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Gravity Waves under Nonuniform Pressure over a Free Surface. Exact Solutions

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Abstract—Plane periodic oscillations of an infinitely deep fluid are studied in the case of a nonuniform pressure distribution over its free surface. The fluid flow is governed by an exact solution of the Euler equations in the Lagrangian variables. The dynamics of an oscillating standing soliton are described, together with the scenario of the soliton evolution and the birth of a wave of an anomalously large amplitude against the background of the homogeneous Gerstner undulation (freak wave model). All the flows are nonuniformly vortical.

Keywords: Lagrangian coordinates, exact solutions, Ptolemaic flows, freak waves.

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Only two exact solutions are known in water wave theory [1]. The first solution obtained by Gerstner describes the trochoidal gravity waves on deep water. The second solution was derived by Crapper and pertains to the case of steady capillary waves. In this study we derive and analyze the exact solutions for unsteady vortical free-surface oscillations maintained by a nonuniform, periodically varying pressure on it.

We will consider unsteady plane vortical fluid motions which pertain to the class of Ptolemaic flows [2–4] and include the Gerstner waves as a particular case. The Ptolemaic solutions were previously used for describing the dynamics of a solitary vortex region in an external potential flow [2–4], the azimuthal waves on the surface of a rotating cavity [5], and the MHD flows in a homogeneous longitudinal magnetic field [6]. In this study they are used for investigating fluid flows with complicated dynamics of the free boundary. Emphasis is placed on an analysis of the formation of a wave of an anomalously large amplitude on the surface.

In recent years, the formation of waves, whose height can be by a factor of three and more greater than the mean undulation amplitude (freak waves), has been actively studied by means of numerical methods within the framework of the complete fluid dynamics equations (see overview in [7]). The consideration is made under the assumption of fluid flow potentiality and, except for individual cases, without regard for the wind effect; in this case the pressure on the free surface can be taken to be constant. In [8] the wind effect is taken into account by imposing the linear relation between the pressure and the local surface steepness for the profile regions on which the latter is greater than a certain threshold value. Thus, the wind effect is modeled by a nonuniform pressure distribution over the surface.

Below we construct and analyze the examples of the birth of a finite-amplitude elevation for two cases, namely, against the background of a fluid in rest at infinity and a Gerstner wave. All the flows considered are nonuniformly vortical. The pressure on the free surface varies in antiphase with the profile elevation.

1. PTOLEMAIC FLOWS

In the Lagrangian variables the system of equations of the two-dimensional inviscid fluid dynamics is equivalent to the conditions of the conservation of two Jacobians [2–4]

$$\frac{D(W,\, W^*)}{D(\chi,\, \chi^*)} = \frac{D(W_0,\, W_0^*)}{D(\chi,\, \chi^*)} = D_0(\chi,\, \chi^*),$$
$$\frac{D(W_t,\, W^*)}{D(\chi,\, \chi^*)} = \frac{D(W_{t0},\, W_0^*)}{D(\chi,\, \chi^*)} = \frac{i}{2} D_0 \Omega(\chi,\, \chi^*). \tag{1.1}$$

Here, $W = X + iY$ ($W^* = X - iY$), X and Y are the Cartesian coordinates of a fluid particle, the asterisks denote complex conjugate values, $\chi = a + ib$ ($\chi^* = a - ib$), a and b are the Lagrangian coordinates of a fluid particle, W_t is the time derivative, and W_0 is the value at the initial moment of time. The function D_0 characterizes the relation between the initial particle positions X_0 and Y_0 with the Lagrangian variables; it cannot be zero in the flow region. The first equation of system (1.1) is the continuity equation and the second equation represents the vorticity Ω conservation condition for the fluid particles.

Substituting the function
$$W = G(\chi)e^{i\delta t} + F(\chi^*)e^{i\mu t} \tag{1.2}$$
into Eqs. (1.1) we can convince ourselves that it is the exact solution of the two-dimensional fluid dynamics equations; here, G and F are some functions analytical in the flow domain, while δ and μ are arbitrary real numbers. The functions G and F are in a considerable degree arbitrary, since the only restriction on their choice is the requirement of the constancy of the sign of D_0, that is, for the sake of definiteness, the condition
$$D_0 = |G'|^2 - |F'|^2 \geq 0. \tag{1.3}$$

The trajectories of the fluid particles described by Eq. (1.2) are epicycloids (hypocycloids). In the Ptolemaic World System these were the orbits of planets; in this connection, the flows of this type are called Ptolemaic [2–4].

We will assume that in the Lagrangian coordinates the domain of the wave motion of the fluid is associated with the lower half-space $b \leq 0$ and the flow is described by the following expression
$$W = G(\chi) + F(\chi^*)e^{-i\omega t} \quad (\operatorname{Im}\chi \leq 0). \tag{1.4}$$

The flow belongs to the family of Ptolemaic flows (1.2); for the sake of convenience we let $\mu = -\omega$ in the exponent. The function $G(\chi)$ must be single-valued, that is, its derivative cannot turn to zero in the flow domain. This is fulfilled if inequality (1.3) holds: the positiveness of the sign of D_0 invokes the single-valuedness of $G(\chi)$.

In the case in which $\delta = 0$ the fluid particles rotate in circle about their equilibrium positions. The rotation radius is determined by the absolute value of the function F. At a depth the particles are at rest, so that the following condition is fulfilled
$$|F| \to 0, \quad \text{as} \quad b \to -\infty.$$

Since the function F is analytical, its absolute value is maximum on the free boundary. Therefore, the maximum value of the oscillations of individual fluid particles necessarily corresponds to the particles on the free boundary.

We will determine the pressure distribution over the free surface associated with the wave solution (1.4). The expression for the pressure can be written in the form:
$$\frac{p - p_0}{\rho} = -g(\operatorname{Im}G + \operatorname{Im}Fe^{-i\omega t}) + \frac{1}{2}\omega^2|F|^2 + \operatorname{Re}\left(e^{i\omega t}\int \omega^2 G' F^*\, d\chi\right).$$

As can be seen from this expression, at $b = 0$ the pressure depends on the horizontal Lagrangian coordinate a and periodically varies with time. The form of the nonuniform pressure distribution over the free surface is determined by the functions G and F. Expression (1.4) determines a class of exact solutions describing the free surface dynamics under a nonuniform and periodically varying pressure. In all examples the pressure nonuniformity will be preassigned on a limited interval. At infinity the pressure tends to a constant value p_0.

The nonlinear waves thus determined do not perform mass transfer: the fluid particles move along circles and the drift flow is absent. The vorticity of flow (1.4) is determined by the expression

$$\Omega = \frac{2\omega|F'|^2}{|G'|^2 - |F'|^2}$$

and, by virtue of condition (1.3), is always of the same sign. The vorticity Ω is maximum at the free boundary and decreases down to zero at the bottom.

The profile of the free boundary of the wave can include singular points (cusps). The necessary condition for the occurrence of this singular point is the appearance of the vertical tangent on the profile, which corresponds for this solution to the vanishing of the Jacobian D_0. This is possible only on the free surface, at points satisfying the relation $|G'(a)| = |F'(a)|$. Moreover, the Jacobian D_0 can vanish only at discrete moments of time.

We will consider the Ptolemaic solution (1.4) of the form:

$$W = \chi + iA\exp\left[i(k\chi^* - \omega t)\right].$$

This relation specifies the Gerstner waves in which the fluid particles move along circles, $A\exp kb$ in radius (the b isolines are associated with the same oscillation amplitudes). At any moment of time the free surface of the wave under consideration is a trochoid. The wave remains invariable and travels at a velocity $c = \omega k^{-1}$. Its amplitude is A and the wavenumber is k. Only the solutions with the wave amplitudes $A \le k^{-1}$ have the physical meaning, otherwise the profile is self-intersecting. If $A = k^{-1}$ the crests on the profile become sharp. The Gerstner wave vorticity is as follows:

$$\Omega = \frac{2k^3A^2c\exp(2kb)}{1 - k^2A^2\exp(2kb)}.$$

It diminishes rapidly with the depth. The dispersion equation for the Gerstner waves takes the form $\omega^2 = gk$, as for the linear potential waves on deep water. When this condition is fulfilled, the pressure on the wave profile remains constant.

2. OSCILLATING STANDING SOLITON

The linear function G and the exponential function F correspond to the simplest choice. This is the unique case, when solution (1.4) describes a steady wave on the water. The arbitrariness in choosing the function F (G being linear) makes it possible to consider a wide range of possible initial shapes of the free surface profiles.

We will consider the Ptolemaic solution of the form:

$$W = \chi + \frac{\beta}{(\chi^* + i)^n}e^{-i\omega t}, \qquad \beta > 0, \qquad n \ge 2. \tag{2.1}$$

It describes the dynamics of a solitary elevation. In this case, the function F has a pole of the order n. It corresponds to the value $b = 1$ and, for this reason, lies outside the flow domain. In this formula the quantities χ and β are dimensionless. In accordance with Eq. (1.3), the permissible parameter range is given by the condition $\beta \le 1/n$.

The pressure on the free surface for flow (2.1) can be written in the form:

$$\frac{p - p_0}{\rho} = -g\,\mathrm{Im}\,\frac{\beta\exp(-i\omega t)}{(a + i)^n} + \frac{\beta\omega^2}{2(a^2 + 1)^n} - \frac{\omega^2\beta}{n - 1}\mathrm{Re}\,\frac{\exp(i\omega t)}{(a - i)^{n-1}}.$$

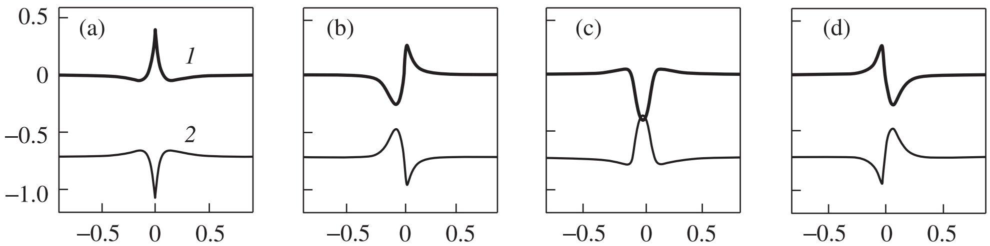

Fig. 1. Profile (*1*) and pressure (*2*) for a solitary elevation without a cusp: $F = 0.4/(\chi^* + i)$; (a–d) relate to $\omega t = \pi/2$, π, $3\pi/2$, and 2π.

It diminishes with distance from the soliton and it is equal to a constant value at infinity. In Eq. (2.1) $n \geq 2$. If n is taken to be unity, then the pressure at infinity increases in accordance with the logarithmic law; this solution is physically meaningless.

In Fig. 1 the free surface evolution pattern and the pressure curve for different moments of time are presented for the parameter values $n = 2$ and $\beta = 0.4$. The pressure curve is plotted for the normalized pressure $p_n = p/(\rho g L)$, where L is a length scale. To make the analysis more convenient it is displaced downward at a constant level.

At the moment $t = \pi/(2\omega)$ the free surface takes the shape of a solitary splash with two small depressions on its periphery. Within the quarter-period a symmetric re-oscillation is formed on the surface; as compared with the initial soliton profile its maximum is displaced rightwards and decreases in amplitude. Within the next quarter-period there appears a solitary depression with two small side splashes on the profile and, finally, within the total oscillation period the profile returns to its initial shape. The surface pressure varies in antiphase with the free boundary profile. This unsteady structure can be regarded as an oscillating standing soliton.

Figure 2 corresponds to the case in which $\beta = 1/2$. This is the limiting permissible value of the parameter β. In this case, the parametric representation of the free boundary shape is as follows:

$$X_s = a + \frac{1}{2(a^2 + 1)}\left(\cos\omega t - \frac{2a}{a^2 + 1}\sin\omega t\right),$$

$$Y_s = -\frac{1}{2(a^2 + 1)}\left(\sin\omega t + \frac{2a}{a^2 + 1}\cos\omega t\right).$$

At the moments $t = (\pi/2 + 2\pi l)/\omega$, where l is an integer, cusps are formed at the point on the free surface profile corresponding to the Lagrangian coordinate $a = 0$. At these moments cusps also arise at the same point on the pressure profile. In the antiphase $t = (3\pi/2 + 2\pi l)/\omega$, when the splash transforms into a depression, neither the profile nor the pressure distribution have a cusp.

In the example considered the free surface shape evolution is in good agreement with the breather profile obtained numerically within the framework of the Euler equations in [9]. The oscillating soliton has a smooth depression and a sharper splash at the moments $t = 2\pi l/\omega$ and it transforms with time into a symmetric profile with two depressions and a sharper crest. We emphasize that, as distinct from [9], solution (2.1) is vortical. Its vorticity is determined by the equation

$$\Omega = \frac{2\omega n^2\beta^2}{\left[a^2 + (1 - b)^2\right]^{n+1} - n^2\beta^2}.$$

As distinct from the Gerstner waves, in which Ω is expressed in terms of exponentially decaying functions, the soliton vorticity depends on the coordinates as a polynomial and decreases fairly slowly with distance from it.

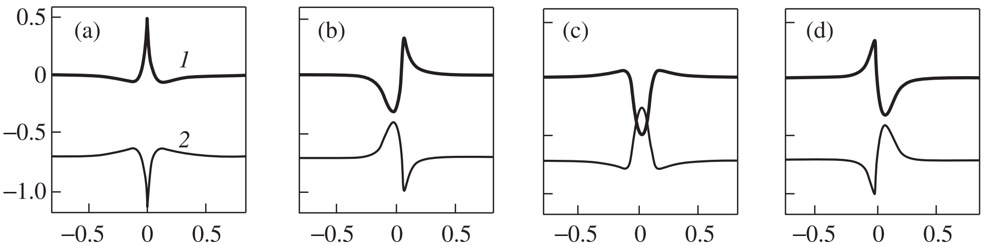

Fig. 2. Profile (*1*) and pressure (*2*) for a solitary elevation with a cusp: $F = 0.5/(\chi^* + i)$; (a–d) relate to $\omega t = \pi/2$, π, $3\pi/2$, and 2π.

3. SOLITON ON THE BACKGROUND OF A UNIFORM UNDULATION

In solution (1.4) with a given linear function G the function F can be taken rather arbitrarily. For this function the superposition principle holds: if F is chosen in the form of the sum of two functions, then the free boundary profile corresponds to the superimposition of the profiles determined by the individual terms. We will take solution (1.4) in the form:

$$W = \chi + \left(iA\exp(ik\chi^*) + \frac{\beta}{(\chi^* + i)^n}\right)\exp(-i\omega t). \tag{3.1}$$

In the expression for the function F the first term corresponds to the Gerstner wave and the second term to the soliton solution (the quantity k is assumed to be dimensionless). Expression (3.1) describes a solitary elevation against the background of a steady undulation. A simple analytical estimation for the range of the permissible values of the parameter β leads to the inequality

$$\beta \leq (1 - Ak)/n. \tag{3.2}$$

More accurate constraints on the value of β can be obtained using numerical calculations. In Fig. 3 the boundaries of the parameter range for β as a function of kA are plotted for the cases $k = 1$, 2, and 3 and $n = 2$. All these curves are convex arcs based on the segment given by Eq. (3.2). The parameters corresponding to these curves give the solutions with a single profile cusp at one moment of time during an oscillation period.

In Fig. 4 we have plotted the free surface profile (*1*) and the pressure distribution (*2*) for the case in which $A = 0.04$, $k = 1$, $\beta = 0.48$, and $n = 2$. For these parameter values at the moment $t = \pi/2\omega$ the elevation height is by a factor of 12 greater than the uniform undulation level. This crest has no a cusp. Generally, the evolution of this soliton is similar with the oscillating soliton behavior in the absence of uniform undulation (Figs. 1 and 2). Far from the soliton the pressure disturbance vanishes as $1/a$. It is chiefly concentrated in the soliton region. The pressure varies in antiphase with the profile height.

Figure 4 demonstrating the dynamics of a solitary elevation on the background of the Gerstner wave can be interpreted as a case of the formation and breakdown of a wave with an anomalously large amplitude (freak wave). In this case the sharp splash region corresponds to a local pressure minimum. The law of pressure variation during a total period in the soliton region is fairly complicated but the existence of the solution of form (3.1) demonstrates the fundamental possibility of the formation of extremal waves under the action of the pressure nonuniformly distributed over the surface.

As in the soliton problem without undulation (Figs. 1 and 2), ahead of the elevation there is a depression growing during a half period. The solitary splash of a fairly large amplitude with a depression ahead of the wave corresponds to the description of the so-called Lavrenov wave [10]. The depression enhances the destructive effect of the wave, since a boat plows in the depression, whereupon it is subjected to the impact of the anomalously high wave.

Adding a finite number of terms having poles outside the flow region to the function F makes it possible to construct the solution for a wave train with any number of maxima given beforehand. Obviously, the greater the number of the terms taken into account the more complicated the behavior of the surface pressure.

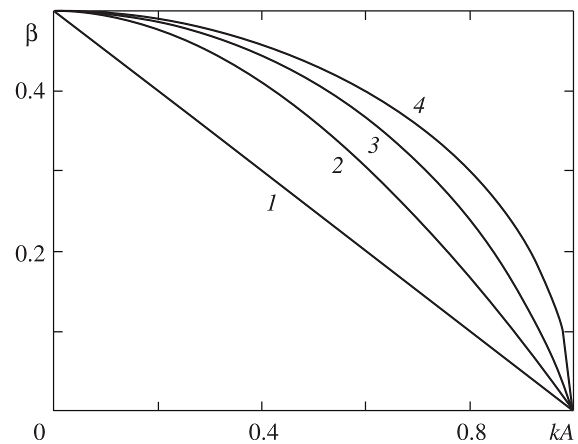

Fig. 3. Boundary of the permissible parameter range for a solitary elevation on the background of a uniform undulation; (*1*) $\beta = (1 - kA)/2$ and (*2–4*) relate to $k = 1$, 2, and 3.

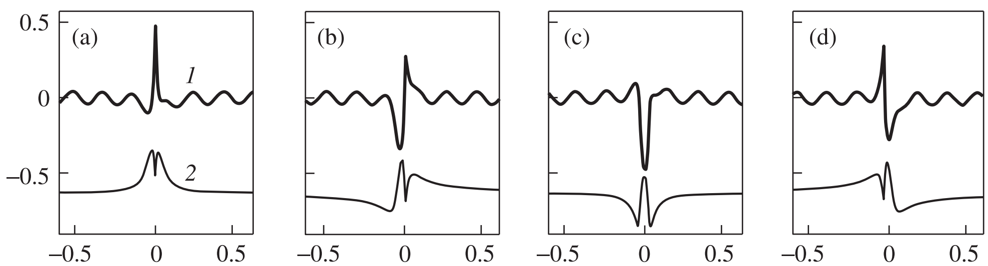

Fig. 4. Profile (*1*) and pressure (*2*) for a solitary elevation on the background of a uniform undulation, $F = 0.04i\exp(i\chi^*) + 0.48/(\chi^* + i)^2$; (a–d) relate to $\omega t = \pi/2$, π, $3\pi/2$, and 2π.

4. MODEL OF THE FREAK WAVE BIRTH

Qualitatively, all the solutions considered above do not give a fairly adequate description of a freak wave, since in all these cases the motion is periodic and a solitary splash transforms with time in a depression, whose depth is similar in value with the splash height.

However, it is possible to construct a solution of Eq. (1.2) with another qualitative behavior. The selection of a suitable, other-than-linear function $G(\chi)$ makes it possible to compensate the depression formed by the term with the function $F(\chi^*)$. The solution of this kind models the appearance of an anomalously high wave against the background of homogeneous undulation.

We will consider the Ptolemaic solution of the form:

$$W = \chi + \frac{\beta_1}{(\chi - i)^n} + \left[iA\exp(ik\chi^*) + \frac{\beta_2}{(\chi^* + i)^n}\right]e^{-i\omega t}. \tag{4.1}$$

The pole of the function $G(\chi)$ corresponds to the point $b = 1$ and is located outside the flow region. As distinct from the case of a solitary elevation, the parameters β_1 and β_2 are not necessarily real. Now the analytical estimate of the range of permissible values of the parameters is as follows:

$$|\beta_1| + |\beta_2| \le (1 - Ak)/n.$$

Letting $|\beta_1| = |\beta_2| = \beta > 0$ we can rewrite this condition as follows: $\beta \le (1 - Ak)/(2n)$. In solution (4.1) the range of permissible β values is half as large as in the case of the solitary elevation (3.2).

In order for the additional term in the function $G(\chi)$ could offset the depression we must take n to be even and to let

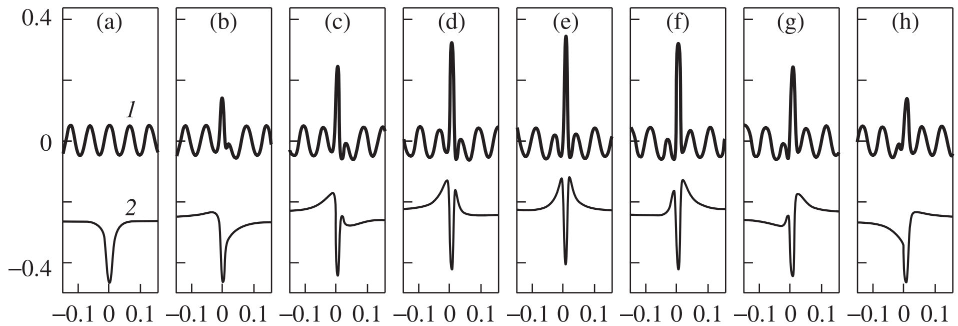

Fig. 5. Profile (*1*) and pressure (*2*) for the model of the birth of an anomalously high wave on the background of a uniform undulation, $W = \chi - 0.2i/(\chi - i)^2 + [0.05i\exp(i\chi^*) + 0.2i/(\chi^* + i)^2]e^{-i\omega t}$; (a–h) relate to $\omega t = 0, \pi/4, \pi/2, 3\pi/4, 5\pi/4, 3\pi/2$, and $7\pi/4$.

$$\beta_1 = (-1)^m ri, \qquad r > 0, \qquad m = n/2.$$

If $n = 2$, then $\beta_1 = -ri$ and the limiting height of the disturbance is 2β.

At the greatest permissible value $\beta = (1 - Ak)/(2n)$ the ratio of the maximum profile height to the uniform undulation amplitude amounts to $(1/A - k)/n$ and can increase without bound. If the quantity kA tends to unity, the maximum disturbance height vanishes. Thus, the smaller the Gerstner wave steepness the higher the wave of the anomalous amplitude.

Figure 5 presents the dynamics of a solitary elevation for the case in which $A = 0.05$, $k = 1$, $\beta_1 = -0.2i$, $\beta_2 = 0.2i$, and $n = 2$. At the moment $t = 0$ there is no a disturbance and the wave profile faithfully copies the Gerstner wave profile. Further on, with time a small disturbance appears, grows and reaches a maximum at the moment $t = \pi/\omega$, and thereupon diminishes down to a small amplitude and vanishes. This motion is periodically repeated, the anomalous-height wave being at rest. The ratio of the maximum height of the splash to the uniform undulation amplitude is about five.

The distinctive feature of the pressure distribution is that during the entire period the absolute pressure minimum is realized at the point of the freak wave onset, the "depression" amplitude being almost invariant. With time the depression becomes more narrow and two splashes of lower level appear on its sides (relative to the constant value determined by the Gerstner wave). This pressure behavior is due to the choice of the value $k = 1$. For other-than-unity values of k the pressure behavior is more complicated and oscillating. However, the pattern of the onset of an anomalously high wave from the Gerstner undulation remains qualitatively the same.

The solutions constructed make it possible to suppose that, in principle, one of the possible mechanisms of the appearance of freak waves is the presence of pressure differences on the free surface of the ocean. The low-pressure regions are capable to form splashes of a fairly high amplitude against the background of a uniform undulation.

Summary. Within the framework of the Lagrangian description of inviscid incompressible flows a family of exact two-dimensional solutions modeling the onset of solitary splashes under a nonuniform pressure distribution on the free surface is studied. A scenario of the birth of a wave of an anomalously large amplitude against the background of the Gerstner wave is analyzed.

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