabriel, A. Roiter, *Representations of finite-dimensional algebras*, Springer, Berlin, 1997.
13. D. Happel, *On the derived category of a finite-dimensional algebra*, Commentarii Mathematici Helvetici **62** (1987), 339–389.
14. D. Hernandez, B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, Preprint 2011, arXiv:1109.0862
15. B. Leclerc, P. Plamondon, *Nakajima varieties and repetitive algebras*, Preprint 2012, arXiv:1208.3910
16. A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64. MR MR1162487 (93d:16014)

Giovanni Cerulli Irelli:
Mathematisches Institut, Universität Bonn, Bonn, Germany 53115
E-mail address: cerulli.math@googlemail.com

Evgeny Feigin:
Department of Mathematics,
National Research University Higher School of Economics,
Russia, 117312, Moscow, Vavilova str. 7
and
Tamm Department of Theoretical Physics, Lebedev Physics Institute
E-mail address: evgfeig@gmail.com

Markus Reineke:
Fachbereich C - Mathematik, Bergische Universität Wuppertal, D - 42097 Wuppertal
E-mail address: reineke@math.uni-wuppertal.de