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# FREQUENCY SPECTRUM OF EXACT SOLUTIONS OF THE TWO-DIMENSIONAL HYDRODYNAMIC EQUATIONS

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*We show that the discrete frequency spectrum of a plane hydrodynamic flow of ideal incompressible liquid with localized trajectories of the liquid particles can contain only one, two, or an infinite number of harmonics.*

## 1. INTRODUCTION

When explaining or interpreting the properties of physical systems (or solutions of the corresponding equations), the spectral approach is often useful. Its use is mostly justified for the linear systems with constant parameters [1]. At the same time, the spectral method is also actively employed when analyzing nonlinear problems. In the approximations corresponding to a weak nonlinearity, the problems of three-wave interactions and explanation of the modulation-instability nature by an increase in the amplitude of oscillations at the side frequencies have already become classical. For strongly nonlinear phenomena, the theory of turbulence is a demonstrative example of using the spectral method.

In this work, we consider the issue of whether the exact solution of the equations of two-dimensional hydrodynamics can have a finite discrete set of time frequencies. This problem is related to the discovery of the Ptolemaic flows, i.e., a class of exact solutions to the equations of the two-dimensional hydrodynamics of an ideal liquid [2]. The general solution for such flows contains two arbitrary frequencies.

The Gerstner waves (trochoidal waves in deep water), as well as the Kirchhoff vortex (an elliptic region with homogeneous vorticity, which uniformly rotates round its center) belong to the Ptolemaic flows [3]. In both cases, the liquid particles rotate over the circumference, i.e., one of the frequencies in the general formula for the exact Ptolemaic solution is equal to zero. The one-frequency Ptolemaic flows were used for describing a single vortex region in the surrounding potential flow [2] and solving the problem of oscillations of a free liquid surface under the action of a nonuniform pressure distribution over the surface [4–6]. The examples of two-frequency Ptolemaic flows, i.e., epicycloidal waves on the surface of a uniformly rotating circular cavity and hypocycloidal waves on the plasma-cylinder surface in a uniform homogeneous longitudinal magnetic field are developed in [7] and [8], respectively. We would like to know whether there exist exact solutions in the case of three or more discrete frequencies.

In addition to the two-frequency nature, the fact that the analytical expression contains two somewhat arbitrary functions of the spatial coordinates is another important property of the analytical expression describing the Ptolemaic flows. This allows one to find the whole sets of exact solutions for certain types of the physical problems. Thereby, the possible generalization of the Ptolemaic solutions to the case of a more complicated spectrum of the hydrodynamic flow is of both applied and general theoretical interest.

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TABLE 1.

| $G(\chi)$ | $F(\bar{\chi})$ | $\lambda$ | $\mu$ | Flow type or name |
|---|---|---|---|---|
| $\chi$ | $iA\exp(ik\bar{\chi})$ | $0$ | $\neq 0$ | Gerstner waves [3] |
| $\alpha\exp(ik\chi)$ | $\beta\exp(-ik\bar{\chi})$ | $\neq 0$ | $0$ | Kirchhoff vortex [3] |
| $\alpha\exp(ik\chi)$ | $\lvert F'\rvert \le 1$, $\lvert\chi\rvert \le 1$ | $\neq 0$ | $0$ | Abrashkin–Yakubovich vortices [2, 11] |
| $\alpha\exp(ik\chi)$ | $\beta\exp(ikn\bar{\chi})$, $n = 2, 3, \ldots$ | $\neq 0$ | $\neq 0$ | hypocycloidal vortex in a plasma [8] |
| $\alpha\exp(ik\chi)$ | $\beta\exp(-ikn^2\bar{\chi})$, $n = \pm 1, \pm 2, \ldots$ | $\neq 0$ | $n\lambda$ | epicycloidal waves inside the cylindrical cavity [7] |
| $\chi - \dfrac{i\beta}{(\chi - ia)^2}$ | $iA\exp(ik\bar{\chi}) + \dfrac{i\beta}{(\bar{\chi} + i\alpha)^2}$ | $= 0$ | $\neq 0$ | rogue wave against the background of uniform waves [4, 5] |
| $\chi - \dfrac{i\beta}{(\chi - i)^2}$ | $\dfrac{i\beta}{(\bar{\chi} + i)^2}$ | $= 0$ | $\neq 0$ | breather on calm water [6] |

The mathematical part of this study is based on the original method which assumes transition to the complex Lagrangian coordinates. In the hydrodynamic problems, the Lagrangian variables are used rather rarely because of the more complicated (nonlinear) form of the hydrodynamic equations in the Lagrangian form compared with their Euler analog. However, in a number of particular problems, the method of the complex Lagrangian coordinates turns out to be efficient [2, 4–8]. Therefore, analysis of its advantages and disadvantages is appropriate and necessary.

From the physical viewpoint, the invariability of the spectral set of discrete frequencies during the motion of an ideal medium means that either the energy of each mode is preserved or the total mode energy is constantly redistributed among various modes. The latter situation was, in particular, observed in the numerical Fermi–Pasta–Ulam experiment, in which the oscillations in a chain of particles connected by elastic springs were simulated [9]. The returnability effect (i.e., quasiperiodic time dependence of the energy of eigenmodes), which was discovered in [9], is explained by that the system is close to the completely integrable one. In this regard, the considered problem can be reformulated as follows: is the system of equations of two-dimensional hydrodynamics integrable for a discrete restricted spectrum if the number of frequencies exceeds two?

In Sec. 2 of this work, a type of the Ptolemaic solution is presented and various flows, which are described using this solution, are reviewed. Section 3 deals with a proof of the negative answer to the above-formulated question and discusses some pertinent consequences.

## 2. PTOLEMAIC FLOWS

The two-dimensional equations of hydrodynamics in the Lagrangian variables have the form [9, 10]:

$$\frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(X_a Y_b - X_b Y_a) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(X_{ta} X_b - X_{tb} X_a + Y_{ta} Y_b - Y_{tb} Y_a) = 0. \tag{2}$$

Here, $X$ and $Y$ are the Cartesian coordinates of a liquid particle, $a$ and $b$ are its Lagrangian coordinates, and the subscript $t$ denotes the time differentiation. Equation (1) is the continuity equation, while Eq. (2) is the condition of conservation of vorticity for liquid particles.

In Eqs. (1) and (2) written in the Lagrangian form, we pass to the complex variables $\chi = a + ib$ and $\bar{\chi} = a - ib$ and the complex coordinate $W = X + iY$ ($\bar{W} = X - iY$) of the trajectory. Then the system of equations (1) and (2) can be written in the form of two conservation laws, i.e., the time independence of the two Jacobians [2]

$$\frac{\partial}{\partial t}\frac{D(W,\bar{W})}{D(\chi,\bar{\chi})}=0, \qquad \frac{\partial}{\partial t}\frac{D(W_t,\bar{W})}{D(\chi,\bar{\chi})}=0. \tag{3}$$

By direct substitution into Eq. (3), it can be verified that the expression

$$W=G(\chi)\exp(i\lambda t)+F(\bar{\chi})\exp(i\mu t), \tag{4}$$

where $G$ and $F$ are analytic functions and $\lambda$ and $\mu$ are arbitrary real numbers, is an exact solution to the equations of two-dimensional hydrodynamics [2]. The functions $G$ and $F$ are arbitrary to a considerable degree since the only restriction imposed on their choice is the requirement that the Jacobian $D$ (see Eq. (1)) is nonzero in the flow region, which guarantees the unambiguity of the functions $X(a,b,t)$ and $Y(a,b,t)$. Therefore, Eq. (4) describes a rather wide class of vortex flows. In the complex coordinates, the Jacobian $D$ is equal to $|G'|^2-|F'|^2$.

The epicycloids (hypocycloids) are the liquid-particle trajectories for flows (4). Moving along them, the particles circumscribe a circle whose center, in turn, moves along another circumference. Such flows are called Ptolemaic by analogy with the planet orbits in the Ptolemaic world picture.

Ptolemaic solutions (4), which include two arbitrary analytic functions, describe a rather wide class of two-dimensional vortex motions. Table 1 presents various examples of the flows, which can be described by Ptolemaic solutions (4). In all the flows, excepting two, one of the frequencies ($\lambda$ or $\mu$) is equal to zero and liquid particles travel along the circumference. For the vortex in the plasma and the waves inside the cavity, the particle trajectories depend on two frequencies. In the former case, they are hypocycloids, while in the latter case they can be both hypocycloids ($n<0$) and epicycloids ($n>0$).

However, separation of solution (4) into one- and two-frequency solutions is still rather conventional and refers to considering the flow field in the laboratory reference frame. Uniform rotation of the fluid as a whole with the angular velocity $\omega$ is characterized in the expression for the trajectory $W$ by the common multiplier $\exp(i\omega t)$, and the multipliers of the functions $G$ and $F$ can be changed by choosing the appropriate reference frame. Using such a procedure, it is impossible to change only the frequency difference $\lambda-\mu$, but it is possible to make one of the terms in Eq. (4) time-independent. In other words, solution (4) becomes one-frequency in the reference frame rotating with the angular velocity $\lambda$ (or the angular velocity $\mu$).

In what follows, all the reasoning is made for a resting reference frame.

## 3. THE UNIQUENESS OF THE PTOLEMAIC FLOWS

With allowance for the type of solution (4), one may ask whether the Ptolemaic solutions can be generalized by adding the epicycles. Let us analyze whether the solutions of the hydrodynamic equations can exist in the form of the finite series

$$W=\sum_{k=1}^{N}W_k(\chi,\bar{\chi})\exp(i\lambda_k t), \tag{5}$$

where the number $N>2$ of the terms is finite and all $\lambda_k$ are the real constants. Although the frequency spectrum of Eq. (5) is discrete and limited in width, it is not a too strict restriction. Therefore, according to Kotel'nikov's sampling theorem, a continuous signal with limited spectrum can effectively be represented by the sum of signals at discrete frequencies. Nevertheless, Eq. (5) cannot be an exact solution of hydrodynamic equations for $N>2$. Let us prove this.

Let us successively consider the possibility of existence of the solution in the cases $N=2,3$, and finally for an arbitrary finite number of frequencies $\lambda_k$.

Let $N=2$. In this case, the Jacobians of Eq. (3) are time-independent only if

$$W_{1\chi}\bar{W}_{2\bar{\chi}}=W_{1\bar{\chi}}\bar{W}_{2\chi}. \tag{6}$$

Note that

$$\bar{W}_{2\chi} = (\bar{W}_2)_\chi, \qquad \bar{W}_{2\bar{\chi}} = (\bar{W}_2)_\chi.$$

Hence, Eq. (6) is equivalent to the equation

$$\frac{D(W_1, \bar{W}_2)}{D(\chi, \bar{\chi})} = [W_1, \bar{W}_2] = 0. \tag{7}$$

In what follows, the brackets are used to denote the Jacobian.

If Jacobian (7) is equal to zero, then

$$W_1 = f(\bar{W}_2),$$

where $f$ is a certain analytic function. Exactly, this situation is realized for the Ptolemaic flows.

The above analysis, in particular, shows that for a finite set of functions of the type $W_k \exp(i\lambda_k t)$, the terms containing the maximum and minimum indices in the exponential (let them be $\lambda_1$ and $\lambda_N$) are related as

$$W_1 = f(\bar{W}_N).$$

Let us proceed to the case $N = 3$. We substitute the expression

$$W = \sum_{k=1}^{3} W_k \exp(i\lambda_k t) \tag{8}$$

into the Jacobian of the continuity equation and equate all the time-oscillating terms to zero. Two cases are possible.

1. All the frequency differences are not equal to one another. Then we write

$$[W_1, \bar{W}_2] = 0, \qquad [W_1, \bar{W}_3] = 0, \qquad [W_2, \bar{W}_3] = 0.$$

It follows from the first condition that the coordinate $\bar{W}_2$ is a function of $W_1$, while the second condition indicates that $\bar{W}_3$ is a function of $W_1$. But from the former and latter equalities, we obtain that the coordinate $\bar{W}_1$ is a function of $W_1$. This is possible only if $W_1$ is the complex function of one real parameter. Obviously, $W_2$ and $W_3$ are also functions of only the same real parameter. We adopt this parameter as the Lagrangian variable $a$. As a result, we can state that the trajectory $W$ depends only on the variable $a$ and, therefore, the flow described by Eq. (8) with nonequidistant spectrum cannot be two-dimensional. This result was obtained without using the condition of conservation of vorticity of a liquid particle (the Jacobian of the second equation in system (3)).

2. Let now the frequencies in the triplet be equidistant, i.e.,

$$\lambda_2 - \lambda_1 = \lambda_3 - \lambda_2.$$

In this case, it is necessary to already use both hydrodynamic equations. It follows from the continuity equation that

$$[W_1, \bar{W}_3] = 0 \text{ and } [W_1, \bar{W}_2] + [W_2, \bar{W}_3] = 0,$$

while the equation of motion yields

$$\lambda_1[W_1, \bar{W}_2] + \lambda_2[W_2, \bar{W}_3] = 0.$$

Then, as in the case 1, we obtain

$$[W_1, \bar{W}_2] = 0, \qquad [W_1, \bar{W}_2] = 0, \qquad [W_2, \bar{W}_3] = 0, \text{ if } \lambda_1 \neq \lambda_2.$$

Therefore, the trajectory $W$ again depends only on the variable $a$.

Let us finally consider the case of an arbitrary number of terms. According to the above remark, $[W_1, \bar{W}_N] = 0$ and, thus, $W_1 = f(\bar{W}_N)$. Let us arrange all the frequency differences in order of their lack of increase from the maximum one (equal to $\lambda_N - \lambda_1$) to that which is the closest to $(\lambda_N - \lambda_1)/2$, but exceeds the latter. The Jacobians consisting of the function pairs, which correspond to the frequency pairs included in the above-mentioned differences, should be equal to zero. This immediately follows from the continuity equation if equal frequency differences are absent and the motion and continuity equations if the pairs of equal frequency differences are present (see the discussion of the triplet).

Therefore, the functions $W_k$ with the frequencies located on the frequency axis on the one side of the frequency $(\lambda_N - \lambda_1)/2$ functionally depend on the complex-conjugate functions $\bar{W}$, which correspond to the frequencies from the other half of the frequency interval. It is easily seen that any two functions in a pair from one half of the frequency interval are functionally related to each other (without complex conjugation).

Let us consider the function $W_p$ with the frequency $\lambda_p$, which is the closest to the middle of the frequency interval. Let it for definiteness be located in the same half of the interval as $W_1$. Then, according to the last statement, $W_1$ is a function of $W_p$, i.e., $W_1 = f(W_p)$, since they both refer to the same half of the frequency interval. However, according to the continuity equation (see the discussion of the duplet and the triplet), we have $W_1 = h(\bar{W}_p)$. The last two equalities mean that the coordinate $W_j$ is a function of only one real parameter $a$. Obviously, the remaining quantities $W_j$ are also functions of only this real parameter. It is adopted as the Lagrangian variable $a$. As a result, we arrive at the statement that the complex function $W$ depends on only one Lagrangian variable $a$ and is independent of $b$, which contradicts the initial assumption on its form.

If the frequency $\lambda_p$ is located in the other half of the frequency spectrum, all the above reasoning holds true, but the function $W_N$ should be used instead of the function $W_1$.

## 4. DISCUSSION

The solution in the form of Eq. (5) cannot be generalized to the case of three or more frequencies. The requirements of the limited nature and discreteness of the frequency spectrum of the plane flow are sufficiently strict. The general results of this type with reference to the Euler equation are unknown to us.

According to the scenario of transition to the Landau–Hopf turbulence, after the successive development of a number of unstable regimes, the hydrodynamic-flow spectrum can be represented as a finite set of incommensurable frequencies. The superposition of the oscillations defined by these frequencies, generally speaking, yields a complicated nonrecurrent motion pattern, which is proposed to be identified with turbulence. Such a flow is described by Eq. (5) if we adopt that the ratio of the frequencies $\lambda_k$ of at least one pair is irrational. The feature of such a flow is that the liquid particles move within a limited region and are not drifted by a flow. It is believed that the probability of realization of the Landau–Hopf scenario is extremely low because of the oscillation-synchronization phenomenon [12]. Our study confirms that it is in principle impossible for the two-dimensional plane flows with the localized trajectories of liquid particles.

Equation (5) specifies the flows with periodic or quasiperiodic (in the case of the frequency incommensurability) trajectories of liquid particles. The term describing the mean (shear) flow is absent in Eq. (5), as well as the circular rotation of liquid, which is described by the expression $W = \chi \exp[if(|\chi|)t]$, where $f\,|\chi|$ is the angular velocity of the liquid-particle rotation. This observation opens up two possible directions of generalization of the Ptolemaic flows, namely, the inclusion of a time-linear term to the formula for the complex coordinate $W$ of the trajectory and introduction of the dependence of the frequencies $\lambda$ and $\mu$ on the Lagrangian coordinates to Eq. (4). However, both of them yield no results, as far as we have found.

Nevertheless, the Ptolemaic flows can be generalized to the case of three-dimensional geometry [13, 14]. The frequency spectrum of the generalized Ptolemaic flows already contains four frequencies. The trajectories of liquid particles are the windings over the toroidal surfaces, which are formed by rotations of the differently spatially oriented ellipses round the vertical axis. The feature of such flows is that they can be represented as the superposition of three-frequency motion in the horizontal plane and the vertical one-frequency oscillation. Therefore, the three-frequency regime is conventionally realized for the function $W$, but this occurs due to adding motion along the third coordinate. Passage to the three-dimensional flows allows one to increase the number of time harmonics in exact-solution spectrum. This switches the search for the exact solutions of the Ptolemaic type to the three-dimensional flows.

## 5. CONCLUSIONS

It has been shown that among all possible two-dimensional flows which contain a finite set of time (coordinate-independent) frequencies, only the two-frequency (Ptolemaic) flow satisfies the equations of hydrodynamics of an ideal liquid.

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