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Local and Global Stability of Compact Leaves and Foliations

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The equivalence of the local stability of a compact foliation to the completeness and the quasi analyticity of its pseudogroup is proved. It is also proved that a compact foliation is locally stable if and only if it has the Ehresmann connection and the quasianalytic holonomy pseudogroup. Applications of these criterions are considered. In particular, the local stability of the complete foliations with transverse rigid geometric structures including the Cartan foliations is shown. Without assumption of the existence of an Ehresmann connection, the theorems on the stability of the compact leaves of conformal foliations are proved. Our results agree with the results of other authors.

Key words: foliation, compact foliation, Ehresmann connection for a foliation, holonomy pseudogroup, local stability of leaves.

Mathematics Subject Classification 2010: 57R30, 53D22.

Introduction

The notion of the stability of leaves of foliations was introduced by Ehresmann and Reeb, the founders of the theory of foliations.

Remind that a subset of the foliated manifold is called *saturated* if it can be represented as a union of some leaves of the foliation.

Definition 1. *A leaf L of a foliation $(M, \mathcal{F})$ of codimension q is said to be locally stable in the sense of Ehresmann and Reeb if there exists a family of its saturated neighbourhoods $\{W_k | k \in \mathbb{N}\}$ satisfying the following conditions:*

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1) *there exists a submersion* $f_1 : W_1 \to L$ *such that for every* $k \in \mathbb{N}$ *the triplet* (W_k, f_k, L)*, where* $f_k = f_1|_{W_k}$ *is a locally trivial fibration with a q-dimensional disk* D^q *as the standard fiber such that its fibers are transversal to the leaves of the foliation* $(W_k, \mathcal{F}|_{W_k})$*;*

2) *for any point* $x \in L$*, the set* $\{W_k \bigcap f_1^{-1}(x) \,|\, k \in \mathbb{N}\}$ *forms a base of the topology of the fiber* $f_1^{-1}(x)$ *at* x*.*

According to well-known Reeb's theorem [23]), any compact leaf of the foliation with finite holonomy group is locally stable.

Remind that a foliation is said to be *compact* if every its leaf is compact. For a compact foliation $(M, \mathcal{F})$, the local stability of a leaf L by Definition 1 is equivalent to the existence for L a basis of saturated neighborhoods of L in M.

G. Reeb [20] proved that the leaf space of every smooth compact foliation of codimension one is Hausdorff. In [11], D. Epstein proved that any leaf of a compact foliation $(M, \mathcal{F})$ has finite holonomy group iff the leaf space $M/\mathcal{F}$ is Hausdorff.

In [16], K. Millett made the following conjecture:

Every leaf of a compact foliation on a compact manifold has finite holonomy group.

Due to the Reeb theorem mentioned above, the Millett conjecture is called *the problem on the local stability* of compact foliations.

R. Edwards, K. Millett and D. Sullivan [8] and independently E. Vogt [24] proved that in the case of codimension $q = 2$ the Millett conjecture is valid. For the one-dimensional compact foliations on the closed 3-manifolds it was proved earlier by D. Epstein [10].

If the foliated manifold M is not compact, then the analog of the Millett conjecture is not true in general for the compact foliations $(M, \mathcal{F})$ of codimension $q = 2$. Now it is known that, generally speaking, for $q = 3$ the Millett conjecture is not valid. The first counterexample was constructed by Sullivan. He found a smooth unstable flow on a closed 5-manifold [22], in which each orbit is periodic. This example shows that some additional hypothesis of a global character on M is required in general.

The examples of the unstable compact foliations were also constructed by D. Epstein and E. Vogt [12], Thurston [22] and others.

After the Sullivan counterexample, there appeared a number of works containing criterions and sufficient conditions for the Millett conjecture to be true.

According to Epstein's assertion mentioned above, the validity of the Millett conjecture is equivalent to the property of the leaf space of the foliation to be Hausdorff. Decesaro and Nagano [7] stated that the Hausdorff separation property for the topology of the leaf space is equivalent to the boundedness of the volume of leaves function near every given leaf with respect to any Riemannian metric on M.

H. Rummer [21] proved that the local stability of a compact foliation $(M, \mathcal{F})$ on a compact manifold M is equivalent to the existence of a Riemannian metric on M such that, with this metric, each leaf is a minimal Riemannian submanifold.

A survey of the results on the Millett conjecture can be found in [8] and [14].

Remark that according to the papers [11], [30] and [17], a compact foliation $(M, \mathcal{F})$ is locally stable iff there exists a complete bundle-like metric on M with respect to $(M, \mathcal{F})$, that is equivalent to the existence of a natural structure of a smooth q-dimensional orbifold on the leaf space $M/\mathcal{F}$.

The compact foliations $(M, \mathcal{F})$ with a Hausdorff separation property for the leaf space $M/\mathcal{F}$ are referred to the *Hausdorff foliations* and are studied in [6]. As indicated above, the Hausdorff foliations are locally stable.

A. Gogolev [15] and P.D. Carrasco [5] studied the partially hyperbolic diffeomorphisms with compact center foliation, where the local stability of this foliation plays an important role.

Definition 2. *A pseudogroup $\mathcal{H}$ of local diffeomorphisms of a manifold N is said to be quasianalytic if the existence of an open connected subset V in N such that $h|_V = id_V$ for an element $h \in \mathcal{H}$ implies $h = id_{D(h)}$, where $D(h)$ is the connected domain of definition of h that contains V.*

For instance, holonomy pseudogroups of G-foliations are quasianalytic. Let us emphasize that the holonomy pseudogroup of every foliation with transverse rigid geometry in the sense of [36] is quasianalytic.

Definition 3. *A pseudogroup $\mathcal{H}$ of local diffeomorphisms of a manifold N is called complete if for every pair of the points x and x' on N there exist the open neighbourhoods U and U' such that:*

if $y \in U$ and $y' = \gamma(y) \in U'$ for some $\gamma \in \mathcal{H}$ there exists a prolongation $h \in \mathcal{H}$ of the local diffeomorphism γ to the entire neighbourhood U.

Here we prove the following new criterion of the stability of compact foliations.

Theorem 1. *A compact foliation $(M, \mathcal{F})$ of arbitrary codimension q is locally stable if and only if the holonomy pseudogroup of this foliation is complete and quasianalytic.*

In Subsection 1.4. we remind the definition of (G, X)-foliations.

Corollary 1. *Any compact (G, X)-foliation is locally stable.*

The notion of the Ehresmann connection for the foliations was introduced by Blumenthal and Hebda [1] as a natural generalization of the notion of the Ehresmann connection for the submersions (the definition is contained in Sec. 2).

In this work, we give a detailed proof and apply the following criterion of the local stability of compact foliations formulated by us (without proving) in [32].

Theorem 2. *For the compact foliation $(M, \mathcal{F})$ of arbitrary codimension q to be locally stable, it is necessary and sufficient that the following two conditions should hold:*

1) *the holonomy pseudogroup of the foliation $(M, \mathcal{F})$ is quasianalytic;*
2) *there exists an Ehresmann connection for $(M, \mathcal{F})$.*

Accentuate that Theorems 1 and 2 were proved by us without assumption of the compactness of foliated manifolds.

The effectiveness of the second criterion is confirmed by the following statement obtained as an application of Theorem 2.

Theorem 3. *Each compact foliation on an n-manifold of an arbitrary codimension q, $0 < q < n$, belonging to at least one of the following classes:*

1) *complete foliations with transverse rigid geometry in the sense of [36];*
2) *transversally holomorphic foliations with Ehresmann connections;*
3) *transversally real analytic foliations admitting an Ehresmann connection;*
4) *G-foliations with Ehresmann connection,*

is locally stable.

R e m a r k 1. The class of foliations with transverse rigid geometry introduced in [36] contains the Cartan foliations with effective transverse Cartan geometries as well as the foliations admitting transversely complete, transversely transitive foliated system of differential equations with the unique solution property in the sense of Wolak [26].

Observe that each holonomy group of a compact foliation $(M, \mathcal{F})$ can be linearized iff $(M, \mathcal{F})$ is a locally stable foliation. Therefore, applying Theorem 2, we obtain the following statement.

Corollary 2. *Let $(M, \mathcal{F})$ be a compact foliation with quasi analytic holonomy pseudogroup. Then each holonomy group can be linearized iff $(M, \mathcal{F})$ admits an Ehresmann connection.*

In particular, each holonomy group of every foliation satisfying Theorem 3 can be linearized.

The holonomy group $\Gamma(L, x)$ of a leaf L of the foliation $(M, \mathcal{F})$ at the point $x \in L$, usually used in the foliation theory [23], consists of the germs of local holonomy diffeomorphisms along the loops based at x of a transversal at the point $x \in L$. We will call it the *germinal* holonomy group to distinguish from the

$\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L,x)$ for the foliation $(M,\mathcal{F})$ with Ehresmann connection $\mathfrak{M}$ (see Sec. 2).

As an application of Theorem 2, we obtain the following theorem ([31], Theorem 7.4) on the global stability of a compact leaf with finite holonomy group (and also with finite fundamental group).

Theorem 4. *Let $(M,\mathcal{F})$ be a foliation with quasianalytic holonomy pseudogroup and Ehresmann connection. If there exists a compact leaf with finite germinal holonomy group (or finite fundamental group), then each leaf of this foliation is compact and has finite germinal holonomy group (or finite fundamental group, correspondingly), and $(M,\mathcal{F})$ is a locally stable foliation.*

Theorem 4 is an analog of the famous Reeb theorem [23] on the global stability of a compact leaf with finite fundamental group for the C^r foliations, $r \geq 2$, of codimension one on the compact manifolds.

Corollary 3. *Let $(M,\mathcal{F})$ be a foliation belonging to at least one of the classes* 1)–4) *in Theorem 3. If there exists a compact leaf with finite germinal holonomy group, then every leaf has finite germinal holonomy group, and $(M,\mathcal{F})$ is a locally stable compact foliation.*

In particular, if $(M,\mathcal{F})$ is a complete conformal foliation of codimension $q > 2$, then the main result of the Blumenthal paper [2] follows from Corollary 3. In the case when M is compact, the statement analogous to Corollary 3 was proved by Wolak for the complete G-foliations of finite type [27].

It is known (see, for example, [1]) that for a totally geodesic foliation $(M,\mathcal{F})$ of codimension q on a Riemannian manifold (M,g) with complete induced metric on leaves the orthogonal q-dimensional distribution $\mathfrak{M}$ is an Ehresmann connection. Therefore the following assertion follows from Theorem 4.

Corollary 4. *Let $(M,\mathcal{F})$ be a totally geodesic foliation on the Riemannian manifold (M,g). If the induced metric on the leaves is complete and the holonomy pseudogroup is quasianalytic, then the existence of a compact leaf with finite germinal holonomy group implies the compactness of every leaf and the finiteness of its germinal holonomy group, i.e., the compactness and local stability of this foliation.*

Let $(M,\mathcal{F})$ be an arbitrary smooth foliation, where $\mathcal{F} = \{L_\alpha \,|\, \alpha \in A\}$. Recall the construction of the graph of the foliation given by Winkelnkemper in [25]. Take any two points x and y from a leaf L_α. Denote by $A(x,y)$ the set of piecewise smooth paths in the leaf L_α connecting x with y. Two paths h and g from $A(x,y)$ are said to be equivalent $h \sim g$ if the loop, which is equal to the product $h \cdot g^{-1}$ of the paths h and g^{-1}, defines the trivial element of the germinal holonomy group $\Gamma(L_\alpha, x)$. The equivalent class containing the path h is denoted by $< h >$. The set $G(\mathcal{F})$ of triplets of the form $(x, < h >, y)$, where $x \in M$, $y \in L(x)$, $h \in A(x, y)$, is said to be *the graph of the foliation* $(M, \mathcal{F})$, and the maps

$$p_1 : G(\mathcal{F}) \to M : (x, < h >, y) \mapsto x, \; p_2 : G(\mathcal{F}) \to M : (x, < h >, y) \mapsto y$$

are called the *canonical projections*. One can define a structure of the smooth $(n + p)$-manifold on the graph $G(\mathcal{F})$, where n is a dimension of M and p is a dimension of the foliation $(M, \mathcal{F})$. The topological space of $G(\mathcal{F})$ is not Hausdorff in general. The family

$$\mathbb{F} = \{\mathbb{L} = p_1^{-1}(L_\alpha) \,|\, \alpha \in A\}$$

forms *the induced foliation* on the graph $G(\mathcal{F})$. Winkelnkemper [25] proved a criterion of the property of the graph $G(\mathcal{F})$ to be Hausdorff (see Sec. 4, Prop. 2).

Suppose that a foliation $(M, \mathcal{F})$ admits an Ehresmann connection $\mathfrak{M}$. Replacing the germinal holonomy group $\Gamma(L, x)$ of each leaf L with the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ in the definition of the graph of a foliation, we obtain the definition of *the graph* $G_{\mathfrak{M}}(\mathcal{F}) = \{(x, \{h\}, y)\}$ *of the foliation with Ehresmann connection.* We proved that $G_{\mathfrak{M}}(\mathcal{F})$ is equipped in a natural way with the structure of the smooth manifold [31] (see also [33, 34]). We showed that the topological space of $G_{\mathfrak{M}}(\mathcal{F})$ is always Hausdorff unlike the one of the graph $G(\mathcal{F})$. The map

$$\beta : G_{\mathfrak{M}}(\mathcal{F}) \to G(\mathcal{F}) : (x, \{h\}, y) \mapsto (x, < h >, y)$$

is a local diffeomorphism. Both graphs $G(\mathcal{F})$ and $G_{\mathfrak{M}}(\mathcal{F})$ are equipped with a groupoid structure, and β is a groupoid epimorphism.

A holonomy vanishing cycle for a foliation $(M, \mathcal{F})$ [29] is a mapping $c : S^1 \times [0, 1] \to M$ such that for any $t \in [0, 1]$ the loop $c_t = c|_{S^1 \times \{t\}}$ belongs to a leaf of the foliation, and for every $t > 0$ the loop c_t induces the trivial element of the corresponding germinal holonomy group unlike c_0, which induces a nontrivial element of the germinal holonomy group of the leaf containing it.

The following theorem sums up our results on the local stability of compact foliations along with the known results of other authors.

Theorem 5. *For any compact foliation* $(M, \mathcal{F})$ *of an arbitrary codimension* q *on an* n*-dimensional manifold* M*, where* $0 < q < n$*, the following conditions are equivalent:*

1) *the foliation* $(M, \mathcal{F})$ *is locally stable;*

2) *the holonomy pseudogroup of the foliation* $(M, \mathcal{F})$ *is complete and quasi-analytic;*

3) *the foliation* $(M, \mathcal{F})$ *has an Ehresmann connection and a quasianalytic holonomy pseudogroup;*

4) *the foliation* $(M, \mathcal{F})$ *has an Ehresmann connection, and its graph* $G(\mathcal{F})$ *is Hausdorff;*

5) *the foliation* $(M, \mathcal{F})$*, where* $\mathcal{F} = \{L_\alpha \,|\, \alpha \in A\}$*, admits an Ehresmann connection* $\mathfrak{M}$ *such that the holonomy groups* $H_{\mathfrak{M}}(L_\alpha)$ *and* $\Gamma(L_\alpha)$*,* $\forall \alpha \in A$*, are isomorphic in a natural way;*

6) *there exists an Ehresmann connection* $\mathfrak{M}$ *for* $(M, \mathcal{F})$ *such that the map defined above,* $\beta : G_{\mathfrak{M}}(\mathcal{F}) \to G(\mathcal{F})$*, is a groupoid isomorphism;*

7) *all the fibres of the canonical projections* $p_i : G(\mathcal{F}) \to M$*,* $i = 1, 2$*, are compact;*

8) *the induced foliation* $(G(\mathcal{F}), \mathbb{F})$ *on the graph* $G(\mathcal{F})$ *is compact;*

9) *the foliation* $(M, \mathcal{F})$ *is Riemannian;*

10) *the leaf space* $M/\mathcal{F}$ *is Hausdorff;*

11) *the leaf space* $M/\mathcal{F}$ *of the foliation* $(M, \mathcal{F})$ *carries a structure of a smooth orbifold such that the quotient map is a morphism in the category of orbifolds;*

12) *there exists a complete bundle-like metric with respect to the foliation* $(M, \mathcal{F})$*;*

13) *for every Riemannian metric* g *on* M *the function for the volume of the leaves is locally bounded;*

14) *on* M *there exists the Riemannian metric with respect to which every leaf is a minimal submanifold;*

15) *all the germinal holonomy groups of* $(M, \mathcal{F})$ *are finite;*

16) *the foliation* $(M, \mathcal{F})$ *has no holonomy vanishing cycles.*

Corollary 5. *A compact foliation* $(M, \mathcal{F})$ *on a compact manifold is locally stable iff its graph* $G(\mathcal{F})$ *is compact.*

Without assumption of the compactness of the foliation $(M, \mathcal{F})$, the equivalence of conditions 3) and 5), 6) was proved in [33] while the equivalence of 4) and 16) was proved in ([29], Theorem 1).

Let $(M, \mathcal{F})$ be a transversally complete G-foliation of finite type on a compact manifold M. In this case Theorems 3 and 5 imply Theorem 1 from [28], according to which the foliation $(M, \mathcal{F})$ is compact iff its orbit space is a smooth orbifold.

Basing on the results from [37], we obtain the following two theorems on the stability of leaves of conformal foliations without assumption of the existence of the Ehresmann connection.

Theorem 6. *Any compact conformal foliation of codimension* $q > 2$ *is locally stable.*

Theorem 7. *Let* $(M, \mathcal{F})$ *be a conformal foliation of codimension* $q > 2$ *on a compact manifold* M*. If there exists a compact leaf with finite germinal holonomy* *group, then every leaf of this foliation is compact with finite germinal holonomy group, i.e., $(M, \mathcal{F})$ is a locally stable foliation.*

R e m a r k 2. The analogous theorem for the holomorphic foliations on compact complex Kaehler manifolds was proved by Pereira in [19].

1. The Holonomy Pseudogroups of Foliations

1.1. The Definition of a Foliation by an N-cocycle

Suppose that there are given:

1) an n-dimensional manifold M and a possibly disconnected q-dimensional manifold N, where $0 < q < n$;

2) an open locally finite covering $\{U_i | i \in J\}$ of the manifold M;

3) submersions with connected fibres $f_i : U_i \to V_i$ on $V_i \subset N$;

4) diffeomorphism $\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ between the open subsets in the manifold N satisfying the equality $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$ for any $i, j \in J$ such that $U_i \cap U_j \neq \emptyset$.

Condition 4) implies the equality $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ for $U_i \cap U_j \cap U_k \neq \emptyset$ and $\gamma_{ii} = id|_{U_i}$.

The maximal (by inclusion) N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ satisfying conditions 1)–4) defines a new topology τ on M called the *leaf topology*, which has as a base the set of all fibres of the submersions f_i. The connected components of the topological space (M, τ) form a division of M denoted by $\mathcal{F} = \{L_\alpha | \alpha \in A\}$, and the pair $(M, \mathcal{F})$ is called a foliation *given by an N-cocycle* $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$ with leaves L_α, $\alpha \in A$. The manifold N is said to be a transversal manifold. As every N-cocycle belongs to the unique maximal N-cocycle, to define the foliation $(M, \mathcal{F})$ it is sufficient to take any N-cocycle satisfying conditions 1)–4).

1.2. Transverse Geometric Structures

Let $(M, \mathcal{F})$ be a foliation given by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j \in J}$. A holonomy invariant, i.e., invariant under all local diffeomorphisms γ_{ij}, $i, j \in J$, geometric structure on the manifold N is called the *transverse geometric structure* of this foliation.

A foliation admitting a G-structure as a transverse geometric structure, where G is a subgroup of the Lie group $GL(R, q)$, is said to be the *G-foliation.* If there exists a natural number k such that the kth prolongation of the G-structure on N is the e-structure, then it is said that $(M, \mathcal{F})$ is a *G-foliation of finite type.* The G-foliation is called the *$\nabla - G$-foliation* if the transverse manifold N admits a holonomy invariant G-connection [27]. These foliations were studied by Molino, who called them the foliations with transversally projectable G-connection.

A foliation admitting a Cartan geometry, as a transverse geometric structure, is called a *Cartan* foliation ([3, 35]).

In particular, a foliation $(M, \mathcal{F})$ of codimension q is said to be *Riemannian* if it is given by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$, and the manifold N admits a Riemannian metric g_N such that all $\{\gamma_{ij}\}$ are local isometries. By now Riemannian foliations have been most deeply studied [17].

1.3. Holonomy Pseudogroup of the Foliation

Remind the main notions ([4, 17], Appendix D).

Definition 4. *Let N be a smooth q-dimensional manifold which can be disconnected. A smooth pseudogroup of the local transformations $\mathcal{H}$ on N is a set of the diffeomorphisms $h : D(h) \to R(h)$ between the open subsets of N satisfying the following axioms:*

1. *Let $g, h \in \mathcal{H}$ and $R(h) \subset D(g)$, then $g \circ h \in \mathcal{H}$.*
2. *If $h \in \mathcal{H}$, then $h^{-1} \in \mathcal{H}$.*
3. *$id_N \in \mathcal{H}$.*
4. *Let $h \in \mathcal{H}$ and let $W \subset D(h)$ be an open subset, then $h|_W \in \mathcal{H}$.*
5. *If $h : D(h) \to R(h)$ is a diffeomorphism between the open subsets of N, and for any $w \in D(h)$ there is a neighbourhood W in $D(h)$ such that $h|_W \in \mathcal{H}$, then $h \in \mathcal{H}$.*

Definition 5. *Let A be a family of the local diffeomorphisms of N containing id_N. The pseudogroup obtained by adding h^{-1} for each h from A and restricting local diffeomorphisms on the open subsets, compositions and unions of the elements from A, is called the pseudogroup generated by A.*

Definition 6. *Let $(M, \mathcal{F})$ be a foliation given by an N-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}_{ij\in J}$. The pseudogroup generated by the local diffeomorphisms γ_{ij} of the manifold N is called the holonomy pseudogroup of this foliation and is denoted by $\mathcal{H} = \mathcal{H}(M, \mathcal{F})$.*

1.4. (G, X)-foliations

Let X be a connected manifold and G be some group of the diffeomorphisms of X. The group G acts on X quasianalytically if no element of G, except the identity, fixes a nonempty open set in X.

Definition 7. *Assume that the group G of diffeomorphisms of a connected manifold X acts on X quasianalytically. A foliation $(M, \mathcal{F})$ given by an X-cocycle $\xi = \{U_i, f_i, \{\gamma_{ij}\}\}_{i,j\in J}$ is called the (G, X)-foliation if for any $U_i \cap U_j \neq \emptyset$, $i, j \in J$, there exists an element $g \in G$ such that $\gamma_{ij} = g|_{f_j(U_i\cap U_j)}$.*

Remark that a holonomy pseudogroup of any (G, X)-foliation is complete and quasianalytic.

2. Ehresmann Connection for Foliations

2.1. A Vertical-Horizontal Homotopy

Remind the notion of the Ehresmann connection introduced by R.A. Blumenthal and J.J. Hebda [1]. We use the term *a vertical-horizontal homotopy* introduced previously by Hermann. All mappings are supposed to be piecewise smooth.

Let $(M, \mathcal{F})$ be a foliation of arbitrary codimension $q \geq 1$. A distribution $\mathfrak{M}$ on the manifold M is called *transversal* to the foliation $\mathcal{F}$ if for any $x \in M$ the equality $T_xM = T_x\mathcal{F} \oplus \mathfrak{M}_x$ holds, where $\oplus$ stands for the direct sum of the vector spaces. The vectors from $\mathfrak{M}_x$, $x \in M$, are called horizontal. A piecewise smooth curve σ is horizontal (or $\mathfrak{M}$-horizontal) if each of its smooth segments is an integral curve of the distribution $\mathfrak{M}$. The distribution $T\mathcal{F}$ tangent to the leaves of the foliation $(M, \mathcal{F})$ is called vertical. In other words, a curve h is vertical if h is contained in a leaf of the foliation $(M, \mathcal{F})$.

A *vertical-horizontal homotopy* (v.h.h.) is a piecewise smooth map $H : I_1 \times I_2 \to M$, where $I_1 = I_2 = [0, 1]$, such that for any $(s, t) \in I_1 \times I_2$ the curve $H|_{I_1 \times \{t\}}$ is horizontal and the curve $H|_{\{s\} \times I_2}$ is vertical. The pair of the curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called *the base of the v.h.h. H*. Two paths (σ, h) with the common origin $\sigma(0) = h(0)$, where σ is a horizontal path and h is a vertical one, are called an *admissible pair of paths.*

A distribution $\mathfrak{M}$ transversal to a foliation $(M, \mathcal{F})$ is called an *Ehresmann connection for* $(M, \mathcal{F})$ if for any admissible pair of paths (σ, h) there exists a v.h.h. with a base (σ, h).

Let $\mathfrak{M}$ be an Ehresmann connection for the foliation $(M, \mathcal{F})$. Then for any admissible pair of the paths (σ, h) there exists the unique v.h.h. H with the base (σ, h). We say that $\widetilde{\sigma} := H|_{I_1 \times \{1\}}$ is the result of the *translation of the path σ along h with respect to the Ehresmann connection* $\mathfrak{M}$. It is denoted by $\sigma \overset{h}{\longrightarrow\!\!>} \widetilde{\sigma}$.

In the similar way, we define the v.h.h. and translations for the cases when I_1 and I_2 are replaced with half-intervals.

Let $(M, \mathcal{F})$ be a Riemannian foliation of codimension q. Then there exists a bundle-like metric g_M on M relatively $(M, \mathcal{F})$. Denote by $\mathfrak{M}$ the q-dimensional distribution complementary (by orthogonality) to $T\mathcal{F}$ on the Riemannian manifold (M, g_M). It is known that the completeness of the Riemannian metric g_M guarantees that the distribution $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, \mathcal{F})$.

2.2. Holonomy Groups of Foliations with Ehresmann Connections

Let $(M, \mathcal{F})$ be a foliation with an Ehresmann connection $\mathfrak{M}$. Take any point $x \in M$. Denote by Ω_x the set of horizontal curves with the origin at x. An action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set Ω_x is defined in the following way:

$$\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \widetilde{\sigma},$$

where $[h] \in \pi_1(L, x)$, and $\widetilde{\sigma}$ is the result of the translation of $\sigma \in \Omega_x$ along h relatively $\mathfrak{M}$. Let $K_{\mathfrak{M}}(L, x)$ be the kernel of the action Φ_x, i.e.,

$$K_{\mathfrak{M}}(L, x) = \{\alpha \in \pi_1(L, x) \mid \alpha(\sigma) = \sigma, \forall \sigma \in \Omega_x\}.$$

The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/K_{\mathfrak{M}}(L, x)$ is the $\mathfrak{M}$-*holonomy group of the leaf* L [1]. Due to the linear connectedness of the leaves, the $\mathfrak{M}$-holonomy groups at different points of the same leaf are isomorphic.

Let $\Gamma(L, x)$ be the germinal holonomy group of the leaf L. Then there exists a unique group epimorphism $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the equality

$$\chi \circ \mu = \nu, \tag{1}$$

where $\mu : \pi_1(L, x) \to H_{\mathfrak{M}}(L, x)$ is the quotient map and $\nu([h]) := <h>$ is a germ of the holonomy diffeomorphism of a transverse q-dimensional disk along the loop h at the point x.

Let us emphasize that the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ has *a global character* in contrast to the germinal holonomy group $\Gamma(L, x)$ which has a local-global character: global along the leaves and local along the transverse directions.

Lemma 1. *Let* $(M, \mathcal{F})$ *be a foliation with an Ehresmann connection* $\mathfrak{M}$. *If there exists a leaf* L_0 *with the trivial* $\mathfrak{M}$*-holonomy group* $H_{\mathfrak{M}}(L_0, x_0)$, *then for any other leaf* L *there is a horizontal curve* σ *such that* $x_0 = \sigma(0) \in L_0$, $y_0 = \sigma(1) \in L$, *and there is defined a regular covering map* $f_\sigma : L_0 \to L$ *which takes a point* $x \in L_0$ *to the point* $y := \widetilde{\sigma}(1)$, *where* h *is a path in* L_0 *connecting* x_0 *with* x *and* $\sigma \xrightarrow{h}> \widetilde{\sigma}$. *Moreover, the group of the deck transformations of* $f_\sigma : L_0 \to L$ *is isomorphic to the* $\mathfrak{M}$*-holonomy group* $H_{\mathfrak{M}}(L, y_0)$ *of the leaf* L.

P r o o f. The existence of a horizontal curve σ connecting any two leaves of a foliation with an Ehresmann connection was shown in [1]. The triviality of the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L_0, x_0)$ of the leaf L_0 implies independence of the definition of $f_\sigma(x)$ from the choice of the path h connecting x_0 with x. Using the Ehresmann connection $\mathfrak{M}$ it is not difficult to check that $f_\sigma : L_0 \to L$ is the covering map.

Consider an element $g = \mu([h]) \in H_{\mathfrak{M}}(L, y_0)$, where $[h] \in \pi_1(L, y_0)$. Take an arbitrary point $z \in f_\sigma^{-1}(y_0)$ and put $g(z) := \hat{h}(1)$, where $\hat{h}$ is the path with the origin at $z = \hat{h}(0)$ covering the path h via $f_\sigma : L_0 \to L$. Thus, an action of the group $H_{\mathfrak{M}}(L, y_0)$ on the fibre $f_\sigma^{-1}(y_0)$ is defined. It is easy to check that $H_{\mathfrak{M}}(L, y_0)$ acts on $f_\sigma^{-1}(y_0)$ simply transitively and the group $H_{\mathfrak{M}}(L, y_0)$ is isomorphic to the group of the deck transformations of $f_\sigma : L_0 \to L$. ∎

3. Two Graphs of a Foliation with an Ehresmann Connection

3.1. Graph $G_{\mathfrak{M}}(F)$

The graph $G_{\mathfrak{M}}(F)$ of a foliation $(M, \mathcal{F})$ with an Ehresmann connection was introduced by the author in [31] (see also [32–34].

Let $(M, \mathcal{F})$ be a foliation of arbitrary dimension k on an n-manifold M, and $q = n - k$ be the codimension of this foliation. Suppose that the foliation $(M, \mathcal{F})$ admits an Ehresmann connection $\mathfrak{M}$.

Consider the set Ω_x of the $\mathfrak{M}$-horizontal curves with the origin at $x \in M$.

Take any points x and y in a leaf L of $(M, \mathcal{F})$. Introduce an equivalence relation ρ on the set $A(x, y)$ of the vertical paths in L connecting x with y. The paths h and f from $A(x, y)$ are called ρ-equivalent if the loop $h \cdot f^{-1}$ generates the trivial element of the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$. In other words, the paths h and f are ρ-equivalent iff they define the same translations of the $\mathfrak{M}$-horizontal curves from Ω_x relatively to the Ehresmann connection $\mathfrak{M}$. The ρ-equivalence class containing h is denoted by $\{h\}$.

The set of the ordered triplets $(x, \{h\}, y)$, where x and y are the points of an arbitrary leaf L of the foliation $(M, \mathcal{F})$, and $\{h\}$ is a class of the ρ-equivalent paths from x to y in L, is called *the graph of the foliation* $(M, \mathcal{F})$ *with the Ehresmann connection* $\mathfrak{M}$ and is denoted by $G_{\mathfrak{M}}(\mathcal{F})$.

A chart (U, φ) of the manifold M is said to be $\mathfrak{M}$*-fibred with the center at the point* x if:

1) $\varphi(U) = R^n$, $\varphi(x) = \{0\} \in R^n$, and φ maps each connected component of the intersection $U \cap L_\alpha$ of U with an arbitrary leaf L_α, which is called the local leaf in U, onto some leaf of the trivial foliation $F = \{R^k \times \{c\} \mid c \in R^q\}$ of the coordinate space R^n;

2) if L_x is the local leaf in U containing x, then for any $z \in L_x$ the submanifold $D_z := \varphi^{-1}(\{0\} \times R^q)$ is a q-dimensional transverse disk at the point z formed by the points of some smooth curves from Ω_z;

3) if h is an arbitrary path in the local leaf L_x, then for any admissible pair of the paths (σ, h) such that $\sigma(0) = h(0) = z$ and $\sigma(I_1) \subset D_z$, the v.h.h. H with the base (σ, h) is equal to $\varphi^{-1} \circ H_0$, where H_0 is the standard v.h.h. of the product $R^n = R^p \times R^q$ with the base $(\varphi \circ \sigma, \varphi \circ h)$.

It is not difficult to show that at any point x there exists an $\mathfrak{M}$-fibred chart.

Similarly to the proof of the theorem on the continuity of foliations from [18], it is easy to show that for any point $(a, \{h\}, b)$ of the graph $G_{\mathfrak{M}}(\mathcal{F})$ there exist $\mathfrak{M}$-fibred charts (U, φ) and (V, ψ) with the centers at a and b, respectively, and the transversal disks D_a and D_b having the following properties:

1) for any $c \in R^q$, the local leaves $\varphi^{-1}(R^p \times \{c\})$ and $\psi^{-1}(R^p \times \{c\})$ belong to the same leaf of the foliation $(M, \mathcal{F})$;

2) if $\sigma \in \Omega_a$, and $\sigma(I) \subset D_a$, with $\sigma \xrightarrow{h}> \widetilde{\sigma}$, then $\widetilde{\sigma} \in \Omega_b$ $\widetilde{\sigma}(I) \subset D_b$.

Define an open neighbourhood V_z of a point $z = (a, \{h\}, b)$ in $G_{\mathfrak{M}}(\mathcal{F})$ using the $\mathfrak{M}$-fibred charts (U, φ) and (V, ψ) indicated above. Let x be a point in U and L_x be a local leaf in U containing x. Then there exists an $\mathfrak{M}$-curve $\sigma : I \to D_a$ connecting a with $x_0 := L_x \cap D_a$. Let $h \overset{\sigma}{\longrightarrow}> \widetilde{h}$ and $y_0 := \widetilde{h}(1)$. Take any point y from the local leaf passing through y_0. Connect x with x_0 in L_x by a path t_x, and y with y_0 in L_y by a path t_y. Put $\hat{h} := t_x \cdot \widetilde{h} \cdot t_y^{-1}$. Consider the set $V_{z,h}$ of all points $\hat{z} := \{(x, \{\hat{h}\}, y)\}$ obtained. The contractibility of the local leaves implies that $\hat{z}$ is independent from the choice of the paths connecting t_x and t_y. If h' is another path from $\{h\}$, then the result of the translation of σ along h' coincides with the curve $\widetilde{\sigma}$ obtained by translating σ along h. The definition of V_z is given by $V_z := V_{z,h}$.

A coordinate map $\chi_z : V_z \to R^{n+p}$ is given by the equality

$$\chi_z(x, \{\hat{h}\}, y) = (\varphi(x), pr \circ \psi(y)),$$

where $pr : R^n \cong R^p \times R^q \to R^p$ is the canonical projection onto the first factor. The pair (V_z, χ_z) is a coordinate chart, and the family of the charts $\{(V_z, \chi_z) \mid z \in G_{\mathfrak{M}}(\mathcal{F})\}$ is an atlas of the manifold of dimension $2n - q$, which defines a smooth structure on $G_{\mathfrak{M}}(\mathcal{F})$.

The maps

$$p_1 : G_{\mathfrak{M}}(\mathcal{F}) \to M : (x, \{h\}, y) \mapsto x, \;\; p_2 : G_{\mathfrak{M}}(\mathcal{F}) \to M : (x, \{h\}, y) \mapsto y$$

are called the *canonical projections*. The graph $G_{\mathfrak{M}}(\mathcal{F})$, equipped with the binary operation $(y, \{h_1\}, z) * (x, \{h_2\}, y) := (x, \{h_1 \cdot h_2\}, z)$ and the canonical projections p_1 and p_1, becomes a smooth $\mathfrak{M}$-holonomy groupoid.

In [31] (see also [33] and [34]), the author proved the following properties of the graph $G_{\mathfrak{M}}(\mathcal{F})$, its canonical projections and the induced foliation:

$$\mathbb{F} := \{p_1^{-1}(L_\alpha) \mid L_\alpha \in (\mathcal{F})\}.$$

Proposition 1. 1. *The graph $G_{\mathfrak{M}}(\mathcal{F})$ of a foliation $(M, \mathcal{F})$ with an Ehresmann connection $\mathfrak{M}$ equipped with the smooth structure as indicated above becomes a Hausdorff manifold. The canonical projections p_1 and p_2 determine locally trivial fibrations with the common typical fibre Y.*

2. *For any point $x \in M$ there is defined a regular covering map $p_x : Y \to L(x)$, and the group of its deck transformations is isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$.*

3. *The diagonal action of the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ on the product $Y \times Y$ is free and properly discontinuous, so it defines a regular covering whose deck transformation group Ψ is isomorphic to $H_{\mathfrak{M}}(L, x)$. Moreover, the base of this covering $(Y \times Y)/\Psi$ is diffeomorphic to the leaf $\mathbb{L} := p_1^{-1}(L)$.*

4. *The map* $\Phi : G_{\mathfrak{M}}(\mathcal{F}) \to G(\mathcal{F}) : (x, \{h\}, y) \to (x, < h >, y)$ *is an epimorphism of groupoids.*

R e m a r k 3. For a foliation $(M, \mathcal{F})$ with an Ehresmann connection $\mathfrak{M}$ the existence of a manifold Y, which satisfies the statement 2 in Proposition 1, was proved [1] in another way.

3.2. Winkelnkemper's Criterion

Let $(M, \mathcal{F})$ be a smooth foliation and $G(\mathcal{F})$ be its graph (the definition of which was reminded in Introduction). As it is known, the topological space of $G(\mathcal{F})$ is not Hausdorff in general. We remark that the criterion of the property of the graph $G(\mathcal{F})$ to be Hausdorff proved by Winkelnkemper [25] can be reformulated as follows.

Proposition 2. *The topological space of the graph* $G(\mathcal{F})$ *of the foliation* $(M, \mathcal{F})$ *is Hausdorff iff the holonomy pseudogroup of this foliation is quasi analytic.*

4. Quasi Analyticity of Holonomy Pseudogroups

4.1. Criterion of Isomorphism between Holonomy Groups of Foliation with an Ehresmann Connection

Proposition 3. *Let* $(M, \mathcal{F})$ *be a foliation with an Ehresmann connection* $\mathfrak{M}$. *Then the group epimorphism* $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ *satisfying equality* (1) *is the group isomorphism if and only if the holonomy pseudogroup* $\mathcal{H}(M, \mathcal{F})$ *is quasianalytic.*

P r o o f. Use the notations from Sec. 2. As proved by Blumenthal and Hebda that if a foliation $(M, \mathcal{F})$ is formed by the fibres of the submersion $p : M \to B$, then $\mathfrak{M}$ is an Ehresmann connection for $(M, \mathcal{F})$ iff $\mathfrak{M}$ is an Ehresmann connection for the submersion $p : M \to B$. Let U_i and a submersion $f_i : U_i \to V_i$ belong to the N-cocycle defining the foliation $(M, \mathcal{F})$. Consider an admissible pair of the paths σ, h in U_i such that the translation of σ along h is realized in U_i, and $\sigma \xrightarrow{h}> \tilde{\sigma}$. Then $\tilde{\sigma}$ is the $\mathfrak{M}$-horizontal lift of the path $f_i \circ \sigma$ into the point $h(1)$ of relativity f_i.

Consider any $\sigma \in \Omega_x$ and $[h] \in \pi_1(L, x)$. Let $\sigma \xrightarrow{h}> \widetilde{\sigma}$. Then $\Phi_x([h], \sigma) = \widetilde{\sigma}$. Cover the loop $h(t)$, $t \in I_2$, by a finite chain of the fibred neighbourhoods $U_1, \ldots, U_k$ from the N-cocycle defining the foliation $(M, \mathcal{F})$. Let $f_i : U_i \to V_i$ be the corresponding submersions and $0 = t_0 < t_1 < \ldots < t_k = 1$ be the division of the segment I_1 such that $h([t_{i-1}, t_i]) \subset U_i$, $\forall i = 1, \ldots, k$. Let us follow to ([17], Appendix D) and consider V_i as a transversal q-dimensional disk embedded to U_i

and $h(t_i) \in V_i \subset U_i$. Then a composition $\gamma_{(i+1)i} \circ \gamma_{i(i-1)}$, $i = 1, \ldots, k-1$, is the local holonomic diffeomorphism along the path $h|_{[t_{i-1}, t_{i+1}]}$ of a neighbourhood of the point $h(t_{i-1})$ in V_{i-1} to the conforming neighbourhood of the point $h(t_{i+1})$ in V_{i+1}. A local diffeomorphism $\gamma := \gamma_{1k} \circ \gamma_{k(k-1)} \circ \ldots \circ \gamma_{32} \circ \gamma_{21}$ from the holonomy pseudogroup $\mathcal{H}(M, \mathfrak{F})$ is defined at some neighbourhood of the point $v = f_1(x)$ belonging to $V_1 \subset N$.

A set of the germs $\{\gamma\}_v$ at $v \in N$ of the local diffeomorphisms γ obtained as shown above, when $[h]$ runs over $\pi_1(L, x)$, is a group which can be interpreted as the germinal holonomy group $\Gamma(L, x)$ of the leaf L.

Assume that the holonomy pseudogroup $\mathcal{H}(M, \mathfrak{F})$ is quasi analytic. To prove that χ is a group isomorphism, it is sufficient to show that any element $[h] \in \pi_1(L, x)$ from the kernel $Ker(\nu)$ of the epimorphism $\nu : \pi_1(L, x) \to \Gamma(L, x)$ belongs to the kernel $K_{\mathfrak{M}}(L, x)$ of the epimorphism $\mu : \pi_1(L, x) \to H_{\mathfrak{M}}(L, x)$, i.e., each curve $\sigma \in \Omega_x$ is fixed by the action of $[h]$ via the map Φ_x. Let $h \overset{\sigma|_{[0,s]}}{\longrightarrow}> h_s$, $\forall s \in [0, 1]$, $x_s = h_s(0)$.

Suppose that $[h] \in Ker(\nu)$, which is equivalent to the triviality of the germ of the diffeomorphism γ at v, i.e., $\{\gamma\}_v = \{id_{V_1}\}_v$. Therefore there exists the number $\delta > 0$ such that $f_1 \circ \sigma|_{[0,\delta]}$ is a curve in the neighbourhood $V \subset V_1$ of v, where $\gamma|_V = id_V$. Thus, $\widetilde{\sigma}|_{[0,\delta]} = \sigma|_{[0,\delta]}$. Moreover, for any $s \in [0, \delta]$ the path h_s belongs to $Ker(\nu)$.

Consider the set

$$A := \{a \in I_1 \mid h_s \in Ker(\nu), \forall s \in [0, a]\}.$$

Then $[0, \delta] \subset A$, and hence A is a nonempty set. Applying the previous arguments to the path $[h_\delta]$ and repeating them, we can see that A is an open subset of I_1.

Let us show that A is a closed subset of I_1. In the opposite case, there exists the number $\varepsilon > 0$, $\varepsilon \notin A$, such that $[0, \varepsilon) \subset A$. Hence, $\sigma(s) = \widetilde{\sigma}(s)$ for any $s \in [0, \varepsilon)$. Due to the continuity of the paths σ and $\widetilde{\sigma}$, the equality $\sigma(\varepsilon) = \widetilde{\sigma}(\varepsilon)$ is valid, so h_ε is a loop at the point x_ε. There is a neighbourhood U_j containing x_ε from the N-cocycle defining the foliation $(M, \mathfrak{F})$. Let $f_j : U_j \to V_j$ be a submersion from this N-cocycle and $v_\varepsilon = f_j(x_\varepsilon) \in V_j$. By covering the curve $h_\varepsilon(t)$, $t \in I_2$, by a finite chain of fibred neighbourhoods from N-cocycle, in the same way as above, we get a local holonomic diffeomorphism γ_ε, $\gamma_\varepsilon(v_\varepsilon) = v_\varepsilon$ of some neighbourhood V_ε of v_ε belonging to the holonomy pseudogroup $\mathcal{H}(M, \mathfrak{F})$. The choice of ε implies the existence of an open subset $W \subset V_\varepsilon$ such that $\gamma|_W = id_W$, and $v_\varepsilon \in \overline{W}$, where $\overline{W}$ is the closure of W in N. Due to quasi analyticity of the pseudogroup $\mathcal{H}(M, \mathfrak{F})$, the equality $\gamma_\varepsilon = id_D$ is valid in the entire connected domain of $D = D(\gamma_\varepsilon)$ of γ_ε containing v_ε. As $V_\varepsilon \subset D$, so $\gamma_\varepsilon|_{V_\varepsilon} = id_{V_\varepsilon}$. Therefore $h_\varepsilon \in Ker(\nu)$, hence $\varepsilon \in A$. The contradiction with the assumption shows that A is a closed subset of I_1. As I_1 is connected, the nonempty open-closed subset A coincides with it, i.e., $A = I_1$.

Thus, $Ker(\nu) \subset K_{\mathfrak{M}}(L,x) = Ker(\mu)$, therefore equality (1) implies the triviality of the kernel of the epimorphism $\chi : H_{\mathfrak{M}}(L,x) \to \Gamma(L,x)$, i.e., χ is a group isomorphism.

Let us show the converse statement. Assume that $\chi : H_{\mathfrak{M}}(L,x) \to \Gamma(L,x)$ is a group isomorphism for any point $x \in M$. Then, according to Prop. 1, the map $\beta : G_{\mathfrak{M}}(\mathcal{F}) \to G(\mathcal{F})$ is a diffeomorphism. Therefore the property of the graph $G_{\mathfrak{M}}(\mathcal{F})$ to be Hausdorff implies the same property of $G(\mathcal{F})$. Hence, by Winkelnkemper's criterion (Prop. 2), the holonomy pseudogroup of the foliation $(M,\mathcal{F})$ must be quasi analytic. ∎

4.2. Proof of Theorem 1

Let $(M,\mathcal{F})$ be any locally stable compact foliation. As well known, on M there is a complete bundle-like metric with respect to $(M,\mathcal{F})$. Therefore, according to ([17], Appendix D, Prop. 2.6), the holonomy pseudogroup of the Riemannian foliation is complete. It is also quasi analytic because its elements are local isometries.

On the other hand, suppose that the holonomy pseudogroup $\mathcal{H} = \mathcal{H}(M,\mathcal{F})$ of a compact foliation $(M,\mathcal{F})$ is complete and quasi analytic. Assume that there exists a leaf $L = L(x)$ with infinite holonomy group. Then there is a submersion $f_i : U_i \to V_i$ from the N-cocycle defining $(M,\mathcal{F})$ such that $x \in U_i$, and the group of germs of local diffeomorphisms from the stationary pseudogroup $\mathcal{H}_w = \{h \in \mathcal{H} \,|\, h(w) = w\}$ at $w = f_i(x)$ is infinite. The completeness of the holonomy pseudogroup $\mathcal{H}$ implies the existence of an open neighbourhood U of w in V_i on which every $h \in \mathcal{H}_w$ is defined.

By the theorem of Epstein–Millett–Tischler [13], the union of all leaves with trivial holonomy is a dense G_δ-subset in M. Thus there is a leaf L_0 without holonomy intersecting the open subset $f_i^{-1}(U)$. Let $y \in L_0 \cap f_i^{-1}(U)$ and $v = f_i(y) \in U$.

Notice that the leaf L_0 of $(M,\mathcal{F})$ is compact if and only if the orbit $\mathcal{H} \cdot v$ of the point v is finite. Thus the orbit $\mathcal{H}_w \cdot v$ is finite. Hence there is an infinite sequence of the elements $\{g_n\}$, $n \in \mathbb{N}$, from $\mathcal{H}_w$ belonging to different germs at w such that $g_n(v) = v$. By the definition of v, for each element g_n there exists a neighbourhood W_n, where $g_n|_{W_n} = id_{W_n}$. Therefore the quasi analyticity of $\mathcal{H}$ implies $g_n|_U = id_U$. This contradicts to the property of g_n to define different germs at w. Hence the holonomy group $\Gamma(L,x)$ is finite.

Thus, all leaves have finite holonomy groups and $(M,\mathcal{F})$ is a locally stable foliation. ∎

4.3. Proof of Theorem 2

Let $(M, \mathcal{F})$ be a compact foliation with quasi analytic holonomy pseudogroup admitting an Ehresmann connection $\mathfrak{M}$. Use the notations introduced in Sec. 2. By the theorem of Epstein–Millett–Tischler [13], the union of all leaves without holonomy is the G_δ-subset of the manifold M. Therefore the foliation $(M, \mathcal{F})$ has a leaf $L_0 = L_0(x_0)$, $x_0 \in M$, with the trivial holonomy group $\Gamma(L_0, x_0)$. According to Prop. 3, due to the quasi analyticity of the holonomy pseudogroup of the foliation $(M, \mathcal{F})$, for any point $x \in M$ the map $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ is a group isomorphism. Hence the leaf $L_0 = L_0(x_0)$ has the trivial $\mathfrak{M}$-holonomy group.

By Lemma 1, for any leaf $L = L(x)$, there exists a regular covering map $f : L_0 \to L$, with the group of deck transformations isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$ of the leaf L. As both leaves L_0 and L are compact, so $f : L_0 \to L$ is a finitely sheeted covering. It implies the finiteness of the group $H_{\mathfrak{M}}(L, x)$. Therefore the holonomy group $\Gamma(L, x)$, which is isomorphic to $H_{\mathfrak{M}}(L, x)$, is also finite.

Thus, according to the Reeb theorem on the local stability of a compact leaf with finite holonomy group, all the leaves of $(M, \mathcal{F})$ are locally stable.

Let us prove the converse statement. Suppose that a compact foliation $(M, \mathcal{F})$ of codimension q is locally stable. As well known, in this case $(M, \mathcal{F})$ is the Riemannian foliation. Hence its holonomy pseudogroup $\mathcal{H}(M, \mathcal{F})$ consists of local isometries. Therefore $\mathcal{H}(M, \mathcal{F})$ is quasi analytic. Moreover, there exists a complete bundle-like metric g relatively $(M, \mathcal{F})$ (see, for instance, [30]). Hence the q-dimensional orthogonal distribution $\mathfrak{M}$ is an Ehresmann connection for the foliation $(M, \mathcal{F})$ [1]. ■

5. Proof of Theorems 3–7

5.1. Proof of Theorem 3

As it was shown in ([36], Prop. 2), the completeness of a foliation admitting a transverse rigid geometry implies the existence of an Ehresmann connection for this foliation. Therefore it is sufficient to observe that the holonomy pseudogroup of every foliation mentioned in Theorem 3 is quasi analytic and to apply Theorem 2. ■

5.2. Proof of Theorem 4

Suppose that a foliation $(M, \mathcal{F})$ satisfies conditions 1) and 2) of Theorem 2. Assume that there exists a compact leaf L' with the finite germinal holonomy group $\Gamma(L', x')$. According to [13], there is a leaf L_0 having the trivial germinal holonomy group. As the conditions of Prop. 3 are satisfied, $\chi : H_{\mathfrak{M}}(L, x) \to$

$\Gamma(L, x)$ is the group isomorphism for any $x \in M$. Hence the $\mathfrak{M}$-holonomy group of the leaf L_0 is also trivial. By Lemma 1, there exists a regular covering map $f_0 : L_0 \to L'$, and the group of deck transformations of this map is isomorphic to the group $H_{\mathfrak{M}}(L', x') \cong \Gamma(L', x')$. Therefore the group $H_{\mathfrak{M}}(L', x')$ is finite and the leaf L_0 is compact. With accordance to Lemma 1, the leaf L_0 covers each leaf L of this foliation, and the group of the deck transformations is $H_{\mathfrak{M}}(L, x) \cong \Gamma(L, x)$. Hence every leaf L is compact and has finite germinal holonomy group $\Gamma(L, x)$.

If there exists a compact leaf L' with the finite fundamental group $\pi_1(L', x')$, then, by analogy, we can show that all leaves have the same compact universal covering space. Therefore each leaf L is compact and it has the finite fundamental group. ■

5.3. Proof of Theorem 5

By Theorems 1 and 2, conditions 1), 2) and 3) are equivalent.

We remark that Winkelnkemper's criterion for the graph $G(\mathcal{F})$ of a foliation $(M, \mathcal{F})$ to be Hausdorff [25], which was reformulated by us as Prop. 2, implies the equivalence of 3) and 4).

Suppose that 4) is true. As 4) is equivalent to 3), it follows from the proof of Theorem 2 that the map $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ is the group isomorphism for every point $x \in M$, i.e. 4) $\Rightarrow$ 5). Using Prop. 1 it is not difficulty to show that 5) $\Leftrightarrow$ 6).

Assume that 6) holds. According to Prop. 1, the $\mathfrak{M}$-holonomy groupoid $G_{\mathfrak{M}}(\mathcal{F})$ is always Hausdorff. The isomorphism of the holonomy groupoids $G_{\mathfrak{M}}(\mathcal{F})$ and $G(\mathcal{F})$ is the diffeomorphism between them. Therefore the topological space of $G(\mathcal{F})$ is also Hausdorff. By Prop. 2, it is equivalent to the quasi analyticity of the holonomy pseudogroup of $(M, \mathcal{F})$, i.e., 6) implies 4). Thus, the first six conditions are equivalent.

Now suppose that 6) is valid, i.e., $G(\mathcal{F}) \cong G_{\mathfrak{M}}(\mathcal{F})$. According to the first statement of Prop. 1, the canonical projection $p_1 : G(\mathcal{F}) \to M$ is a locally trivial fibration with the standard fibre Y. Consequently, any fibre $p_1^{-1}(x)$ over $x \in M$ is diffeomorphic to Y. By the mentioned above result of [13], there exists a leaf $L_0 = L_0(x_0)$ with the trivial germinal holonomy group. By the definition of the graph $G(\mathcal{F})$, the manifold $p_1^{-1}(x_0)$ is diffeomorphic to a leaf L_0 of $(M, \mathcal{F})$. Due to the compactness of the foliation $(M, \mathcal{F})$, the leaf L_0 is compact. Hence $Y \cong p_1^{-1}(x_0)$ is compact. Therefore each fibre $p_1^{-1}(x)$ is also compact, i.e., 6) implies 7).

Notice that an arbitrary leaf $\mathbb{L} = p_1^{-1}(L)$, where $L = L(x)$, of the induced foliation is diffeomorphic to the quotient manifold $(p_1^{-1}(x) \times p_1^{-1}(x))/\Psi$, and the group Ψ is isomorphic to the holonomy group $\Gamma(L, x)$ of the leaf L. The quotient map $p_1^{-1}(x) \times p_1^{-1}(x) \to \mathbb{L}$ is a regular covering mapping with the group of the deck transformations Ψ . If condition 7) is valid, then any leaf $\mathbb{L}$ of the induced foliation $(G(\mathcal{F}), \mathbb{F})$ is compact since it is the image of the compact space $p_1^{-1}(x) \times p_1^{-1}(x)$ under a continuous map. Thus, 7) $\Rightarrow$ 8).

To prove the implication 8) $\Rightarrow$ 1), suppose that the induced foliation $(G(\mathcal{F}), \mathbb{F})$ is compact, i.e., each its leaf $\mathbb{L}$ is compact. Note that the map $f : \mathbb{L} \to L \times L$, taking a point $z = (x, \{h\}, y)$ from $\mathbb{L}$ to the point $(x, y) \in L \times L$, is a regular covering map with the group of the deck transformations isomorphic to the holonomy group $\Gamma(L, x)$ of the leaf L. The map $f : \mathbb{L} \to L \times L$ is a finitely sheeted covering, because it is a covering map of one compact manifold onto another compact manifold. Hence every leaf L is compact with the finite holonomy group $\Gamma(L, x)$. Therefore, in conformity with Reeb's theorem, the foliation $(M, \mathcal{F})$ is locally stable, i.e., 8) $\Rightarrow$ 1).

The equivalence of conditions 9)–16) and 1) follows from the works given in Introduction. ∎

5.4. Proof of Theorem 6

Being compact, the foliation $(M, \mathcal{F})$ does not admit an attractor. According to our result ([37], Theorem 2), in this case the conformal foliation $(M, \mathcal{F})$ of codimension $q > 2$ must be a compact Riemannian foliation. Therefore it is locally stable. ∎

5.5. Proof of Theorem 7

Consider a conformal foliation $(M, \mathcal{F})$ of codimension $q > 2$ on a compact manifold M. It follows from Theorem 4 proved by us in [37] that if there exists a compact leaf with a finite germinal holonomy group, then $(M, \mathcal{F})$ is a Riemannian foliation. Due to the compactness of M, there is a complete bundle-like metric g with respect to $(M, \mathcal{F})$. Therefore, the orthogonal q-dimensional distribution $\mathfrak{M}$ is an Ehresmann connection for $(M, \mathcal{F})$. Thus, the required assertion follows from Theorem 2. ∎

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