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MATHEMATICS

Asymptotics of Solutions of Emden–Fowler-Type Equations

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We consider an equation of the form

$$y^{(n)} = p(x)|y|^{\sigma}\operatorname{sgn} y, \quad n \geq 2, \quad \sigma > 1, \\ y = y(x), \quad p(x) \in C^0, \quad x, y \in R^1, \quad p(x) \neq 0. \tag{1}$$

For $n = 2$ and $p(x) = \pm x^{\beta}, x > 0, \beta = \text{const}$, this it the well-known Emden–Fowler equation (see, e.g., [1]), which is related to the study of certain physical processes.

Definition 1. A solution $y(x)$ of Eq. (1) is said to be right (left) continuable if it is defined in a neighborhood of $+\infty$ $(-\infty)$.

Definition 2. A nontrivial solution $y(x)$ of Eq. (1) is said to be right (left) oscillating if, for any x belonging to its domain, there exists an $x_1 > x$ $(x_1 < x)$ such that $y(x_1) = 0$.

We refer to nontrivial solutions which are not continuable (not oscillating) in some direction as noncontinuable (nonoscillating) in this direction.

Below we recall some notions used in power geometry [2, 3]. Suppose that, as $x \to +\infty$, a solution $y(x)$ of Eq. (1) has the form

$$y(x) = c_r x^r (1 + o(x^{-\gamma})), \quad r, c_r, \gamma = \text{const}, \\ c_r \neq 0, \quad \gamma > 0. \tag{2}$$

Then the expression

$$y = c_r x^r \tag{3}$$

is called the power asymptotics of solution (2).

If a solution of Eq. (1) as $x \to +\infty$ has the form

$$y(x) = c_r x^r \ln^q x(1 + o(\ln^{-\gamma} x)), \\ r, q, \gamma, c_r = \text{const}, \quad q, c_r \neq 0, \quad \gamma > 0, \tag{4}$$

then the expression

$$y = c_r x^r \ln^q x \tag{5}$$

is called the power-logarithmic asymptotics of solution (4).

This paper considers Eq. (1) in which $p(x)$ has a power asymptotics as $x \to +\infty$:

$$p(x) = a_{\beta} x^{\beta}(1 + o(x^{-\varepsilon})), \quad \beta, \varepsilon, a_{\beta} = \text{const}, \\ a_{\beta} \neq 0, \quad \varepsilon > 0. \tag{6}$$

Under this condition, we describe all power asymptotics of right continuable solutions of Eq. (1) and present solutions having power-logarithmic asymptotics. The results carry over in a natural way to left extendable solutions of Eq. (1). For $n = 2$ and $p(x) = ax^{\beta}$, $x > 0$, $a \neq 0$, this problem was studied in detail in [1].

We consider the following conditions on the number β in (6):

(i) Condition β_0:

$$-\beta < n; \tag{7}$$

(ii) Condition β_k, where $k \in \{1, 2, \ldots, n-1\}$:

$$n + (k-1)(\sigma - 1) < -\beta < n + k(\sigma - 1); \tag{8}$$

(iii) Condition β_n:

$$-\beta > n + (n-1)(\sigma - 1). \tag{9}$$

Theorem 1. *If, for some* $k \in \{0, 1, \ldots, n\}$, *the parameters* β *and* a_{β} *in* (6) *satisfy condition* β_k *and the inequality* $(-1)^{n-k} a_{\beta} > 0$, *then Eq.* (1) *has the following solutions with power asymptotics:*

$$y(x) = \pm c_r x^r (1 + o(x^{-\gamma})), \quad r = \frac{-\beta - n}{\sigma - 1}, \\ c_r = (r(r-1)\ldots(r-n+1)a_{\beta}^{-1})^{\frac{1}{\sigma-1}}, \quad \gamma = \text{const} > 0, \tag{10}$$

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it also has solutions of the form

$$y(x) = d_m x^m(1 + o(x^{-\gamma})), \quad d_m, \gamma = \text{const}, \quad d_m \neq 0, \quad \gamma > 0, \quad m \in \{0, 1, \ldots, k-1\}, \quad k \geq 1. \tag{11}$$

Under the conditions of the theorem, Eq. (1) *has no right continuable solutions with other power asymptotics.*

Remark. The existence of solutions (11) of Eq. (1) under the conditions of Theorems 1–4 presented here was proved in [4] (see also Theorem 2 in [5]).

Let us prove the existence of solutions of the form (10). We shall consider only a positive solution. Following [2], we make the change $y = x^r z$, $r = \dfrac{-\beta - n}{\sigma - 1}$, $t = \ln x$ in (1). The equation takes the form

$$\sum_{j=0}^{n} b_j z_t^{(n-j)} = a_\beta z^\sigma (1 + o(e^{-\varepsilon t})), \quad b_j = \text{const}, \quad 0 \leq j \leq n. \tag{12}$$

Consider the truncated equation

$$b_n z = a_\beta z^\sigma. \tag{13}$$

Note that $b_n = (-1)^n(-r)(-r+1)\ldots(-r+n-1)$; therefore, it follows from the assumption of the theorem that $b_n a_\beta > 0$. Hence, Eq. (13) has the solution

$$z = z_0 = (b_n a_\beta^{-1})^{\frac{1}{\sigma-1}}.$$

The change $z = w + z_0$ reduces Eq. (12) with small w to the form

$$\sum_{j=0}^{n} b_j w^{(n-j)} = a_\beta z_0^\sigma (h(\eta)(1 + o(e^{-\varepsilon t})) + o(e^{-\varepsilon t})), \quad \eta = \frac{w}{z_0}, \quad h(\eta) = (1+\eta)^\sigma - 1. \tag{14}$$

Setting $w^{(i)} = u_{i+1}$ for $0 \leq i \leq n-1$ and $u = (u_1, u_2, \ldots, u_n)^{\mathrm{T}}$, we obtain the following system of equations for the function $u(t)$ in a small neighborhood of zero:

$$\dot{u} = Au + F(u) + G(u)f(t) + f(t), \quad \|f(t)\| = o(e^{-\varepsilon t}), \quad F(u), G(u) \in C^\infty, \quad F(0) = 0, \quad F'(0) = 0, \quad G(0) = 0. \tag{15}$$

For sufficiently small $0 < \gamma < \varepsilon$, this system of equations has the solution $u = u(t), \|u(t)\| = o(e^{-\gamma t})$ as $t \to +\infty$ (see, e.g., [6]). It follows that $y = x^r(z_0 + u_1(\ln x)) = c_r x^r(1 + o(x^{-\gamma}))$ is a solution of Eq. (1) of the required form (10).

Now, let us prove the absence of solutions with different power asymptotics except those of the form (11). Suppose that there exists a right continuable solution which has the form

$$y(x) = ax^q(1 + o(x^{-\delta})), \quad a = \text{const} \neq 0, \quad \delta > 0, \tag{16}$$

where $q \neq r$, in a neighborhood of $+\infty$. Without loss of generality, we assume that $a > 0$.

First, suppose that $q > r$. Suppose also that $a_\beta > 0$ in (6). For large x, we obtain

$$y^{(n)}(x) > Dx^{\mu+q-n}, \quad D = \text{const} > 0, \quad \mu = (q-r)(\sigma-1) > 0. \tag{17}$$

Since all $y^{(i)}(x), 0 \leq i \leq n-1$, are monotone, it follows from (17) that

$$y(x) > D_1 x^{\mu+q}, \quad D_1 = \text{const} > 0,$$

which contradicts (16).

In the case where $a_\beta < 0$ in (6), the argument is the same with the only constraint $r \leq n-1$. Thus, the case $q > r$ is impossible.

Now, suppose that $q < r$. Then, for large x, we have

$$\left|p(x)y^{\sigma-1}(x)\right| \leq Dx^{\mu-n}, \quad D = \text{const} > 0, \quad \mu = (q-r)(\sigma-1) < 0. \tag{18}$$

Moreover, $y(x)$ satisfies the linear equation

$$y^{(n)} = p_1(x)y, \quad p_1(x) = p(x)y^{\sigma-1}(x). \tag{19}$$

However, according to [7], under condition (18), the functions

$$y_m(x) = x^m(1 + o(x^{-\delta})), \quad \delta > 0, \quad m \in \{0, 1, \ldots, n-1\},$$

form a fundamental system of solutions of Eq. (19) and, therefore, $y(x)$ is a linear combination of these functions: $y(x) = \sum_{m=0}^{n-1} d_m y_m(x)$. Taking into account (16), we see that $d_m = 0$ for $m > q$ and $y(x) = x^h(d_h + o(x^{-\delta_1}))$, where $h \in \{0, 1, \ldots, n-1\}$, $h \leq q < r$, and $\delta_1 > 0$. Thus, $y(x)$ has the form (11). It is easy to show that if a solution of Eq. (1) has the form (16), then, at $q = r$, we have $a = \pm c_r$, where c_r is the constant in (10).

Theorem 2. *If, for some* $k \in \{0, 1, \ldots, n-1\}$, *the parameters* β *and* a_β *in* (6) *satisfy the conditions*

$$(-1)^{n-k} a_\beta > 0, \quad -\beta = n + k(\sigma - 1), \tag{20}$$

then Eq. (1) *has the following solutions with power-logarithmic asymptotics:*

$$y(x) = \pm c_k x^k \ln^q x(1 + o(\ln^{-\gamma} x)), \quad q = \frac{1}{1-\sigma}, \quad c_k = \left(\frac{(\sigma-1)|a_\beta|}{k!(n-k-1)!}\right)^q, \quad \gamma > 0. \tag{21}$$

Equation (1) *has no right continuable solutions with power asymptotics except those of the form* (11).

The existence of solutions of Eq. (1) which have the form (11) was mentioned in the remark.

Let us prove the existence of solutions of the form (21).

We consider only positive solutions. The change $y = x^k z$, $t = \ln x$ in (1) reduces the equation to the form

$$\sum_{j=0}^{n-1} b_j z_t^{(n-j)} = a_\beta z^\sigma (1 + o(e^{-\varepsilon t})), \qquad b_j = \text{const}, \quad 0 \le j \le n-1. \tag{22}$$

To find the required solution of the obtained equation, it is expedient to consider the Newton polyhedron of the equation $\sum_{j=0}^{n-1} b_j z_t^{(n-j)} = a_\beta z^\sigma$. The right edge of this polyhedron corresponds to the truncated equation

$$b_{n-1} z' = a_\beta z^\sigma. \tag{23}$$

Note that $b_{n-1} = (-1)^{n+k-1} k!(n-k-1)!$; therefore, it follows from the assumption of the theorem that $b_{n-1} a_\beta < 0$. Hence, Eq. (23) has the solution

$$z = z_0(t) = ct^q, \quad c = \left(\frac{(1-\sigma)a_\beta}{b_{n-1}}\right)^q, \quad q = \frac{1}{1-\sigma}. \tag{24}$$

The change $z = w + z_0$, $z_0 = z_0(t)$, reduces Eq. (22) to the form

$$\sum_{j=0}^{n-1} b_j w^{(n-j)} = \frac{\sigma b_{n-1} w}{(1-\sigma)t} + a_\beta z_0^\sigma ((1+\eta)^\sigma - 1 - \sigma\eta) + a_\beta z_0^\sigma (1+\eta)^\sigma o(e^{-\varepsilon t}) + h(t), \qquad \eta = \frac{w}{z_0}, \quad h(t) = -\sum_{j=0}^{n-2} b_j z_0^{(n-j)}. \tag{25}$$

Next, it can be proved that this equation has a solution of the form $w(t) = o(t^{q-\gamma})$, where $\gamma > 0$ is a number. This means that Eq. (1) has a solution of the form (21). The absence of solutions with power asymptotics except those of the form (11) is proved in the same way as in the preceding theorem.

Theorem 3. *Suppose that, for some* $k \in \{0, 1, \ldots, n\}$, *the parameters* β *and* a_β *in* (6) *satisfy the inequality* $(-1)^{n-k} a_\beta < 0$ *and one of the following conditions:*

$$-\beta \le n, \quad \text{if} \quad k = 0, \tag{26}$$

$$n + (k-1)(\sigma - 1) < -\beta \le n + k(\sigma - 1), \quad \text{if} \quad k \in \{1, 2, \ldots, n-1\}, \tag{27}$$

$$-\beta > n + (n-1)(\sigma - 1), \quad \text{if} \quad k = n, \tag{28}$$

then Eq. (1) *has no nontrivial right continuable right nonoscillating solutions except those of the form* (11).

The existence of solutions of Eq. (1) which have the form (11) under the conditions of the theorem has already been mentioned. The absence of other nontrivial right continuable right nonoscillating solutions (independently of the presence of some asymptotics) under the conditions of the theorem follows from Theorem 4 stated below. To state Theorem 4, we define the following conditions on the function $p(x)$ in (1):

(i) Condition β_0^*:

$$|p(x)|^{-1} \le cx^n, \quad c = \text{const} > 0, \quad x \ge x_0 > 0; \tag{29}$$

(ii) Condition β_k^* with $k \in \{1, 2, \ldots, n-1\}$:

$$c_1 x^{n+(k-1)(\sigma-1)+\delta} \le |p(x)|^{-1} \le c_2 x^{n+k(\sigma-1)}, \qquad c_{1,2}, \delta = \text{const} > 0, \quad x \ge x_0 > 0; \tag{30}$$

(iii) Condition β_n^*:

$$|p(x)|^{-1} \ge cx^{n+(n-1)(\sigma-1)+\delta}, \qquad c, \delta = \text{const} > 0, \quad x \ge x_0 > 0. \tag{31}$$

Theorem 4. *If, for some* $k \in \{0, 1, \ldots, n\}$, *the function* $p(x)$ *in* (1) *satisfies one of the conditions* β_k^* *and the inequality* $(-1)^{n-k} p(x) < 0$, *then Eq.* (1) *has no nontrivial right continuable right nonoscillating solutions except those of the form* (11).

The existence of solutions of the form (11) under the conditions of this theorem was mentioned in the remark. Let us prove the absence of other right continuable sign-preserving solutions. Suppose that, on the contrary, such a solution $y(x)$ exists. Without loss of generality, we assume that $y(x) > 0$ for $x \ge x_0 > 0$.

It is easy to show that, for all $k \le j \le n-1$, we have $y^{(j)} \to 0$ as $x \to +\infty$.

Let us show that if $k \in \{1, 2, \ldots, n\}$, then

$$y(x) \le Dx^{k-1}, \quad D = \text{const} > 0, \quad x \ge x_0 > 0. \tag{32}$$

First, suppose that $k = n$. Then $p(x) < 0$, i.e., the function $y^{(n-1)}(x)$ decreases and, therefore, $y(x) \le Dx^{n-1}$, where $D = \text{const} > 0$ and $x \ge x_0 > 0$. If $1 \le k \le n-1$, then, since the functions $y^{(j)}(x)$, $k \le j \le n-1$, monotonically tend to zero as $x \to +\infty$, it follows that, at large x, we have $y^{(j+1)}(x) y^{(j)}(x) < 0$, $k \le j \le n-1$. But $\operatorname{sgn} y^{(n)}(x) = (-1)^{n-k-1}$ by the assumption of the theorem; therefore, $y^{(k)}(x) < 0$. Thus, the function $y^{(k-1)}(x)$ decreases and $y(x) \le Dx^{k-1}$ for $x \ge x_0 > 0$, where $D = \text{const} > 0$. This proves estimate (32).

Note that the function $y(x)$ satisfies the linear equation (19). It follows from (30), (31), and estimate (32) that, for $k \in \{1, 2, \ldots, n\}$, we have

$$|p_1(x)| \le D_1 x^{-n-\delta}, \quad x \ge x_0 > 0, \quad \delta, D_1 = \text{const} > 0. \tag{33}$$

But under condition (33), Eq. (19) has a fundamental system of solutions consisting of functions of the form

$$y_m(x) = x^m (1 + o(x^{-\gamma})), \quad \gamma > 0, \quad m \in \{0, 1, \ldots, n-1\}.$$

Therefore, $y(x)$ is a linear combination of such functions, and (32) implies that the solution $y(x)$ has the form (11).

It remains to consider the case $k = 0$. In this case, we have $\operatorname{sgn} y^{(n)}(x) = (-1)^{n-1}$; therefore, at large x, $\operatorname{sgn} y^{(j)}(x) = (-1)^{j-1}$. But then $y(x) < 0$, which contradicts the assumption that the solution under consideration is positive. The obtained contradiction proves the absence of nontrivial right continuable right nonoscillating solutions at $k = 0$.

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