Differential Bargaining Games as Microfoundations for Production Function *

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Abstract In the present paper the game theory is applied to an important open question in economics: providing microfoundations for often-used types of production function. Simple differential games of bargaining are proposed to model a behavior of workers and capital-owners in processes of formation of possible factor prices and participants' weights (moral-ethical assessments). These games result, correspondingly, in a factor price curve and a weight curve – structures dual to a production function. Ultimately, under constant bargaining powers of the participants, the Cobb-Douglas form of the production function is received.

Keywords: bargaining, differential games, production factors, choice of technology, duality, production function.

1. Introduction

It is well known that the acceptance of concrete types of production functions in economics, such as the Cobb-Douglas and the CES forms, was rather occasional and till now not enough attempts have been made to explain and justify the wide used types of production function – e.g. (Matveenko, 1997; Acemoglu, 2003; Jones, 2005; Lagos, 2006; Nakamura, 2009; Matveenko, 2010; Dupuy, 2012). In the paper models resulting in the Cobb-Douglas production function are constructed on base of differential games of bargaining and by use of dual relations in production and distribution. A simple differential game of price bargaining is introduced as a benchmark and then is modified to a differential game of bargaining for prices of capital and labor and to a differential game of weights (moral-ethical assessments of the factor owners. Each of these three differential games exploits one or another of duality relations existing in the economy (cf. (Cornes, 1992)).

One of the duality relations used in the paper is usually represented as the duality between the production function $Y = F(K, L)$ and the cost function $C(p_K, p_L, Y)$. The first of these functions shows the maximal output in dependence on production factors: capital and labor, while the second one shows the minimal cost in dependence on prices of the production factors and the output. We study a similar duality by use of the well-known representation of the production function by use of the Euler theorem:

$$F(K.L) = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L = p(x)x,$$

where $x = (K, L)'$ is the vector of production factors (capital and labor) and $p(x) = (\partial F/\partial K, \partial F/\partial L)$ is the corresponding price vector (the vector of marginal

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products). There exists a set Π of the price vectors corresponding the production function, such that the Euler theorem can be written in the "extremal" version:

$$F(K, L) = \min_{p \in \Pi} px, \tag{1}$$

which means that the production function represents a result of a choice of the price vector from the set Π. Let $M = \{x : F(x) = 1\}$ be the unit level line of the production function F. A conjugate problem for (1) is the problem of a choice of the bundle of production factors $x = (K, L)$ from the set M to provide a unit output with minimal cost:

$$F^*(p) = \min_{x \in M} px.$$

Rubinov (Rubinov and Glover, 1998; Rubinov, 2000) introduced some other types of duality using instead of the usual inner product its analogues, such as Leontief function $\min_{i=1,\dots,n} l_i x_i$. Notably, the latter is similar to the inner product but uses the idempotent operation of summation: $\bigoplus = \min$. Matveenko (1997; 2010) and Jones (2005) found a representation for neoclassical production functions which reminds (1) but uses the Leontief function as an inner product; in the two factor case:

$$F(K, L) = \max_{l \in \Psi} \min\{l_K K, l_L L\}.$$

In Section 2 we introduce the benchmark differential game of price bargaining. In Section 3 a differential game of factor price curve formation is considered. In Section 4 a differential game of weight curve formation is studied which, together with the model in Section 3, provides a foundation for the Cobb-Douglas production function. Section 5 concludes.

2. Benchmark differential game of price bargaining

The term *bargain* relates both to a process of bargaining and to a result of this process. Both sides of bargaining are being studied in the bargaining theory – a special chapter of the game theory, However, traditionally, the bargaining theory deals more with results of bargains rather than with processes of bargaining. Nash (1950) proposed a system of axioms leading to a so called symmetric Nash bargaining solution; later an asymmetric solution was found and axiomatized . For the reviews of the axiomatic approach in the bargaining see (Roth, 1979; Thomson, 1994; Serrano, 2008). The models of processes of bargaining are usually based on assumptions concerning economic benefits gained by participants under one or other running of the process of bargaining (see (Muthoo, 1999)). For example, a participant can bear some costs connected with the duration of the bargaining process. In practice, however, in many cases the course of a bargaining process depends in much not on expectations of economic benefits by participants but on their skills to bargain (see (Schelling, 1956; Blainey, 1988)). These skills can be associated with bargaining powers of the participants. The notion of bargaining power is often used in game theory, though, different authors put different sense into this notion . In this Section we propose a simple differential game as a model of a bargaining process. In different versions of the game the bargaining powers of the players are either given exogenously or are defined endogenously in the game itself.

In the benchmark example of bargaining (Muthoo, 1999) an object is on sale (e.g. a house). A seller (player S) wishes to sell the house for a price exceeding

$\bar{p}_S^0$(the latter is the minimal price acceptable for player S). A buyer (player B) is ready to purchase the house for a price not exceeding $\bar{p}_B^0$ (the maximal acceptable price for player B). Here $\bar{p}_B^0 > \bar{p}_S^0$, what ensures the possibility of the bargain. The seller starts from a start price, $p_S(0) > \bar{p}_S^0$, and then decreases her price, while the buyer simultaneously starts from a price $p_B(0) < \bar{p}_B^0$ and then increases her price. It is assumed, naturally, that $p_B(0) < p_S(0)$. A price trajectory $p_B(t), p_S(t)$ formed in continuous time stops at a moment T when $p_B(T) = p_S(T)$. It follows that $p_B(t) < p_S(t)$ for $t \in [0, T)$. The selling price will be referred as p^* . A surplus of the selling price over (under) the minimal (maximal) admissible price of a player can be considered as the player's utility:

$$u_S = p^* - \bar{p}_S^0, u_B = \bar{p}_B^0 - p^*. \tag{2}$$

A set Ω of all possible pairs of utilities on plane (u_B, u_S) is

$$\Omega = \{(u_B, u_S) : u_B + u_S = \bar{p}_B^0 - \bar{p}_S^0, u_B, u_S \geq 0\}.$$

A simplest model of price bargaining appears under an assumption that each player $i = B, S$ changes her price with a constant velocity equal to the bargaining power of her opponent. A strong opponent forces the player to change her price faster:

$$p_S(t) = p_S(0) - b_B t, p_B(t) = p_B(0) + b_S t.$$

The game stops at the moment T which is found from equation:

$$p_S(0) - b_B T = p_B(0) + b_S T,$$

i.e. at the moment

$$T = \frac{p_S(0) - p_B(0)}{b_S + b_B}$$

when the selling price is:

$$p^* = p_S(T) = p_B(T) = \frac{b_S}{b_S + b_B} p_S(0) + \frac{b_B}{b_S + b_B} p_B(0). \tag{3}$$

So, the selling price is the convex combination of the start prices proposed by the players summed with weights equal to their relative bargaining powers. If each player i knows the minimal (maximal) price accessible for the opponent and establishes it as her start price, then the play stops at the moment:

$$T = \frac{\bar{p}_B^0 - \bar{p}_S^0}{b_S + b_B}$$

with the selling price:

$$p^* = \frac{b_S}{b_S + b_B} \bar{p}_B^0 + \frac{b_B}{b_S + b_B} \bar{p}_S^0$$

and with the utilities of the players equal to

$$u_i = \frac{b_i}{b_S + b_B} (\bar{p}_B^0 - \bar{p}_S^0), i = B, S. \tag{4}$$

Theorem 1. *Price p^* corresponds the asymmetric Nash bargaining solution of the bargaining problem under utilities (2) and bargaining powers b_S, b_B .*

Proof. The asymmetric Nash bargaining solution is here a solution of the problem of maximization of the function $u_B^{b_B} u_S^{b_S}$ on the set Ω . The first order optimality condition, $b_B u_S + b_S u_B$, and the constraint, $u_B + u_S = \bar{p}_B^0 - \bar{p}_S^0$, define the asymmetric Nash bargaining solution is found, which coincides with (4)).

The case when the players change prices under constant growth rates (rather than constant velocities) is similar. Since the growth rate of price is the velocity of changing the logarithm of the price, an equation similar to (3) is fulfilled: the bargaining stops under a price the logarithm of which is equal to the convex combination of the logarithms of the start prices with weights equal to the relative bargaining powers of the players.

In a more complex case the velocity of changing price by a player depends on the actions of her opponent. If the seller decreases her price slowly then the buyer also increases her price slowly because she does not want the game to stop on a too high price. Similarly, if the buyer increases her price slowly then the seller decreases her price slowly. Let the growth rates of price change, $g_i = \frac{\dot{p}_i}{p_i}, i = B, S$, be constant. The bargaining power of player i can be defined as the value inverse to $|g_i|$:

$$b_B = \frac{1}{g_B}, b_S = -\frac{1}{g_S}.$$

Then

$$\frac{g_B}{g_S} = -\frac{b_S}{b_B},$$

i.e.

$$\frac{dp_B}{dp_S}\frac{p_S}{p_B} = -\frac{b_B}{b_S} = const,$$

The game interpretation of this differential equation is the following. Each player i chooses a control g_i , and the controls are connected by the relation:

$$g_B \geq |g_S|\frac{b_S}{b_B},$$

which means that in the bargaining process the faster the seller decreases her price the faster the buyer increases hers. Moreover, a higher bargaining power of the buyer relaxes this constraint (this means a lower degree of reaction to the opponent's actions), and a higher bargaining power of the seller reinforces the constraint. At the same time the seller is limited by the opposite constraint:

$$g_S \geq |g_B|\frac{b_B}{b_S},$$

which means that the faster the buyer increases her price the faster the seller decreases hers. An increased bargaining power of the buyer forces the seller to diminish her price faster, and an increased own bargaining power allows the seller to diminish her price slower. Simultaneous fulfillment of inequalities (7) and (8) implies the Equation (6).

3. Bargaining for production factor prices and corresponding choice of technologies

In the just described benchmark differential game the players change their proposals concerning the same price. Now we turn to differential games in which the interests of the players relate to different prices. At each moment of time one of the players *attacks*, another one *defends*. Only the attacker is satisfied by the direction of her price change while the defender hinders changes in her price.

In the present Section the following pair of dual objects will be under consideration:

(i) a neoclassical production function $F(K, L)$ which is characterized by its factor curve: $M = \{(K, L) : F(K, L) = 1\}$, i.e. the set of bundles of resources allowing the unit output, and

(ii) the factor price curve $\Pi = \{(p_K, p_L)\}$ i.e. the set of such bundles of prices under which the unit output under unit costs is possible.

3.1. Usual causality

Given production function $F(K, L)$, the price curve Π can be found from the following system of equations:

$$F(K, L) = 1, \tag{5}$$

$$p_K K + p_L L = 1, \tag{6}$$

$$\frac{\partial F/\partial K}{\partial F/\partial L} = \frac{p_K}{p_L}. \tag{7}$$

Equations (5) and (6) are conditions of the unit output under unit costs. Equation (7) is a condition of efficiency of production; it can be interpreted as a condition of output maximization under given costs.

The system (5)-(6) establishes a one-to-one correspondence between points of the factor curve, M, and points of the factor price curve, Π. Indeed, by the Euler theorem, the Equation (9) can be written as

$$\frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L = 1, \tag{8}$$

then Equations (6)-(7) imply:

$$\frac{\partial F}{\partial K} = p_K, \frac{\partial F}{\partial L} = p_L. \tag{9}$$

In particular, for the Cobb-Douglas production function, $F(K, L) = AK^{\alpha} L^{1-\alpha}$, the system (9) takes the form:

$$\alpha K^{\alpha-1} L^{1-\alpha} = p_K,$$

$$(1-\alpha) AK^{\alpha} L^{1-\alpha} = p_L.$$

Excluding the ratio K/L from these two equations we find the factor price curve:

$$B p_K^{\alpha} p_L^{1-\alpha} = 1$$

where $B = A^{-1} \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)}$.

For the CES function $F(K, L) = (\alpha (A_K K)^p + (1-\alpha)(A_L L)^p)^{\frac{1}{p}}$ where $p \in (-\infty, 0) \cup (0, 1)$ the system (13) takes the form:

$$\alpha A_K^p (\alpha A_K^p + (1-\alpha) K^{-p} (A_L L)^p)^{\frac{1}{p}-1} = p_K,$$

$$(1-\alpha) A_L^p (\alpha (A_K K)^p L^{-p} + (1-\alpha) A_L^p)^{\frac{1}{p}-1} = p_L.$$

Excluding $K^{-p}L^p$ from these equations we receive, after some transformations, the following equation of the factor price curve in form of CES function:

$$B\left[\beta\left(\frac{p_K}{A_K}\right)^q + (1-\beta)\left(\frac{p_L}{A_L}\right)^q\right]^{1/q} = 1,$$

where

$$B = (\alpha^{\frac{1}{1-p}} + (1-\alpha)^{\frac{1}{1-p}})^{\frac{p-1}{p}}, \beta = \frac{\alpha^{\frac{1}{1-p}}}{\alpha^{\frac{1}{1-p}} + (1-\alpha)^{\frac{1}{1-p}}}, q = \frac{p}{1-p}.$$

3.2. Reversed causality

Usual causality, considered in the previous Subsection, presupposes that the prices are primarily determined by the physical side of production - physical technologies and available bundles of production resources. However, another direction of causality is possible: institutions, reflected by the prices, can define which products have to be produced and by use of which technologies. We propose now a model in which the factor price curve, Π , is defined in a pure institutional way. This model belongs to a class of island models - such where partially independent segments of a market are considered.

There are two types of agents: workers and entrepreneurs. A single product is produced in a continuum of segments – islands, some of them are "inhabited" by the agents of both types. On each of the inhabited islands in each moment of time there are definite prices of labor and capital in terms of the product. In random moments of time from randomly chosen islands either a part of workers or a part of entrepreneurs moves to an uninhabited island. Since this moment the prices in the inhabited island are fixed. After that a part of the other social group also moves to the "new" island and there the groups start bargaining about the factor prices. Those who have come first possess an advantage and try to increase their factor price - they attack. Those who have come later try not to allow their factor price to fall too much - they defend. As start prices in the bargaining process the groups use the prices in the "old" island at the moment when the first group left. It is assumed that the social groups always have constant bargaining powers, b_K, b_L . Weakening this assumption is left for a future research. Opposed to the case of the selling/purchasing bargaining game considered in Section 2, now the prices relate to different goods (labor and capital). The attacker, a, is interested in maximizing the growth rate of her factor price while the defender, d, is interested in minimizing (the module of) the growth rate of her factor price. In the simplest case, similarly to the case considered in Section 2, it can be assumed that players have constant growth rates of their factor prices, $g_i = \dot{p}_i/p_i$, where $g_a > 0$ for the attacker; $g_d < 0$ for the defender; and the price growth rates are linked with the bargaining powers by the equation:

$$|g_d| = \frac{b_a}{b_d} g_a.$$

According to this equation, a higher relative bargaining power b_d/b_a of the defender allows her to reach a slower decline in her factor price, i.e. a smaller $|g_d|$. Vice versa, an increase in the bargaining power of the attacker forces the defender to agree to a larger decline in her factor price. Equation (14) describing the price change process turns into:

$$\frac{dp_a}{dp_d} = -\frac{b_d}{b_a} = const,$$

which can be written as

$$\frac{dp_K}{p_K} b_K = -\frac{dp_L}{p_L} b_{L.}$$

Solving this differential equation we receive the price curve Π :

$$p_K^{b_K} p_L^{b_L} = C. \tag{10}$$

If initially the price vector belongs the curve Π given a constant C then the vector stays in the same curve further.

To describe the strategic behavior of the players in more details, let the attacker's problem be to maximize her price growth rate, g_a , under the following constraint:

$$|g_d| \geq g_a \frac{b_a}{b_d},$$

and, correspondingly, let the defender's problem be to minimize the module of her price growth rate, g_d, under (17). The inequality (17) means that the attacker forces the defender to increase her price reduction rate. An increased bargaining power of the attacker reinforces this constraint, while an increase in the bargaining power of the defender relaxes it. There exists a continuum of Nash equilibria, (g_a, g_d) , and all of them satisfy the equation

$$\frac{g_a}{|g_d|} = \frac{b_d}{b_a}$$

This equation, independently on which player (K or L) is the attacker, leads to the price curve (10). Now let us show in what way the price curve (16) leads to the Cobb-Douglas type of production function. We will use the representation of neoclassical production function by use of a menu of Leontief technologies (Matveenko, 1997; Matveenko, 2010; Jones, 2005). Matveenko (2010) has shown that to each neoclassical production function $F(K, L)$ a unique technological menu Ψ corresponds which consists of effectiveness coefficients of the Leontief function and is such that

$$F(K, L) = \max_{l \in \Psi} \min\{l_K K, l_L L\}.$$

Moreover, there exists a simple one-to-one correspondence between the points $(K, L) \in M$ of the factor curve and the points $l \in \Psi$ of the technological menu:

$$(l_K, l_L) \in \Psi \Leftrightarrow \left(\frac{1}{l_K}, \frac{1}{l_L}\right) = (\tilde{K}, \tilde{L}) \in M.$$

The function

$$F^{\circ}(l_K, l_L) = \frac{1}{F\left(\frac{1}{l_K}, \frac{1}{l_L}\right)}$$

is referred to as a *conjugate (polar) function*. Representation (18) follows from the following Lemma.

Lemma 1. *Let $F(x_1, x_2, ..., x_n)$ be an increasing positively homogeneous of 1st power function of n positive variables, M - its unit level set, and Ψ - the unit level set of the conjugate function:*

$$M = \{x : F(x_1, x_2, ..., x_n) = 1\},$$

$$\Psi = \{l : F\left(\frac{1}{l_1}, \frac{1}{l_2}, ..., \frac{1}{l_n}\right) = 1\}.$$

Then

$$F(x_1, x_2, ..., x_n) = \max_{l \in \Psi} \min\{l_1 x_1, l_2 x_2, ..., l_n x_n\}.$$

See proof in (Matveenko, 2010).

When a pair of prices is defined on an island, the island chooses a suitable technology on base of one or another pure economic criterion (efficiency) or an institutional criterion (fairness). We assume that the whole set ("cloud") of available Leontief technologies is extensive enough to include all those technologies which any islands would choose to use. The technological menu Ψ is narrower and consists of those technologies which would be chosen. Below three mechanisms of choice are identified resulting in the same technological menu Ψ and the factor curve .

Mechanism A. Given factor prices p_K^0, p_L^0 , an island chooses such Leontief technology (l_K, l_L) which guarantees receiving factor shares equal to the relative bargaining powers of the social groups : $\alpha = \frac{b_K}{b_K + b_L}$ for the capital and $1 - \alpha = \frac{b_L}{b_K + b_L}$ for the labor. For this technology, such volumes of factors $\tilde{K}, \tilde{L}$ exist for which:

$$l_K \tilde{K} = l_L \tilde{L} = 1 - \alpha = 1, p_K^0 \tilde{K} = \alpha, p_L^0 \tilde{L} = 1 - \alpha.$$

Such kind of choice of the Leontief technologies by all islands results in the following factor curve:

$$M = \{(K, L) : p_K K = \alpha, p_L L = 1 - \alpha, (p_K, p_L) \in \Pi\} = \{(K, L) : AK^\alpha L^{1-\alpha}\},$$

where $A = \frac{C}{\alpha^\alpha (1-\alpha}^{1-\alpha}$. Thus, the Leontief technologies chosen by all the islands define the Cobb-Douglas production function: $F(K, L) = AK^\alpha L^{1-\alpha}$.

Mechanism B. Given factor prices p_K^0, p_L^0 , an island chooses such Leontief technology $(l_K, l_L) = (\frac{1}{K^0}, \frac{1}{L^0})$ which is *competitive* in the sense that, under this technology, the cost of the unit production on the island is equal to 1, while the cost on any other island is greater than 1. So, the usage of this technology is profitable only on the present island. In other words,

$$p_K^0 K^0 + p_L^0 L^0 = 1 < p_K K^0 + p_L L^0$$

for any bundle of prices $p_K, p_L \in \Pi, (p_K, p_L) \neq (p_K^0, p_L^0)$. It follows that p_K^0, p_L^0 is a solution for the problem:

$$\min_{(p_K, p_L) \in \Pi} (p_K K^0 + p_L L^0).$$

The first order optimality condition for this problem is:

$$\frac{p_L^0 L^0}{p_K^0 K^0} = \frac{1 - \alpha}{\alpha},$$

and we come to the Mechanism A.

Mechanism C. Given factor prices p_K^0, p_L^0 , an island chooses a Leontief technology (l_K, l_L) (or, what is equivalent, $(K, L) \in M$) ensuring fulfillment of a fairness principle:

$$\max_{(K,L)\in M} \min\{\frac{p_K^0 K}{b_K}, \frac{p_L^0}{b_L}\},$$

which is analogous to the Rawlsian maximin principle: a gain of the most hurt agent has to be maximized. Here the gain of an agent is her revenue but with account of her bargaining power: a participant's gain increases if her relative bargaining power increases. The solution is characterized by the equation:

$$\frac{p_K^0 \tilde{K}}{b_K} = \frac{p_L^0 \tilde{L}}{b_L},$$

hence,

$$\frac{p_L^0 \tilde{L}}{p_K^0 \tilde{K}} = \frac{1-\alpha}{\alpha},$$

and again we come to the Mechanism A.

4. Differential game of weights formation

In this Section we provide a microfoundation for the Mechanism A decribed in Section 3. We propose a differential game in which the players (workers and capital-owners) form a weight curve - a set of possible assessments (weights); the curve is used by an arbiter to choose a vector of weights in a concrete bargain.

Three common features present in many real bargains and negotiations. Firstly, it is a presence of an arbiter in which role often a community acts, in a framework of which the bargainers interact. Examples are so called 'international community', including governments and elites of countries, and different international organizations; a 'collective' or a union in a firm; a local community; a 'scientific community', etc. The community acts as an arbiter realizing a control for bargains in such way that unfair, from the point of view of the arbiter, bargains are less possible, at least as routine ones. An outcome of an unfair bargain can be, with a help of the arbiter, revised, if not formally than through a conflict. Such conflicts rather often arise, both on a local and on a national levels, as well as in international relations. Secondly, bargains inside a fixed set of participants are often not 'one-shot' but represent a routine repeated process in which a 'public opinion' of the community is important; and the latter is being formed along with the bargains. Usually it is unknown in advance what concrete bargains will take place and in what time, and the process of formation of the public opinion processes uninterruptedly to prepare it for future bargains. The public opinion can be modeled as a set of the vectors of weights - the moral-ethical assessments which can be used by the arbiter as coefficients for the participants' utilities . Possibilities of formation of public opinion are limited both by possibilities of access to media and by image-making abilities of the participants. Thirdly, the moral-ethical assessments formed by participants are usually not univalent, but allow a variance: the public opinion practically always can stress both positive and negative features of a participant; concrete weights differ in different concrete bargains depending on circumstances. Thus, it can be useful to speak not about a single vector of weights but rather about a curve (in case of two participants) or a surface of admissible assessments.

Thus, the public opinion can be modeled as a weight curve (or a weight surface). In its approval or disapproval of a possible result of a concrete bargain the arbiter acts in accordance with a Rawlsian-type maximin principle, paying attention to the most infringed participant, but taking into account admissible vectors of weights for utilities, the set of which is formed in advance by the participants.

We consider a two stage game. On the first stage, two players (workers and capital-owners) form a curve $\Lambda = (\lambda_K, \lambda_L$ consisting of vectors of admissible reputational assessments (*weights*). On the second stage, for a concrete bargain, an arbiter (community) chooses an admissible pair of weights from the weight curve and divides the product Y among the players ($Y = Y_K + Y_L$) to achieve the maximin

$$\max_{y \in \Omega} \max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\}, \tag{11}$$

where

$$\Omega = \{y = (Y_K, Y_L) : Y = Y_K + Y_L\}$$

is the set of outputs.

Let us describe the first stage of the game in detail. A player's gain depends negatively on her weight and depends positively on the opponent's weight . Hence, each player is interested in decreasing her weight and in increasing the opponent's weight. However, in the process of the weight curve formation, the player i would agree to a decrease in the opponent's weight in some part of Λ at the expense of an increase in her own weight, as far as the opponent similarly temporizes in another part of Λ. Since the system of weights is essential only to within a multiplicative constant, the players can start the formation of the weight curve Λ from an arbitrary pair of weights and then construct parts of the curve to the left and to the right of the initial point. The player who attacks maximizes, at each moment of time, the module of her weight's growth rate while the defender minimizes her weight's growth rate. This takes place under the following constraint:

$$|g_a| \leq g_d \frac{b_a}{b_d}, \tag{12}$$

which means that a higher bargaining power of the attacker helps her to enlarge the constraint, while an increase in the bargaining power of the defender makes the constraint stricter.

In equilibrium (12) is fulfilled as an equality. Thus, the constancy of the bargaining powers of the participants implies :

$$\frac{d\lambda_L}{d\lambda_K} \frac{\lambda_K}{\lambda_L} = -\frac{b_K}{b_L} = const. \tag{13}$$

The more the bargaining power of a player is the better reputational assessment she gains for herself. Solving the differential equation (13) we receive the weight curve λ :

$$\lambda_K^{b_K} \lambda_L^{b_L} = C = const.$$

Now we turn to the second stage of the game.

Lemma 2. *For each outcome, the following equality is valid:*

$$max_{\lambda \in \Lambda} min\{\lambda_K Y_K, \lambda_L Y_L\} = AY_K^{\frac{b_K}{b_K+b_L}} Y_L^{\frac{b_L}{b_K+b_L}},$$

where $A = const$.

Proof. It follows from Lemma 1 when it is applied to the set Λ.

According to (14), the solution of the arbiter's problem (11) is none other than the asymmetric Nash bargaining solution.

It is easily seen that the players receive shares proportional to their bargaining powers. This provides a support to the Mechanism A described in Section 3. This mechanism, as we have seen there, generates the Cobb-Douglas production function. Notice, that a constancy of bargaining powers can explain a constancy of factor shares in some countries on a definite stage of their development.

5. Conclusion

In this paper a new approach is proposed for understanding a relation between a physical side of economy (resources and technologies) and its institutional side (distributional relations between social groups). The idea of the models presented here is that the distributional behavior can be described by a differential game of bargaining.

Three differential games are proposed to describe a behavior of economic agents in processes of prices and weights formation. In the benchmark model of price bargaining players are interested in changing the same price in opposite directions. It is shown that under some conditions this game leads to the Nash bargaining solution. This benchmark game is modified to games in which players change (different) prices of their owned resources or change weights (moral-ethical assessments). One of these games describes bargaining of workers and capital-owners for their factor prices. In another game the same players bargain for weights (moral-ethical assessments); these weights enter a Rawlsian-type criterion which is used by an arbiter (community) in concrete bargains. These games result in construction of structures – a price curve in one case and a weight curve in another – which are dual to the production function. Ultimately, under constant bargaining powers of the participants, these games lead to the Cobb-Douglas form of production function.

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