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Hereditary completeness for systems of exponentials and reproducing kernels

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Abstract

We solve the spectral synthesis problem for exponential systems on an interval. Namely, we prove that any complete and minimal system of exponentials $\{e^{i\lambda_n t}\}_{n\in N}$ in $L^2(-a, a)$ is hereditarily complete up to a one-dimensional defect. This means that for any partition $N = N_1 \cup N_2$ of the index set, the orthogonal complement to the system $\{e^{i\lambda_n t}\}_{n\in N_1} \cup \{e'_n\}_{n\in N_2}$, where $\{e'_n\}$ is the system biorthogonal to $\{e^{i\lambda_n t}\}$, is at most one-dimensional. However, this one-dimensional defect is possible and, thus, there exist nonhereditarily complete exponential systems. Analogous results are obtained for systems of reproducing kernels in de Branges spaces. For a wide class of de Branges spaces we construct nonhereditarily complete systems of reproducing kernels, thus answering a question posed by N. Nikolski.

Keywords: Spectral synthesis; Hereditary completeness; Paley–Wiener space; de Branges spaces

1. Introduction and main results

1.1. Hereditary completeness in general setting

A system of vectors $\{x_n\}_{n\in N}$ in a separable Hilbert space H is said to be *exact* if it is both *complete* (i.e., $\overline{\text{Span}}\{x_n\} = H$) and *minimal* (i.e., $\overline{\text{Span}}\{x_n\}_{n\neq n_0} \neq H$ for any n_0). Given an exact

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system we consider its (unique) *biorthogonal* system $\{x_n'\}_{n\in N}$ which satisfies $(x_m, x_n') = \delta_{mn}$. Then to every element $x \in H$ we associate its formal Fourier series

$$x \sim \sum_{n\in N} (x, x_n') x_n.$$

A natural condition is that this correspondence is one-to-one: no nonzero vector generates zero series, in other words the biorthogonal system $\{x_n'\}$ is also complete. Another important property is the possibility to reconstruct the vector x from its Fourier series:

$$x \in \overline{\operatorname{Span}}\, \{(x, x_n') x_n\}.$$

If this holds, we say that the system $\{x_n\}_{n\in N}$ is *hereditarily complete*. We will use an equivalent description: for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, the system

$$\{x_n\}_{n\in N_1} \cup \{x_n'\}_{n\in N_2}$$

is complete in H. Equivalence of this property to the hereditary completeness is immediate (for details, see [20, Lemma 3.1]). In the opposite situation (i.e., when $\{x_n\}$ and $\{x_n'\}$ are complete, but $\{x_n\}$ is not hereditarily complete) we say that the system is nonhereditarily complete.

The importance of this notion is related to the spectral synthesis problem for linear operators. If $\{x_n\}$ is the sequence of eigenfunctions and root functions of some compact operator (with trivial kernel), then the hereditary completeness of $\{x_n\}$ is equivalent to the possibility of the so-called *spectral synthesis* for this operator, i.e., its restriction to any invariant subspace is complete (see [20] or [15, Chapter 4]).

The condition that the biorthogonal system $\{x_n'\}$ is also complete in H is by no means automatic and corresponding examples can be easily constructed. It is less trivial to give examples of the situations where both $\{x_n\}$ and $\{x_n'\}$ are complete, but the system $\{x_n\}$ fails to be hereditarily complete. In fact, first examples go back to Hamburger [13] who constructed a compact operator with a complete set of eigenvectors, whose restriction to an invariant subspace is a nonzero Volterra operator (and, hence, is not complete). Further examples of nonhereditarily complete systems were found by Markus [20] and Nikolski [21], while a general approach to constructing nonhereditarily complete systems was developed by Dovbysh, Nikolski and Sudakov [9,10]. Any nonhereditarily complete system gives an example of an exact system which is not a summation basis. On the other hand, uniform minimality and closeness to an orthonormal system may be combined with nonhereditary completeness [10].

1.2. Hereditary completeness for exponential systems

It is natural to study the problem of hereditary completeness for special systems in functional spaces, e.g. those which appear as families of eigenvectors and root vectors of a certain operator. Exponential systems form an important class in this respect. Let $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ and let $e_\lambda(t) = \exp(i\lambda t)$. We consider the exponential system $\{e_\lambda\}_{\lambda\in\Lambda}$ in $L^2(-a, a)$, $a > 0$. It was shown by Young [25] that, in contrast to the general situation, for any exact system of exponentials its biorthogonal system is complete. Another approach to this problem was suggested in [12], where it is shown that any exact system of exponentials is the system of eigenfunctions of the differentiation operator $i\frac{d}{dx}$ in $L^2(-a, a)$ with a certain generalized boundary condition.

Applying the Fourier transform $\mathcal{F}$ one reduces the problem for exponential systems in $L^2(-\pi, \pi)$ to the same problem for systems of reproducing kernels in the Paley–Wiener space

$\mathcal{P}W_\pi = \mathcal{F}L^2(-\pi,\pi)$. Recall that the reproducing kernel of $\mathcal{P}W_\pi$ corresponding to a point $\lambda \in \mathbb{C}$ is of the form

$$K_\lambda(z) = \frac{\sin \pi(z-\overline{\lambda})}{\pi(z-\overline{\lambda})}, \qquad f(\lambda) = (f, K_\lambda)_{\mathcal{P}W_\pi}.$$

Hereditary completeness of exponential systems is a particular case of the following problem posed by Nikolski: whether there exist nonhereditarily complete systems of reproducing kernels in the model subspaces of the Hardy space (for the theory of model spaces see [22]; the Paley–Wiener space and de Branges spaces are such spaces up to a canonical unitary equivalence). Let us also recall a related result by Olevskii [23]: there exists an orthonormal basis $\{\varphi_n\}$ in $L^2(-\pi,\pi)$ consisting of trigonometric polynomials, for which the approximation of functions f by the sums $\sum_{n:\,(f,\varphi_n)\neq 0} c_n\varphi_n$ fails in the metric of $C[-\pi,\pi]$ or $L^p(-\pi,\pi)$, $p>2$.

We completely solve the problem of hereditary completeness for exponential systems. Namely, we show that hereditary completeness holds up to a possible one-dimensional defect.

Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in the Paley–Wiener space $\mathcal{P}W_\pi$. Then the biorthogonal system $\{g_\lambda\}_{\lambda\in\Lambda}$ is given by

$$g_\lambda(z) = \frac{G(z)}{G'(\lambda)(z-\lambda)},$$

where G is the so-called generating function of the set Λ. By the above-mentioned result of Young, $\{g_\lambda\}_{\lambda\in\Lambda}$ is also an exact system. It is well known that G is a function of exponential type π and has only simple zeros at the points of Λ.

Theorem 1.1. *If $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in the Paley–Wiener space $\mathcal{P}W_\pi$, then for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the orthogonal complement in $\mathcal{P}W_\pi$ to the system*

$$\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2} \tag{1.1}$$

is at most one-dimensional.

Moreover, there are certain obstacles for the existence of this exceptional one-dimensional complement. This cannot happen when the sequence Λ_1 has non-zero upper density. Given a sequence Λ set

$$D_+(\Lambda) = \limsup_{r\to\infty} \frac{n_r(\Lambda)}{2r},$$

where $n_r(\Lambda)$ is the usual counting function of the sequence Λ, $n_r(\Lambda) = \operatorname{card}\{\lambda \in \Lambda, |\lambda| \le r\}$.

Theorem 1.2. *Let $\Lambda \subset \mathbb{C}$, let the system $\{K_\lambda\}_{\lambda\in\Lambda}$ be exact in $\mathcal{P}W_\pi$, and let the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfy $D_+(\Lambda_1) > 0$. Then the system (1.1) is complete in $\mathcal{P}W_\pi$.*

Surprisingly, the one-dimensional defect for exponential systems is still possible.

Theorem 1.3. *There exist a system of exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$, $\lambda_n \in \mathbb{R}$, which is complete and minimal in $L^2(-\pi,\pi)$, but is not hereditarily complete.*

Thus, hereditary completeness may fail even for exponential systems (reproducing kernels of the Paley–Wiener space), which answers the question of Nikolski. Further counterexamples will be discussed in the next subsection.

1.3. Reproducing kernels of the de Branges spaces

The above results may be extended to the de Branges spaces. Let E be an entire function in the Hermite–Biehler class, that is E has no zeros on $\mathbb{R}$, and

$$|E(z)| > |E^*(z)|, \quad z \in \mathbb{C}_+,$$

where $E^*(z) = \overline{E(\overline{z})}$ and $\mathbb{C}_+$ stands for the upper half-plane. With any such function we associate the *de Branges space* $\mathcal{H}(E)$ which consists of all entire functions F such that F/E and F^*/E restricted to $\mathbb{C}_+$ belong to the Hardy space $H^2 = H^2(\mathbb{C}_+)$. The inner product in $\mathcal{H}(E)$ is given by

$$(F, G)_E = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2}\, dt.$$

The reproducing kernel of the de Branges space $\mathcal{H}(E)$ corresponding to the point $w \in \mathbb{C}$ is given by

$$K_w(z) = \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\overline{w} - z)}.$$

The Hilbert spaces of entire functions $\mathcal{H}(E)$ were introduced by L. de Branges [8] in connection with inverse spectral problems for differential operators. These spaces are also of great interest from the function theory point of view. The Paley–Wiener space $\mathcal{P}W_a$ is the de Branges space corresponding to $E(z) = \exp(-iaz)$.

An important characteristics of the de Branges space $\mathcal{H}(E)$ is its phase function, that is, an increasing C^∞-function φ such that $E(t)\exp(i\varphi(t)) \in \mathbb{R},\ t \in \mathbb{R}$ (thus, essentially, $\varphi = -\arg E$ on $\mathbb{R}$). Clearly, for $\mathcal{P}W_a$, $\varphi(t) = at$. If $\varphi' \in L^\infty(\mathbb{R})$ (in which case we say that φ has sublinear growth), the space $\mathcal{H}(E)$ shares certain properties with the Paley–Wiener spaces.

A crucial property of the de Branges spaces is the existence of orthogonal bases of reproducing kernels corresponding to real points [8]. For $\alpha \in [0, \pi)$ we consider the set of points $t_n \in \mathbb{R}$ such that

$$\varphi(t_n) = \alpha + \pi n, \quad n \in \mathbb{Z}. \tag{1.2}$$

Thus, $\{t_n\}$ is the zero set of the function $e^{i\alpha}E - e^{-i\alpha}E^*$. It should be mentioned that the points t_n may exist not for all $n \in \mathbb{Z}$ (e.g., the sequence $\{t_n\}$ may be one-sided, that is, t_n may exist only for $n \ge n_0$). If the points t_n are defined by (1.2), then the system of reproducing kernels $\{K_{t_n}\}$ is an orthogonal basis for $\mathcal{H}(E)$ for each $\alpha \in [0, \pi)$ except, may be, one (α is an exceptional value if and only if $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$). One should think of the sequence $\{t_n\}$ as a spectral characteristic of the space $\mathcal{H}(E)$.

The completeness of a system biorthogonal to an exact system of reproducing kernels was studied in [3,11]. In particular, it was shown in [11] that such biorthogonal systems are always complete when $\varphi' \in L^\infty(\mathbb{R})$. The following extension of this result is obtained in [3]: if, for some $N > 0$, $\varphi'(t) = O(|t|^N)$, $|t| \to \infty$, then either $e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$ for some $\alpha \in [0, \pi)$, or any system biorthogonal to an exact system of reproducing kernels is complete in $\mathcal{H}(E)$.

The method of the proof of Theorem 1.1 extends to the case of the de Branges spaces with sublinear growth of the phase function.

Theorem 1.4. *Let $\mathcal{H}(E)$ be a de Branges space such that $\varphi' \in L^\infty(\mathbb{R})$. If the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ is exact in $\mathcal{H}(E)$, then for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the* *orthogonal complement in $\mathcal{H}(E)$ to the system*

$$\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2} \tag{1.3}$$

is at most one-dimensional.

A crucial step in the proofs of Theorems 1.1 and 1.4 is the use of expansions of functions in $\mathcal{P}W_\pi$ or in $\mathcal{H}(E)$ with respect to *two different* orthogonal bases of reproducing kernels. At first glance it may look like an artificial trick; however it should be noted that the existence of two orthogonal bases of reproducing kernels is a property which characterizes de Branges spaces among all Hilbert spaces of entire functions (see [5,6]). Therefore, we believe this method to be intrinsically connected with the deep and complicated geometry of de Branges spaces.

As in the Paley–Wiener case, there are obstacles to the existence of the one-dimensional complement. Here we give just a result for one-component inner functions E^*/E (see, for instance, [1]) of special type.

Theorem 1.5. *Let $\mathcal{H}(E)$ be a de Branges space such that $\varphi' \in L^\infty(\mathbb{R})$,*

$$\sup_x \frac{|\varphi(2x)|}{|\varphi(x)|+1} < \infty,$$

and

$$\left|\frac{\varphi'(a)}{\varphi'(b)}\right| \le c \quad \text{if } \frac{1}{2} \le \frac{\varphi(a)}{\varphi(b)} \le 2.$$

Let $\Lambda \subset \mathbb{R}$, let the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ be exact in $\mathcal{H}(E)$, and let the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfy

$$D_+^\varphi(\Lambda_1) = \limsup_{r\to\infty} \frac{n_r(\Lambda)}{\varphi(r)-\varphi(-r)} > 0.$$

Then the system (1.3) is complete in $\mathcal{H}(E)$.

Furthermore, we show that nonhereditary completeness for reproducing kernels is possible in many de Branges spaces. Namely, we construct such examples under some mild restrictions on the spectrum $\{t_n\}$ (including, e.g., all power growth spectra $|t_n| = |n|^\gamma$, $\gamma > 0$, $n \in \mathbb{N}$ or $n \in \mathbb{Z}$).

Theorem 1.6. *Let $\{t_n\}$ be a sequence of real points such that $t_n < t_{n+1}$ and $|t_n| \to \infty, n \to \infty$. Assume that for some $N > 0, c > 0$, we have*

$$c|t_n|^{-N} \le t_{n+1} - t_n = o(|t_n|), \quad |n| \to \infty. \tag{1.4}$$

Then there exists a de Branges space $\mathcal{H}(E)$ such that $\{t_n\}$ is the zero set of the function $E + E^* \notin \mathcal{H}(E)$ and there is an exact system of reproducing kernels $\{K_\lambda\}$ in $\mathcal{H}(E)$ such that its biorthogonal system is complete, but the original system $\{K_\lambda\}$ is nonhereditarily complete.

We also mention here that recently Burnol [7] studied the hereditary completeness property of the system $\left\{\frac{\zeta(s)}{(s-\lambda)^k}\right\}$, where λ are nontrivial zeros of the Riemann zeta function and $1 \le k \le m_\lambda$, m_λ being the multiplicity of λ. He showed that this system is complete and minimal in some associated space of analytic functions, and, moreover, that this system is hereditarily complete up to a possible one-dimensional defect. It is not known whether this one-dimensional defect is really possible, but in view of our results, the presence of this complement seems to be a sufficiently general phenomenon.

The above counterexamples admit an operator-theoretic interpretation. It was recently shown in [4, Theorem 2.5] that any exact system of reproducing kernels in a de Branges space is unitarily equivalent to a system of eigenvectors of some rank one perturbation of a compact self-adjoint operator:

Let $\mathcal{H}(E)$ be a de Branges space such that $e^{i\alpha}E - e^{-i\alpha}E^* \notin \mathcal{H}(E)$ for any $\alpha \in \mathbb{R}$, and let $\{t_n\}$ be the zero set of $E + E^*$, $t_n \neq 0$. Put $s_n = t_n^{-1}$ and let A be a compact selfadjoint operator with the spectrum $\{s_n\}$. Then for any exact system $\{K_\lambda\}_{\lambda\in\Lambda}$ of reproducing kernels in $\mathcal{H}(E)$ there exists a bounded rank-one perturbation L of A (i.e., $Lx = Ax + (x, b)\,a$) such that the system $\{K_\lambda\}_{\lambda\in\Lambda}$ is unitarily equivalent to the system of eigenvectors of L, while its biorthogonal is unitarily equivalent to the system of eigenvectors of the adjoint operator L^*.

Thus, we have the following corollary of Theorem 1.6.

Corollary 1.7. *Let $\{s_n\}$ be a sequence of real numbers such that $s_n \searrow 0, n \geq 0,\ n \to \infty$, while $s_n \nearrow 0, n < 0,\ n \to -\infty$. Assume also that for some $N > 0, c > 0$, we have*

$$c|s_n|^N \leq |s_{n+1} - s_n| = o(|s_n|), \quad |n| \to \infty.$$

Let A be a compact selfadjoint operator with the spectrum $\{s_n\}$. Then there exists a rank one perturbation L of A (with trivial kernel) such that both L and L^* have complete sets of eigenvectors, but L does not admit spectral synthesis.

Throughout the paper the notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \leq CV(z)$ holds for all z in the set in question, which may be a Hilbert space, a set of complex numbers, or a suitable index set. We write $U(z) \asymp V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

2. Preliminaries

Note that if $\Lambda = \Lambda_1 \cup \Lambda_2$, and one of the sets Λ_1 or Λ_2 is finite, then the corresponding system (1.1) is complete by a simple Hilbert space argument. Therefore, from now on we exclude the case when one of the sets Λ_1, Λ_2 is finite.

Let $h \in \mathcal{P}W_\pi$ be a function orthogonal to the system (1.1). Assume that $\Lambda \cap \mathbb{Z} = \emptyset$ and write the expansion of the vector h with respect to the Shannon–Kotelnikov–Whittaker orthonormal basis $K_n(z) = \frac{\sin\pi(z-n)}{\pi(z-n)}$, $n \in \mathbb{Z}$,

$$h(z) = \sum_{n\in\mathbb{Z}} \overline{a_n} K_n(z) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} \overline{a_n}(-1)^n \frac{\sin\pi z}{z-n},$$

where $a_n = \overline{h(n)}$ and $\|h\|^2 = \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$. For simplicity, in this section, we write $\sum_n$ instead of $\sum_{n\in\mathbb{Z}}$.

The fact that h is orthogonal to $\left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda\in\Lambda_1}$ is equivalent to

$$\left(\frac{G(z)}{z-\lambda}, h\right) = \sum_n \frac{a_n G(n)}{n-\lambda} = 0, \quad \lambda \in \Lambda_1, \tag{2.1}$$

while $(h, K_\lambda) = 0, \lambda \in \Lambda_2$, implies that

$$\sum_n \frac{\overline{a_n}(-1)^n}{\lambda - n} = 0, \quad \lambda \in \Lambda_2. \tag{2.2}$$

Without loss of generality we may assume that h does not vanish at integers, that is, $a_n \neq 0,\ n \in \mathbb{Z}$. Otherwise, since the zero set of h is discrete, there exists $\alpha \in (0, 1)$ such that $h(n+\alpha) \neq 0,\ n \in \mathbb{Z}$, and we can expand h with respect to the basis $\{K_{n+\alpha}\}$.

Now let G_2 be an entire function of genus 1 with the zero set Λ_2, and let $G_1 = G/G_2$. The function G_2 is defined uniquely up to an exponential factor $e^{\gamma_1+\gamma z}$. Note that the zeros of G satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$. Therefore, we may choose γ such that $G_2^*/G_2 = B_1/B_2$ for some Blaschke products B_1 and B_2. Hence G_1^*/G_1 is the ratio of two Blaschke products as well, since G^*/G is of this form for any generating function G of a complete minimal system of reproducing kernels.

We can rewrite conditions (2.1)–(2.2) as

$$\sum_n \frac{a_n G(n)}{z-n} = \frac{\pi G_1(z) S_1(z)}{\sin \pi z}, \tag{2.3}$$

$$\sum_n \frac{\overline{a_n}(-1)^n}{z-n} = \frac{\pi G_2(z) S_2(z)}{\sin \pi z}, \tag{2.4}$$

where S_1 and S_2 are some entire functions.

The pairs (S_1, S_2) of entire functions satisfying (2.3)–(2.4) parametrize all functions orthogonal to (1.1). We will denote the set of such pairs by $\Sigma(\Lambda_1, \Lambda_2)$. Note that *the function* $S_2 = h/G_2$ *does not depend on the choice of the orthogonal basis* $\{K_{n+\alpha}\}$ (we will use this fact repeatedly), while S_1 will depend on the choice of the basis.

Comparing the residues at n in (2.3)–(2.4) we get

$$S_1(n) = (-1)^n a_n G_2(n), \qquad G_2(n) S_2(n) = \overline{a_n}. \tag{2.5}$$

Put $S = S_1 S_2$. Then

$$S(n) = S_1(n) S_2(n) = (-1)^n |a_n|^2. \tag{2.6}$$

Lemma 2.1. *The function* $G_1 S_1$ *is in* $\mathcal{P}W_\pi + z\mathcal{P}W_\pi$.

Proof. If w is a zero of $G_1 S_1$, then it follows from (2.3) and the inclusion $\{G(n)(1+|n|)^{-1}\}_n \in \ell^2$ that

$$\frac{\pi G_1(z) S_1(z)}{z-w} = \sin \pi z \sum_n \frac{a_n G(n)}{(n-w)(z-n)} \in \mathcal{P}W_\pi. \quad \square \tag{2.7}$$

In what follows we denote by $\mathcal{P}W_\pi + \mathbb{C}\sin \pi z$ the class of functions of the form $f + c \sin \pi z$, where $f \in \mathcal{P}W_\pi$, $c \in \mathbb{C}$.

Lemma 2.2. *Let* $h \in \mathcal{P}W_\pi$ *be orthogonal to some system of the form* (1.1) *and let* $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. *Then* $S \in \mathcal{P}W_\pi + \mathbb{C}\sin \pi z$.

Proof. Consider the function $Q \in \mathcal{P}W_\pi$ which solves the interpolation problem $Q(n) = (-1)^n |a_n|^2,\ n \in \mathbb{Z}$ (where a_n are the coefficients in the expansion $h = \sum_n \overline{a}_n K_n$) and put $\tilde{S} = S - Q$. Then $\tilde{S}$ vanishes on $\mathbb{Z}$ and so $\tilde{S}(z) = H(z)\sin \pi z$. It remains to show that H is a constant. Note that $G_2 S_2 = h \in \mathcal{P}W_\pi$ and, by Lemma 2.1, $G_1 S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$. Hence,

$$GS \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}, \tag{2.8}$$

and, since $G \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$ and $GQ \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}$, also

$$G(z)\tilde{S}(z) = G(z)H(z)\sin\pi z \in \mathcal{P}W_{2\pi} + z\mathcal{P}W_{2\pi}.$$

We may divide by $\sin\pi z$, and so

$$GH \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi.$$

Since G is an entire function of exponential type π, we conclude that H is of zero exponential type. Now if H has at least one zero z_1, we conclude that $\frac{H(z)G(z)}{z-z_1} \in \mathcal{P}W_\pi$ which contradicts the fact that Λ is a uniqueness set for the Paley–Wiener space. Thus, H is a constant. □

Lemma 2.3. *Let $h \in \mathcal{P}W_\pi$ be orthogonal to some system of the form (1.1) and let $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Then both functions S_1/S_1^* and S_2/S_2^* are ratios of two Blaschke products.*

Proof. The zero sets of S_1 and S_2 satisfy the Blaschke condition in $\mathbb{C}_+$ and in $\mathbb{C}_-$ since $G_1S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$ and $h = G_2S_2 \in \mathcal{P}W_\pi$. Thus, it remains to show that S_1/S_1^* and S_2/S_2^* have no exponential factors. By Lemma 2.2 we know that S satisfies this property. Indeed, if $c \neq 0$ this is obvious, whereas if $c = 0$, then the function S coincides with the function $Q \in \mathcal{P}W_\pi$ which is real on $\mathbb{R}$ and has at least one zero in each interval $(n, n+1)$. So the size of the conjugate indicator diagram of the function GS equals 4π. Hence, the size of the conjugate indicator diagram both for G_1S_1 and for G_2S_2 equals 2π. Since $G_2S_2 \in \mathcal{P}W_\pi$, we obtain that $G_2S_2/(G_2^*S_2^*)$ is a ratio of two Blaschke products. By the construction of G_2, the same is true for S_2 and, hence, for S_1. □

Lemma 2.4. *If $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$, then also $(S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$.*

Proof. By Lemma 2.3, S_1^*/S_1 is of the form B_1/B_2 for some Blaschke products B_1 and B_2. We consider the following representation

$$\frac{\pi G_1(z)S_1(z)}{\sin\pi z}\cdot\frac{S_1^*(z)}{S_1(z)} = \sum_n \frac{a_nG(n)}{z-n}\cdot\frac{S_1^*(n)}{S_1(n)} + H(z), \tag{2.9}$$

where H is an entire function (which holds since the residues at integers coincide). On the other hand, $G_1S_1^* \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$, whence $|H(z)| \lesssim 1 + |z|$ and so H is a polynomial of degree at most 1. Finally, (2.3) implies that $e^{-\pi|y|}|G_1(iy)S_1(iy)| \to 0$, $|y| \to \infty$. Since the function S_1^*/S_1 is reciprocal to itself at conjugate points, we conclude that $\min(|H(iy)|, |H(-iy)|) \to 0$, $|y| \to \infty$, and so $H \equiv 0$.

Set $b_n = a_nS_1^*(n)/S_1(n)$. We can use an analogous argument to show that

$$\frac{\pi G_2(z)S_2^*(z)}{\sin\pi z} = \sum_n \frac{\overline{a_n}(-1)^n}{z-n}\cdot\frac{S_2^*(n)}{S_2(n)}. \tag{2.10}$$

Since $S(n) = S_1(n)S_2(n) \in \mathbb{R}$ and so $S_2^*(n)/S_2(n) = S_1(n)/S_1^*(n)$, we get

$$\overline{a_n}(-1)^nS_2^*(n)/S_2(n) = \overline{b_n}(-1)^n.$$

Thus, the pair (S_1^*, S_2^*) corresponds to the sequence $\{b_n\}$ in Eqs. (2.3) and (2.4). This means that $(S_1^*, S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$. □

By Lemma 2.4, if $(S_1, S_2) \in \Sigma(\Lambda_1, \Lambda_2)$, then $(S_1 + S_1^*, S_2 + S_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$ and $(iS_1 - iS_1^*, -iS_2 + iS_2^*) \in \Sigma(\Lambda_1, \Lambda_2)$. Thus, in what follows we may assume that the functions S_1 and S_2 are real on $\mathbb{R}$. In this case we have an immediate corollary from (2.6).

Corollary 2.5. *If S_1 and S_2 are real on $\mathbb{R}$, then each open interval $(n, n+1)$, $n \in \mathbb{Z}$, contains exactly one zero of S, and S has no other zeros.*

Proof. Since S is real on $\mathbb{R}$ and changes the sign at $n \in \mathbb{Z}$, it has at least one zero in every interval $(n, n+1)$. Choosing a zero in each interval we construct the (principal value) canonical product S_0. Then $S = S_0 H$ for some entire function H of zero exponential type which is real on $\mathbb{R}$. Clearly, $|S_0(iy)| \gtrsim |y|^{-1} e^{\pi|y|}$, $|y| \to \infty$. By Lemma 2.2 we have $S \in \mathcal{P}W_\pi + \mathbb{C}\sin\pi z$. Hence, $|H(iy)| \lesssim |y|, |y| \to \infty$, which implies that H is a polynomial of degree at most 1. Since the signs of $S(n)$ interchange, S cannot have two zeros in any of the intervals $(n, n+1)$. Thus, H is a constant. □

3. Proofs of Theorems 1.1 and 1.2

We are now ready to prove the main positive results on hereditary completeness for exponential systems.

3.1. Completeness up to a one-dimensional defect

Proof of Theorem 1.1. Without loss of generality assume that $\Lambda \cap \mathbb{Z} = \emptyset$. Let $f = \sum_{n\in\mathbb{Z}} \overline{a_n} K_n$ and $h = \sum_{n\in\mathbb{Z}} \overline{b_n} K_n$ be two linearly independent vectors orthogonal to (1.1), and let (S_1, S_2) and (T_1, T_2) be the corresponding pairs of entire functions from $\Sigma(\Lambda_1, \Lambda_2)$. Since, by Lemma 2.4, the pairs $(S_1+S_1^*, S_2+S_2^*)$, $(iS_1-iS_1^*, -iS_2+iS_2^*)$, $(T_1+T_1^*, T_2+T_2^*)$, and $(iT_1-iT_1^*, -iT_2+iT_2^*)$ also belong to $\Sigma(\Lambda_1, \Lambda_2)$, we may assume from the very beginning that the pairs (S_1, S_2) and (T_1, T_2) are linearly independent and the functions S_1, S_2, T_1, and T_2 are real on $\mathbb{R}$.

Using Eqs. (2.5) for S and T we get

$$S_1(n)T_2(n)\overline{G_2(n)} = T_1(n)S_2(n)\overline{G_2(n)} = (-1)^n G_2(n) a_n b_n,$$

and hence,

$$S_1(n)T_2(n) = S_2(n)T_1(n) = \beta_n a_n b_n,$$

with $|\beta_n| = 1$.

Denote by Q the function in $\mathcal{P}W_\pi$ which solves the interpolation problem $Q(n) = \beta_n a_n b_n$. Then

$$T_1(z)S_2(z) = Q(z) + a(z)\sin\pi z, \qquad S_1(z)T_2(z) = Q(z) + b(z)\sin\pi z,$$

for some entire functions a and b. We show now that a and b are constants.

Note that the functions $S = S_1 S_2$ and $T = T_1 T_2$ are in $\mathcal{P}W_\pi + \mathbb{C}\sin\pi z$ by Lemma 2.2. Furthermore, the pair $(S_1 + T_1, S_2 + T_2)$ corresponds to the vector $f + h$ while the pair $(S_1 + iT_1, S_2 - iT_2)$ corresponds to the vector $f + ih$. Applying again Lemma 2.2 we obtain that $U = (S_1 + T_1)(S_2 + T_2)$ and $V = (S_1 + iT_1)(S_2 - iT_2)$ are in $\mathcal{P}W_\pi + \mathbb{C}\sin\pi z$. Hence the functions

$$S_1T_2 + S_2T_1 = U - S - T, \qquad i(S_2T_1 - S_1T_2) = V - S - T$$

belong to $\mathcal{P}W_\pi + \mathbb{C}\sin\pi z$. Thus, $S_1T_2, S_2T_1 \in \mathcal{P}W_\pi + \mathbb{C}\sin\pi z$, and we conclude that a and b are constants.

Assume that $a \neq 0$. Let us denote by $\mathcal{M}$ the set of $m \in \mathbb{Z}$ such that in the interval $[m-1/2, m+1/2)$ there exists a zero of S_2, and let us denote this zero (or one of these zeros) by s_m. Then

$$Q(s_m) + a(-1)^m \sin \pi(s_m - m) = 0,$$

whence

$$\sum_{m\in\mathcal{M}} |s_m - m|^2 \asymp \sum_{m\in\mathcal{M}} \sin^2 \pi(s_m - m) \asymp \sum_{m\in\mathcal{M}} |Q(s_m)|^2 < \infty.$$

On the other hand, the zeros of S_2 do not depend on the choice of the basis, they are the zeros of h/G_2. Expanding with respect to another basis (say, $\{K_{n+\delta}\}$ with small $\delta > 0$) we conclude that $\sum_{m\in\mathcal{M}} |s_m - m - \delta|^2 < \infty$. This is obviously wrong.

Thus, for some choice of the basis $\{K_{n+\delta}\}$, we have proved that $a = b = 0$ and all a_n, b_n are nonzero. Therefore, $S_1T_2 = T_1S_2 = Q$, the functions S_1 and S_2 have no common zeros, and the same is true for T_1, T_2. We conclude that the zero sets of S_2 and T_2 coincide, and, thus, $f = ch$ for some constant c, a contradiction. □

3.2. *Proof of Theorem 1.2*

The following proposition plays the key role in the proof of Theorem 1.2. In Section 5 we prove a slightly stronger result which applies to general de Branges spaces (see Proposition 5.4). We prefer, however, to include an elementary proof to make the exposition concerning exponential systems more self-contained.

Proposition 3.1. *Let $S \in \mathcal{P}W_\pi + \mathbb{C}\sin \pi z$ be a real entire function with real zeros $\mathcal{Z}_S$ interlacing with $\mathbb{Z}$. If $\sum_{n\in\mathbb{Z}} |S(n)| < \infty$, then for every $\delta > 0$ we have*

$$L_\delta := \lim_{N\to\infty} \frac{1}{N} \operatorname{card}\left\{|k| \le N : \operatorname{dist}(\mathcal{Z}_S \cap [k, k+1], \mathbb{Z}) > \delta\right\} = 0.$$

Proof. Let $S(n) = (-1)^n c_n$. Without loss of generality we may assume that $c_n > 0$ and $\sum_{n\in\mathbb{Z}} c_n = 1$. Then $S(z)/\sin \pi z$ is a Herglotz function in $\mathbb{C}_+$ and

$$\frac{S(z)}{\sin \pi z} = b + \sum_{n\in\mathbb{Z}} \frac{c_n}{z-n}$$

for some $b \in \mathbb{R}$. Set $s(x) = \sum_{n\in\mathbb{Z}} \frac{c_n}{x-n}$.

Case 1. If $b \neq 0$, then

$$\lim_{x\in\mathcal{Z}_S, |x|\to\infty} \operatorname{dist}(x, \mathbb{Z}) = 0.$$

Indeed, suppose that for some $\delta > 0$, there exists a sequence $\{x_n\} \subset \mathcal{Z}_S$, $|x_n| \to \infty$, $n \to \infty$, such that $\operatorname{dist}(x_n, \mathbb{Z}) \ge \delta$. Since $s(x_n) \to 0$, we obtain that $b = 0$, which is absurd.

Case 2. Suppose that $b = 0$. Fix two positive numbers $\delta < 1/4$ and $\eta < \delta^3$ and choose M so that $\sum_{|n|\le M} c_n > 1 - \eta$.

Now let the integer N be so large that $\delta N > M$. Put

$$E_N = \left\{x \in \mathbb{R} : \left|\sum_{n\in\mathbb{Z}} \frac{c_n}{x-n}\right| \ge \frac{1}{N}\right\}.$$

By Boole's lemma (see, e.g., [18]), $|E_N| = 2N$ (by $|E|$ we denote the Lebesgue measure of the set E).

Next, set

$$F_N = \left\{ x \in \mathbb{R} : \left| \sum_{|n|>M} \frac{c_n}{x-n} \right| \geq \frac{\delta}{2N} \right\}.$$

Then

$$|F_N| \leq \frac{4N\eta}{\delta}.$$

Let $J_N = [-N - \delta N - M, N + \delta N + M]$. Since

$$\left| \sum_{|n|\leq M} \frac{c_n}{x-n} \right| \leq \frac{1}{(1+\delta)N}, \quad x \notin J_N,$$

we have, for $x \in E_N \setminus J_N$,

$$\left| \sum_{|n|>M} \frac{c_n}{x-n} \right| \geq \frac{1}{N} - \frac{1}{(1+\delta)N} = \frac{\delta}{(1+\delta)N},$$

and so $x \in F_N$. We conclude that $E_N \setminus J_N \subset F_N$.

Consider the family $\mathcal{I}_N$ of the intervals of the form $I_k = [k, k+1] \subset J_N$ with $|k| \geq M + \delta N$ satisfying the following two properties:

$$(I_k^* \cap E_N) \setminus F_N \neq \emptyset, \qquad I_k^* = [k+\delta, k+1-\delta]; \tag{3.1}$$

$$|I_k \cap F_N| < \delta. \tag{3.2}$$

We will show that, for sufficiently large N, we have

$$\operatorname{card} \mathcal{I}_N \geq (1 - A_1\delta)|J_N|, \tag{3.3}$$

where A_1 is some absolute (numeric) constant. In what follows, symbols A_1, A_2, etc., will denote different absolute constants.

If $(I_k^* \cap E_N) \setminus F_N = \emptyset$ (i.e., the interval I_k^* does not satisfy (3.1)), then $I_k^* \subset (J_N \setminus E_N) \cup F_N$ and

$$\begin{aligned} |(J_N \setminus E_N) \cup F_N| &\leq |J_N| - |J_N \cap E_N| + |F_N| \\ &= |J_N| - |E_N| + |E_N \setminus J_N| + |F_N| \\ &\leq 2N + 2\delta N + 2M - 2N + \frac{8N\eta}{\delta} \leq A_2\delta N. \end{aligned}$$

Hence, for the number N_1 of those intervals I_k^* which do not satisfy (3.1), we have the estimate

$$N_1(1 - 2\delta) \leq A_3\delta N.$$

On the other hand, for the number N_2 of those intervals I_k which do not satisfy (3.2), we get $N_2\delta \leq \frac{4N\eta}{\delta}$, and so $N_2 \leq \frac{4N\eta}{\delta^2} \leq A_4\delta N$, since $\eta < \delta^3$. Thus, for sufficiently large N,

$$\operatorname{card} \mathcal{I}_N \geq 2N - N_1 - N_2 \geq 2N - A_5\delta N.$$

The latter inequality implies (3.3).

Now, if $I_k \in \mathcal{I}_N$, then there exists a point $y \in (I_k^* \cap E_N) \setminus F_N$ and so we have

$$\left|\sum_{|n|\le M} \frac{c_n}{y-n}\right| \ge \frac{1}{N} - \frac{\delta}{2N} > \frac{1}{2N}.$$

For any $x \in I_k^*$ using the fact that $|k| \ge M + \delta N$ we get

$$\left|\sum_{|n|\le M} \frac{c_n}{x-n} - \sum_{|n|\le M} \frac{c_n}{y-n}\right| \le \sum_{|n|\le M} \frac{c_n|x-y|}{|(x-n)(y-n)|} \le \frac{1}{\delta^2 N^2} \le \frac{1}{4N} \tag{3.4}$$

for sufficiently large N, and hence,

$$\left|\sum_{|n|\le M} \frac{c_n}{x-n}\right| \ge \frac{1}{4N}, \quad x \in I_k^*.$$

Suppose that for some $w \in I_k^*$ we have $s(w) = \sum_{n\in\mathbb{Z}} \frac{c_n}{w-n} = 0$. Then

$$\left|\sum_{|n|>M} \frac{c_n}{w-n}\right| \ge \frac{1}{4N} > \frac{\delta}{N}.$$

So $w \in F_N$ and, moreover, since the function under the modulus sign is monotone on I_k we obtain that either $[k, w] \subset F_N$ or $[w, k+1] \subset F_N$, which is impossible due to (3.2).

Thus, the zeros of s (and hence of S) on $I_k \in \mathcal{I}_N$ are in $I_k \setminus I_k^*$. It follows from (3.3) that

$$L_\delta = \limsup_{N\to\infty} \frac{1}{N} \operatorname{card}\left\{|k| \le N : \operatorname{dist}(\mathcal{Z}_S \cap [k, k+1], \mathbb{Z}) > \delta\right\} \le A\delta$$

for some absolute constant A. Since L_δ is a non-increasing nonnegative function of δ on $(0, 1/4)$, it follows that $L_\delta \equiv 0$. □

Proof of Theorem 1.2. Assume that there is a nontrivial function h orthogonal to the system (1.1) such that $D_+(\Lambda_1) > 0$. Denote by $\mathcal{Z}_1$ and $\mathcal{Z}_2$ the zero sets of S_1 and S_2, respectively.

Since $G_1 S_1 \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$, by the Levinson theorem (see, for instance, [16, Section IIIH3]) we have

$$D(\Lambda_1 \cup \mathcal{Z}_1) = \lim_{r\to\infty} \frac{n_r(\Lambda_1 \cup \mathcal{Z}_1)}{2r} \le \pi,$$

and so

$$D_-(\mathcal{Z}_1) = \liminf_{r\to\infty} \frac{n_r(\mathcal{Z}_1)}{2r} < \pi.$$

Since S is of exponential type π, we have $D_+(\mathcal{Z}_2) > 0$.

The function $S_2 = h/G_2$ does not depend on the choice of the basis, and replacing if necessary the basis $\{K_n\}$ by the basis $\{K_{n+\alpha}\}$ we may find α such that for a subsequence $\tilde{\mathcal{Z}}_2$ of $\mathcal{Z}_2$ with positive upper density we have $\operatorname{dist}(\tilde{\mathcal{Z}}_2, \mathbb{Z} + \alpha) \ge 1/4$. Without loss of generality assume that this holds for $\alpha = 0$. Construct the function S_1 corresponding to this basis by formula (2.3). Then for $S = S_1 S_2$ we have $\sum_{n\in\mathbb{Z}} |S(n)| < \infty$. Note that by Corollary 2.5 the zeros of S interlace with $\mathbb{Z}$. By Proposition 3.1 all zeros of S except the set of zero density are close to $\mathbb{Z}$, and we come to a contradiction. □

4. An example of a nonhereditarily complete exponential system

In this section we prove Theorem 1.3. As before, we pass to the equivalent problem in the Paley–Wiener space and construct a nonhereditarily complete system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}W_\pi$.

We deduce Theorem 1.3 from the following statement.

Proposition 4.1. *There exist a sequence $\{a_n\} \in \ell^1(\mathbb{Z})$ such that $a_n > 0$, and an infinite sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$, $n_{k+1} > 2n_k$, $k \ge 1$, such that the functions*

$$h(z) = \pi^{-1} \sin \pi z \sum_{n\in\mathbb{Z}} \frac{a_n}{z-n}, \qquad S(z) = \pi^{-1} \sin \pi z \sum_{n\in\mathbb{Z}} \frac{a_n^2}{z-n}$$

vanish at the points $s_k = n_k + 1/2$, $k \in \mathbb{N}$, and $a_{n_k} = \alpha_k k^{-2}$ with $\alpha_k \in (1/2, 3)$, $k \in \mathbb{N}$.

Proposition 4.1 is proved using standard fixed point arguments of nonlinear analysis. We postpone its (rather technical) proof and show first how Theorem 1.3 follows from Proposition 4.1.

Proof of Theorem 1.3. We have seen in Section 2 that if $\{K_\lambda\}_{\lambda\in\Lambda}$ is a complete minimal system in $\mathcal{P}W_\pi$ with a generating function G and $\Lambda = \Lambda_1 \cup \Lambda_2$, then we may construct entire functions G_1 and G_2 with zero sets Λ_1 and Λ_2 respectively such that each of the functions G_1^*/G_1 and G_2^*/G_2 is a ratio of two Blaschke products and $G = G_1G_2$. Once such functions G_1 and G_2 are chosen, we have seen that the system (1.1) is not complete in $\mathcal{P}W_\pi$ if and only if there exists a nonzero sequence $\{a_n\} \in \ell^2$ and entire functions S_1 and S_2 satisfying Eqs. (2.3)–(2.4).

We first choose S_1, S_2 and G_2, and finally construct G_1 as a perturbation of S_2. Let $\{a_n\} \in \ell^1$ and $\{n_k\} \subset \mathbb{N}$ be the sequences from Proposition 4.1. As in Proposition 4.1 put

$$h(z) = \pi^{-1} \sin \pi z \sum_{n\in\mathbb{Z}} \frac{a_n}{z-n}, \tag{4.1}$$

$$S(z) = \pi^{-1} \sin \pi z \sum_{n\in\mathbb{Z}} \frac{a_n^2}{z-n}.$$

Note that for the functions h and S we have

$$\frac{|h(iy)|}{e^{\pi|y|}} \asymp \frac{1}{|y|}, \qquad \frac{|S(iy)|}{e^{\pi|y|}} \asymp \frac{1}{|y|}, \quad |y| \ge 1. \tag{4.2}$$

Denote by S_2 the genus zero canonical product with the zeros $n_k + \frac{1}{2}$. Then we may represent h and S as

$$h = G_2 S_2, \quad S = S_1 S_2$$

for some entire functions S_1 and S_2. Since $a_n > 0$ for any $n \in \mathbb{Z}$, the function $h(z)/\sin \pi z$ is a Herglotz function and so all the zeros of h (and thus of G_2) are simple and real.

We need to show that there exists an entire function G_1 with simple real zeros (different from the zeros of G_2) such that $G = G_1G_2$ is the generating function of some complete and minimal system of reproducing kernels, and

$$\frac{\pi G_1(z) S_1(z)}{\sin \pi z} = \sum_{n\in\mathbb{Z}} \frac{a_n(-1)^n G(n)}{z-n} \tag{4.3}$$

(this equation is actually Eq. (2.3) for the sequence $\{(-1)^n a_n\}_n$). Note that (4.1) gives us (2.4) for the same sequence.

Put

$$G_1(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{n_k + \frac{1}{2} - k^2}\right).$$

Then an easy estimate of the infinite products gives us

$$\frac{|G_1(n)|}{|S_2(n)|} \asymp \frac{\left|n + k^2 - n_k - \frac{1}{2}\right|}{\left|n - n_k - \frac{1}{2}\right|}, \qquad \frac{n_{k-1}+n_k}{2} \le n \le \frac{n_k + n_{k+1}}{2},$$

whence, in particular,

$$\frac{|G_1(n_k)|}{|S_2(n_k)|} \asymp k^2 \quad \text{and} \quad \frac{|G_1(n)|}{|S_2(n)|} \lesssim |n|^{1/2}, \quad n \ne 0.$$

Since $G = hG_1/S_2$, we have

$$|G(n_k)| = \frac{|h(n_k)G_1(n_k)|}{|S_2(n_k)|} \asymp a_{n_k}k^2 \asymp 1$$

(recall that in Proposition 4.1, $|h(n_k)| = a_{n_k} = \alpha_k k^{-2}$, $\alpha_k \in (1,3)$), and

$$|G(n)| \lesssim |n|^{1/2}|h(n)|, \quad n \ne 0.$$

Hence, $\{G(n)\}_{n\in\mathbb{Z}} \notin \ell^2$, and thus $G \notin \mathcal{P}W_\pi$. However, $\left\{\frac{G(n)}{|n|+1}\right\}_{n\in\mathbb{Z}} \in \ell^2$, and, using the fact that

$$\frac{|G(iy)|}{e^{\pi|y|}} \asymp \frac{1}{|y|}, \quad |y| \ge 1, \tag{4.4}$$

we conclude that $\frac{G(z)}{z-\lambda} \in \mathcal{P}W_\pi$ for any zero λ of G. Now let us turn to the formula (4.3). Comparing the residues at $n \in \mathbb{Z}$ we have $S_1(n)S_2(n) = (-1)^n a_n^2$ and $S_2(n)G_2(n) = a_n(-1)^n$ whence $G_1(n)S_1(n) = a_nG(n)$. Therefore, the residues in the left and the right-hand sides of (4.3) coincide. Hence,

$$\frac{\pi G_1(z)S_1(z)}{\sin \pi z} = \sum_{n\in\mathbb{Z}} \frac{a_n(-1)^nG(n)}{z-n} + H(z)$$

for some entire function H. By the standard growth arguments H is of zero exponential type. Note also that $G_1S_1 = SG_1/S_2$ whence, by (4.2) and the fact that $|G_1(iy)| \asymp |S_2(iy)|$, $|y| \to \infty$, we get

$$\frac{|G_1(iy)S_1(iy)|}{e^{\pi|y|}} \asymp \frac{1}{|y|}, \quad |y| \to \infty.$$

Thus, $H(iy) \to 0$, $|y| \to \infty$, whence $H \equiv 0$ and (4.3) is proved.

It remains to show that G is the generating function of a complete and minimal system of reproducing kernels. We have already seen that $G \notin \mathcal{P}W_\pi$ but $G \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi$. Assume now that the zero set Λ of G is not a uniqueness set for $\mathcal{P}W_\pi$. Then there exists a nonzero function T of zero exponential type such that $TG \in \mathcal{P}W_\pi$. Hence, $e^{i\pi z}TG \in H^2(\mathbb{C}_+)$, $e^{-i\pi z}T^*G^* \in$

$H^2(\mathbb{C}_-)$, and it follows from (4.4) that

$$\frac{|T(iy)|}{|y|} \asymp \frac{|T(iy)G(iy)|}{e^{\pi|y|}} \lesssim |y|^{-1/2}, \quad |y| \ge 1.$$

Thus T is a constant function whence $T \equiv 0$. $\square$

Proof of Proposition 4.1. We will construct the sequence a_n as follows: let a_0 be an arbitrary positive number, $a_n = n^{-2}$ for $n \ne 0$ and for $n \ne n_k, n_k+1, n_k+2$, while

$$a_{n_k} = 2r_{2k-1}k^{-2}, \qquad a_{n_k+1} = r_{2k}k^{-2}, \qquad a_{n_k+2} = 3k^{-2}$$

for some free parameters r_{2k-1} and r_{2k}. Here n_k is some very sparse subsequence of positive integers. The sparseness condition is to be specified later. Thus the coefficients a_{n_k} have a much slower decay than all other coefficients.

Using basic tools of nonlinear analysis we will show that it is possible to find parameters $r_{2k-1}, r_{2k} \in (1/2, 3/2)$, $k \in \mathbb{N}$, such that

$$h(s_k) = S(s_k) = 0, \quad k \in \mathbb{N}, \ s_k = n_k + \frac{1}{2}. \tag{4.5}$$

Denote by $\mathcal{N}$ the set $\bigcup_k \{n_k\} \cup \{n_k+1\}$. Clearly, (4.5) is equivalent to the system of equations

$$\sum_{l=1}^{\infty} \left(\frac{2r_{2l-1}}{l^2(s_k - n_l)} + \frac{r_{2l}}{l^2(s_k - n_l - 1)} \right) = -\sum_{n \notin \mathcal{N}} \frac{a_n}{s_k - n} \tag{4.6}$$

and

$$\sum_{l=1}^{\infty} \left(\frac{4r_{2l-1}^2}{l^4(s_k - n_l)} + \frac{r_{2l}^2}{l^4(s_k - n_l - 1)} \right) = -\sum_{n \notin \mathcal{N}} \frac{a_n^2}{s_k - n}. \tag{4.7}$$

Multiply Eq. (4.6) by $k^2/2$ and (4.7) by $k^4/2$. Using the fact that $s_k = n_k + 1/2$ and that $a_{n_k+2} = 3k^{-2}$, we may single out the diagonal part which will form the main contribution to the equations:

$$\begin{aligned} &2r_{2k-1} - r_{2k} + \sum_{l \ne k} \left(\frac{k^2 r_{2l-1}}{l^2(s_k - n_l)} + \frac{k^2 r_{2l}}{2l^2(s_k - n_l - 1)} \right) \\ &\quad = 1 - \sum_{n \notin \mathcal{N}, n \ne n_k+2} \frac{k^2 a_n}{2(s_k - n)}, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &4r_{2k-1}^2 - r_{2k}^2 + \sum_{l \ne k} \left(\frac{2k^4 r_{2l-1}^2}{l^4(s_k - n_l)} + \frac{k^4 r_{2l}^2}{2l^4(s_k - n_l - 1)} \right) \\ &\quad = 3 - \sum_{n \notin \mathcal{N}, n \ne n_k+2} \frac{k^4 a_n}{2(s_k - n)}. \end{aligned} \tag{4.9}$$

Diagonal part of the map. Denote by r the vector $(r_j)_{j=1}^{\infty}$ and consider the nonlinear mapping $D : \ell^\infty \to \ell^\infty$,

$$(Dr)_{2k-1} = 2r_{2k-1} - r_{2k},$$
$$(Dr)_{2k} = 4r_{2k-1}^2 - r_{2k}^2.$$

Thus, D is a block-diagonal mapping and the solution of the equation $D(r) = y$ is given by

$$(D^{-1}y)_{2k-1} = r_{2k-1} = \frac{y_{2k} + y_{2k-1}^2}{4y_{2k-1}}, \qquad (D^{-1}y)_{2k} = r_{2k} = \frac{y_{2k} - y_{2k-1}^2}{2y_{2k-1}}, \quad y_{2k-1} \neq 0. \tag{4.10}$$

If we set $r_j^\circ \equiv 1$, then $D(r^\circ) = y^\circ$ with $y_{2k-1}^\circ = 1$, $y_{2k}^\circ = 3$. Next, for any $y \in \ell^\infty$ such that $\|y - y^\circ\|_\infty < 1/2$ there exists a unique solution $r \in \ell^\infty$ of the equation $D(r) = y$.

Moreover, it is easy to see from the form (4.10) of the block-diagonal mapping D^{-1} that there exists an absolute constant $A_0 > 0$ such that

$$\|D^{-1}(y) - D^{-1}(z)\|_\infty \le A_0\|y - z\|_\infty \tag{4.11}$$

for all $y, z \in \ell^\infty$ such that $\|y - y^\circ\|_\infty < 1/2$, $\|z - y^\circ\|_\infty < 1/2$.

We need one more estimate for the mapping D^{-1}. Put $F(u) = \frac{u_2+u_1^2}{4u_1}$, $u = (u_1, u_2)$. Then an elementary estimate gives us

$$\begin{aligned}&|(F(u + \Delta u) - F(u)) - (F(v + \Delta v) - F(v))|\\&\quad \le A_1\|\Delta u - \Delta v\|_\infty + A_2\|\Delta v\|_\infty\|u - v\|_\infty\end{aligned}$$

for some absolute constants A_1 and A_2 whenever $u_1, v_1 \in (1/2, 3/2), u_2, v_2 \in (2, 4)$ and $\|\Delta u\|_\infty \le 1/10$, $\|\Delta v\|_\infty \le 1/10$. An analogous estimate holds for $F(u) = \frac{u_2-u_1^2}{2u_1}$. Hence, taking into account formula (4.10) for D^{-1}, we conclude that

$$\begin{aligned}&\|(D^{-1}(y + \Delta y) - D^{-1}(y)) - (D^{-1}(z + \Delta z) - D^{-1}(z))\|_\infty\\&\quad \le A_1\|\Delta y - \Delta z\|_\infty + A_2\|\Delta z\|_\infty\|y - z\|_\infty\end{aligned} \tag{4.12}$$

for all $y, z \in \ell^\infty$ such that $\|y - y^\circ\|_\infty < 1/2$, $\|z - z^\circ\|_\infty < 1/2$, and $\|\Delta y\|_\infty < 1/10$, $\|\Delta z\|_\infty < 1/10$.

Finally we will need the following obvious estimate: there exists an absolute constant $A_3 > 0$ such that

$$\|D(r) - D(s)\|_\infty \le A_3\|r - s\|_\infty, \qquad \|r\|_\infty \le 10, \qquad \|s\|_\infty \le 10. \tag{4.13}$$

Sparseness conditions on $\{n_k\}$. Now we impose the *first sparseness condition* on the sequence n_k:

$$\sum_{n\notin\mathcal{N}, n\neq n_k+2} \left|\frac{k^2 a_n}{2(s_k - n)}\right| + \sum_{n\notin\mathcal{N}, n\neq n_k+2} \left|\frac{k^4 a_n}{2(s_k - n)}\right| < \frac{1}{200(A_0 + 1)}, \quad k \in \mathbb{N} \tag{4.14}$$

(where A_0 is the constant from (4.11)). Since $a_n = n^{-2}, n \notin \mathcal{N} \cup \{n_l + 2\}_{l=1}^\infty$, we have $|a_n| \asymp n_k^{-2}$, $n \in [n_k/2, 2n_k]$, $n \neq n_k, n_k + 1, n_k + 2$, and so the terms

$$\left|\frac{k^2 a_n}{2(s_k - n)}\right|, \qquad \left|\frac{k^4 a_n}{2(s_k - n)}\right|$$

may be made arbitrarily small when n_k grows sufficiently fast. For example, we may take $n_k = M2^k$ with a sufficiently large constant M.

Let us consider the vector $y^* \in \ell^\infty$ defined by

$$y_{2k-1}^* = 1 - \sum_{n\notin\mathcal{N}, n\neq n_k+2} \frac{k^2 a_n}{2(s_k - n)}, \qquad y_{2k}^* = 3 - \sum_{n\notin\mathcal{N}, n\neq n_k+2} \frac{k^4 a_n}{2(s_k - n)}.$$

By (4.14), $\|y^* - y^\circ\|_\infty < (200(A_0+1))^{-1}$. Hence, there exists r^* such that $D(r^*) = y^*$ and, by (4.11), $\|r^* - r^\circ\|_\infty < A_0 \cdot (200(A_0+1))^{-1} < 1/200$.

Next we define the mapping W corresponding to the nondiagonal part of the Eqs. (4.8)–(4.9):

$$(Wr)_{2k-1} = \sum_{l\neq k}\left(\frac{k^2 r_{2l-1}}{l^2(s_k - n_l)} + \frac{k^2 r_{2l}}{2l^2(s_k - n_l - 1)}\right),$$
$$(Wr)_{2k} = \sum_{l\neq k}\left(\frac{2k^4 r_{2l-1}^2}{l^4(s_k - n_l)} + \frac{k^4 r_{2l}^2}{2l^4(s_k - n_l - 1)}\right).$$

Choosing the sequence n_k sufficiently sparse (again $n_k = M2^k$ will do the job) we may achieve our *second and third sparseness conditions*:

$$\|W(r)\|_\infty \le \frac{1}{200(1 + A_0 + A_2A_3)}, \qquad \|r\|_\infty \le 10, \tag{4.15}$$

and

$$\|W(r) - W(s)\|_\infty \le \frac{\|r - s\|_\infty}{200A_1}, \qquad \|r\|_\infty \le 10, \qquad \|s\|_\infty \le 10, \tag{4.16}$$

where A_1, A_2 and A_3 are constants from (4.12)–(4.13).

Application of the fixed point theorem. Eqs. (4.8)–(4.9) are equivalent to

$$D(r) + W(r) = y^*.$$

Consider the mapping

$$T(r) = r^* + r - D^{-1}(D(r) + W(r)).$$

We show that T is a contractive mapping on the ball $B = \{\|r - r^\circ\|_\infty \le 1/100\}$. Then there exists $r \in B$ such that $T(r) = r$ which is equivalent to $D^{-1}(D(r) + W(r)) = r^*$, whence $D(r) + W(r) = D(r^*) = y^*$.

1. *T is well-defined on B*. Clearly, we have $\|D(r) - D(r^\circ)\|_\infty < 1/4$ *and* $\|W(r)\|_\infty < 1/200$ when $\|r - r^\circ\| \le 1/100$. Thus, $\|D(r) + W(r) - y^\circ\|_\infty < 1/2$ and so $D^{-1}(D(r) + W(r))$ is well-defined.
2. *$T(B) \subset B$*. We have already seen that $\|r^* - r^\circ\|_\infty < 1/200$. Then
$$\|T(r) - r^\circ\|_\infty \le \|r^* - r^\circ\|_\infty + \|D^{-1}(D(r)) - D^{-1}(D(r) + W(r))\|_\infty < \frac{1}{200} + A_0\|W(r)\|_\infty < \frac{1}{100}$$
by (4.11) and (4.15).
3. *T is a contraction on B*. Let $r, s \in B$. Then
$$T(r) - T(s) = \big(D^{-1}(D(s) + W(s)) - D^{-1}(D(s))\big) - \big(D^{-1}(D(r) + W(r)) - D^{-1}(D(r))\big).$$

By (4.12) applied to $y = D(s)$, $\Delta y = W(s)$, and $z = D(r)$, $\Delta z = W(r)$, we have

$$\|T(r) - T(s)\|_\infty \le A_1\|W(s) - W(r)\|_\infty + A_2\|W(r)\|_\infty\|D(s) - D(r)\|_\infty$$
$$\le A_1\|W(s) - W(r)\|_\infty + A_2A_3\|W(r)\|_\infty\|r - s\|_\infty \le \frac{\|r - s\|_\infty}{100}.$$

We used estimate (4.13) in the second inequality and (4.15) and (4.16) in the last one. Thus, T is a contractive mapping from B to B and we conclude that T has a fixed point. $\square$

5. Extensions to the de Branges spaces

5.1. Preliminary remarks

We start with a general construction of functions biorthogonal to a system of reproducing kernels. Let $\mathcal{H}(E)$ be a de Branges space, and let φ be the corresponding phase function. As usual, we write $E = A - iB$. To avoid inessential difficulties we will always assume that $A \notin \mathcal{H}(E)$. The reproducing kernel of $\mathcal{H}(E)$ can be written as

$$K_w(z) = \frac{\overline{A(w)}B(z) - \overline{B(w)}A(z)}{\pi(z - \overline{w})}.$$

Let $\Lambda \subset \mathbb{C}$ be such that the system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ of the space $\mathcal{H}(E)$ is exact. Then there exists the generating function, that is, an entire function $G \in \mathcal{H}(E) + z\mathcal{H}(E)$, such that $GH \notin \mathcal{H}(E)$ for any nontrivial entire function H, and vanishing exactly on the set Λ. The biorthogonal system to $\{K_\lambda\}_{\lambda\in\Lambda}$ is given by

$$g_\lambda(z) := \frac{G(z)}{G'(\lambda)(z - \lambda)}.$$

We will assume that $\{g_\lambda\}_{\lambda\in\Lambda}$ is also an exact system in $\mathcal{H}(E)$ (recall that this is the case, e.g., when $\varphi' \in L^\infty(\mathbb{R})$ [11] or when φ' has at most power growth and $\Theta = E^*/E$ has no finite derivative at ∞ [3]).

Denote by $T = \{t_n\}$ the zero set of A (assume that $T \cap \Lambda = \emptyset$) and recall that the functions

$$\frac{A(z)}{z - t_n} = \pi i \frac{K_{t_n}(z)}{E(t_n)}$$

form an orthogonal basis in $\mathcal{H}(E)$ [8, Theorem 22] and $\|\frac{A(z)}{z-t_n}\|^2 = \pi\varphi'(t_n)$. Then every $h \in \mathcal{H}(E)$ can be written as

$$h(z) = A(z)\sum_n \frac{\overline{a}_n \mu_n^{1/2}}{z - t_n}, \quad \{a_n\} \in \ell^2, \tag{5.1}$$

where $\mu_n = 1/\varphi'(t_n)$,

$$\sum_n \frac{\mu_n}{1 + t_n^2} < \infty.$$

Let $h \in \mathcal{H}(E)$ be orthogonal to $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$. Then

$$\sum_n \frac{\overline{a}_n \mu_n^{1/2}}{z - t_n} = \frac{G_2(z)S_2(z)}{A(z)} \tag{5.2}$$

for some entire function S_2. As in the Paley–Wiener case we assume that G_2 is an entire function which vanishes exactly on Λ_2 and $G_2^*/G_2 = B_1/B_2$ for some Blaschke products B_1 and B_2. On the other hand, since $h \perp g_\lambda$, $\lambda \in \Lambda_1$, we obtain

$$\sum_n \frac{G(t_n)}{E(t_n)} \frac{a_n \mu_n^{1/2}}{z - t_n} = -i \frac{G_1(z)S_1(z)}{A(z)} \tag{5.3}$$

for some entire function S_1 (argue as in the Paley–Wiener case). Comparing the residues we get

$$S_1(t_n)G_1(t_n) = i\frac{a_n\mu_n^{1/2}A'(t_n)G(t_n)}{E(t_n)}, \tag{5.4}$$

and

$$S_2(t_n)G_2(t_n) = \overline{a}_n\mu_n^{1/2}A'(t_n). \tag{5.5}$$

Hence, for $S = S_1S_2$, we have

$$S(t_n) = i|a_n|^2\mu_n(A'(t_n))^2/E(t_n).$$

Since $iA'(t_n) = E(t_n)\varphi'(t_n)$ (the phase function φ is chosen in such a way that $\varphi(t_n) = \pi/2 + \pi n$), we get

$$S(t_n) = |a_n|^2A'(t_n). \tag{5.6}$$

In what follows we need the following theorem due to M.G. Krein (see, e.g., [14, Chapter I, Section 6]): *If an entire function F is of bounded type both in $\mathbb{C}_+$ and in $\mathbb{C}_-$, then F is of finite exponential type. If, moreover, F is in the Smirnov class both in $\mathbb{C}_+$ and in $\mathbb{C}_-$, then F is of zero exponential type*. Recall that a function f analytic in $\mathbb{C}_+$ is said to be of *bounded type*, if $f = g/h$ for some functions $g, h \in H^\infty(\mathbb{C}_+)$. If, moreover, h may be taken to be outer, we say that f is in *the Smirnov class in* $\mathbb{C}_+$.

In particular, any analytic function f such that $\operatorname{Im} f > 0$ in $\mathbb{C}_+$ is in the Smirnov class. In what follows we use the fact that if we put $\Theta = E^*/E$, then Θ is inner, and both $A/E = 1 + \Theta$ and $E/A = (1 + \Theta)^{-1}$ are in the Smirnov class. Another useful observation is that if G is a generating function of some exact system of reproducing kernels, then both G/E and G^*/E are of the form Bh, where B is a Blaschke product and h is outer in $\mathbb{C}_+$. Indeed, if G/E has an exponential factor, i.e., $G(z)/E(z) = e^{iaz}B(z)h(z)$, where $a > 0$ and h is outer, then the function

$$z \mapsto E(z)\frac{e^{iaz} - 1}{z}B(z)h(z)$$

belongs to $\mathcal{H}(E)$ and vanishes at Λ.

From now on we assume that φ is of *tempered growth*, that is,

$$\varphi'(t) = O(|t|^N), \quad |t| \to \infty, \tag{5.7}$$

for some N. It follows from (5.7) that, for any $F \in \mathcal{H}(E)$,

$$\frac{|F(x)|}{|E(x)|} \le \frac{\|K_x\|_E\|F\|_E}{|E(x)|} = \left(\frac{\varphi'(x)}{\pi}\right)^{1/2}\|F\|_E \lesssim (|x| + 1)^{N/2}, \quad x \in \mathbb{R}.$$

Using the same arguments as in the proof of Lemma 2.1 we get $G_1S_1 \in \mathcal{H}(E) + z\mathcal{H}(E)$. Hence,

$$GS \in \mathcal{P}_{\frac{N}{2}+1} \cdot \mathcal{H}(E^2), \tag{5.8}$$

where $\mathcal{P}_M$ is the set of polynomials of degree at most M.

Arguing analogously to the proof of Lemma 2.2 we obtain the following growth restriction.

Lemma 5.1. *Assume that φ satisfies (5.7). Let $h \in \mathcal{H}(E)$ be orthogonal to some system $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$ and let (S_1, S_2) be the corresponding pair. Then $S \in \mathcal{P}_M \cdot \mathcal{H}(E)$ for some M depending only on N and*

$$\left|\frac{S(iy)}{A(iy)}\right| \gtrsim \frac{1}{|y|^K}, \quad |y| \to \infty, \tag{5.9}$$

for some $K > 0$.

Proof. By (5.6) we have

$$\frac{|S(t_n)|}{|E(t_n)|(\varphi'(t_n))^{1/2}} = |a_n|^2(\varphi'(t_n))^{1/2} \lesssim |a_n|^2|t_n|^{N/2},$$

and, dividing out sufficiently many zeros $s_1, \dots, s_M$ of S we obtain that

$$\sum_n \frac{|\tilde{S}(t_n)|^2}{|E(t_n)|^2\varphi'(t_n)} < \infty, \qquad \tilde{S}(z) = \frac{S(z)}{(z-s_1)\cdots(z-s_M)}.$$

Now let Q be the (unique) function in $\mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = \tilde{S}(t_n)$. Using (5.8) and an analogous estimate for GQ, we obtain that $G(\tilde{S} - Q) \in \mathcal{P}_M \cdot \mathcal{H}(E^2)$. Since $\tilde{S} - Q$ vanishes on $\{t_n\}$, we have $G(\tilde{S} - Q) = GAH \in \mathcal{P}_M \cdot \mathcal{H}(E^2)$ for some entire function H. We want to show that H is a polynomial of degree at most $M + 1$, whence $\tilde{S} = Q + AH \in \mathcal{P}_{M+1} \cdot \mathcal{H}(E)$.

By the remarks after the formulation of Krein's theorem, $(GA)/E^2$ and $(G^*A)/E^2$ are of the form Bh, where B is a Blaschke product and h is outer in $\mathbb{C}_+$. Since $GAH = g \in \mathcal{P}_M \cdot \mathcal{H}(E)$, we see that $H = \frac{g}{E^2} \cdot \frac{E^2}{GA}$ is in the Smirnov class in $\mathbb{C}_+$ and the same holds for H^*. Then, by Krein's theorem, H is of zero exponential type.

If H has at least $M + 2$ zeros, then dividing them out we obtain an entire function $\tilde{H}$ such that $GA\tilde{H} \in \mathcal{H}(E^2)$ and $|G(iy)\tilde{H}(iy)|/|E(iy)| = o(y^{-1})$, $|y| \to \infty$ (we use the fact that $|A(iy)|/|E(iy)| \gtrsim y^{-1}$, $y \to +\infty$). Let v_n be such that $\varphi(v_n) = \pi n$ (thus, $\{v_n\}$ is the support of another orthogonal family of reproducing kernels). Since $|A(v_n)| = |E(v_n)|$, we conclude that

$$G(v_n)\tilde{H}(v_n)/E(v_n) \in L^2(\nu), \quad \nu = \sum_n (\varphi'(v_n))^{-1}\delta_{v_n}.$$

Now it remains to apply [8, Theorem 26] to conclude that $G\tilde{H} \in \mathcal{H}(E)$, a contradiction to the fact that G is the generating function of a complete system of kernels.

We have shown that $S = H_1(Q + AH_2)$ for some polynomials H_1, H_2. It follows from the representation of functions in $\mathcal{H}(E)$ (formula (5.1)) that $Q(iy)+A(iy)H_2(iy) \sim A(iy)H_2(iy)$ for any $Q \in \mathcal{H}(E)$ and any nonzero polynomial H_2. Thus, in this case $|S(iy)| \gtrsim |A(iy)|$, $|y| \to \infty$, and (5.9) is trivial. In the case when $H_2 \equiv 0$ and $S = H_1 Q$ we use that the function Q is the solution of the interpolation problem

$$\begin{aligned} Q(t_n) &= \frac{S(t_n)}{(t_n - s_1)\cdots(t_n - s_M)} \\ &= \frac{A'(t_n)|a_n|^2}{(t_n - s_1)\cdots(t_n - s_M)} = A'(t_n)|a_n|^2\left(\frac{1}{t_n^M} + \frac{b_n + ic_n}{t_n^{M+1}}\right), \end{aligned}$$

where $\{b_n\}_n$ and $\{c_n\}_n$ are bounded sequences, and assume without loss of generality that M is even and $t_n \neq 0$. Then

$$\frac{Q(z)}{A(z)} = \sum_n \frac{|a_n|^2}{z - t_n}\left(\frac{1}{t_n^M} + \frac{b_n + ic_n}{t_n^{M+1}}\right),$$

and

$$-\mathrm{Im}\,\frac{Q(iy)}{A(iy)} = \sum_n \frac{|a_n|^2}{y^2 + t_n^2}\left(\frac{y}{t_n^M} + \frac{b_n y}{t_n^{M+1}} + \frac{c_n}{t_n^M}\right).$$

All the sums in the brackets except, possibly, a finite number are positive when $y \to +\infty$ and negative when $y \to -\infty$. Expanding the right-hand side in powers of $1/y$, we deduce (5.9). □

It follows from (5.9) that S^*/S is a ratio of two Blaschke products, i.e., has no exponential factor. We show now that the same is true for each of the functions S_2^*/S_2 and S_1^*/S_1. Suppose that $G_2^*S_2^*/(G_2S_2)$ is not a ratio of Blaschke products, i.e., let $G_2^*S_2^*/(G_2S_2) = e^{ibz}B_1/B_2$, where B_1 and B_2 are meromorphic Blaschke products and $b \in \mathbb{R}$. Assume that $b > 0$ (the case $b < 0$ is analogous). Then the function $e^{icz}S_2G_2,\ 0 < c \le b$, is also in $\mathcal{H}(E)$ and formulas (5.2) and (5.3) will hold also for the functions $e^{icz}S_2G_2$ and $e^{-icz}S_1G_1,\ 0 < c < b$, with $\{e^{-ict_n}a_n\}_n$ in place of $\{a_n\}_n$. Hence, $(S_1e^{-icz}, S_2e^{icz}) \in \Sigma(\Lambda_1, \Lambda_2)$ and $((1 + e^{-icz})S_1, (1 + e^{icz})S_2) \in \Sigma(\Lambda_1, \Lambda_2)$. Now, by Lemma 5.1, the function $\tilde{S}(z) = S(z)(1+e^{icz})(1+e^{-icz})$ belongs to $\mathcal{P}_M \cdot \mathcal{H}(E)$, whence $\tilde{S}/A$ is of Smirnov class in the upper half-plane. However, this contradicts to (5.9). Thus, S_2/S_2^* and S_1/S_1^* are ratios of Blaschke products.

Now by an argument, analogous to that in the proof of Lemma 2.4, the pair (S_1^*, S_2^*) also corresponds to some function orthogonal to $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$. Thus, we may always find functions S_1, S_2 which are real on $\mathbb{R}$. By (5.6), the function S changes its sign at adjacent points t_n (as usual we assume that the basis is chosen in such a way that all coefficients a_n are nonzero), and thus, there is a zero of S in each of the intervals (t_n, t_{n+1}). We have an analogue of Corollary 2.5.

Lemma 5.2. *Assume that φ satisfies (5.7). If a pair (S_1, S_2) corresponds to a function $h \in \mathcal{H}(E)$ orthogonal to some system $\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{K_\lambda\}_{\lambda\in\Lambda_2}$ and S_1 and S_2 are real on $\mathbb{R}$, then $S = S_0H$, where S_0 has exactly one zero in any interval (t_n, t_{n+1}) and H is a polynomial of degree bounded by $M = M(N)$.*

5.2. Proof of Theorem 1.4

Without loss of generality assume that φ is unbounded both from below and from above, and $\Lambda \cap \{t_n\} = \emptyset$, where $\varphi(t_n) = \pi n,\ n \in \mathbb{Z}$. Let f and h be orthogonal to the system (1.3),

$$f(z) = A(z)\sum_n \frac{\overline{a}_n\mu_n^{1/2}}{z - t_n}, \qquad h(z) = A(z)\sum_n \frac{\overline{b}_n\mu_n^{1/2}}{z - t_n}, \quad \{a_n\},\ \{b_n\} \in \ell^2.$$

Let (S_1, S_2) and (T_1, T_2) be the corresponding pairs of entire functions such that S_1, S_2, T_1 and T_2 are real on $\mathbb{R}$. Using Eqs. (5.4)–(5.5) in the same way as in the proof of Theorem 1.1, we obtain

$$S_1(t_n)T_2(t_n) = T_1(t_n)S_2(t_n) = a_nb_n|E(t_n)|\varphi'(t_n)\beta_n,$$

where $|\beta_n| = 1$. The hypothesis $\sup_n \varphi'(t_n) < \infty$ implies that

$$\sum_n \frac{|S_1(t_n)T_2(t_n)|^2}{|E(t_n)|^2\varphi'(t_n)} = \sum_n a_n^2 b_n^2 < \infty.$$

Since $\{K_{t_n}\}$ is an orthogonal basis in $\mathcal{H}(E)$ and $\|K_{t_n}\|_E^2 = |E(t_n)|^2\varphi'(t_n)/\pi$, we conclude that there exists a unique function $Q \in \mathcal{H}(E)$ which solves the interpolation problem $Q(t_n) = a_n b_n |E(t_n)|\varphi'(t_n)\beta_n$. Then

$$T_1(z)S_2(z) = Q(z) + a(z)A(z), \qquad S_1(z)T_2(z) = Q(z) + b(z)A(z),$$

for some entire functions a and b. We show now that a and b are polynomials.

Note that by Lemma 5.1 the functions $S = S_1 S_2$ and $T = T_1 T_2$ as well as $(S_1 + T_1)(S_2 + T_2)$ and $(S_1 + iT_1)(S_2 - iT_2)$ are in $\mathcal{P}_M \cdot \mathcal{H}(E)$. Hence, the functions $S_1 T_2$ and $S_2 T_1$, and, consequently, the functions $S_1 T_2 - Q$ and $S_2 T_1 - Q$ are in $\mathcal{P}_M \cdot \mathcal{H}(E)$.

Now assume that $F = AH$ for some entire function H, and $F \in \mathcal{P}_M \cdot \mathcal{H}(E)$. First, since E/A and F/E are in the Smirnov class in $\mathbb{C}_+$, we conclude that, by Krein's theorem, H is of zero exponential type. We claim that H must be a polynomial.

Indeed, if H has at least M zeros z_j, then dividing F by $\prod_{j=1}^M (z - z_j)$ we obtain a function in $\mathcal{H}(E)$ which vanishes on $\{t_n\}$ and, thus, is identically zero. Applying this argument to $S_1 T_2 - Q$ and $S_2 T_1 - Q$ we conclude that a and b are polynomials.

Now assume that $a \neq 0$. Let us denote by s_m the zero of S_2 such that $|\varphi(s_m) - \varphi(t_m)| \leq \pi/2$ whenever such a zero exists. Then

$$Q(s_m) + a(s_m)A(s_m) = 0.$$

Note that $\{t_m\}$ is separated sequence (i.e., $\inf_{n \neq m} |t_n - t_m| > 0$) and so s_m is the union of two separated sequences. By a simple variant of Carleson embedding theorem for the de Branges spaces with $\varphi' \in L^\infty(\mathbb{R})$ (an explicit statement may be found in [2, Theorem 5.1], though the proof may be recovered already from [24, Theorem 2]) we have

$$\sum_m \frac{|Q(s_m)|^2}{|E(s_m)|^2} < \infty$$

for any $Q \in \mathcal{H}(E)$, whence

$$\sum_m \frac{|A(s_m)|^2}{|E(s_m)|^2} < \infty.$$

By the definition of the phase function, $|A(s_m)| = |E(s_m)\sin(\varphi(s_m) - \varphi(t_m))|$. Thus, we obtain that

$$\sum_m \sin^2(\varphi(s_m) - \varphi(t_m)) \asymp \sum_m (\varphi(s_m) - \varphi(t_m))^2 < \infty.$$

To complete the proof we apply once again the argument with the shift of the basis. The zeros of S_2 do not depend on the choice of the basis. Expanding with respect to another basis, say $\{K_{\tilde{t}_n}\}$, with $\varphi(\tilde{t}_n) = \delta + \pi n$ for some small δ, we get that $\sum_m (\varphi(s_m) - \varphi(\tilde{t}_m))^2 < \infty$. However, $|\varphi(t_m) - \varphi(\tilde{t}_m)| = \delta$ and we come to a contradiction.

Thus, we have proved that $a = b = 0$, and so $S_1 T_2 = T_1 S_2 = Q$. Since S_1 has no common zeros with S_2 (we choose the basis so that all a_n are nonzero) we conclude that the zero sets of S_2 and T_2 coincide, and, thus, f is proportional to h.

Theorem 5.3. *If φ is of tempered growth, then the orthogonal complement to the system (1.3) is always finite dimensional, with a bound on the dimension depending only on N from (5.7).*

Proof. By Lemma 5.2, there exists $M = M(N)$ such that for any pair (S_1, S_2) which corresponds to a function f in the orthogonal complement to (1.3) and is real on $\mathbb{R}$, we have $S = S_0 H$, $H \in \mathcal{P}_M$. In particular, any interval (t_n, t_{n+1}) contains at most $M + 1$ zeros of S.

Now assume that the orthogonal complement to (1.3) contains at least $M + 3$ linearly independent vectors $f_{j,0}$, $j = 1, \dots, M + 3$, such that the corresponding functions $S_{1,j,0}$, $S_{2,j,0}$ are real on $\mathbb{R}$. Considering linear combinations (with real coefficients) $f_{j,1} = f_{j,0} - \alpha_j f_{M+3,0}$, $j = 1, \dots, M + 2$, we may achieve that the functions $S_{1,j,1}$ corresponding to $f_{j,1}$ have a common zero at $x_1 \in (t_0, t_1)$. Repeating this procedure we obtain a nonzero function $f_{M+2,1}$ in the orthogonal complement to (1.3) such that the corresponding function $S_{1,M+2,1}$ vanishes at $M + 2$ distinct points $x_1, \dots, x_{M+2} \in (t_0, t_1)$ which gives a contradiction. □

5.3. Density results

Let a pair (S_1, S_2) correspond to a function $h \in \mathcal{H}(E)$ orthogonal to some system $\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{K_\lambda\}_{\lambda \in \Lambda_2}$ and let S_1 and S_2 be real on $\mathbb{R}$. We show that most of the zeros of S are in a certain sense close to the set $\{t_n\}$ (the support of a de Branges orthogonal basis). Thus, the zeros of S_2 which do not depend on the choice of the basis form a small proportion of the zeros of S (see Corollary 5.5).

By Lemma 5.2, $S = S_0 H$, where S_0 has exactly one zero in each of the intervals (t_n, t_{n+1}) and H is a polynomial. Moreover, by (5.6) we have $\{S(t_n)/A'(t_n)\} \in \ell^1$, whence $\{S_0(t_n)/A'(t_n)\} \in \ell^1$. By Lemma 5.1 we have $S \in \mathcal{P}_M \cdot \mathcal{H}(E)$ for some M, whence S_0/A grows at most polynomially along $i\mathbb{R}_+$. Since the zeros of A and S_0 interlace, the function S_0/A is a Herglotz function and thus has a representation

$$\frac{S_0(z)}{A(z)} = az + b + \sum_n \frac{c_n}{z - t_n}, \quad \{c_n\} \in \ell^1. \tag{5.10}$$

We will show that in this case the zeros of S_0 (and S) must be necessarily close (in some sense) to the points t_n. The case when $a \neq 0$ or $b \neq 0$ should be treated exactly as in Proposition 3.1. The remaining case follows from the following proposition (apparently, known to experts).

Proposition 5.4. *Let $t_n \in \mathbb{R}, n \in \mathbb{Z}$, $t_n \to \pm\infty$, $n \to \pm\infty$, and let $\mu_n > 0$, $\sum_n \mu_n = M < \infty$. Let A be an entire function which is real on $\mathbb{R}$ and has only simple real zeros at the points $\{t_n\}$. Define an entire function B by the Herglotz representation*

$$\frac{B(z)}{A(z)} = \sum_n \frac{\mu_n}{z - t_n}.$$

Denote by s_n the zero of B in (t_n, t_{n+1}). Then

$$\sum_{s_n > 0} \frac{t_{n+1} - s_n}{s_n} < \infty, \qquad \sum_{s_n < 0} \frac{s_n - t_n}{|s_n|} < \infty. \tag{5.11}$$

Proof. The zeros of B are simple and interlace with the zeros of A. Since $\operatorname{Im} \frac{B}{A} > 0$ in $\mathbb{C}_+$, the function $E = A - iB$ is in the Hermite–Biehler class and so we can define the de Branges space

$\mathcal{H}(E)$. The measure $\mu = \sum_n \mu_n \delta_{t_n}$ is a corresponding Clark measure for which the embedding operator $\frac{1}{\pi E}\mathcal{H}(E) \to L^2(\mu)$ is unitary.

Consider the inner function $\Theta = E^*/E$. Since $2A/E = 1+\Theta$ and $2B/E = -i(\Theta - 1)$, we have

$$i\frac{1-\Theta(z)}{1+\Theta(z)} = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z} \sim i\frac{M}{y}, \quad z = iy,\ y \to +\infty.$$

Hence,

$$\frac{1+\Theta(iy)}{1-\Theta(iy)} \sim \frac{y}{M}, \quad y \to +\infty. \tag{5.12}$$

It is well known that the function Θ may be reconstructed from the sets $\{t_n\} = \{\Theta = 1\}$ and $\{s_n\} = \{\Theta = -1\}$ by the formula

$$\log \frac{\Theta+1}{\Theta-1} = c + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) f(t)dt,$$

where

$$f(t) = \begin{cases} -1/2, & t \in (t_n, s_n), \\ 1/2, & t \in (s_n, t_{n+1}), \end{cases}$$

and $c \in \mathbb{R}$ (essentially, this is a very special case of the Krein spectral shift formula [17], see also [19, Section 6.1]). Then, by (5.12), we have

$$\int_{\mathbb{R}} \frac{(1-y^2)t}{(t^2+y^2)(t^2+1)} f(t)dt = \mathrm{Re} \int_{\mathbb{R}} \left(\frac{1}{t-iy} - \frac{t}{t^2+1}\right) f(t)dt = \log y + O(1),$$
$$y \to +\infty.$$

A direct computation shows, however, that

$$\int_{\mathbb{R}} \frac{(y^2-1)|t|}{(t^2+y^2)(t^2+1)} |f(t)|dt = \log y + O(1), \quad y \to +\infty,$$

whence

$$\int_{\{t:\ tf(t)>0\}} \frac{(y^2-1)\,tf(t)}{(t^2+y^2)(t^2+1)} dt = O(1), \quad y \to +\infty,$$

and therefore

$$\int_{\{t:\ tf(t)>0\}} \frac{tf(t)}{t^2+1} dt < \infty.$$

Since $tf(t) > 0$ for $t \in (s_n, t_{n+1})$, $s_n > 0$, or $t \in (t_n, s_n)$, $s_n < 0$, we have

$$\sum_{s_n>0} \int_{s_n}^{t_{n+1}} \frac{dt}{t} = \sum_{s_n>0} \ln \frac{t_{n+1}}{s_n} < \infty, \qquad \sum_{s_n<0} \int_{t_n}^{s_n} \frac{dt}{|t|} = \sum_{s_n<0} \ln \frac{|t_n|}{|s_n|} < \infty.$$

The latter convergences are obviously equivalent to (5.11). $\square$

As a corollary we immediately obtain a slightly refined version of Proposition 3.1. Moreover, if $t_n = n$, $n \in \mathbb{Z}$, $A(z) = \sin \pi z$, and $S = S_1 S_2$ is the function arising from the possible one-dimensional defect in the Paley–Wiener space, then

$$\sum_{s\in\mathcal{Z}_2}\frac{1}{|s|}<\infty.$$

Indeed, the zero set $\mathcal{Z}_2$ of the function S_2 does not depend on the choice of the basis, therefore applying Proposition 5.4 to $t_n = n$ and $t_n = n+\delta$ (e.g., $\delta = \frac{1}{2}$), $n \in \mathbb{Z}$, we obtain

$$\sum_{s\in\mathcal{Z}_2,\,s>0}\frac{[s]+1-s}{s}<\infty,\qquad \sum_{s\in\mathcal{Z}_2,\,s>0}\frac{[s-\delta]+1+\delta-s}{s}<\infty,$$

whence $\sum_{s\in\mathcal{Z}_2,\,s>0} s^{-1}<\infty$. The convergence for $s<0$ is analogous.

Under natural regularity conditions, Proposition 5.4 implies the following closeness of the sequences $\{t_n\}$ and $\{s_n\}$.

Corollary 5.5. *Let $A, B, \{t_n\}$ and $\{s_n\}$ be as in Proposition 5.4. Put $I_n = [t_n, t_{n+1}]$. Assume that $|I_k| \asymp |I_n|,\ n \le k \le 2n$, with the constants independent on k, n, and that $|t_{an}| \ge \rho|t_n|$ with some $a \ge 2$, $\rho > 1$. Then for any $\delta > 0$ the set $\mathcal{N}$ of indices n such that $t_n > 0$ and $t_{n+1} - s_n \ge \delta|I_n|$ (respectively, $t_n < 0$ and $s_n - t_n \ge \delta|I_n|$) has zero density.*

Proof. Note that $|t_k| \asymp |t_n|,\ n \le k \le an$. If the upper density of $\mathcal{N}$ is positive, then there exists a sequence $M_j \to \infty$ such that

$$\begin{aligned}\sum_{n\in[M_j,aM_j]\cap\mathcal{N}}\frac{t_{n+1}-s_n}{s_n} &\gtrsim \sum_{n\in[M_j,aM_j]\cap\mathcal{N}}\frac{t_{n+1}-t_n}{t_n}\\ &\gtrsim \sum_{n\in[M_j,aM_j]}\frac{t_{n+1}-t_n}{t_n}\gtrsim \log\frac{t_{aM_j}}{t_{M_j}}\ge\log\rho,\end{aligned}$$

and the first series in (5.11) diverges, a contradiction. □

Arguing as in the proof of Theorem 1.2 we deduce Theorem 1.5 from Corollary 5.5.

5.4. Nonhereditarily complete systems of reproducing kernels in de Branges spaces

In this section we prove Theorem 1.6, i.e., we construct a de Branges space $\mathcal{H}(E)$ and a complete and minimal system of reproducing kernels $\{K_\lambda\}_{\lambda\in\Lambda}$ such that its biorthogonal system is also complete, but the system $\{K_\lambda\}_{\lambda\in\Lambda}$ is not hereditarily complete.

We have already seen that the existence of a nonhereditarily complete system of reproducing kernels generated by some function G in the de Branges space $\mathcal{H}(E)$ is equivalent to the solvability of the equations

$$\begin{aligned}&\sum_n\frac{\overline{a}_n\mu_n^{1/2}}{z-t_n}=\frac{G_2(z)S_2(z)}{A(z)},\\ &\sum_n\frac{G(t_n)}{E(t_n)}\cdot\frac{a_n\mu_n^{1/2}}{z-t_n}=-i\frac{G_1(z)S_1(z)}{A(z)}\end{aligned}\tag{5.13}$$

for some nonzero $\{a_n\}\in\ell^2$ and some entire functions S_1 and S_2. If all the above objects are found, then $h = G_2S_2$ is orthogonal to the corresponding system. The corresponding equations will be constructed as small perturbations of an orthogonal expansion in a de Branges space with respect to a reproducing kernels basis.

Let the sequence $\{t_n\}$ satisfy (1.4). Without loss of generality we may assume that $t_n \ge 0,\ n \ge 0$ and $t_n < 0,\ n < 0$. It follows from (1.4) that $|t_n| \asymp |t_{n+1}|$ and $|t_n| \gtrsim |n|^\gamma$, $|n| \to \infty$, with some $\gamma > 0$.

We construct the space $\mathcal{H}(E)$ and the functions G_1, G_2, S_1 and S_2 in the reverse order. Namely, we start with the construction of the function S. First choose two sequences of positive integers $n_k, l_k \to \infty$ with the following properties:

$$2t_{n_k} < t_{n_k+l_k} < \frac{t_{n_{k+1}}}{2} \quad \text{and} \quad k(t_{n_k+1} - t_{n_k}) \le t_{n_k}/100, \quad k \in \mathbb{N}.$$

Let $a_n \in \mathbb{R}$ be such that

$$|a_{n_k}| = |a_{n_k+1}| = |a_{n_k+l_k}| = |a_{n_k+l_k+1}| = k^{-1},$$

and let $|a_n| = (|n|+1)^{-1}$ for all other values of n. Note that $|a_n| \gtrsim |t_n|^{-M}$ for some $M > 0$. The signs of a_n will be specified later on. Let A be a canonical Hadamard product (of finite genus) whose zeros are simple and coincide with $\{t_n\}$ (thus, A is real on $\mathbb{R}$). Define the entire function S by

$$\frac{S(z)}{A(z)} = \sum_n \frac{a_n^2}{z - t_n}.$$

Then S has exactly one zero z_n in each interval (t_n, t_{n+1}).

We write S as the product

$$S = S_1 S_2 = T_0 T_1 S_2,$$

where T_0 is the canonical product with the zeros $s_k = z_{n_k}$ in intervals (t_{n_k}, t_{n_k+1}) and S_2 is a canonical product with the zeros $z_{n_k+l_k}$ in $(t_{n_k+l_k}, t_{n_k+l_k+1})$, $k \in \mathbb{N}$. Next we construct h. We will construct it as $h = \tilde{T}_0 T_1 S_2$ where $\tilde{T}_0$ is a perturbation of the function T_0 such that

$$\frac{h(z)}{A(z)} = \sum_n \frac{c_n |a_n|}{z - t_n}, \tag{5.14}$$

$$\sum_n c_n^2 = \infty, \qquad \sum_{t_n \ne 0} \frac{c_n^2}{t_n^2} < \infty. \tag{5.15}$$

Condition (5.14) means that

$$\frac{S(z)}{A(z)} \cdot \frac{\tilde{T}_0(z)}{T_0(z)} = \sum_n \frac{\tilde{T}_0(t_n)}{T_0(t_n)} \cdot \frac{a_n^2}{z - t_n},$$

and $c_n = |a_n| \tilde{T}_0(t_n)/T_0(t_n)$. Let us show that all these conditions may be satisfied.

Assume that $|s_k - t_{n_k}| > |s_k - t_{n_k+1}|$. Then we shift the zero s_k of T_0 in the following way:

$$\tilde{s}_k = t_{n_k+1} - k|s_k - t_{n_k+1}|\rho_k.$$

(Analogously, if $|s_k - t_{n_k}| \le |s_k - t_{n_k+1}|$, we put

$$\tilde{s}_k = t_{n_k} - k|s_k - t_{n_k}|\rho_k;$$

in what follows we consider only the first situation.) Let $\tilde{T}_0$ be the canonical product with the zeros $\tilde{s}_k$.

By hypothesis (1.4) we may choose $\rho_k \in (1,2)$ such that

$$\operatorname{dist}(\tilde{s}_k, \{t_n\}_{n\neq n_k+1}) \gtrsim |t_{n_k}|^{-N} \tag{5.16}$$

for some $N > 0$, $\tilde{s}_k \in (t_{n_k+1}/2, t_{n_k+1})$ and zero sets of $\tilde{T}_0$ and $T_1 S_2$ do not intersect. An easy estimate of the infinite products shows that with such choice of zeros for $\tilde{T}_0$ we have

$$\left|\frac{\tilde{T}_0(x)}{T_0(x)}\right| \asymp \left|\frac{x-\tilde{s}_k}{x-s_k}\right|, \quad x \in \left(\frac{t_{n_k}+t_{n_{k-1}}}{2}, \frac{t_{n_k}+t_{n_{k+1}}}{2}\right).$$

Then we obtain

$$|c_{n_k+1}| \asymp \left|\frac{\tilde{T}_0(t_{n_k+1})}{T(t_{n_k+1})}\right| \cdot |a_{n_k+1}| \asymp \left|\frac{t_{n_k+1}-\tilde{s}_k}{t_{n_k+1}-s_k}\right| \cdot k^{-1} \asymp 1,$$

whence the first series in (5.15) diverges. Moreover, it is easy to see that

$$\left|\frac{\tilde{T}_0(t_n)}{T_0(t_n)}\right| \gtrsim 1, \quad t_n \in \left[t_{n_{k-1}}, \frac{t_{n_k}}{2}\right] \cup [t_{n_k+1}, t_{n_{k+1}}],$$

while

$$\left|\frac{\tilde{T}_0(t_n)}{T_0(t_n)}\right| \gtrsim \frac{\operatorname{dist}(\tilde{s}_k, \{t_n\}_{n\neq n_k+1})}{t_n}, \quad t_n \in \left[\frac{t_{n_k}}{2}, t_{n_k}\right].$$

Thus, by (5.16), we have

$$|t_n|^{-N-1} \lesssim \left|\frac{\tilde{T}_0(t_n)}{T_0(t_n)}\right| \lesssim k, \quad n_{k-1} \le n \le n_k, \tag{5.17}$$

and $\left|\frac{\tilde{T}_0(t_n)}{T_0(t_n)}\right| \asymp 1$ for $n \le 0$. Hence,

$$\begin{aligned} &t_n^{-N-1}|a_n| \lesssim |c_n| \lesssim |a_n| k \lesssim 1, \quad n_{k-1} \le n \le n_k, \\ &|c_n| \asymp |a_n|, \quad n \le 0, \end{aligned} \tag{5.18}$$

and, thus, the second condition in (5.15) is satisfied (note that $k = o(t_{n_k}),\ k \to \infty$).

Moreover, $|\tilde{T}_0(iy)/T_0(iy)| \asymp 1$, and so both terms in (5.14) tend to zero along $i\mathbb{R}$. We conclude that the interpolation formula holds.

Next we introduce a de Branges space $\mathcal{H}(E)$. Put $\mu_n = c_n^2$ and $\mu = \sum_n \mu_n \delta_{t_n}$. By (5.15), $\int (1+t^2)^{-1} d\mu(t) < \infty$, and we can define a meromorphic inner function Θ by the formula

$$\frac{1-\Theta(z)}{1+\Theta(z)} = \frac{1}{i}\int \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\mu(t), \quad z \in \mathbb{C}_+.$$

Then $\Theta = E^*/E$ for some entire function E in the Hermite–Biehler class. We may assume that E does not vanish on $\mathbb{R}$. Moreover, since the zero set of $E + E^*$ coincides with $\{t_n\}$, we may choose E so that $E + E^* = 2A$. Now, if we choose the signs of a_n so that $\operatorname{sign} a_n = \operatorname{sign} c_n$, formula (5.14) becomes

$$\frac{h(z)}{A(z)} = \sum_n \frac{a_n|c_n|}{z-t_n} = \sum_n \frac{a_n\mu_n^{1/2}}{z-t_n}.$$

Hence, $h \in \mathcal{H}(E)$.

We have $h = \tilde{T}_0 T_1 S_2$. Put $G_2 = \tilde{T}_0 T_1$. Then $h = G_2 S_2$, and it remains to construct G_1 such that G is the generating function of a complete and minimal system of reproducing kernels in $\mathcal{H}(E)$ and such that (5.13) is satisfied.

We will construct G_1 as a small perturbation of S_2 as we did above. We need to satisfy $G \notin \mathcal{H}(E)$, $G \in \mathcal{H}(E) + z\mathcal{H}(E)$ and (5.13) which is rewritten as

$$\frac{S(z)}{A(z)} \cdot \frac{G_1(z)}{S_2(z)} = i \sum_n \frac{G_1(t_n)}{S_2(t_n)} \cdot \frac{h(t_n)}{E(t_n)} \cdot \frac{a_n\mu_n^{1/2}}{z-t_n}. \tag{5.19}$$

Note that in any de Branges space we have $iA'(t_n) = E(t_n)\varphi'(t_n) = E(t_n)\mu_n^{-1}$. Then (5.19) simplifies to

$$\frac{S(z)}{A(z)} \cdot \frac{G_1(z)}{S_2(z)} = \sum_n \frac{G_1(t_n)}{S_2(t_n)} \cdot \frac{h(t_n)}{A'(t_n)|c_n|} \cdot \frac{a_n}{z-t_n}.$$

The residues, obviously, coincide.

Applying the above construction to S_2 in place of T_0 (i.e., shifting the zeros $z_{n_k+l_k}$) we construct G_1 (again we may assume that G_1 has no common zeros with $\tilde{T}_0 T_1$) so that

$$\left|\frac{G_1(t_n)}{S_2(t_n)}\right| \lesssim k, \quad l_k + n_k \le n \le n_{k+1} + l_{k+1} \tag{5.20}$$

and

$$\left|\frac{G_1(t_{n_k+l_k+1})}{S_2(t_{n_k+l_k+1})}\right| \cdot |a_{n_k+l_k+1}| \asymp 1.$$

Note that $|h(t_n)| = |A'(t_n)| \cdot |a_n| \cdot \mu_n^{1/2} = |E(t_n)| \cdot |a_n| \cdot |c_n|^{-1}$. Then

$$\left|\frac{G(t_n)}{E(t_n)}\right| = \left|\frac{G_1(t_n)}{S_2(t_n)}\right| \cdot |a_n| \cdot |c_n|^{-1}.$$

Hence, in particular,

$$\left|\frac{G(t_{n_k+l_k+1})}{E(t_{n_k+l_k+1})}\right| \asymp |c_{n_k+l_k+1}|^{-1},$$

whence, $\|G/E\|^2_{L^2(\mu)} = \sum_n |G(t_n)|^2|E(t_n)|^{-2}|c_n|^2 = \infty$. Thus, $G \notin \mathcal{H}(E)$. However, by (5.20),

$$\sum_{t_n \ne 0} \frac{|G(t_n)|^2 c_n^2}{t_n^2|E(t_n)|^2} \lesssim \sum_{t_n \ne 0} \frac{a_n^2}{t_n^2}\left|\frac{G_1(t_n)}{S_2(t_n)}\right|^2 < \infty,$$

whence $\frac{G(z)}{(z-\lambda)E(z)} \in L^2(\mu)$ for the zeros λ of G. Also $|G_1(iy)/S_2(iy)| \asymp 1$, so

$$\left|\frac{G(iy)}{A(iy)}\right| \asymp \left|\frac{S(iy)}{A(iy)}\right| \asymp |y|^{-1}, \quad |y| \to \infty, \tag{5.21}$$

and by [8, Theorem 26], $G \in \mathcal{H}(E) + z\mathcal{H}(E)$. Estimate (5.21) also yields the interpolation formula (5.19).

It remains to show that G is the generating function of a complete and minimal system of kernels such that its biorthogonal is also complete.

To prove the first statement, we use that, by the construction, $S/G = T_0 S_2/(\tilde{T}_0 G_1)$ is a Smirnov class function both in the upper and the lower half-planes, while A/S is a Herglotz function and, thus, also a Smirnov class function. Hence, if $GH \in \mathcal{H}(E)$, then an application of Krein's theorem (Section 5.1) yields that H is of zero exponential type. Then it follows from (5.21) that H is a polynomial, which contradicts the fact that $G \notin \mathcal{H}(E)$.

Finally, by (5.18), $|c_n| \gtrsim |t_n|^{-N-1}|a_n|$, thus $\mu_n \gtrsim |t_n|^{-M}$ and also $\sum_n \mu_n = \infty$. Then, by [3, Theorem 1.2], the system biorthogonal to $\{K_\lambda : G(\lambda) = 0\}$ is also complete. This completes the construction of the example (and, thus, the proof of Theorem 1.6).

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