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EQUIVARIANT COBORDISM OF FLAG VARIETIES AND OF SYMMETRIC VARIETIES

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Abstract. We obtain an explicit presentation for the equivariant cobordism ring of a complete flag variety. An immediate corollary is a Borel presentation for the ordinary cobordism ring. Another application is an equivariant Schubert calculus in cobordism. We also describe the rational equivariant cobordism rings of wonderful symmetric varieties of minimal rank.

1. Introduction

Let k be a field of characteristic zero, and G a connected reductive group split over k. Recall that a smooth *spherical variety* is a smooth k-scheme X with an action of G and a dense orbit of a Borel subgroup of G. Well-known examples of spherical varieties include flag varieties, toric varieties, and wonderful compactifications of symmetric spaces. In this paper, we study the equivariant cobordism rings of the following two classes of spherical varieties: the flag varieties and the wonderful symmetric varieties of minimal rank (the latter include wonderful compactifications of semisimple groups of adjoint type such as complete collineations).

The equivariant cohomology and the equivariant Chow groups of these two classes of spherical varieties have been extensively studied before in [1], [30], [5], [6], and [7]. Based on the theory of algebraic cobordism by Levine and Morel [29], and the construction of equivariant Chow groups by Edidin–Graham [11] and

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Totaro [38], the equivariant cobordism was initially introduced in [10] for smooth varieties. It was subsequently developed into a complete theory of equivariant oriented Borel–Moore homology for all k-schemes in [22]. An equivalent version of this equivariant cobordism has also been recently studied by Heller and Malagón–López [15]. For other related works on the algebraic cobordism of varieties with group actions, we refer the reader to [28], [34], and [31].

Similarly to equivariant cohomology, equivariant cobordism is a powerful tool for computing ordinary cobordism of the varieties with a group action. The techniques of equivariant cobordism have been recently exploited to give explicit descriptions of the ordinary cobordism rings of smooth toric varieties in [25], and that of the flag bundles over smooth schemes in [24].

In this paper, we give an explicit Borel type presentation of the equivariant cobordism ring of a complete flag variety. Such a presentation for the equivariant K-theory is due to Kostant–Kumar ([21, Thm. 4.4]). The analogous results for the equivariant Chow ring (with rational coefficients) and the singular cohomology ring are, respectively, due to Brion ([5, Prop. 2]) and Holm–Sjamaar ([16, Prop. 2.2]).

The ordinary cobordism rings of such varieties have been recently described by Hornbostel–Kiritchenko [18] and Calmès–Petrov–Zainoulline [8]. Let $B \subset G$ be a Borel subgroup containing a split maximal torus T. In Theorem 5.1, we obtain an explicit presentation for $\Omega^*_T(G/B)$ tensored with $\mathbb{Z}[t_G^{-1}]$, where t_G is the torsion index of G (see Section 4 for a definition). As a consequence, one immediately obtains an expression for the ordinary cobordism rings of complete flag varieties (tensored with $\mathbb{Z}[t_G^{-1}]$) using a simple relation between the equivariant and the ordinary cobordism (cf. [23, Thm. 3.4]). We also outline an equivariant Schubert calculus in $\Omega^*_T(G/B)$ (see Subsection 5.2).

To compute $\Omega^*_T(G/B)$, we first prove some comparison theorems which relate the equivariant algebraic and complex cobordism rings of cellular varieties over the field of complex numbers (see Section 3). In particular, we give a Borel type presentation for the equivariant complex cobordism $MU^*_T(G/B)$ (see Section 4). The highlight of our proof is that it uses only elementary techniques of equivariant geometry and does not use any computation of the non-equivariant cobordism or cohomology. An interesting problem is to find a purely algebraic proof of our presentation of $\Omega^*_T(G/B)$.

In Section 6, we describe the rational T-equivariant cobordism rings of wonderful symmetric varieties of minimal rank by purely algebraic arguments. Again, this implies a description for their ordinary cobordism rings. In particular, one gets a presentation for the cobordism ring of the wonderful compactification of an adjoint semisimple group. The main ingredient of the proof is the localization theorem for the equivariant cobordism rings for torus action [23, Thm. 7.8]. Once we have this tool, the final result is obtained by adapting the argument of Brion–Joshua [7], who described the equivariant Chow ring. As it turns out, similar steps can be followed to compute the equivariant cobordism ring of any regular compactification of a symmetric space of minimal rank.

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2. Recollection of equivariant cobordism

In this section, we recollect the basic definitions and properties of equivariant cobordism that we shall need in the sequel. For more details, see [22]. Let k be a field of characteristic zero and let G be a linear algebraic group over k.

Let $\mathcal{V}_k$ denote the category of quasi-projective k-schemes and let $\mathcal{V}_k^S$ denote the full subcategory of smooth quasi-projective k-schemes. The category of quasi-projective k-schemes with an algebraic G-action and G-equivariant maps is denoted by $\mathcal{V}_G$ and the corresponding subcategory of smooth schemes is denoted by $\mathcal{V}_G^S$. In this paper, a *scheme* will always mean an object of $\mathcal{V}_k$ and a G-scheme will mean an object of $\mathcal{V}_G$. For all the definitions and properties of algebraic cobordism that are used in this paper, we refer the reader to [29]. All representations of G will be finite-dimensional. Let $\mathbb{L}$ denote the Lazard ring (which is the same as the cobordism ring $\Omega^*(k)$).

We recall the notion of a *good pair* from [22, § 4, p.106]. For integer $j \geq 0$, let V_j be a G-representation, and $U_j \subset V_j$ an open G-invariant subset such that the codimension of the complement is at least j. The pair (V_j, U_j) is called a *good pair* corresponding to j for the G-action if G acts freely on U_j and the quotient U_j/G is a quasi-projective scheme. Quotients U_j/G approximate algebraically the *classifying space* B_G (which is not algebraic), while U_j approximate the *universal space* E_G. It is known that such good pairs always exist (see [11, Lemma 9] or [38, Remark 1.4]).

Let X be a smooth G-scheme. For each $j \geq 0$, choose a good pair (V_j, U_j) corresponding to j. For $i \in \mathbb{Z}$, set

$$\Omega^i_G(X)_j = \frac{\Omega^i\left(X \overset{G}{\times} U_j\right)}{F^j\Omega^i\left(X \overset{G}{\times} U_j\right)}. \tag{2.1}$$

Then it is known ([22, Lemma 4.2, Remark 4.6]) that $\Omega^i_G(X)_j$ is independent of the choice of the good pair (V_j, U_j). Moreover, there is a natural surjective map $\Omega^i_G(X)_{j'} \twoheadrightarrow \Omega^i_G(X)_j$ for $j' \geq j \geq 0$. Here, $F^\bullet\Omega^*(X)$ is the coniveau filtration on $\Omega^*(X)$, i.e., $F^j\Omega^*(X)$ is the set of all cobordism cycles $x \in \Omega^*(X)$ such that x dies in $\Omega^*(X \setminus Y)$, where $Y \subset X$ is closed of codimension at least j (cf. [10, Sect. 3]).

Definition 2.1. Let X be a smooth k-scheme with a G-action. For any $i \in \mathbb{Z}$, we define the *equivariant algebraic cobordism* of X to be

$$\Omega^i_G(X) = \varprojlim_j \Omega^i_G(X)_j. \tag{2.2}$$

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^i_G(X)$ can be non-zero for any $i \in \mathbb{Z}$. We set

$$\Omega^*_G(X) = \bigoplus_{i \in \mathbb{Z}} \Omega^i_G(X).$$

Clearly, if G is trivial, then the G-equivariant cobordism reduces to the ordinary one.

Remark 2.2. If X is a G-scheme of dimension d, which is not necessarily smooth, one defines (cf. [22, Def. 4.4]) the equivariant cobordism of X by

$$\Omega_i^G(X)_j = \varprojlim_j \frac{\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}\left(X \overset{G}{\times} U_j\right)}, \tag{2.3}$$

where $g = \dim(G)$ and $l_j = \dim(U_j)$. Here, $F_\bullet\Omega_*(X)$ is the niveau filtration on $\Omega_*(X)$ such that $F_j\Omega_*(X)$ is the union of the images of the natural $\mathbb{L}$-linear maps $\Omega_*(Y) \to \Omega_*(X)$ where $Y \subset X$ is closed of dimension at most j. It is known (cf. [22, Remark 4.7]) that if X is smooth of dimension d, then $\Omega_G^i(X) \cong \Omega_{d-i}^G(X)$. Since we shall be dealing mostly with the smooth schemes in this paper, we do not need this definition of equivariant cobordism.

It is known that $\Omega_G^*(X)$ satisfies all the properties of a multiplicative oriented cohomology theory like the ordinary cobordism. In particular, it has pull-backs, projective push-forward, Chern class of equivariant bundles, external and internal products, homotopy invariance, and projection formula. We refer to [22, Thm. 5.2] for further details.

The G-equivariant cobordism group $\Omega_G^*(k)$ of the ground field k is denoted by $\Omega^*(B_G)$ and is called the cobordism ring of the *classifying space* of G. We shall often write it as $S(G)$. We also recall the following result, which gives a simpler description of the equivariant cobordism and which will be used throughout this paper.

Theorem 2.3 (([22, Thm. 6.1]))**.** *Let $\{(V_j, U_j)\}_{j\geq 0}$ be a sequence of good pairs for the G-action such that*

(i) $V_{j+1} = V_j \oplus W_j$ *as representations of* G *with* $\dim(W_j) > 0$,
(ii) $U_j \oplus W_j \subsetneq U_{j+1}$ *as* G*-invariant open subsets, and*
(iii) $\operatorname{codim}_{V_{j+1}}(V_{j+1} \setminus U_{j+1}) > \operatorname{codim}_{V_j}(V_j \setminus U_j)$.

Then for any smooth scheme X with a G-action, and any $i \in \mathbb{Z}$,

$$\Omega_G^i(X) \xrightarrow{\cong} \varprojlim_j \Omega^i\left(X \overset{G}{\times} U_j\right).$$

Moreover, such a sequence $\{(V_j, U_j)\}_{j\geq 0}$ of good pairs always exists.

For the rest of this paper, a *sequence of good pairs* $\{(V_j, U_j)\}_{j\geq 0}$ will always mean a sequence as in Theorem 2.3.

2.1. Torus equivariant cobordism of a point

Let $G = T$ be a split torus. The cobordism ring $S(T)$ was described in [22, Example 6.6]. Throughout the paper, we will use the following more invariant description.

Let $\widehat{T}$ denote the character lattice of T, and $\operatorname{Sym}(\widehat{T})$ the symmetric algebra (over the integers) of $\widehat{T}$. Consider the graded algebra $\operatorname{Sym}(\widehat{T}) \otimes \mathbb{L}$ (with respect to the total grading, that is, the degree of an element $a \otimes b$ is the sum of the degrees of a and b in $\mathrm{Sym}(\widehat{T})$ and $\mathbb{L}$, respectively). Then $S(T)$ is canonically isomorphic to

$$S(T) = \bigoplus_{i\in\mathbb{Z}} S^i(T),$$

where

$$S^i(T) := \varprojlim_j \left(\mathrm{Sym}^j(\widehat{T}) \otimes \mathbb{L}\right)^i.$$

The isomorphism sends a character $\chi \in \widehat{T}$ to the first T-equivariant Chern class of the T-equivariant line bundle L_χ on $\mathrm{Spec}\,(k)$. In particular, if $\chi_1, \ldots, \chi_n$ is a basis in $\widehat{T}$, then $S(T)$ is isomorphic to the graded power series ring $S = \mathbb{L}^{\mathrm{gr}}[[c_1^T(L_{\chi_1}), \ldots, c_1^T(L_{\chi_n})]]$.

Recall that for a graded ring R, the *graded* power series ring $R^{\mathrm{gr}}[[x_1, \ldots, x_n]]$ consists of all finite linear combinations of homogeneous (with respect to the total grading) power series (e.g., if R has no terms of negative degree then $R^{\mathrm{gr}}[[x_1, \ldots \ldots, x_n]]$ is just a ring of polynomials).

2.2. Equivariant cobordism of the variety of complete flags in k^n

As an example illustrating the definition of equivariant cobordism, we now compute $\Omega_T^*(G/B)$ for $G = \mathrm{GL}_n(k)$ directly by definition. Note that the same result can be obtained by less computationally involved arguments (see Section 4 where we compute $\Omega_T^*(G/B)$ for all reductive groups G and also [17] for the theorem on flag bundles).

We identify the points of the complete flag variety $X = G/B$ with *complete flags* in k^n. A *complete flag* F is a strictly increasing sequence of subspaces

$$F = \{\{0\} = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^n = k^n\}$$

with $\dim(V^i) = i$. There are n natural line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on X, that is, the fiber of $\mathcal{L}_i$ at the point F is equal to V^i/V^{i-1}. These bundles are equivariant with respect to the left action of the diagonal torus $\mathrm{GL}_n(k)$ on X, namely, $\mathcal{L}_i$ corresponds to the character χ_i of the diagonal torus $T \subset \mathrm{GL}_n(k)$ given by the i-th entry of T. For each $i = 1, \ldots, n$, consider also the T-equivariant line bundle L_i on $\mathrm{Spec}\,(k)$ corresponding to the character χ_i.

Theorem 2.4. *There is the following ring isomorphism*

$$\Omega_T^*(X) \simeq \mathbb{L}^{\mathrm{gr}}[[x_1, \ldots, x_n; t_1, \ldots, t_n]]/(s_i(x_1, \ldots, x_n) - s_i(t_1, \ldots, t_n), i = 1, \ldots, n),$$

where $s_i(x_1, \ldots, x_n)$ denotes the i-th elementary symmetric function of the variables $x_1, \ldots, x_n$. The isomorphism sends x_i and t_i, respectively, to the first T-equivariant Chern classes $c_1^T(\mathcal{L}_i)$ and $c_1^T(L_i)$.

Proof. First, note that $\Omega_T^*(X) = \Omega_B^*(X)$ by [22, Prop. 8.1], where B is a Borel subgroup in G (we choose B to be the subgroup of the upper-triangular matrices). For $N > n$, we can approximate the classifying space B_B by partial flag varieties $\mathbb{F}_{N,n} := \mathbb{F}(N-n, N-n+1, \ldots, N-1, N)$ consisting of all flags

$$F = \{V^{N-n} \subsetneq V^{N-n+1} \subsetneq \ldots \subsetneq V^{N-1} \subsetneq k^N\}.$$

We choose exactly this approximation because its cobordism ring is easier to compute via projective bundle formula than the cobordism ring of the dual flag variety $\mathbb{F}(1, 2, \ldots, n; N)$ (for cohomology rings, this difference does not show up, since for the additive formal group law, the Chern classes of dual vector bundles are the same up to a sign). Approximate E_B by the variety $E_N := \mathrm{Hom}^{\circ}(k^N, k^n)$ of all projections of k^N onto k^n. Note that $\{(\mathrm{Hom}(k^N, k^n), E_N)\}_{N \geq n}$ is a sequence of good pairs (as in Theorem 2.3) for the action of GL_n.

Denote by $\mathcal{E}$ the tautological quotient bundle of rank n on $\mathbb{F}_{N,n}$ (i.e., the fiber of $\mathcal{E}$ at the point F is equal to k^N/V^{N-n}). For the complete flag variety X, we have that $X \times^B E_N$ is the flag variety $\mathbb{F}(\mathcal{E})$ relative to the bundle $\mathcal{E}$. Points of $\mathbb{F}(\mathcal{E})$ can be identified with complete flags in the fibers of $\mathcal{E}$. Hence, we can compute the cobordism ring of $X \times^B E_N$ by the formula for the cobordism rings of relative flag varieties [18, Thm. 2.6]. We get

$$\Omega^*(X \times^B E_N) = \Omega^*(\mathbb{F}(\mathcal{E})) \simeq \Omega^*(\mathbb{F}_{N,n})[x_1, ..., x_n]/I,$$

where I is the ideal generated by the relations $s_k(x_1, .., x_n) = c_k(\mathcal{E})$ for $1 \leq k \leq n$. The isomorphism sends x_i to the first Chern class of the line bundle $\mathcal{L}_i \times^B E_N$ on $X \times^B E_N$.

By the repeated use of the projective bundle formula (as in the proof of [18, Thm. 2.6]) we get that

$$\Omega^*(\mathbb{F}_{N,n}) \simeq \mathbb{L}^{\mathrm{gr}}[t_1, \ldots, t_n]/(h_N(t_n), h_{N-1}(t_{n-1}, t_n), \ldots, h_{N-n+1}(t_1, \ldots, t_n)),$$

where t_i is the first Chern class of the i-th tautological line bundle on $\mathbb{F}_{N,n}$ (whose fiber at the point F is equal to $V^{N-n+i}/V^{N-n+i-1}$), and $h_k(t_i, \ldots, t_n)$ denotes the sum of all monomials of degree k in $t_i, \ldots, t_n$.

It is easy to deduce from the Whitney sum formula that $c_k(\mathcal{E}) = s_k(t_1, \ldots, t_n)$. Passing to the limit, we get that $\Omega^i_B(X) := \varprojlim_N \Omega^i(X \times^B E_N)$ consists of all homogeneous power series of degree i in $t_1, \ldots, t_n$ and $x_1, \ldots, x_n$ modulo the relations $s_k(x_1, \ldots, x_n) = s_k(t_1, \ldots, t_n)$ for $1 \leq k \leq n$. Indeed, all relations between $t_1, \ldots, t_n$ in $\Omega^*(\mathbb{F}_{N,n})$ are in degree greater than i if $N > i + n - 1$. □

3. Algebraic and complex cobordism

In this section and in Section 4, we shall assume our ground field to be the field of complex numbers $\mathbb{C}$. A *scheme* will mean a quasi-projective scheme over $\mathbb{C}$. To describe the equivariant algebraic cobordism ring of flag varieties, we first describe the equivariant complex cobordism and then use comparison results between the algebraic and complex cobordism. Our main goal in this section is to establish such comparison results.

For a scheme X, let $H^*(X, A)$ denote the singular cohomology of the analytic space $X(\mathbb{C})$ with coefficients in an abelian group A. Let $MU^*(X, A)$ denote $MU^*(X) \otimes_{\mathbb{Z}} A$, where $MU^*(-)$ is the complex cobordism, a generalized cohomology theory on the category of CW-complexes.

Recall from [33, §2] that $X \mapsto MU^*(X(\mathbb{C}))$ is an example of an oriented cohomology theory on $\mathcal{V}^S_{\mathbb{C}}$. In fact, it is the universal oriented cohomology theory in the category of CW-complexes. This cohomology theory is multiplicative in the sense that it has external and internal products. One knows that $X \mapsto H^*(X, \mathbb{Z})$ is also an example of a multiplicative oriented cohomology theory on $\mathcal{V}^S_{\mathbb{C}}$.

3.1. Equivariant complex cobordism

Recall ([22, Sect. 7]) that if G is a complex linear algebraic group and X is a finite CW-complex with a G-action, then its Borel style *equivariant complex cobordism* is defined as

$$MU^*_G(X) := MU^*\left(X \overset{G}{\times} E_G\right), \tag{3.1}$$

where $E_G \to B_G$ is a universal principal G-bundle. It is known that $MU^*_G(X)$ is independent of the choice of this universal bundle.

Definition 3.1. Let $\mathcal{U} = \{(V_j, U_j)\}_{j \geq 0}$ be a sequence of good pairs for G-action. For a linear algebraic group G acting on a scheme X and for any $i \in \mathbb{Z}$, we define

$$MU^i_G\left(X, \mathcal{U}\right) := \varprojlim_{j \geq 0} MU^i\left(X \overset{G}{\times} U_j\right) \tag{3.2}$$

and set $MU^*_G\left(X, \mathcal{U}\right) = \bigoplus_{i \in \mathbb{Z}} MU^i_G\left(X, \mathcal{U}\right)$. We also set

$$\Omega^i_G\left(X, \mathcal{U}\right) := \varprojlim_{j \geq 0} \Omega^i\left(X \overset{G}{\times} U_j\right) \text{ and } \Omega^*\left(X, \mathcal{U}\right) = \bigoplus_{i \in \mathbb{Z}} \Omega^i_G\left(X, \mathcal{U}\right). \tag{3.3}$$

It is easy to check as in [22, Thm. 5.2] that $MU^*_G(-, \mathcal{U})$ and $\Omega^*_G\left(-, \mathcal{U}\right)$ have all the functorial properties of the equivariant cobordism. In particular, both are contravariant functors on $\mathcal{V}^S_G$ and $\Omega^*_G\left(-, \mathcal{U}\right)$ is also covariant for projective maps. Moreover, the pull-back and the push-forward maps commute with each other in a fiber diagram of smooth and projective morphisms.

Lemma 3.2. *Let $\mathcal{U} = \{(V_j, U_j)\}_{j \geq 1}$ be a sequence of good pairs for the G-action and let X be a smooth G-scheme such that $H^*_G(X, \mathbb{Z})$ is torsion-free. There is an isomorphism $MU^i_G(X) \to MU^i_G\left(X, \mathcal{U}\right)$ of abelian groups for any $i \in \mathbb{Z}$.*

Proof. Since $\mathcal{U}$ is a sequence of good pairs for the G-action, the codimension of the complement of U_j in the G-representation V_j is at least j. In particular, the pair (V_j, U_j) is $(j-1)$-connected. Taking the limit, we see that $E_G = \bigcup_{j \geq 0} U_j$ is contractible and hence $E_G \to E_G/G$ is a universal principal G-bundle and we can take $B_G = E_G/G$. Since $X(\mathbb{C})$ has the homotopy type of a finite CW-complex, we see that $X_G = X \overset{G}{\times} E_G$ has a filtration by finite subcomplexes

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_i \subset \cdots \subset X_G$$

with $X_j = X \overset{G}{\times} U_j$ and $X_G = \bigcup_{j \geq 0} X_j$. This yields the Milnor exact sequence

$$0 \to \varprojlim_{j \geq 0}{}^1\, MU^{i-1}\left(X_j\right) \to MU^i_G\left(X\right) \to \varprojlim_{j \geq 0} MU^i\left(X_j\right) \to 0. \tag{3.4}$$

Since $H^*_G\left(X, \mathbb{Z}\right) = H^*\left(X_G, \mathbb{Z}\right)$ is torsion-free, it follows from [27, Cor. 1] that the first term in this exact sequence is zero. This proves the lemma. $\square$

3.2. Comparison theorem

Recall from [12, Example 1.9.1] that a scheme (or an analytic space) L is called *cellular*, if it has a filtration $\varnothing = L_{n+1} \subsetneq L_n \subsetneq \cdots \subsetneq L_1 \subsetneq L_0 = L$ by closed subschemes (subspaces) such that each $L_i \setminus L_{i+1}$ is a disjoint union of affine spaces $\mathbb{A}^{r_i}$ (*cells*). It follows from the Bruhat decomposition that varieties G/B are cellular with cells labelled by elements of the Weyl group. Using the above definition and the fact that a vector bundle over an affine space is also an affine space (Quillen-Suslin theorem), one checks that a vector bundle over a cellular scheme is also cellular. The following elementary and folklore result yields more examples of cellular schemes.

Lemma 3.3. *Let X be a scheme with a filtration $\varnothing = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_1 \subsetneq X_0 = X$ by closed subschemes such that each $X_i \setminus X_{i+1}$ is a cellular scheme. Then X is also a cellular scheme.*

Proof. It follows from our assumption that X_n is cellular. It suffices to prove by induction on the length of the filtration of X that, if $Y \hookrightarrow X$ is a closed immersion of schemes such that Y and $U = X \setminus Y$ are cellular, then X is also cellular. Consider the cellular decompositions

$$\varnothing = Y_{l+1} \subsetneq Y_l \subsetneq \cdots \subsetneq Y_1 \subsetneq Y_0 = Y,$$
$$\varnothing = U_{m+1} \subsetneq U_m \subsetneq \cdots \subsetneq U_1 \subsetneq U_0 = U$$

of Y and U. Set

$$X_i = \begin{cases} Y \cup U_i & \text{if } 0 \leq i \leq m+1 \\ Y_{i-m-1} & \text{if } m+2 \leq i \leq m+l+2 \ . \end{cases}$$

It is easy to verify that $\{X_i\}_{0 \leq i \leq m+l+2}$ is a filtration of X by closed subschemes such that $X_i \setminus X_{i+1}$ is a disjoint union of affine spaces over $\mathbb{C}$. □

Let T be a split torus of rank n and let $\mathcal{U} = \{(V_j, U_j)\}_{j \geq 1}$ be the sequence of good pairs for T-action such that each $(V_j, U_j) = (V'_j, U'_j)^{\oplus n}$, where V'_j is the j-dimensional representation of $\mathbb{G}_m$ with all weights -1, and U'_j is the complement of the origin and T acts on V_j diagonally.

Definition 3.4. A scheme (or a scheme over any other field) X with an action of T is called *T-equivariantly cellular*, if there is a filtration $\varnothing = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_1 \subsetneq X_0 = X$ by T-invariant closed subschemes such that each $X_i \setminus X_{i+1}$ is isomorphic to a disjoint union of representations V_i's of T.

It is obvious that a T-equivariantly cellular scheme is cellular in the usual sense. It follows from a theorem of Bialynicki-Birula [2] that if X is a smooth projective scheme with a T-action such that the fixed point locus X^T is isolated, then X is T-equivariantly cellular. In particular, a complete flag variety G/B or a smooth projective toric variety is T-equivariantly cellular.

Proposition 3.5. *Let $\mathcal{U} = \{(V_j, U_j)\}_{j \geq 1}$ be as above, and X a smooth scheme with a T-action such that it is T-equivariantly cellular. Then the natural map*

$$\Omega^*_T(X, \mathcal{U}) \to MU^*_T(X, \mathcal{U})$$

is an isomorphism.

Proof. For any scheme Y with T action, we set $Y^j = Y \overset{T}{\times} U_j$ for $j \geq 1$. Consider the T-equivariant cellular decomposition of X as in Definition 3.4 and set $W_i = X_i \setminus X_{i+1}$. It follows immediately that X^j has a filtration

$$\varnothing = (X^j)_{n+1} \subsetneq (X^j)_n \subsetneq \cdots \subsetneq (X^j)_1 \subsetneq (X^j)_0 = X^j,$$

where $(X^j)_i = (X_i)^j = X_i \overset{T}{\times} U_j$ and thus $(X^j)_i \setminus (X^j)_{i+1} = (W_i)^j$.

Since $U_j/T \cong \left(\mathbb{P}^{j-1}\right)^n$ is cellular and since $(W_i)^j = W_i \overset{T}{\times} U_j \to U_j/T$ is a disjoint union of vector bundles, it follows that each $(X^j)_i = (W_i)^j$ is cellular. We conclude from Lemma 3.3 that X^j is cellular. In particular, the map $\Omega^*(X^j) \to MU^*(X^j)$ is an isomorphism (cf. [18, Thm. 6.1]). The proposition now follows by taking the limit over $j \geq 1$. $\square$

Lemma 3.6. *Let X be a T-equivariantly cellular scheme. Then $H^*_T(X, \mathbb{Z})$ is torsion-free.*

Proof. Let $\mathcal{U} = \{(V_j, U_j)\}_{j \geq 1}$ be a sequence of good pairs for T-action as above. Since $H^i_T(X, \mathbb{Z}) \xrightarrow{\cong} H^i(X^j, \mathbb{Z})$ for $j \gg 0$, it suffices to show that $H^*(X^j, \mathbb{Z})$ is torsion-free for any $j \geq 0$. But we have shown in Proposition 3.5 that each X^j is cellular and hence $H^*(X^j, \mathbb{Z})$ is a free abelian group. $\square$

Theorem 3.7. *Let X be a smooth scheme with an action of a split torus T. Assume that X is T-equivariantly cellular. Then there is a degree-doubling map*

$$\Phi^{\text{top}}_X : \Omega^*_T(X) \to MU^*_T(X)$$

which is a ring isomorphism.

Proof. It follows from Lemma 3.6 and [22, Prop. 7.5] that there is a ring homomorphism $\Phi^{\text{top}}_X : \Omega^*_T(X) \to MU^*_T(X)$.

We now choose a sequence $\{(V_j, U_j)\}_{j \geq 1}$ of good pairs for the T-action as in Proposition 3.5. It follows from [22, Thm. 6.1] that $\Omega^i_T(X) \xrightarrow{\cong} \Omega^i_T(X, \mathcal{U})$ for each $i \in \mathbb{Z}$. Since $H^*_T(X, \mathbb{Z}) = H^*\left(X \overset{T}{\times} E_G, \mathbb{Z}\right)$ is torsion-free by Lemma 3.6, Lemma 3.2 implies that $MU^i_T(X) \xrightarrow{\cong} MU^i_T(X, \mathcal{U})$. The theorem now follows from Proposition 3.5. $\square$

Let G be a connected reductive group with a maximal torus T and a Borel subgroup B containing T. Since we have already noted that the flag variety G/B is T-equivariantly cellular, the following is an immediate consequence of Theorem 3.7.

Corollary 3.8. *There is a ring isomorphism*

$$\Phi^{\text{top}}_{G/B} : \Omega^*_T(G/B) \xrightarrow{\cong} MU^*_T(G/B).$$

For any character $\alpha \in \widehat{T}$, let x_α denote the first T-equivariant Chern class $c^T_1(L_\alpha) \in S$ of the associated T-equivariant line bundle L_α on Spec $(\mathbb{C})$ (see Subsection 2.1). Let Σ denote the root system of (G, T), and W the Weyl group. The following description of $\Omega^*_T(G/B)$ as a subring of $S^{|W|}$ follows immediately from Corollary 3.8, [14, Thm. 3.1] and the isomorphism $S \xrightarrow{\cong} MU^*(B_T)$ ([26]).

Theorem 3.9. *The inclusion* $\iota : (G/B)^T \hookrightarrow G/B$ *of the fixed point locus induces a ring isomorphism*

$$\Omega^*_T(G/B) \xrightarrow[\cong]{\iota^*} \big\{(f_w)_{w\in W} \in S^{|W|} \mid f_w \equiv f_{s_\alpha w} \pmod{x_\alpha} \,\forall \alpha \in \Sigma, \forall w \in W\big\}.$$

A description of the kind obtained in Theorem 3.9 was first conceived in the paper [13] of Goresky–Kottwitz–MacPherson, who showed that the equivariant singular cohomology of G/B can be described in such a way. Around the same time, Brion [5] showed that the T-equivariant Chow groups of G/B can also be described in a similar way. The results of Goresky–Kottwitz–MacPherson and Brion were subsequently extended to a bigger class of equivariant cohomology theories such as equivariant K-theory and equivariant complex cobordism of Kac–Moody flag varieties by Harada–Henriques–Holm [14, Thm. 3.1]. Note that this description is different from the Borel type description we obtain in the next sections.

4. Equivariant complex cobordism of G/B

In this section, we continue working over the ground field $\mathbb{C}$. Let G be a connected reductive group. We fix a maximal torus T of rank n and a Borel subgroup B containing T. The Weyl group of G is denoted by W. In this section, we compute the equivariant complex cobordism ring $MU^*_T(G/B)$ of the complete flag variety G/B. For this description, we need the following special case of the Leray–Hirsch theorem for a multiplicative generalized cohomology theory.

Theorem 4.1 (Leray–Hirsch). *Let* X *be a (possibly infinite) CW-complex with finite skeleta and let* $F \xrightarrow{i} E \xrightarrow{p} X$ *be a fiber bundle such that the fiber* F *is a finite CW-complex. Assume that there are elements* $\{e_1, \cdots, e_r\}$ *in* $MU^*(E)$ *such that* $\{f_1 = i^*(e_1), \ldots, f_r = i^*(e_r)\}$ *forms an* $\mathbb{L}$*-basis of* $MU^*(F)$ *for each fiber* F *of the fiber bundle. Assume furthermore that* $H^*(X, \mathbb{Z})$ *is torsion-free. Then the map*

$$\begin{aligned} \Psi : MU^*(F) \otimes_{\mathbb{L}} MU^*(X) &\to MU^*(E) \\ \Psi\Big(\sum_{1\le i\le r} f_i \otimes b_i\Big) &= \sum_{1\le i\le r} p^*(b_i) e_i \end{aligned} \tag{4.1}$$

is an isomorphism of $MU^*(X)$*-modules. In particular,* $MU^*(E)$ *is a free* $MU^*(X)$*-module with the basis* $\{e_1, \ldots, e_r\}$.

This result is well known and can be found, for example, in [36, Thm. 15.47] and [20, Thm. 3.1].

4.1. Equivariant complex cobordism of G/B

In what follows, we assume all spaces to be pointed and let $p_X : X \to \mathrm{pt}$ be the structure map. Let $MU^*(B_T) = MU^*_T(\mathrm{pt})$ denote the coefficient ring of the T-equivariant complex cobordism. It is well known ([26]) that $MU^*(B_T)$ is isomorphic to $S(T)$ (which is denoted by S in this text). The isomorphism sends a character χ of T to the first Chern class of the T-equivariant line bundle L_χ on B_T. Note that each character χ of T also gives rise to the G-equivariant line bundle $\mathcal{L}_\chi := G \overset{B}{\times} L_\chi$ on G/B. We will also use that $MU^*(B_T) = MU^*(B_B)$ is isomorphic to $MU^*_G(G/B)$ since $G/B \overset{G}{\times} E_G = E_G/B$ and we can choose $E_G = E_B$.

For any finite CW-complex X with a G-action, consider the fiber bundle

$$G/B \xrightarrow{i_X} X \overset{B}{\times} E_G \xrightarrow{\pi_X} X \overset{G}{\times} E_G,$$

where i_X is the inclusion of the fiber at the base point. Put $i = i_X$ and $\pi = \pi_X$ when X is the base point. This gives rise to the following commutative diagram:

$$\begin{array}{ccccc} MU^*(B_G) & \xrightarrow{\pi^*} & MU^*(B_T) & \xrightarrow{i^*} & MU^*(G/B) \\ {\scriptstyle p^*_{G,X}}\downarrow & & \downarrow{\scriptstyle p^*_{T,X}} & & \| \\ MU^*_G(X) & \xrightarrow[\pi^*_X]{} & MU^*_T(X) & \xrightarrow[i^*_X]{} & MU^*(G/B). \end{array} \tag{4.2}$$

Recall that the *torsion index* of G is defined as the smallest positive integer t_G such that t_G times the class of a point in $H^{2d}(G/B, \mathbb{Z})$ (where $d = \dim(G/B)$) belongs to the subring of $H^*(G/B, \mathbb{Z})$ generated by the first Chern classes of line bundles $\mathcal{L}_\chi$ (e.g., $t_G = 1$ for $G = \mathrm{GL}_n$; see [39] for computations of t_G for other groups). If G is simply connected then this subring is generated by $H^2(G/B, \mathbb{Z})$.

For the rest of this section, an abelian group A will actually mean its extension $A \otimes_{\mathbb{Z}} R$, where $R = \mathbb{Z}[t_G^{-1}]$. In particular, all the cohomology and the cobordism groups will be considered with coefficients in R.

We shall use the following key fact to prove the main result of this section.

Lemma 4.2. *The homomorphism $i^* : MU^*_G(G/B) \to MU^*(G/B)$ is surjective over the ring R.*

Proof. Since $MU^*_G(G/B) \simeq MU^*(B_T) \simeq S$, the image of i^* is the subring of $MU^*(G/B)$ generated by the first Chern classes of line bundles $\mathcal{L}_\chi$. To prove surjectivity of i^*, we have to show that $MU^*(G/B)$ is generated by the first Chern classes.

Since G/B is cellular, the cobordism ring $MU^*(G/B)$ is a free $\mathbb{L}$-module. Choose a basis $\{e_w\}_{w \in W}$ in $MU^*(G/B)$ such that all e_w are homogeneous (e.g., take resolutions of the closures of cells). Consider the homomorphism

$$\varphi : MU^*(G/B) \to MU^*(G/B) \otimes_{\mathbb{L}} R.$$

Since $H^*(G/B, R)$ is torsion free, we have the isomorphism $MU^*(G/B) \otimes_{\mathbb{L}} R \simeq H^*(G/B, R)$. Note that $H^*(G/B, R)$ is generated by the first Chern classes by definition of the torsion index, and the homomorphism φ takes the Chern classes to the Chern classes. Hence, there exist homogeneous polynomials $\{\varrho_w\}_{w \in W}$, where $\varrho_w \in \mathrm{Sym}(\widehat{T}) \otimes R \subset S$ such that $\varphi(e_w) = \varphi(i^*(\varrho_w))$. Then the set of cobordism classes $\{i^*(\varrho_w)\}_{w \in W}$ is a basis over $\mathbb{L}$ in $MU^*(G/B, R)$. Indeed, consider the transition matrix A from the basis $\{e_w\}_{w \in W}$ to this set (order e_w and ϱ_w so that their degrees decrease). The elements of A are homogeneous elements of $\mathbb{L}$ and

$A \otimes_{\mathbb{L}} R$ is the identity matrix. By degree arguments, it follows that the matrix A is upper-triangular and the diagonal elements are equal to 1, so A is invertible.

Hence, $MU^*(G/B)$ has a basis consisting of polynomials in the first Chern classes and the homomorphism i^* is surjective over R. □

To compute $MU_T^*(G/B)$ and $MU_T^*(X)$, we can now apply the same strategy as in the cohomology case (see, e.g., [4, Prop. 1])

By Lemma 4.2, we can choose polynomials $\{\varrho_w\}_{w\in W}$ in $MU_G^*(G/B) \simeq MU^*(B_T) = S$ such that $\{i^*(\varrho_w)\}_{w\in W}$ form an $\mathbb{L}$-basis in $MU^*(G/B)$. Set $\varrho_{w,X} = p_{T,X}^*(\varrho_w)$ for each $w \in W$. Define $\mathbb{L}$-linear maps

$$\begin{gathered} s : MU^*(G/B) \to S, \quad s_X : MU^*(G/B) \to MU_T^*(X) \\ s(i^*(\varrho_w)) = \varrho_w \quad \text{and} \quad s_X(i^*(\varrho_w)) = \varrho_{w,X}. \end{gathered} \tag{4.3}$$

Note that maps i_X and i are W-equivariant. In particular, the map s is also W-equivariant.

Lemma 4.3. *Let X be a finite CW-complex with a G-action such that $H_T^*(X, R)$ is torsion-free.*

(i) *The map $MU^*(G/B) \otimes_{\mathbb{L}} MU_G^*(X) \to MU_T^*(X)$ which sends (b, x) to $s_X(b) \cdot \pi_X^*(x)$ is an isomorphism of $MU_G^*(X)$-modules. In particular, $MU_T^*(X)$ is a free $MU_G^*(X)$-module with the basis $\{\varrho_{w,X}\}_{w\in W}$.*

(ii) *The map $S \times MU_G^*(X) \to MU_T^*(X)$ which sends (a, x) to $p_{T,X}^*(a) \cdot \pi_X^*(x)$ yields an isomorphism of graded $\mathbb{L}$-algebras*

$$\Psi_X^{\text{top}} : S \otimes_{MU^*(B_G)} MU_G^*(X) \xrightarrow{\cong} MU_T^*(X). \tag{4.4}$$

Proof. It follows from our assumption and [16, Prop. 2.1(i)] that $H_G^*(X, R)$ is torsion-free. Since $i^* = i_X^* \circ p_{T,X}^*$, we conclude from the above construction that $i^*(\varrho_w) = i_X^*\left(p_{T,X}^*(\varrho_w)\right) = i_X^*(\varrho_{w,X})$. Since $\{i^*(\varrho_w)\}_{w\in W}$ form an $\mathbb{L}$-basis of $MU^*(G/B)$, the first statement now follows immediately by applying Theorem 4.1 to the fiber bundle $G/B \xrightarrow{i_X} X \overset{B}{\times} E_G \xrightarrow{\pi_X} X \overset{G}{\times} E_G$.

To prove the second statement, we first notice that $MU_G^*(X) \to MU_T^*(X)$ is a map of $MU^*(= \mathbb{L})$-algebras and so is the map $S \to MU_T^*(X)$. In particular, being the product of these two maps, (4.4) is a morphism of $\mathbb{L}$-algebras. Moreover, it follows from the first part of the lemma that $S \cong MU^*(B_T)$ is a free $MU^*(B_G)$-module with basis $\{\varrho_w\}_{w\in W}$ and $MU_T^*(X)$ is a free $MU_G^*(X)$-module with basis $\{\varrho_{w,X}\}_{w\in W}$. In particular, Ψ_X^{top} takes the basis elements $\varrho_w \otimes 1$ onto the basis elements $\varrho_{w,X}$. Hence, it is an algebra isomorphism. □

We now compute $MU^*(B_G)$.

Proposition 4.4. *The natural map $MU^*(B_G) \to (MU^*(B_T))^W$ is an isomorphism of R-algebras.*

Proof. Note that in the proof of Lemma 4.2, we can choose $\varrho_{w_0} = 1$ (here w_0 is the longest length element of the Weyl group). Then applying Theorem 4.1 to the fiber bundle $G/B \xrightarrow{i} B_T \xrightarrow{\pi} B_G$ (as in the proof of Lemma 4.3 for $X = pt$), we get

$$\Psi(1 \otimes b) = \Psi(i^*(\varrho_{w_0}) \otimes b) = \pi^*(b)\varrho_{w_0} = \pi^*(b) \quad \text{for any } b \in MU^*(B_G), \tag{4.5}$$

where Ψ is as in (4.1). In particular, π^* is the composite map

$$\pi^* : MU^*(B_G) \xrightarrow{1\otimes \mathrm{id}} MU^*(G/B) \otimes_{\mathbb{L}} MU^*(B_G) \xrightarrow{\Psi} MU^*(B_T). \tag{4.6}$$

Hence to prove the proposition, it suffices to show using Theorem 4.1 that the map $1 \otimes \mathrm{id}$ induces an isomorphism $MU^*(B_G) \to \left(MU^*(G/B) \otimes_{\mathbb{L}} MU^*(B_G)\right)^W$ over R.

We first show that the map $MU^*(B_G) \xrightarrow{1\otimes \mathrm{id}} MU^*(G/B) \otimes_{\mathbb{L}} MU^*(B_G)$ is split injective. To do this, we only have to observe from the projection formula for the map $p_{G/B} : G/B \to \mathrm{pt}$ that $p_{G/B_*}\big(\rho \cdot p^*_{G/B}(x)\big) = p_{G/B_*}(\rho) \cdot x = x$, where $\rho \in MU^*(G/B)$ is the class of a point. This gives a splitting of the map $p^*_{G/B}$ and hence a splitting of $1 \otimes \mathrm{id} = p^*_{G/B} \otimes \mathrm{id}$.

To prove the surjectivity, we follow the proof of the analogous result for the Chow groups in [39, Thm. 1.3]. Since the Atiyah-Hirzebruch spectral sequence degenerates over the rationals and since the analogue of our proposition is known for the singular cohomology groups by [39, Thm. 1.3(2)], we see that the proposition holds over the rationals (cf. [22, Thm. 8.8]).

We now let $\alpha : MU^*(G/B) \to \mathbb{L}$ be the map $\alpha(y) = p_{G/B_*}(\rho \cdot y)$ and set $\beta = \alpha \otimes \mathrm{id} : MU^*(G/B) \otimes_{\mathbb{L}} MU^*(B_G) \to MU^*(B_G)$. Set $f^* = p^*_{G/B} \otimes \mathrm{id}$ and $f_* = p_{G/B_*} \otimes \mathrm{id}$. The projection formula as above implies that $f^*\beta f^*(x) = f^*(x)$ for all $x \in MU^*(B_G)$. Thus $f^*\beta(y) = y$ for all y in the image of $1 \otimes \mathrm{id}$. We identify $S \xrightarrow{\cong} MU^*(B_T)$ with $MU^*(G/B) \otimes_{\mathbb{L}} MU^*(B_G)$ over R as in Lemma 4.3 and consider the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\beta} & MU^*(B_G) & \xrightarrow{f^*} & S \\ \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle g} \\ S_{\mathbb{Q}} & \xrightarrow[\beta]{} & MU^*(B_G)_{\mathbb{Q}} & \xrightarrow[f^*]{} & S_{\mathbb{Q}} \end{array} \tag{4.7}$$

where $g : S \to S_{\mathbb{Q}}$ is the natural change of coefficients map.

Let us fix an element $x \in S^W$. Since $g\left(S^W\right) \subseteq (S_{\mathbb{Q}})^W$, it follows from our result over rationals that

$$g\left(f^*\beta(x)\right) = f^*\beta\left(g(x)\right) = g(x).$$

That is, $g\left(x - f^*\beta(x)\right) = 0$. Since S is torsion-free, we must have $x = f^*\beta(x)$ on the top row of (4.7). Since x is an arbitrary element of S^W, we conclude that $S^W \subseteq \mathrm{Image}(f^*)$ over R. $\square$

Combining Lemma 4.3 and Proposition 4.4, we immediately get:

Corollary 4.5. *Let X be a smooth scheme with an action of G such that $H^*_T(X, R)$ is torsion-free. Then, the map*

$$\Psi^{\mathrm{top}}_X : S \otimes_{S^W} MU^*_G(X) \xrightarrow{\simeq} MU^*_T(X)$$

is an isomorphism of R-algebras. In particular, $MU^*_T(G/B)$ is isomorphic to $S \otimes_{S^W} S$.

This extends to cobordism a well-known result for cohomology (see, e.g., [4, Prop. 1(iii)] or [16, Prop. 2.2(ii)]).

5. Equivariant algebraic cobordism of G/B

Let k be any field of characteristic zero. Let G be a connected and reductive group over k. We assume that G has a split maximal torus T contained in a Borel subgroup B. In this section, we prove our main result on the Borel presentation of $\Omega_T^*(G/B)$. We demonstrate how this presentation can be used to define Demazure operators and establish Schubert calculus in $\Omega_T^*(G/B)$.

5.1. Borel presentation of $\Omega_T^*(G/B)$

Using the natural restriction map $r_T^G : \Omega_G^*(G/B) \to \Omega_T^*(G/B)$ ([22, Subsect. 4.1]) and the isomorphisms ([22, Props. 5.4, 8.1])

$$S \cong \Omega_T^*(k) \cong \Omega_B^*(k) \cong \Omega_G^*(G/B),$$

we get the *characteristic* ring homomorphism $\mathbf{c}_{G/B}^{\mathrm{eq}} : S \to \Omega_T^*(G/B)$. On the other hand, the structure map $G/B \to \mathrm{Spec}(k)$ gives another $\mathbb{L}$-algebra map $S \to \Omega_T^*(G/B)$.

Theorem 5.1. *The natural map of S-algebras*

$$\Psi_{G/B}^{\mathrm{alg}} : S \otimes_{S^W} S \to \Omega_T^*(G/B),$$
$$\Psi_{G/B}^{\mathrm{alg}}(a \otimes b) = a \cdot \mathbf{c}_{G/B}^{\mathrm{eq}}(b)$$

is an isomorphism over R.

Proof. Since G contains a split maximal torus, it is a split reductive group over k and hence it is uniquely described by a root system. In particular, there is a split reductive group $G_{\mathbb{Q}}$ over the prime field $\mathbb{Q} \overset{\sigma}{\hookrightarrow} k$ such that $G \cong G_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$. It follows from [23, Thm. 4.7] that $\sigma^* : \Omega_{T_{\mathbb{Q}}}^*(G_{\mathbb{Q}}/B_{\mathbb{Q}}) \xrightarrow{\cong} \Omega_T^*(G/B)$. Since $\Omega_{T_{\mathbb{Q}}}^*(\mathbb{Q}) \xrightarrow{\cong} \Omega_T^*(k)$, it is enough to prove the theorem when $k = \mathbb{Q}$. By using the same argument for the inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}$, we are reduced to proving the theorem when the base field is $\mathbb{C}$.

In this case, we observe that $\mathbf{c}_{G/B}^{\mathrm{eq}}$ is simply the change of group homomorphism, and hence it is the algebraic analogue of the restriction map $MU_G^*(G/B) \xrightarrow{\pi_X^*} MU_T^*(G/B)$ in (4.2). Furthermore, the map $S \to \Omega_T^*(G/B)$ is the algebraic analogue of the map $p_{T,G/B}^*$ in (4.2). Using Corollary 4.5, we get a diagram

$$\begin{array}{ccc} S \otimes_{S^W} S & \xrightarrow{\Psi_{G/B}^{\mathrm{alg}}} & \Omega_T^*(G/B) \\ & \searrow^{\Psi_{G/B}^{\mathrm{top}}} & \downarrow \Phi_{G/B}^{\mathrm{top}} \\ & & MU_T^*(G/B) \end{array} \qquad (5.1)$$

which commutes by the above comparison of the various algebraic and topological maps. The right vertical map is an isomorphism by Corollary 3.8 and the diagonal map is an isomorphism by Corollary 4.5. We conclude that $\Psi_{G/B}^{\mathrm{alg}}$ is an isomorphism too. □

The following consequence of Theorem 5.1 improves [23, Thm. 8.1], which was proven with the rational coefficients. It also improves the computation of the non-equivariant cobordism ring of G/B in [8, Thm. 13.12] (see also [8, Remark 2.21]), where a presentation of $\Omega^*(G/B)$ was obtained in terms of the completion of S with respect to its augmentation ideal.

Corollary 5.2. *There is an R-algebra isomorphism*

$$S \otimes_{S^W} \mathbb{L} \xrightarrow{\cong} \Omega^*(G/B).$$

Proof. This follows immediately from Theorem 5.1 and [23, Thm. 3.4]. □

The following result extends Proposition 4.4 to the algebraic setting and gives the isomorphism $S(G) \simeq MU^*(B_G)$ over R for all reductive groups. The isomorphism $S(G) \simeq MU^*(B_G)$ over the integers was earlier proven in [10] for classical groups.

Proposition 5.3. *The natural map $S(G) \to S^W$ is an isomorphism of R-algebras.*

Proof. We follow the proof of an analogous result for the Chow groups in [39, Thm. 1.3]. For every good pair (V_j, U_j) approximating $B_T = B_B$, consider the fiber bundle

$$G/B \xrightarrow{i} U_j/B \xrightarrow{\pi} U_j/G.$$

By Lemma 4.2, there exists an element $a \in \Omega^d(B_T)$ such that $i^*(a)$ is the class of a point in $\Omega^d(G/B)$. We now show that the push-forward map $\pi_* : \Omega^*(B_T) \to \Omega^{*-d}(B_G)$ sends a to an invertible element of $\Omega^0(B_G)$ (for Chow rings, this is straightforward since $\pi^*(a) = 1$). The projection formula yields

$$\pi_*(a\pi^*([\mathrm{pt}])) = \pi_*(a)[\mathrm{pt}]$$

(here [pt] is the class of a point in $\Omega^*(B_G)$). Since $\pi^*([\mathrm{pt}]) = i_*(1)$, we have that $a\pi^*([\mathrm{pt}]) = i_*(i^*(a))$ is the class of a point in $\Omega^*(B_T)$. Hence, $[\mathrm{pt}] = \pi_*(a)[\mathrm{pt}]$, that is, $\pi_*(a)$ is equal to 1 modulo $F^1\Omega^*(B_G)$ by [29, Thm. 1.2.19, Remark 4.5.6]. Put $b = 1 - \pi_*(a)$. Since $b \in F^1\Omega^*(B_G)$, the power series $1 + b + b^2 + \cdots$ gives a well-defined element of $S(G)$. Indeed, for every variety U_j/G, this series terminates (since b is nilpotent in $\Omega^*(U_j/G)$). It follows that $\pi_*(a)$ is invertible.

The rest of the proof is completely analogous to the proof [39, Thm. 1.3]. We provide the details below for the reader's convenience.

We now show the injectivity of π^*. The projection formula implies

$$\pi_*(\pi^*(x)a) = x\pi_*(a)$$

for all $x \in MU^*(B_T)$. In particular, $\pi^*(x) = 0$ implies $x = 0$, since $\pi_*(a)$ is invertible.

To prove the surjectivity of π^*, we use that the analogue of our proposition holds over the rationals (cf. [22, Thm. 8.6]). Next, observe that for any $x \in MU^*(B_G)$ and $y = \pi^*(x)$, we have $\pi^*(\pi_*(ay)) = \pi^*(\pi_*(a)x) = \pi^*(\pi_*(a))y$. Since $MU^*(B_T)$ is torsion-free, it follows that

$$\pi^*(\pi_*(ay)) = \pi^*(\pi_*(a))y$$

for all $y \in MU^*(B_T)^W$, that is, y is the image of $\pi_*(ay)\pi_*(a)^{-1}$ under π^*. □

5.2. Divided difference operators

Various definitions of *generalized divided difference* (or *Demazure*) *operators* were given in [3] for complex cobordism and in [18], [8] for algebraic cobordism in order to establish Schubert calculus in $MU^*(G/B)$ and $\Omega^*(G/B)$. Corollary 5.2 allows us to compare these definitions. We also outline Schubert calculus in equivariant cobordism using Theorem 5.1.

Let $x_\chi \in S$ denote the first T-equivariant Chern class $c_1^T(L_\chi)$ of the T-equivariant line bundle L_χ on Spec(k) associated with the character χ of T. Recall that the isomorphism $S \simeq \Omega_T^*(k)$ sends χ to x_χ. The Weyl group W_G acts on S: an element $w \in W_G$ sends x_χ to $x_{w\chi}$. For each simple root α, define an $\mathbb{L}$-linear operator ∂_α on the ring S:

$$\partial_\alpha : f \mapsto (1 + s_\alpha)\frac{f}{x_{-\alpha}},$$

where $s_\alpha \in W$ is the reflection corresponding to the root α. One can show that ∂_α is indeed well-defined using arguments of [18, Sect. 5] (in [18] the ring of all power series is considered, but it is easy to check that $\partial_\alpha(f)$ is homogeneous if f is homogeneous). It is also easy to check that ∂_α is S^W-linear. In particular, ∂_α descends to $S \otimes_{S^W} \mathbb{L}$.

The comparison result below follows directly from definitions and Corollary 5.2.

(1) Under the isomorphism $MU^*(B_T) \simeq S$, the operator C_α considered in [3, Prop. 3] coincides with the operator ∂_α.
(2) Under the isomorphism of $S \otimes_{S^W} \mathbb{L} \simeq \Omega^*(G/B)$, the operator ∂_α descends to the operator A_α defined in [18, Sect. 3].
(3) The operator ∂_α coincides with the restriction of the operator C_α given in [8, Def. 3.11] from the ring of all power series to S.

Note that most of the operators considered above also have geometric meaning (see [3], [18], [8] for details). In particular, they were used to compute the *Bott–Samelson classes* in cobordism.

We now define an *equivariant generalized Demazure operator* ∂_α^T on $S \otimes_{S^W} S$:

$$\partial_\alpha^T : f \otimes g \mapsto f \otimes \partial_\alpha(g).$$

It is well-defined since ∂_α is S^W-linear. It follows immediately from Theorem 5.1 that ∂_α^T defines an S-linear operator on $\Omega_T^*(G/B)$. Similarly to the ordinary cobordism, these operators can be used to compute the *equivariant Bott–Samelson classes*. We outline the main steps but omit those details that are the same as for the ordinary cobordism. We use notation and definitions of [18].

Recall that to each sequence $I = \{\alpha_1, \dots, \alpha_l\}$ of simple roots of G, there corresponds a smooth *Bott–Samelson variety* R_I endowed with an action of B such that there is a B-equivariant map $R_I \to G/B$. In particular, each R_I gives rise to the cobordism class $Z_I = [R_I \to G/B]$ as well as to the T-equivariant cobordism class $[Z_I]^T$. The latter can be expressed as follows.

Theorem 5.4.

$$[Z_I]^T = \partial_{\alpha_l}^T \dots \partial_{\alpha_1}^T \left([\mathrm{pt}]^T\right).$$

The key ingredient is the following geometric interpretation of ∂_α^T. Denote by P_α the minimal parabolic subgroup corresponding to the root α.

Lemma 5.5. *The operator* ∂^T_α *is the composition of the change of group homomorphism* $r^{P_\alpha}_T : \Omega^*_{P_\alpha}(G/B) \to \Omega^*_T(G/B)$ *and the push-forward map* $r^T_{P_\alpha} : \Omega^*_T(G/B) \to \Omega^*_{P_\alpha}(G/B)$*:*

$$\partial_\alpha = r^{P_\alpha}_T r^T_{P_\alpha}.$$

Note here that $r^T_{P_\alpha}$ is defined by taking the limit over the push-forward maps on the non-equivariant cobordism groups corresponding to the projective morphism $G/B \overset{B}{\times} U_j \to G/B \overset{P_\alpha}{\times} U_j$ (for a sequence of good pairs $\{(V_j, U_j)\}$ for the action of P_α). Similarly to [18, Cor. 2.3], this lemma follows from the Vishik–Quillen formula [18, Prop. 2.1] applied to $\mathbb{P}^1$-bundles $G/B \overset{B}{\times} U_j \to G/B \overset{P_\alpha}{\times} U_j$. Theorem 5.4 then can be deduced from Lemma 5.5 by the same arguments as in [18, Thm. 3.2].

6. Cobordism ring of wonderful symmetric varieties

The wonderful (or more generally, regular) compactifications of symmetric varieties form a large class of spherical varieties. In fact, much of the study of a very large class of spherical varieties can be reduced to the case of symmetric varieties (cf. [35]). In this section, we compute the rational equivariant cobordism ring of wonderful symmetric varieties of minimal rank (see Theorem 6.4). A presentation for the equivariant cohomology of the wonderful group compactification analogous to Theorem 6.4 below was obtained by Littelmann and Procesi in [30], and the corresponding result for the equivariant Chow ring was obtained by Brion in [6, Thm. 3.1]. This result of Brion was later generalized to the case of wonderful symmetric varieties of minimal rank by Brion and Joshua in [7, Thm. 2.2.1].

Our proof of Theorem 6.4 follows the strategy of [7]. The two new ingredients in our case are the localization theorem for torus action in cobordism (cf. [23, Thm. 7.8]), and a divisibility result (Lemma 6.3) in the ring $S = \Omega^*_T(k)$.

6.1. Symmetric varieties

We now define symmetric varieties and describe their basic structural properties following [7]. Let k be a field of characteristic zero and let G be a connected and split reductive group over k. We assume throughout this section that G is of adjoint type. Let θ be an involutive automorphism of G and let $K \subset G$ be the subgroup of fixed points G^θ. The homogeneous space G/K is called a *symmetric space.*

Let P be a minimal θ*-split* parabolic subgroup of G (a parabolic subgroup P is θ*-split* if $\theta(P)$ is opposite to P), and $L = P \cap \theta(P)$ a θ-stable Levi subgroup of P. Then every maximal torus of L is also θ-stable. We assume that T is such a torus, so that $T = T^\theta T^{-\theta}$ and the identity component $A = T^{-\theta,0}$ is a maximal θ*-split* subtorus of G (a torus is θ*-split* if θ acts on it via the inverse map $g \mapsto g^{-1}$). The rank of such a torus A is called the *rank of the symmetric space* G/K. Since $T^\theta \cap T^{-\theta}$ is finite, we get

$$\mathrm{rk}(G) \le \mathrm{rk}(K) + \mathrm{rk}(G/K). \tag{6.1}$$

One says that the symmetric space G/K is of *minimal rank* if equality occurs in (6.1). This is equivalent to saying that $T^{\theta,0}$ is a maximal torus of K^0 and $T^{-\theta,0}$

is a maximal θ-split torus (here K^0 denotes the identity component of K). Note that K^0 is reductive. Set $T_K = (T \cap K)^0$. For symmetric spaces of minimal rank, the roots of (K^0, T_K) are exactly the restrictions to T_K of the roots of (G, T) ([7, Lemma 1.4.1]). Moreover, the Weyl group of (K^0, T_K) is identified with W^θ.

Let Σ^+ denote the set of positive roots of G with respect to a Borel subgroup B containing T. Let $\Delta_G = \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots which form a basis of the root system, and $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ the set of associated reflections. Since G is adjoint, Δ_G is also a basis of the character group $\widehat{T}$. Recall that $W = W_G$ denotes the Weyl group of G. Let $\Sigma_L \subset \Sigma$ be the set of roots of L, and $\Delta_L \subset \Delta_G$ the subset of simple roots of L.

If G/K is of minimal rank then the image of the restriction map $p : \widehat{T} \to \widehat{A}$ is a reduced root system (denoted by $\Sigma_{G/K}$), and $\Delta_{G/K} := p(\Delta_G \setminus \Delta_L)$ is a basis of $\Sigma_{G/K}$ ([7, Lemma 1.4.3]). This set is also identified with $\{\alpha - \theta(\alpha) | \alpha \in \Delta_G \setminus \Delta_L\}$ under the projection p. Moreover, in the exact sequence

$$1 \to W_L \to W^\theta \xrightarrow{p} W_{G/K} \to 1, \tag{6.2}$$

a representative of the reflection of $W_{G/K}$ associated to the root $\alpha - \theta(\alpha) \in \Delta_{G/K}$ is $s_\alpha s_{\theta(\alpha)}$.

Definition 6.1. A smooth projective G-variety X over k will be called a *wonderful symmetric variety*, if there is a symmetric space G/K such that the following hold.

(1) There is a dense open orbit of G in X isomorphic to G/K.
(2) The complement to this open orbit is the union of $r = \mathrm{rk}(G/K)$ smooth prime divisors $\{X_1, \cdots, X_r\}$ with strict normal crossings.
(3) The G-orbit closures in X are precisely the various intersections of the above prime divisors. In particular, all G-orbit closures are smooth.

A wonderful symmetric variety X as above is said to be of minimal rank, if so is the dense open orbit G/K. It is known from the work of De Concini–Procesi [9] that every symmetric space G/K has a unique G-equivariant compactification (called *wonderful compactification*) that is a wonderful symmetric variety.

Possibly the simplest example of symmetric varieties of minimal rank is when $G = \mathbf{G} \times \mathbf{G}$ where $\mathbf{G}$ is a semisimple group of adjoint type, and θ interchanges the factors. In this case, we have $K = \mathrm{diag}(\mathbf{G})$ and $G/K \cong \mathbf{G}$, where G acts by left and right multiplications. Furthermore, $T = \mathbf{T} \times \mathbf{T}$ where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$. Thus, $T_K = \mathrm{diag}(\mathbf{T})$, $A = \{(x, x^{-1}) | x \in \mathbf{T}\}$, $L = T$, and $W_K = W_{G/K} = \mathrm{diag}(W_{\mathbf{G}}) \subset W_{\mathbf{G}} \times W_{\mathbf{G}} = W$. In this case, the variety X is called the *wonderful group compactification*. For instance, complete collineations arise this way for $\mathbf{G} = \mathrm{PGL}_n$. We refer to [7, Example 1.4.4] for an exhaustive list of symmetric spaces of minimal rank. A well-known example of a wonderful symmetric variety that is not of minimal rank is the space of complete conics.

Let X be a wonderful symmetric variety of minimal rank with a dense open orbit G/K. Let $Y \subset X$ denote the closure of T/T_K in X. It is known that Y is smooth and is the toric variety associated to the fan of the Weyl chambers and their faces of the root datum $(G/K, \Sigma_{G/K})$. Recall here that if $M(G)$ is the root lattice of $(G/K, \Sigma_{G/K})$ with the dual lattice $N(G)$, then every set S of simple roots determines a cone $\sigma_S = \{v \in N(G)_{\mathbb{Q}} | \langle u, v\rangle \geq 0 \; \forall u \in S\}$. This cone is called the Weyl chamber corresponding to S.

Let z denote the unique T-fixed point of the affine T-stable open subset Y_0 of Y given by the positive Weyl chamber of $\Sigma_{G/K}$. It is well known that X has an isolated set of fixed points for the T-action. Moreover, it is also known by [37, §10] that X contains only finitely many T-stable curves. We shall need the following description of the fixed points and T-stable curves.

Lemma 6.2 ([7, Lemma 2.1.1])**.**

(i) *The T-stable points in X (resp. Y) are exactly the points $w \cdot z$, where $w \in W$ (resp. W_K) and these fixed points are parameterized by W/W_L (resp. $W_{G/K}$).*
(ii) *For any $\alpha \in \Sigma^+ \setminus \Sigma^+_L$, there exists a unique irreducible T-stable curve $C_{z,\alpha}$ which contains z and on which T acts through the character α. The T-fixed points in $C_{z,\alpha}$ are z and $s_\alpha \cdot z$.*
(iii) *For any $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/K}$, there exists a unique irreducible T-stable curve $C_{z,\gamma}$ which contains z and on which T acts through its character γ. The T-fixed points in $C_{z,\gamma}$ are exactly z and $s_\alpha s_{\theta(\alpha)} \cdot z$.*
(iv) *The irreducible T-stable curves in X are the W-translates of the curves $C_{z,\alpha}$ and $C_{z,\gamma}$. They are all isomorphic to $\mathbb{P}^1$.*
(v) *The irreducible T-stable curves in Y are the $W_{G/K}$-translates of the curves $C_{z,\gamma}$.*

6.2. Cobordism ring of wonderful symmetric varieties

To prove our main result, we will also need the following result on divisibility in the ring $S = S(T)$. We use notations of Subsection 5.2.

Lemma 6.3. *For any $f \in S$ and any root α, we have*

$$f \equiv s_\alpha(f) \pmod{x_\alpha}. \tag{6.3}$$

Proof. It is enough to check this lemma for all monomials in x_χ for $\chi \in \widehat{T}$.

First, we check the case $f = x_\chi$. We have $s_\alpha\chi = \chi - (\chi, \alpha)\alpha$, where (χ, α) is an integer. Put $m = -(\chi, \alpha)$. We can express $x_\chi - x_{s_\alpha\chi} = x_\chi - x_{\chi+m\alpha}$ as a formal power series $H(x, y) \in \mathbb{L}^{\mathrm{gr}}[[x, y]]$ in $x = x_\chi$ and $y = x_\alpha$ using the universal formal group law. Then $H(x, y)$ is homogeneous and divisible by y [29, (2.5.1)] so that the ratio $H(x, y)/y$ is a homogeneous power series. In particular, $x_\chi - s_\alpha(x_\chi)$ is divisible by x_α.

Next, note that if the lemma holds for f and g, then it also holds for fg, since $fg - s_\alpha(fg) = (f - s_\alpha(f))g + s_\alpha(f)(g - s_\alpha(g))$. In particular, the lemma holds for any monomial in x_χ as desired. □

Theorem 6.4. *Let X be a wonderful symmetric variety of minimal rank. Then the composite map*

$$s^G_T : \Omega^*_G(X) \to (\Omega^*_T(X))^W \to (\Omega^*_T(X))^{W_K} \to (\Omega^*_T(Y))^{W_K} \tag{6.4}$$

is a ring isomorphism with the rational coefficients.

Proof. All the arrows in (6.4) are canonical ring homomorphisms. The isomorphism of the first arrow follows from [22, Thm. 8.6]. We recall here that the proof of [22, Thm. 8.6] is based on a spectral sequence of Hopkins–Morel for the motivic cobordism. Although the result of Hopkins–Morel has not been published yet (however, see [19]), the rational version of their spectral sequence and its degeneration is known and is an immediate consequence of [32, Cor. 10.6(ii)].

Once we know the first isomorphism in (6.4), it suffices to show that the map $(\Omega^*_T(X))^W \to (\Omega^*_T(Y))^{W_K}$ is an isomorphism. We prove this by adapting the argument of [7, Thm. 2.2.1].

Since X has only finitely many T-fixed points and finitely many T-stable curves, it follows from [23, Thm. 7.8] and Lemma 6.2 that $\Omega^*_T(X)$ is isomorphic as an S-algebra to the space of tuples $(f_{w\cdot z})_{w\in W/W_L}$ of elements of S such that

$$f_{v\cdot z} \equiv f_{w\cdot z} \pmod{x_\chi}$$

whenever $v\cdot z$ and $w\cdot z$ lie in an irreducible T-stable curve on which T acts through its character χ. Under this isomorphism, the ring S is identified with the constant tuples (f).

We deduce from this that $(\Omega^*_T(X))^W$ is isomorphic, via the restriction to the T-fixed point z, to the subring of S^{W_L} consisting of those f such that

$$v^{-1}(f) \equiv w^{-1}(f) \pmod{x_\chi} \tag{6.5}$$

for all v, w and χ as above. Using Lemma 6.2, we conclude that $(\Omega^*_T(X))^W$ is isomorphic to the subring of S^{W_L} consisting of those f such that

$$f \equiv s_\alpha(f) \pmod{x_\alpha} \tag{6.6}$$

for $\alpha \in \Sigma^+ \setminus \Sigma^+_L$ and those f such that

$$f \equiv s_\alpha s_{\theta(\alpha)}(f) \pmod{x_\gamma} \tag{6.7}$$

for $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/K}$. However, it follows from Lemma 6.3 that (6.6) holds for all $f \in S$. We conclude from this that $(\Omega^*_T(X))^W$ is isomorphic to the subring of S^{W_L} consisting of those f such that (6.7) holds for $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/K}$.

Doing the similar calculation for Y and using Lemma 6.2 and [23, Thm. 7.8] again, we see that $(\Omega^*_T(Y))^{W_K}$ is isomorphic to the same subring of S. This completes the proof of the theorem. □

Remark 6.5. Since Y is a smooth toric variety, $\Omega^*_T(Y)$ can be explicitly calculated in terms of generators and relations using [25, Thm. 1.1]. Combining this with Theorem 6.4, one gets a simple way of computing the equivariant cobordism ring of wonderful symmetric varieties of minimal rank.

Example 6.6. If $G = \mathrm{PGL}_2 \times \mathrm{PGL}_2$, and θ interchanges both factors, then $G/K \simeq \mathrm{PGL}_2$ admits a unique wonderful compactification $X = \mathbb{P}^3$. Namely, $\mathbb{P}^3$ can be regarded as $\mathbb{P}(\mathrm{End}(k^2))$, where G acts by left and right multiplications.

The toric variety Y is $\mathbb{P}^1$ in this case. The torus $T \subset G$ is two-dimensional, and $S = \mathbb{L}^{\mathrm{gr}}[[x_\alpha, x_\beta]]$, where α and β are simple roots of G. Both $\Omega_T^*(X)$ and $\Omega_T^*(Y)$ can be computed explicitly:

$$\Omega_T^*(X) \simeq \mathbb{L}^{\mathrm{gr}}[[x, x_\alpha, x_\beta]]/((x - x_{\alpha+\beta})(x - x_{\alpha-\beta})(x - x_{-\alpha+\beta})(x - x_{-\alpha-\beta})),$$
$$\Omega_T^*(Y) \simeq \mathbb{L}^{\mathrm{gr}}[[x, x_\alpha, x_\beta]]/((x - x_{\alpha+\beta})(x - x_{-\alpha-\beta})).$$

The Weyl group $W \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ changes signs of α and β. In particular, the nontrivial element of the Weyl group $W_K = \mathrm{diag}(W)$ acts on $\Omega_T^*(Y)$ by $x \mapsto x$, $x_\alpha \mapsto x_{-\alpha}$, $x_\beta \to x_{-\beta}$. It is easy to check directly that $\Omega_T^*(X)^W \simeq \Omega_T^*(Y)^{W_K}$ after tensoring with $\mathbb{Q}$.

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