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HOMOLOGIES OF MODULI SPACE $\mathcal{M}_{2,1}$

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Abstract. We consider the space $\mathcal{M}_{2,1}$ — the open moduli space of complex curves of genus 2 with one marked point. Using language of chord diagrams we describe the cell structure of $\mathcal{M}_{2,1}$ and cell adjacency. This allows us to construct matrices of boundary operators and compute Betty numbers of $\mathcal{M}_{2,1}$ over $\mathbb{Q}$.

1. Introduction

A graph G is correctly embedded into a compact topological oriented surface S, if its complement is homeomorphic to a 2-dimensional disk. If a graph G is correctly imbedded into S and lengths of its edges are given, then the Jenkins-Schtrebel construction allows one to uniquely define a complex structure on S (see [2]). In what follows we will assume that the sum of lengths of edges is 1.

Let $\mathcal{M}_{2,1}$ be the open moduli space of genus 2 complex curves with one marked point. A topological cell complex $\mathcal{M}_{2,1}^{\rm comb}$, which is homeomorphic to the space $\mathcal{M}_{2,1}$, has the following combinatorial description (see [1], Chapter 4, for example). Let us consider the set of all pairwise nonisomorphic graphs, correctly embedded into S_2 — topological compact genus 2 surface. Isomorphism here is the isomorphism of embedded graphs, i.e. isomorphism must preserve the cyclical order of edges under the counterclockwise going around of each vertex.

To a graph Γ with s edges, correctly imbedded into S_2, we correspond the simplex Δ_Γ, which is isometric to standard simplex

$$\{x_1,\ldots,x_s\in\mathbb{R}^s\,|\,x_1+\cdots+x_s=1,\quad x_1,\ldots,x_s>0\}.$$

Here numbers $x_1,\ldots,x_s$ are lengths of edges of Γ. Cells of the highest dimension 8 correspond to correctly imbedded 3-valent graphs (a graph is called 3-valent, if each its vertex has degree 3). A 3-valent graph correctly imbedded in to S_2 has 6 vertices and 9 edges.

If a correctly imbedded graph Γ' is obtained by a contraction of some edge of Γ, then to Γ' some face of Δ_Γ is corresponded.

A correctly imbedded graph Γ may have a nontrivial group of automorphisms. We can define an action of this group on the simplex Δ_Γ also. The cell of the space $\mathcal{M}_{2,1}^{\rm comb}$ is either the simplex Δ_Γ, if the corresponding graph doesn't have any nontrivial automorphisms, or the factor of Δ_Γ by the action of the group.

Thus defined space $\mathcal{M}_{2,1}^{\rm comb}$ is *noncompact*. In this work we will find its homologies over $\mathbb{Q}$.

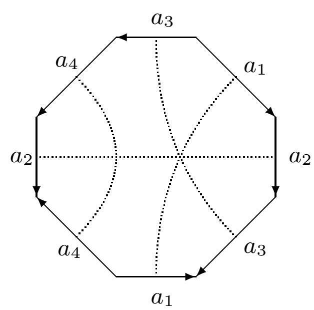
2. Graphs, glueings and diagrams

The topological surface S_2 with correctly imbedded 3-valent graph can be realized as a glueing of 18-gon. Here the embedding correctness is automatically fulfilled and the 3-valency and the genus 2 condition give 9 pairwise nonisomorphic glueings (two glueings of $2n$-gon are isomorphic, if some rotation of $2n$-gon transforms one onto another). The group of automorphisms of a graph is a cyclic group of automorphisms of the corresponding diagram of glueing. Combinatorics of cells of the highest dimension is described in [3].

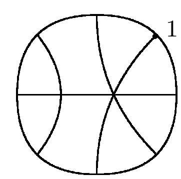
A contraction of an edge of graph corresponds to a contraction of the pair of identified sides of corresponding glueing. Thus cells of dimension 7 correspond to glueings of 16-gons and cells of dimension 6 — to glueings of 14-gons.

We will use schemas of chord diagrams for enumeration of glueings.

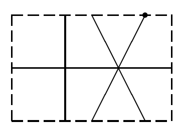
Example 1. Gaussian word $a_1a_2a_3a_1^{-1}a_4a_2^{-1}a_4^{-1}a_3^{-1}$ that describes the glueing of octagon



we will write Gaussian word as string 12314243, and represent the glueing as a chord diagram:



simplifying it to a scheme



The point at diagram denotes the beginning of clockwise numeration of chords.

All glueings we will work with are enumerated in Appendices 1 – 6. A glueing will be denoted $\gamma(k)_l$, where k is the number of edges of corresponding graph and l is the number of this glueing in the list of glueings of $2k$-gons. For each glueing we give its chord diagram, Gaussian word and group of automorphisms (if it is nontrivial). Gaussian word defines the numeration of chords, which will be called *standard*.

All chord schemas, enumerated in Appendices, define a genus 2 curves with correctly embedded graphs. Each diagram $\gamma(k)_i$, $k = 8, 7, 6, 5, 4$ is obtained from some scheme $\gamma(k+1)_j$ by deletion of a chord.

A scheme $\gamma(k)_i$ defines a $(k-1)$-dimensional cell in the space $\mathcal{M}_{2,1}^{\mathrm{comb}}$. Cells that correspond to genus 2 schemas, obtained from $\gamma(k)_i$ by deletion of a chord, constitute the $(k-2)$-dimensional boundary of our cell.

A glueing of $2n$-gon (a chord diagram of glueing) will be called symmetric if it has a nontrivial group of automorphisms. This group is a cyclic group of rotations of $2n$-gon. As sides of $2n$-gon (chords of diagram) are numerated, then generator of group of automorphisms defines a permutation from S_n. We will call a glueing (scheme) *even-symmetric*, if this permutation is even, and *odd-symmetric* in the opposite case.

A cell of the space $\mathcal{M}_{2,1}^{\text{comb}}$ will be called *simple*, if this cell is a simplex. A cell=factorized simplex will be called *special even*, if the corresponding glueing is even symmetric, and *special odd*, if the corresponding glueing is odd symmetric.

The topological space $\mathcal{M}_{2,1}^{\text{comb}}$ consists of (see Appendices 1–6):

- 9 cells of dimension 8: 3 are simple, 5 are simplices, factorized by action of the group $\mathbb{Z}_2$, and one special cell is a simplex factorized by action of the group $\mathbb{Z}_3$ (here all special cells are even);
- 29 cells of dimension 7: 24 cells are simple, 4 are simplices, factorized by action of the group $\mathbb{Z}_2$ (these cells are even), and one special cell is a simplex factorized by action of the group $\mathbb{Z}_4$ (this cell — $\delta(7)_{20}$ is odd);
- 52 cells of dimension 6: 41 cells are simple, 11 are special cells — simplices, factorized by action of the group $\mathbb{Z}_2$ (two of them — $\delta(6)_{39}$ and $\delta(6)_{40}$ are even, all others are odd);
- 45 cells of dimension 5: 37 cells are simple, 5 are simplices factorized by action of the group $\mathbb{Z}_2$, one is a simplex factorized by action of the group $\mathbb{Z}_3$, one is a simplex factorized by action of the group $\mathbb{Z}_4$ and one is a simplex factorized by action of the group $\mathbb{Z}_6$ (two special cells — $\delta(5)_{39}$ and $\delta(5)_{41}$ are even, all others are odd);
- 21 cells of dimension 4: 14 cells are simple, 5 are simplices factorized by action of the group $\mathbb{Z}_2$, one is a simplex factorized by action of the group $\mathbb{Z}_5$ and one is a simplex factorized by action of the group $\mathbb{Z}_{10}$ (one special cell — $\delta(4)_{19}$ is odd, all others are even);
- 4 cells of dimension 3: two are simple, one is a simplex factorized by action of the group $\mathbb{Z}_2$ and one is a simplex factorized by action of the group $\mathbb{Z}_8$ (the cell $\delta(4)_1$ is special even and the cell $\delta(4)_4$ is special odd).

Euler characteristic of open moduli spaces $\mathcal{M}_{2,1}$ is 9-29+52-45+21-4=4. And its "orbifoldic" Euler characteristic is

$$\left(3+\frac{5}{2}+\frac{1}{3}\right)-\left(24+\frac{4}{2}+\frac{1}{4}\right)+\left(41+\frac{11}{2}\right)-\left(37+\frac{5}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}\right)+ \\ +\left(14+\frac{5}{2}+\frac{1}{5}+\frac{1}{10}\right)-\left(2+\frac{1}{2}+\frac{1}{8}\right)=\frac{1}{120}.$$

This result is in agreement with Harer-Zagier formula [4].

3. Homologies

At first we'll explain the notion of induced numeration.

Definition 1. The scheme, obtained by deletion of the i-th chord from a scheme γ will be denoted $\gamma[i]$. We can define a numeration of chords of $\gamma[i]$ in the following way: if a chord c has number $j < i$ in γ, then c has the same number in $\gamma[i]$; if a chord c has number $j > i$ in γ, then c has number $j-1$ in $\gamma[i]$. Thus defined numeration of chords of the scheme $\gamma[i]$ will be called *induced*. We will use notations $\gamma[i]_{\text{ind}}$ and $\gamma[i]_{\text{st}}$ (i.e. standard) when necessary.

If a cell δ corresponds to a scheme γ, then by $\delta[i]$ we will denote the cell that corresponds to scheme $\gamma[i]$.

Remark 1. As $\gamma[i] = \gamma[p]_q$ for some p and q, then scheme $\gamma[i]$ has the standard numeration also. The standard and induced numerations are usually different.

Definition 2. Let $C_k = C_k(\mathcal{M}^{\mathrm{comb}}_{2,1}, \mathbb{Q})$ be the space of k-dimensional chains with rational coefficients. The boundary operator $\partial_k : C_k \to C_{k-1}$ maps each cell $\delta(k)_i$ into chain $\sum_{j=1}^{k+1} \alpha_j \delta(k)_i[j] \in C_{k-1}$. Coefficients α_j are defined by following conditions.

(1) Let cells $\delta(k)_i$ and $\delta(k)_i[j]$ be simple. The identification of schemas $\gamma(k+1)_i[j]_{\mathrm{ind}}$ and $\gamma(k+1)_i[j]_{\mathrm{st}}$ defines some renumeration of chords of the scheme $\gamma(k+1)_i[j]_{\mathrm{ind}}$, i.e. defines a permutation $\sigma \in S_k$ of parity p, $p = 0, 1$. Set $\alpha_j := (-1)^{j+p-1}$.

(2) Let the sell $\delta(k)_i$ be simple and the cell $\delta(k)_i[j]$ be special even — a simplex factorized by action of cyclic group of order m. The identification of schemas $\gamma(k+1)_i[j]_{\mathrm{st}}$ and $\gamma(k+1)_i[j]_{\mathrm{ind}}$ doesn't define the permutation $\sigma \in S_k$ uniquely, but all such permutation have the same parity p. Set $\alpha_j := (-1)^{j+p-1} m$.

(3) Let the sell $\delta(k)_i$ be simple and the cell $\delta(k)_i[j]$ be special odd. Set $\alpha_j := 0$.

(4) Let the cell $\delta(k)_i$ be special even: a cyclic group $\mathbb{Z}_r$ acts on the scheme $\gamma(k+1)_i$ (and on the simplex $\Delta(k)_i$). The j-th chord either belongs to an orbit of cardinality r, or itself is an orbit. In the first case the deletion of this chord gives us a nonsymmetric scheme and the coefficient α_j is defined according to the rule (1) above. In the second case a cyclic group $\mathbb{Z}_q$, where r divides q, acts on the scheme $\gamma(k+1)_i[j]$ (and on the corresponding simplex). If this scheme is even symmetric, then $\alpha_j := (-1)^{n+p-1} q/r$ (where parity is computed in above defined way). If this scheme is odd symmetric, then $\alpha_j := 0$.

(5) Let the cell $\delta(k)_i$ be special odd and a cyclic group $\mathbb{Z}_r$ acts on the scheme $\gamma(k+1)_i$ (and on the simplex $\Delta(k)_i$). If the j-th chord belongs to an orbit of cardinality > 1, then set $\alpha_j := 0$. If this chord itself is an orbit, then the scheme $\gamma(k+1)_i[j]$ is odd symmetric and a cyclic group $\mathbb{Z}_q$, where r divides q, acts on it. Now set $\alpha_j := (-1)^{j+p-1} q/r$ (where parity is computed in above defined way).

Remark 2. The union of special odd cells constitute a subcomplex in $\mathcal{M}^{\mathrm{comb}}_{2,1}$.

Matrices of boundary maps are presented in Appendices.

Theorem 1. $\partial_{k-1} \circ \partial_k = 0$.

Proof. Let γ be a $(k+1)$-scheme. Let us consider chords with numbers i and j. We need to prove that the scheme $\gamma[i][j]$ has zero coefficient in the sum $\partial^2(\gamma)$. If schemas γ, $\gamma[i]$, $\gamma[j]$ and $\gamma[i][j]$ are simple, then this statement is a consequence of the analogous result for simplicial homologies. It means (if we ignore conditions 2-5) that the sign of the passage $\gamma \to \gamma[i] \to \gamma[i][j]$ is opposite to the sign of passage $\gamma \to \gamma[j] \to \gamma[j][i]$. Thus, it remains to take into account symmetry conditions. Let us consider two typical cases.

Schemas γ, $\gamma[i]$ and $\gamma[i][j]$ are simple and scheme $\gamma[j]$ is odd symmetric. Let φ be a generator of symmetry group of the scheme $\gamma[j]$ and $i = i_1, \ldots, i_m$ be the orbit of i-th chord (here m is even and $i_2 = \varphi(i_1)$, $i_3 = \varphi^2(i_1), \ldots$). Symmetry φ defines a new numeration on $\gamma[j]$ such, that i_1-th chord now has number i_2. New induced numeration on $\gamma[i][j]$ after deletion of i_2-th (former i_1-th) chord is the same as old induced numeration on $\gamma[j][i_2]$. It remains to note that in both cases we multiply by $(-1)^{i_2-1}$ and that permutation defined by φ is odd.

Schemas γ and $\gamma[i]$ are simple, scheme $\gamma[j]$ is odd symmetric and scheme $\gamma[i][j]$ is even symmetric. As above, let φ be a generator of symmetry group of the scheme $\gamma[j]$ and $i = i_1, \ldots, i_m$ (m is even) be the orbit of i-th chord. Schemas $\gamma[i_1][j], \ldots, \gamma[i_m][j]$ all are some scheme $\gamma(k)_s$. Let p_k be the parity of renumeration from $\gamma[i_k][j]_{\mathrm{st}}$ to $\gamma[i_k][j]_{\mathrm{ind}}$. Now it is clear that $m/2$ of these p_k are zeroes and $m/2$ are units. □

Theorem 2 is the main result of the work.

Theorem 2. Let M_i be the matrix of the boundary map ∂_i, then

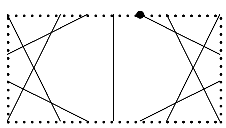
$$\mathrm{rk}(M_8) = 8,\ \mathrm{rk}(M_7) = 20,\ \mathrm{rk}(M_6) = 27,\ \mathrm{rk}(M_5) = 17,\ \mathrm{rk}(M_4) = 3.$$

Thus, Betty numbers are:

$$b_8 = 1,\ b_7 = 1,\ b_6 = 5,\ b_5 = 1,\ b_4 = 1,\ b_3 = 1.$$

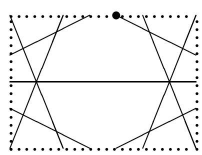
4. Appendix 1. Schemas of nine-edge glueings.

$\gamma(9)_1$:



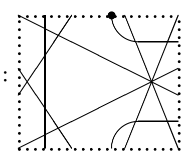
123142345678697895 $\mathbb{Z}_2$

$\gamma(9)_2$:



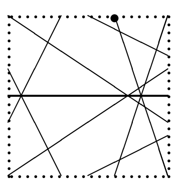
123145235678649789 $\mathbb{Z}_2$

$\gamma(9)_3$:



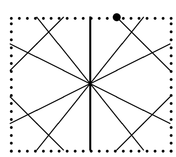
123145623678497589

$\gamma(9)_4$:



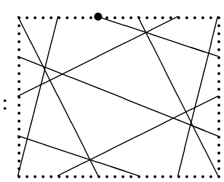
123145672378495869

$\gamma(9)_5$:



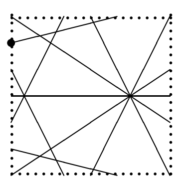
123145647812895697 $\mathbb{Z}_2$

$\gamma(9)_6$:



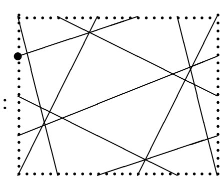
123415624785973689 $\mathbb{Z}_2$

$\gamma(9)_7$:



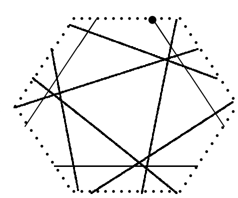
123415672485968379

$\gamma(9)_8$:



123415673859682479 $\mathbb{Z}_2$

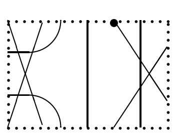
$\gamma(9)_9$:



123451672586937849 $\mathbb{Z}_3$

5. Appendix 2. Schemas of eight-edge glueings.

$\gamma(8)_1$:



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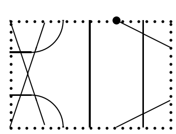
$\gamma(8)_2$:



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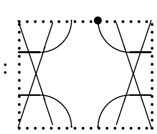
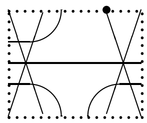
$\gamma(8)_3$:



1231423456758678 $\mathbb{Z}_2$

$\gamma(8)_4:$ 

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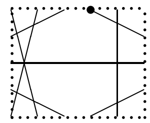
$\gamma(8)_5:$ 

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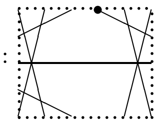
$\gamma(8)_6:$ 

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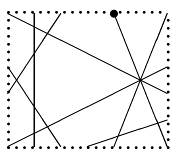
$\gamma(8)_7:$ 

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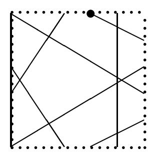
$\gamma(8)_8:$ 

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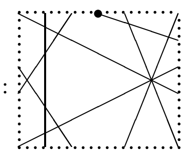
$\gamma(8)_9:$ 

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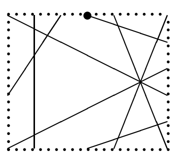
$\gamma(8)_{10}:$ 

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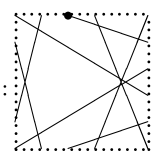
$\gamma(8)_{11}:$ 

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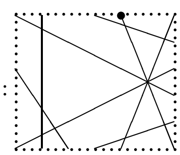
$\gamma(8)_{12}:$ 

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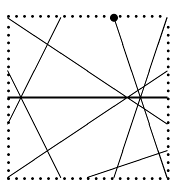
$\gamma(8)_{13}:$ 

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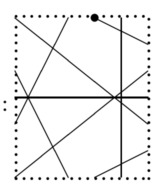
$\gamma(8)_{14}:$ 

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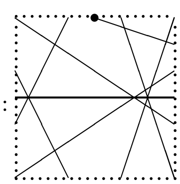
$\gamma(8)_{15}:$ 

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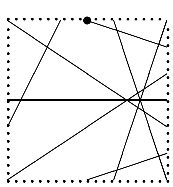
$\gamma(8)_{16}:$ 

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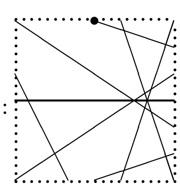
$\gamma(8)_{17}:$ 

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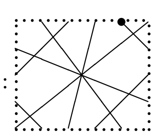
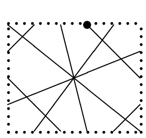
$\gamma(8)_{18}:$ 

1213453672784586 $\mathbb{Z}_2$

$\gamma(8)_{19}:$ 

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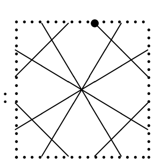
$\gamma(8)_{20}:$ 

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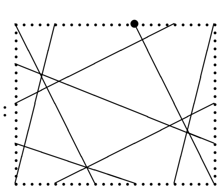
$\gamma(8)_{21}:$ 

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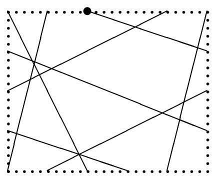
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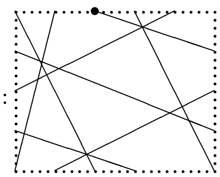
$\gamma(8)_{23}$:

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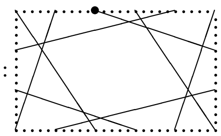
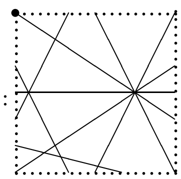

1234152467586378 $\mathbb{Z}_2$

$\gamma(8)_{25}$:

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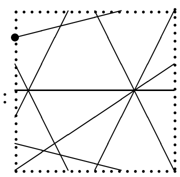
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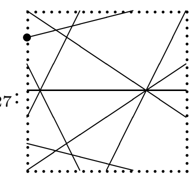
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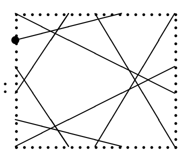
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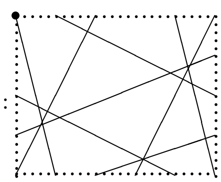
$\gamma(8)_{29}$:

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6. Appendix 3. Schemas of seven-edge glueings.

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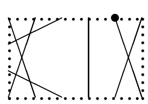
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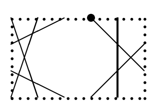
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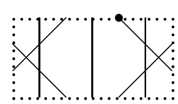
12312345675674 $\mathbb{Z}_2$

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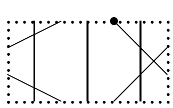
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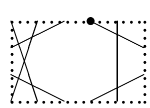
$\gamma(7)_5$:

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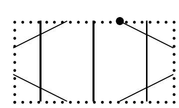
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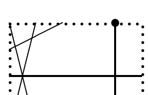
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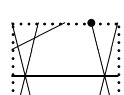
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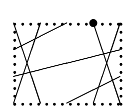
12341245637567 $\mathbb{Z}_2$

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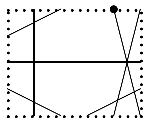
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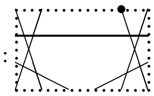
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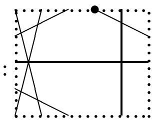
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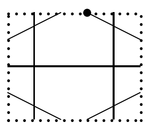
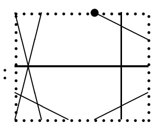

12134245653767 $\mathbb{Z}_2$

$\gamma(7)_{14}$:

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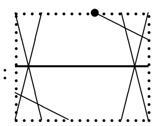
$\gamma(7)_{15}$:

12314235675467 $\mathbb{Z}_2$

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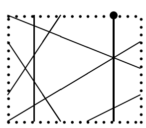
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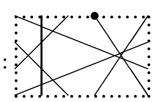
$\gamma(7)_{17}$:

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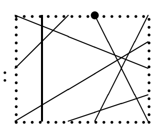
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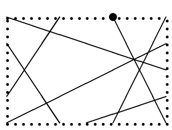
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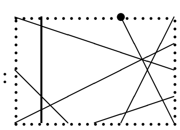
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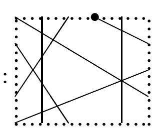
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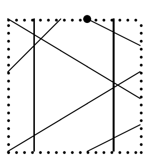
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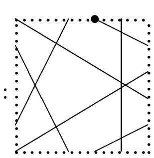
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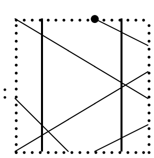
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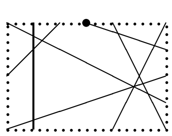
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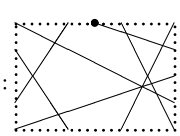
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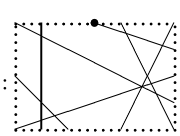
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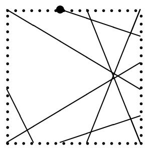
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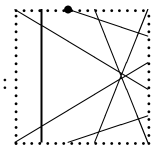
$\gamma(7)_{29}$:

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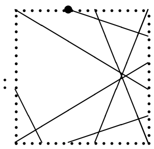
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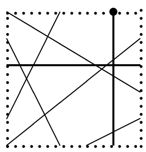
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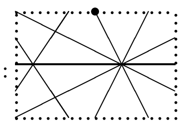
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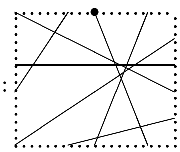
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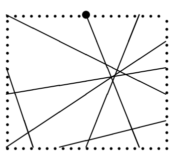
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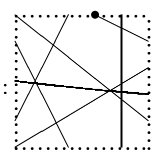
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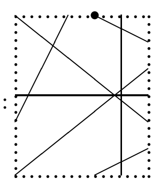
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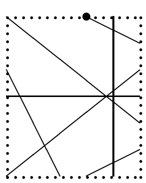
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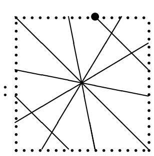
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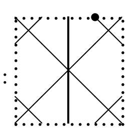
12134356267475 $\mathbb{Z}_2$

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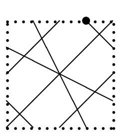
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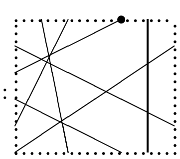
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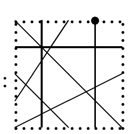
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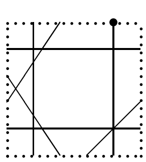
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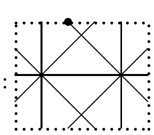
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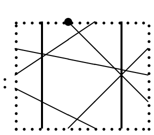
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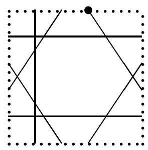
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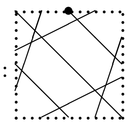
12314536476257 $\mathbb{Z}_2$

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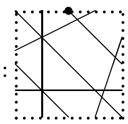
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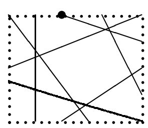
12314256475367

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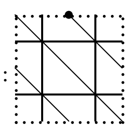
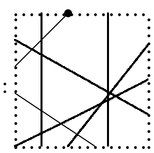

12314526746357 $\mathbb{Z}_2$

$\gamma(7)_{51}$:

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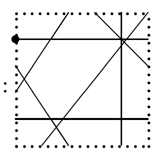
$\gamma(7)_{52}$:

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7. Appendix 4. Schemas of six-edge glueings.

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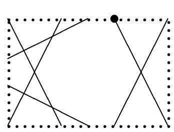
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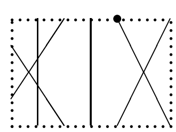
$\gamma(6)_{2}$:

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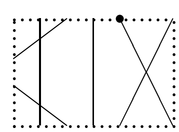
$\gamma(6)_{3}$:

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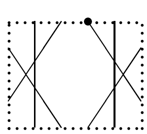
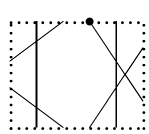
$\gamma(6)_{4}$:

123123456456 $\mathbb{Z}_2$

$\gamma(6)_{5}$:

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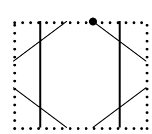
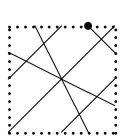
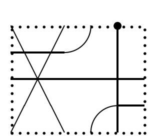
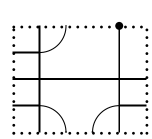
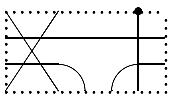
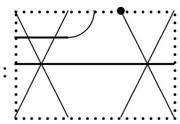

121323454656 $\mathbb{Z}_2$

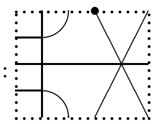
$\gamma(6)_{7}$:

121345326456

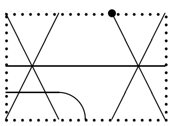
$\gamma(6)_{8}$:

123134526456

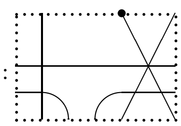
$\gamma(6)_{9}$:

123134542656

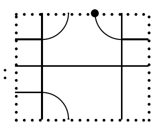
$\gamma(6)_{10}$: 
123134564256

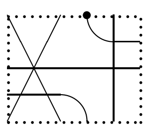
$\gamma(6)_{11}$: 
123124536456

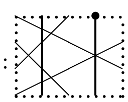
$\gamma(6)_{12}$: 
123124543656

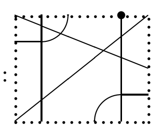
$\gamma(6)_{13}$: 
123124564356

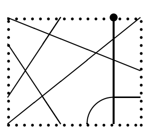
$\gamma(6)_{14}$: 
123412456536

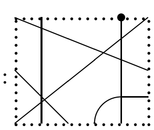
$\gamma(6)_{15}$: 
121324543656

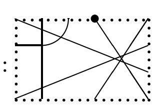
$\gamma(6)_{16}$: 
121324564356

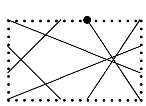
$\gamma(6)_{17}$: 
123145264356

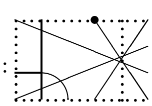
$\gamma(6)_{18}$: 
123414526356

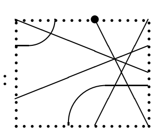
$\gamma(6)_{19}$: 
123414526536

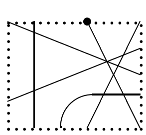
$\gamma(6)_{20}$: 
123414562536

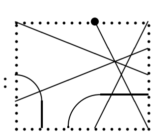
$\gamma(6)_{21}$: 
123412536456

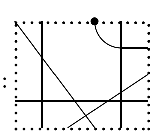
$\gamma(6)_{22}$: 
123412536546

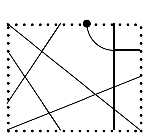
$\gamma(6)_{23}$: 
123412563546

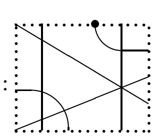
$\gamma(6)_{24}$: 
123451253646

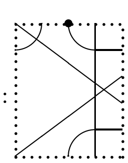
$\gamma(6)_{25}$: 
123451256346

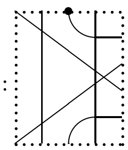
$\gamma(6)_{26}$: 
123451256364

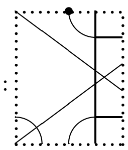
$\gamma(6)_{27}$: 
121342536456

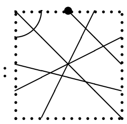
$\gamma(6)_{28}$: 
121342536546

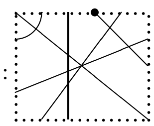
$\gamma(6)_{29}$: 
121342563546

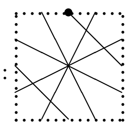
$\gamma(6)_{30}$: 
121345253646

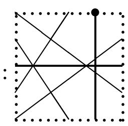
$\gamma(6)_{31}$: 
121345256346

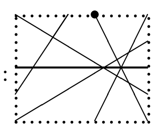
$\gamma(6)_{32}$: 
121345256364

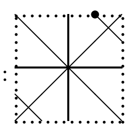
$\gamma(6)_{33}$: 
123145234656

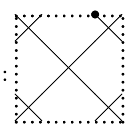
$\gamma(6)_{34}$: 
123145236465

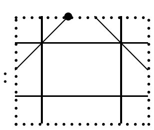
$\gamma(6)_{35}$: 
123145623645 $\mathbb{Z}_2$

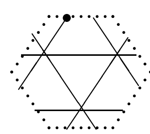
$\gamma(6)_{36}$: 
123415263546

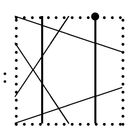
$\gamma(6)_{37}$: 
123451236456

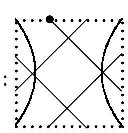
$\gamma(6)_{38}$: 
121345626345 $\mathbb{Z}_2$

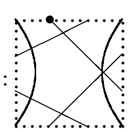
$\gamma(6)_{39}$: 
121343525646 $\mathbb{Z}_4$

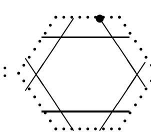
$\gamma(6)_{40}$: 
123425365146

$\gamma(6)_{41}$: 
1234256351461 $\mathbb{Z}_3$

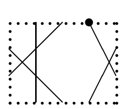
$\gamma(6)_{42}$: 
123145364256

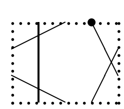
$\gamma(6)_{43}$: 
123413546256 $\mathbb{Z}_2$

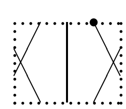
$\gamma(6)_{44}$: 
123413546526

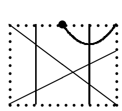
$\gamma(6)_{45}$: 
123143546526 $\mathbb{Z}_6$

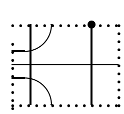
8. Appendix 5. Schemas of five-edge glueings.

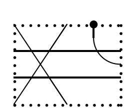
$\gamma(5)_1$: 
1212345345

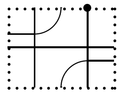
$\gamma(5)_2$: 
1212343545

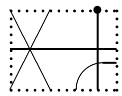
$\gamma(5)_3$: 
1212345453 $\mathbb{Z}_2$

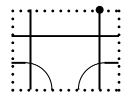
$\gamma(5)_4$: 
1213425345

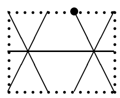
$\gamma(5)_5$: 
1213432545

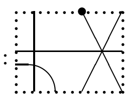
$\gamma(5)_6$: 
1213453245

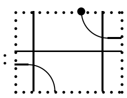
$\gamma(5)_7$: 
1231342545 $\mathbb{Z}_2$

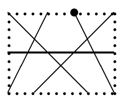
$\gamma(5)_8$: 
1231345345

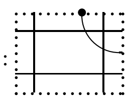
$\gamma(5)_9$: 
1231345425

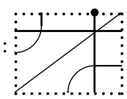
$\gamma(5)_{10}$: 
1231245345 $\mathbb{Z}_2$

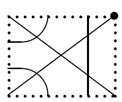
$\gamma(5)_{11}$: 
1231245435

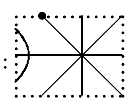
$\gamma(5)_{12}$: 
1213245435 $\mathbb{Z}_2$

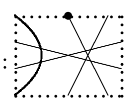
$\gamma(5)_{13}$: 
1231425345

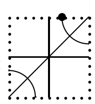
$\gamma(5)_{14}$: 
1231425435

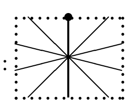
$\gamma(5)_{15}$: 
1234142535

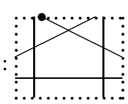
$\gamma(5)_{16}$: 
1234145253

$\gamma(5)_{17}$: 
1234123545

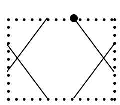
$\gamma(5)_{18}$: 
1234125345

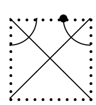
$\gamma(5)_{19}$: 
1213452534 $\mathbb{Z}_2$

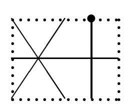
$\gamma(5)_{20}$: 
1234512345 $\mathbb{Z}_{10}$

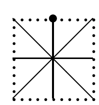
$\gamma(5)_{21}$: 
1231435425 $\mathbb{Z}_5$

9. Appendix 6. Schemas of four-edge glueings

$\gamma(4)_1$: 
12123434 $\mathbb{Z}_2$

$\gamma(4)_2$: 
12132434

$\gamma(4)_3$: 
12134234

$\gamma(4)_4$: 
12341234 $\mathbb{Z}_8$

10. Appendix 7. Transposed matrix of the map $\partial_8 : C_8 \to C_7$

$$\begin{pmatrix}
00100 & 00000 & 00000 & 00000 & 00000 & 0000 \\
00\hat{1}10 & 10000 & 00000 & 00000 & 00000 & 0000 \\
000\hat{1}0 & \hat{1}10\hat{1}1 & \hat{1}1000 & 00000 & 00000 & 0000 \\
00000 & 0000\hat{1} & 1\hat{1}101 & \hat{1}1000 & 00000 & 0000 \\
00000 & 00000 & 00000 & 1\hat{1}000 & 00000 & 0000 \\
00000 & 00010 & 00000 & 00000 & 1\hat{1}\hat{1}\hat{1}0 & 0000 \\
00000 & 00000 & 00\hat{1}0\hat{1} & 00000 & 01100 & 11\hat{1}0 \\
00000 & 0\hat{1}000 & 00000 & 00000 & 00010 & \hat{1}\hat{1}01 \\
00000 & 00000 & 00000 & 00000 & \hat{1}0000 & 001\hat{1}
\end{pmatrix}$$

Remark. To save the space by $\widehat{1}$ we denote the number -1. In what follows we will use the same notation for negative matrix elements.

11. Appendix 8. Transposed matrix of the map $\partial_7 : C_7 \to C_6$

$$
\begin{pmatrix}
1\widehat{1}000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
2000\widehat{1} & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
01000 & 00\widehat{1}000 & \widehat{1}0000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00001 & 0100\widehat{1} & 0\widehat{1}0\widehat{1}0 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
0\widehat{1}000 & 00100 & 10000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 10000 & 01\widehat{1}1\widehat{1} & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00001 & 00010 & 10000 & 1\widehat{1}1\widehat{1}0 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 10000 & 01000 & 0000\widehat{1} & 1\widehat{1}000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 01100 & 00000 & 00100 & 0000\widehat{1} & 001\widehat{1}0 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 01000 & 01000 & 00010 & 00000 & \widehat{1}010\widehat{1} & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00\widehat{1}00 & 01000 & 00001 & 00000 & 0\widehat{1}01\widehat{1} & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 001\widehat{1}1 & 00000 & 00000 & 0\widehat{1}1\widehat{1}0 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 01\widehat{1}10 & 00000 & 1000\widehat{1} & 1\widehat{1}000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 0000\widehat{1} & 1\widehat{1}000 & 01\widehat{1}10 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 000\widehat{1}0 & 00110 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 000\widehat{1}0 & 00110 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00100 & 00000 & 0100\widehat{1} & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 0000\widehat{1} & 00000 & 1000\widehat{1} & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00 \\
00000 & 00000 & 00000 & 0\widehat{1}000 & 00001 & 00000 & 00000 & 00000 & 01101 & 00000 & 00 \\
00000 & 00000 & 00000 & 00000 & \widehat{1}0000 & 10000 & \widehat{1}0000 & 00000 & 10001 & 10\widehat{1}00 & 00 \\
00000 & 00000 & 00000 & 00000 & 10000 & 0\widehat{1}000 & 10000 & 00000 & \widehat{1}1000 & 00110 & 00 \\
00000 & 00000 & 10000 & 00000 & 00000 & 00000 & 00000 & 00000 & 001000 & \widehat{1}00\widehat{1}0 & 00 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 0\widehat{2}010 & 00000 & \widehat{1}\widehat{1}001 & 00000 & 1\widehat{1} \\
00000 & 00000 & 00000 & 100\widehat{1}0 & 00000 & 00000 & 0000\widehat{1} & 00000 & 00000 & \widehat{1}0\widehat{1}00 & 11 \\
00000 & 00000 & 00000 & \widehat{1}0100 & 00000 & 00000 & 00001 & 00000 & 0\widehat{1}100 & 001000 & \widehat{1}0 \\
00000 & 00000 & 00000 & 0000\widehat{1}0 & 00001 & 00000 & 00000 & 00000 & 00101 & 00010 & 01 \\
00000 & 00000 & 00000 & 0100\widehat{1} & 00000 & 00000 & 00000 & 00000 & 0\widehat{1}000 & 00010 & 01
\end{pmatrix}
$$

12. Appendix 8. Matrix of the map $\partial_6 : C_6 \to C_5$

11002	$0\hat{1}100$	01000	00000	00000	00000	00000	00000	00000	00000	00
$00\hat{1}00$	00010	$0000\hat{1}$	00000	00000	00000	00000	00000	00000	00000	00
$000\hat{1}0$	00001	$000\hat{1}0$	00000	00000	00000	00000	00000	00000	00000	00
00000	$\hat{1}0000$	001000	00000	00000	00000	00000	00000	00000	00000	00
$000\hat{2}0$	00000	00000	00000	00000	00000	00000	00000	00000	00000	00
00010	00000	00000	00000	00000	00000	00000	00000	00000	00000	00
00000	01000	01000	00000	00000	$00\hat{1}01$	00000	$0000\hat{1}$	00000	00000	00
00000	$0\hat{1}00\hat{1}$	00000	10110	00000	00000	00000	00000	00000	00000	00
00000	00000	00000	00000	01100	00000	00000	00000	00000	00000	00
00000	01000	00010	$\hat{1}0000$	00001	10000	00000	00000	$00\hat{1}00$	$\hat{1}0000$	00
00000	$00\hat{1}00$	10000	$0\hat{1}10\hat{1}$	00000	00000	00000	00000	$00\hat{1}00$	00000	00
00000	$0000\hat{1}$	00010	00000	$010\hat{1}0$	00000	00000	00000	00000	00000	00
00000	00100	$\hat{1}0000$	01000	00001	$0\hat{1}000$	00000	00000	00000	$000\hat{1}0$	00
00000	00001	$0\hat{1}000$	00011	$\hat{1}0000$	00000	00000	00000	00000	$\hat{1}0010$	00
00000	00000	00000	00000	00110	00000	00000	00000	00000	00000	00
00000	00000	$010\hat{1}0$	00000	10000	11000	00000	00000	00000	00000	00
00000	00000	00000	$\hat{1}0000$	10000	00000	00000	00000	10000	00000	10
00000	00000	00000	10000	01000	00000	$\hat{1}0000$	00000	00000	00100	00
00000	00000	00000	$\hat{1}0000$	00100	00000	10000	00000	00000	$00\hat{1}00$	00
00000	00000	00000	10000	00010	00000	$\hat{1}0000$	00000	00000	$\hat{1}0010$	$\hat{1}\hat{1}$
00000	00000	00000	01100	00001	00000	01000	00000	00000	00000	$1\hat{1}$
00000	00000	00000	$0\hat{1}010$	00000	10000	$0\hat{1}000$	00000	$0000\hat{1}$	00000	01
00000	00000	00000	01001	00000	01000	01000	00000	$\hat{1}0001$	00000	00
00000	00000	00000	00000	11000	00000	00001	00000	$10\hat{1}01$	$\hat{1}0000$	00
00000	00000	00000	00000	$\hat{1}0100$	00000	$0000\hat{1}$	00000	00000	00100	00
00000	00000	00000	00000	10100	00000	00001	00000	00000	$00\hat{1}00$	00
00000	00000	00000	$00\hat{1}\hat{1}0$	00000	001000	00001	$0\hat{1}000$	00000	00000	00
00000	00000	00000	$0010\hat{1}$	$0000\hat{1}$	01020	00110	00000	01000	00000	00
00000	00000	00000	00011	00000	00001	00001	10000	00000	00000	11
00000	00000	00000	00000	$0\hat{1}\hat{1}00$	00000	00000	$000\hat{1}0$	00000	00000	00
00000	00000	00000	00000	$010\hat{1}0$	00000	00000	11001	00000	00000	00
00000	00000	00000	00000	00110	00000	00000	00010	00000	00000	00
00000	00000	00000	00000	$0000\hat{1}$	$\hat{1}0\hat{1}00$	10000	01000	10001	00000	00
00000	00000	00000	00000	00000	$1100\hat{1}$	10000	$\hat{1}0000$	00000	00000	00
00000	00000	00000	00000	00000	00000	00000	$000\hat{1}0$	00000	00000	00
00000	00000	00000	00000	00000	00000	10001	00000	10000	00000	10
00000	00000	00000	00000	00000	00000	00000	00000	00000	00000	00
00000	00000	00000	00000	00000	00000	00000	00000	00000	00000	00
00000	00000	00000	00000	00000	00000	00000	00020	00000	00000	00
00000	00000	00000	00000	00000	00000	00000	00000	$\hat{1}0\hat{1}01$	$000\hat{1}0$	$\hat{1}1$
00000	00000	00000	00000	00000	00000	00000	00000	30000	00300	30
00000	00000	00000	00000	00000	00000	00000	00000	$0010\hat{1}$	$00\hat{1}10$	$0\hat{1}$
00000	00000	00000	00000	00000	00000	00000	00000	00010	00001	00

13. Appendix 9. Matrix of the map $\partial_5 : C_5 \to C_4$

$$
\begin{pmatrix}
00\hat{2}01 & 00\hat{1}0\hat{1} & 10110 & 10000 & 00000 & 00000 & 00000 & 00000 & 00000 \\
00100 & 100\hat{1}0 & 01001 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 \\
0000\hat{2} & \hat{4}0000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 \\
00000 & 01100 & 00010 & 01000 & 000\hat{1}0 & 110\hat{1}0 & 10000 & 00000 & 0\hat{1}000 \\
00000 & 00010 & 00001 & 00000 & 00000 & 00001 & 0\hat{1}000 & 00010 & 00000 \\[2ex]
00000 & 0\hat{1}001 & 00000 & 1\hat{1}10\hat{1} & 00000 & 00000 & \hat{1}01\hat{1}0 & 00000 & 01000 \\
00000 & 000\hat{2}0 & 00000 & 00220 & 00000 & 00000 & 00000 & 00000 & 00000 \\
00000 & 00\hat{1}0\hat{1} & 01000 & 00\hat{1}01 & 11000 & 00000 & 00000 & 00000 & 00010 \\
00000 & 000\hat{1}0 & 00001 & 000\hat{1}\hat{1} & 000\hat{1}\hat{1} & 00000 & 00000 & 00001 & 000\hat{2}0 \\
00000 & 00000 & \hat{2}0\hat{2}00 & 00000 & 20\hat{2}00 & 00000 & 00000 & 00002 & 00000 \\[2ex]
00000 & 00000 & 0\hat{1}0\hat{1}0 & \hat{1}0000 & 01110 & \hat{1}0000 & 00000 & 00000 & 000\hat{1}0 \\
00000 & 00000 & 0000\hat{2} & 00000 & 00002 & 20000 & 00000 & 00000 & 00000 \\
00000 & 00000 & 00000 & 0\hat{2}\hat{1}0\hat{1} & 00010 & 10000 & 00000 & \hat{2}0000 & 11000 \\
00000 & 00000 & 00000 & 010\hat{1}0 & 00001 & 00000 & 00000 & 10002 & 02000 \\
00000 & 00000 & 00000 & 00110 & 0000\hat{1} & \hat{1}0001 & 01000 & 00000 & 00000 \\[2ex]
00000 & 00000 & 00000 & 000\hat{1}\hat{1} & 00011 & 00001 & 01000 & 0000\hat{2} & \hat{1}0010 \\
00000 & 00000 & 00000 & 00000 & 12100 & 010\hat{1}0 & 001\hat{1}00 & 00000 & 00000 \\
00000 & 00000 & 00000 & 00000 & \hat{1}0100 & 0\hat{1}0\hat{1}0 & 00\hat{1}\hat{1}0 & 21000 & 00000 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00100 & 00000 \\
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 0a000 & 00000 \\[2ex]
00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 00000 & 05050
\end{pmatrix}
$$

Here in the row next to the last in position (20,37) instead of "a" must be -10.

14. Appendix 10. Matrix of the map $\partial_5 : C_4 \to C_3$

$$
\begin{pmatrix}
24100 & 0\hat{1}\hat{2}\hat{2}1 & 21000 & 00000 & 0 \\
00000 & 010\hat{2}0 & 0\hat{1}11\hat{1} & 10000 & \hat{1} \\
00010 & 10100 & 10000 & 0\hat{1}000 & 0 \\
00000 & 00000 & 00000 & 00000 & 0
\end{pmatrix}
$$

References

[1] Lando S., Zvonkin A., Graphs on surfaces and their applications. Springer-Verlag, 2004.

[2] Kontzevich M., Intersection theory on the moduli space of curves and matrix Airy function, Comm. Math. Phys. (1992) v. 147, no. 1, 1–23.

[3] Kochetkov Yu., Moduli spaces $\mathcal{M}_{2,1}$ and $\mathcal{M}_{3,1}$, Funct. Anal. Appl., 2010, 44(2), 118-124.

[4] Harer J., Zagier D., The Euler characteristic of the moduli space of curves, Invent. Math., 1986, 85, 457-485.

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