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Mirabolic affine Grassmannian and character sheaves

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Abstract. We compute the Frobenius trace functions of mirabolic character sheaves defined over a finite field. The answer is given in terms of the character values of general linear groups over the finite field, and the structure constants of multiplication in the mirabolic Hall–Littlewood basis of symmetric functions, introduced by Shoji.

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1. Introduction

This note is a sequel to [14]. We make a free use of notations and results thereof. Our goal is to study the mirabolic character sheaves introduced in [3]. According to Lusztig's results, the unipotent character sheaves on GL_N are numbered by the set of partitions of N. For such a partition λ we denote by $\mathcal{F}_\lambda$ the corresponding character sheaf. If the base field is $\mathsf{k} = \mathbb{F}_q$, the Frobenius trace function of a character sheaf $\mathcal{F}_\lambda$ on a unipotent class of type μ is $q^{n(\mu)}K_{\lambda,\mu}(q^{-1})$ where $K_{\lambda,\mu}$ is the Kostka–Foulkes polynomial, and $n(\mu) = \sum_{i\geq 1}(i-1)\mu_i$ (see [5]).

Let $V = \mathsf{k}^N$, so that $\mathrm{GL}_N = \mathrm{GL}(V)$. For a pair (λ, μ) of partitions such that $|\lambda| + |\mu| = N$ the corresponding unipotent mirabolic character sheaf $\mathcal{F}_{\lambda,\mu}$ on $\mathrm{GL}(V)\times V$ was constructed in [3]. On the other hand, the GL_N-orbits in the product of the unipotent cone and V are also numbered by the set of pairs (λ', μ') of partitions such that $|\lambda'|+|\mu'| = N$ (see [14]). In Theorem 2 we compute the Frobenius trace function of a mirabolic character sheaf $\mathcal{F}_{\lambda,\mu}$ on an orbit corresponding to (λ', μ'). The answer is given in terms of certain polynomials $\Pi_{(\lambda',\mu')(\lambda,\mu)}$, the mirabolic analogues of the Kostka–Foulkes polynomials introduced in [12]. More generally, in 5.4 we compute the Frobenius trace functions (on any orbit) of a wide class of Weil mirabolic character sheaves. These trace functions form a basis in the space of $\mathrm{GL}_N(\mathbb{F}_q)$-invariant functions on $\mathrm{GL}_N(\mathbb{F}_q) \times \mathbb{F}_q^N$, and we conjecture that the above class of sheaves exhausts all the irreducible $\mathbb{G}_m$-equivariant Weil mirabolic character sheaves. This would give a positive answer to a question of G. Lusztig.

Recall that the Kostka–Foulkes polynomials are the matrix coefficients of the transition matrix from the Hall–Littlewood basis to the Schur basis of the ring Λ of symmetric functions. Similarly, the polynomials $\Pi_{(\lambda',\mu')(\lambda,\mu)}$ are the matrix coefficients of the transition matrix from a certain *mirabolic Hall–Littlewood* basis of $\Lambda \otimes \Lambda$ (introduced in [12]) to the Schur basis (see 4.2). Recall that Λ is isomorphic to the Hall algebra [8] whose natural basis goes to the basis of Hall–Littlewood polynomials. Similarly, $\Lambda \otimes \Lambda$ is naturally isomorphic to a certain *mirabolic Hall bimodule* over the Hall algebra, and then the natural basis of this bimodule goes to the mirabolic Hall–Littlewood basis (see Section 4). The structure constants of this basis, together with Green's formula for the characters of $\mathrm{GL}_N(\mathbb{F}_q)$, enter the computation of the Frobenius traces of the previous paragraph.

The Hall algebra is also closely related to the spherical Hecke algebra $\mathbf{H}^{\mathrm{sph}}$ of GL_N (the convolution algebra of the affine Grassmannian of GL_N). Similarly, the mirabolic Hall bimodule is closely related to a certain *spherical mirabolic bimodule* over $\mathbf{H}^{\mathrm{sph}}$, defined in terms of convolution of the affine Grassmannian and the *mirabolic affine Grassmannian* (see Section 3). The geometry of the mirabolic affine Grassmannian is a particular case of the geometry of the *mirabolic affine flag variety* studied in Section 2. Both geometries are (mildly) semiinfinite.

Thus all the results of this note are consequences of a single guiding principle which may be loosely stated as follows: the mirabolic substances form a bimodule over the classical ones; this bimodule is usually free of rank one.

However, the affine mirabolic bimodule $\mathcal{R}^{\mathrm{aff}}$ over the affine Hecke algebra $\mathcal{H}^{\mathrm{aff}}$ is not free (see Remark 1). Recall that $\mathcal{H}^{\mathrm{aff}}$ can be realized in the equivariant K-homology of the Steinberg variety. It would be very interesting to find a similar realization of $\mathcal{R}^{\mathrm{aff}}$.

Finally, let us mention that the results of this note are very closely related to the results of [1], though our motivations are rather different. The authors of [1] were primarily interested in the geometry of enhanced nilpotent cone. They proved the parity vanishing of the IC stalks of the orbit closures in the enhanced nilpotent cone, and identified the generating functions of these stalks with Shoji's type-B Kostka polynomials. Since the Schubert varieties in the mirabolic affine Grassmannian are equisingular to the orbit closures in the enhanced nilpotent cone (see 3.7, 3.8), the appearance of Shoji's polynomials in the spherical mirabolic bimodule and in the mirabolic Hall bimodule is an immediate corollary of [1].

2. Mirabolic affine flags

2.1. Notations

We set $\mathbf{F} = \mathsf{k}((t))$ and $\mathbf{O} = \mathsf{k}[[t]]$. Furthermore, $G = \mathrm{GL}(V)$, and $\mathbf{G_F} = G(\mathbf{F})$, $\mathbf{G_O} = G(\mathbf{O})$. The *affine Grassmannian* is $\mathbf{Gr} = \mathbf{G_F}/\mathbf{G_O}$. We fix a flag $F_\bullet \in \mathrm{Fl}(V)$ and its stabilizer Borel subgroup $B \subset G$; it gives rise to an Iwahori subgroup $\mathbf{I} \subset \mathbf{G_O}$. The *affine flag variety* is $\mathbf{Fl} = \mathbf{G_F}/\mathbf{I}$. We set $\mathbf{V} = \mathbf{F}\otimes_\mathsf{k} V$ and $\mathring{\mathbf{V}} = \mathbf{V} - \{0\}$, and $\mathbf{P} = \mathring{\mathbf{V}}/\mathsf{k}^\times$.

It is well known that the $\mathbf{G_F}$-orbits in $\mathbf{Fl}\times\mathbf{Fl}$ are numbered by the affine Weyl group $\mathfrak{S}_N^{\mathrm{aff}}$ formed by all the permutations w of $\mathbb{Z}$ such that $w(i+N) = w(i)+N$ for any $i \in \mathbb{Z}$ (*periodic* permutations). Namely, for a basis $\{e_1,\dots,e_N\}$ of V we set $e_{i+Nj} := t^{-j}e_i$, $i \in \{1,\dots,N\}$, $j \in \mathbb{Z}$; then the following pair $(F^1_\bullet, F^2_\bullet)$ of periodic flags of $\mathbf{O}$-sublattices in $\mathbf{V}$ lies in the orbit $\mathbb{O}_w \subset \mathbf{Fl}\times\mathbf{Fl}$:

$$F^1_k = \langle e_k, e_{k-1}, e_{k-2}, \dots\rangle, \qquad F^2_k = \langle e_{w(k)}, e_{w(k-1)}, e_{w(k-2)}, \dots\rangle \tag{1}$$

(it is understood that $e_k, e_{k-1}, e_{k-2},\dots$ is a *topological* basis of F^1_k).

Following [14, Lemma 2], we define RB^{aff} as the set of pairs (w,β) where $w \in \mathfrak{S}_N^{\mathrm{aff}}$ and $\beta \subset \mathbb{Z}$ are such that if $i \in \mathbb{Z}-\beta$ and $j \in \beta$, then either $i > j$ or $w(i) > w(j)$; moreover, any $i \ll 0$ lies in β, and any $j \gg 0$ lies in $\mathbb{Z}-\beta$.

2.2. $\mathbf{G_F}$-orbits in $\mathbf{Fl}\times\mathbf{Fl}\times\mathbf{P}$

The following proposition is an affine version of [10, 2.11].

Proposition 1. *There is a one-to-one correspondence between the set of $\mathbf{G_F}$-orbits in $\mathbf{Fl}\times\mathbf{Fl}\times\mathring{\mathbf{V}}$ (equivalently, in $\mathbf{Fl}\times\mathbf{Fl}\times\mathbf{P}$) and RB^{aff}.*

Proof. The argument is entirely similar to the proof of Lemma 2 of [14]. It is left to the reader. We only mention that a representative of an orbit corresponding to (w,β) is given by $(F^1_\bullet, F^2_\bullet, v)$ where $(F^1_\bullet, F^2_\bullet)$ are as in (1) and $v = \sum_{k\in\beta} e_k$ (note that this infinite sum makes sense in $\mathbf{V}$). □

2.3. The mirabolic bimodule over the affine Hecke algebra

Let $\mathsf{k} = \mathbb{F}_q$, a finite field with q elements. Then the *affine Hecke algebra* of G is the endomorphism algebra of the induced module $H^{\mathrm{aff}} := \mathrm{End}_{\mathbf{G_F}}(\mathrm{Ind}_{\mathbf{I}}^{\mathbf{G_F}}\mathbb{Z})$. It has the standard basis $\{T_w \mid w \in \mathfrak{S}_N^{\mathrm{aff}}\}$, and the structure constants are polynomial in q, so we may and will view H^{aff} as the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]$-algebra $\mathbf{H}^{\mathrm{aff}}$. Clearly, $H^{\mathrm{aff}} = \mathrm{End}_{\mathbf{G_F}}(\mathrm{Ind}_{\mathbf{I}}^{\mathbf{G_F}}\mathbb{Z})$ coincides with the convolution ring of $\mathbf{G_F}$-invariant functions on $\mathbf{Fl}\times\mathbf{Fl}$.

H^{aff} acts by right and left convolution on the bimodule R^{aff} of $\mathbf{G_F}$-invariant functions on $\mathbf{Fl}\times\mathbf{Fl}\times\mathring{\mathbf{V}}$. For $\tilde{w} \in RB^{\mathrm{aff}}$ let $T_{\tilde{w}} \in R^{\mathrm{aff}}$ stand for the characteristic function of the corresponding orbit in $\mathbf{Fl}\times\mathbf{Fl}\times\mathring{\mathbf{V}}$. Note that the involutions $(F^1_\bullet, F^2_\bullet) \leftrightarrow (F^2_\bullet, F^1_\bullet)$ and $(F^1_\bullet, F^2_\bullet, v) \leftrightarrow (F^2_\bullet, F^1_\bullet, v)$ induce anti-automorphisms of the algebra $\mathbf{H}^{\mathrm{aff}}$ and of the bimodule of $\mathbf{G_F}$-invariant functions on $\mathbf{Fl}\times\mathbf{Fl}\times\mathring{\mathbf{V}}$. These anti-automorphisms send T_w to $T_{w^{-1}}$ and $T_{\tilde{w}}$ to $T_{\tilde{w}^{-1}}$ where $\tilde{w}^{-1} = (w^{-1}, w(\beta))$ for $\tilde{w} = (w,\beta)$.

We are going to describe the right action of H^{aff} on the bimodule R^{aff} in the basis $\{T_{\tilde{w}} \mid \tilde{w} \in RB^{\mathrm{aff}}\}$ (the formulas for the left action would then follow via the above anti-automorphisms). To this end recall that H^{aff} is generated by $T_{s_1}, \dots, T_{s_N}, T_\tau^{\pm 1}$ where T_{s_i} is the characteristic function of the orbit formed by the pairs $(F^1_\bullet, F^2_\bullet)$ such that $F^1_j \neq F^2_j$ iff $j = i \pmod N$; and $\tau(k) = k+1$, $k \in \mathbb{Z}$. Evidently, $T_{\tilde{w}} T_\tau^{\pm 1} = T_{\tilde{w}[\pm 1]}$ where $\tilde{w}[\pm 1]$ is the shift of $\tilde{w}$ by ± 1. The following proposition is an affine version of Proposition 2 of [14], and the proof is straightforward.

Proposition 2. *Let $\tilde{w} = (w, \beta) \in RB^{\mathrm{aff}}$ and let $s = s_i \in \mathfrak{S}_N^{\mathrm{aff}}$, $i \in \{1, \dots, N\}$. Define $\tilde{w}s = (ws, s(\beta))$ and $\tilde{w}' = (w, \beta \triangle \{i+1\})$. Let $\sigma = \sigma(\tilde{w})$ and $\sigma' = \sigma(\tilde{w}s)$ be given by the formula* (6) *of* [14]*. Then*

$$T_{\tilde{w}} T_s = \begin{cases} T_{\tilde{w}s} & \text{if } ws > w \text{ and } i+1 \notin \sigma', \\ T_{\tilde{w}s} + T_{(\tilde{w}s)'} & \text{if } ws > w \text{ and } i+1 \in \sigma', \\ T_{\tilde{w}'} + T_{\tilde{w}'s} & \text{if } ws < w \text{ and } \beta \cap \iota = \{i\}, \\ (q-1) T_{\tilde{w}} + q T_{\tilde{w}s} & \text{if } ws < w \text{ and } i \notin \sigma, \\ (q-2) T_{\tilde{w}} + (q-1)(T_{\tilde{w}'} + T_{\tilde{w}s}) & \text{if } ws < w \text{ and } \iota \subset \sigma, \end{cases} \tag{2}$$

where $\iota = \{i, i+1\}$.

2.4. Modified bases

The formulas (2) being polynomial in q, we may (and will) view the H^{aff}-bimodule R^{aff} as the specialization under $\mathbf{q} \mapsto q$ of the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-bimodule $\mathbf{R}^{\mathrm{aff}}$ over the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra $\mathbf{H}^{\mathrm{aff}}$. We consider a new variable $\mathbf{v}$ with $\mathbf{v}^2 = \mathbf{q}$, and extend the scalars to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$: $\mathcal{H}^{\mathrm{aff}} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]} \mathbf{H}^{\mathrm{aff}}$, $\mathcal{R}^{\mathrm{aff}} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]} \mathbf{R}^{\mathrm{aff}}$.

Recall the basis $\{H_w := (-\mathbf{v})^{-\ell(w)} T_w\}$ of $\mathcal{H}^{\mathrm{aff}}$ (see e.g. [13]), and the Kazhdan–Lusztig basis $\{\underline{\tilde{H}}_w\}$ (*loc. cit.*); in particular, $\underline{\tilde{H}}_{s_i} = H_{s_i} - \mathbf{v}^{-1}$ for s_i ($i = 1, \dots, N$). For $\tilde{w} = (w, \beta) \in RB^{\mathrm{aff}}$, we denote by $\ell(\tilde{w})$ the sum $\ell(w) + \ell(\beta)$ where $\ell(w)$ is the standard length function on $\mathfrak{S}_N^{\mathrm{aff}}$ and $\ell(\beta) = \sharp(\beta \setminus \{-\mathbb{N}\}) - \sharp(\{-\mathbb{N}\} \setminus \beta)$. We introduce a new basis $\{H_{\tilde{w}} := (-\mathbf{v})^{-\ell(\tilde{w})} T_{\tilde{w}}\}$ of $\mathcal{R}^{\mathrm{aff}}$. In this basis the right action of the Hecke algebra generators $\underline{\tilde{H}}_{s_i}$ takes the form:

Proposition 3. *Let $\tilde{w} = (w, \beta) \in RB^{\mathrm{aff}}$ and let $s = s_i \in \mathfrak{S}_N^{\mathrm{aff}}$, $i \in \{1, \dots, N\}$. Define $\tilde{w}s = (ws, s(\beta))$ and $\tilde{w}' = (w, \beta \triangle \{i+1\})$. Let $\sigma = \sigma(\tilde{w})$ and $\sigma' = \sigma(\tilde{w}s)$ be given by the formula* (6) *of* [14]*. Then*

$$H_{\tilde{w}} \underline{\tilde{H}}_s = \begin{cases} H_{\tilde{w}s} - \mathbf{v}^{-1} H_{\tilde{w}} & \text{if } ws > w \text{ and } i+1 \notin \sigma', \\ H_{\tilde{w}s} - \mathbf{v}^{-1} H_{(\tilde{w}s)'} - \mathbf{v}^{-1} H_{\tilde{w}} & \text{if } ws > w \text{ and } i+1 \in \sigma', \\ H_{\tilde{w}'} - \mathbf{v}^{-1} H_{\tilde{w}} - \mathbf{v}^{-1} H_{\tilde{w}'s} & \text{if } ws < w \text{ and } \beta \cap \iota = \{i\}, \\ H_{\tilde{w}s} - \mathbf{v} H_{\tilde{w}} & \text{if } ws < w \text{ and } i \notin \sigma, \\ (\mathbf{v}^{-1} - \mathbf{v}) H_{\tilde{w}} & \\ \quad + (1 - \mathbf{v}^{-2})(H_{\tilde{w}'} + H_{\tilde{w}s}) & \text{if } ws < w \text{ and } \iota \subset \sigma, \end{cases} \tag{3}$$

where $\iota = \{i, i+1\}$.

2.5. Generators

We consider the elements $\tilde{w}_{i,j} = (\tau^j, \beta_i) \in RB^{\rm aff}$ such that $w = \tau^j$ (shift by j), and $\beta_i = \{i, i-1, i-2, \dots\}$ for any $i, j \in \mathbb{Z}$. The following lemma is proved exactly as Corollary 2 of [14].

Lemma 1. *$\mathcal{R}^{\rm aff}$ is generated by $\{\tilde{w}_{i,j} \mid i, j \in \mathbb{Z}\}$ as an $\mathcal{H}^{\rm aff}$-bimodule.*

Remark 1. Let $\mathbf{P_F} \subset \mathbf{G_F}$ be the stabilizer of a vector $v \in \mathring{\mathbf{V}}$. One can see easily that $\mathbf{R}^{\rm aff}|_{\mathbf{q}=q}$ is isomorphic to $\operatorname{End}_{\mathbf{P_F}}(\operatorname{Ind}_{\mathbf{I}}^{\mathbf{G_F}} \mathbb{Z})$ as a bimodule over $\mathbf{H}^{\rm aff}|_{\mathbf{q}=q} = \operatorname{End}_{\mathbf{G_F}}(\operatorname{Ind}_{\mathbf{I}}^{\mathbf{G_F}} \mathbb{Z})$. Let $Z^{\rm aff} \subset \mathcal{H}^{\rm aff}$ stand for the center of $\mathcal{H}^{\rm aff}$. Let $Z^{\rm aff}_{\rm loc}$ stand for the field of fractions of $Z^{\rm aff}$. Let $\mathcal{H}^{\rm aff}_{\rm loc} := \mathcal{H}^{\rm aff} \otimes_{Z^{\rm aff}} Z^{\rm aff}_{\rm loc}$. It is known that $\mathcal{H}^{\rm aff}_{\rm loc} \simeq \operatorname{Mat}_{N!}(\mathbb{Q}) \otimes_{\mathbb{Q}} Z^{\rm aff}_{\rm loc}$. Let $\mathcal{R}^{\rm aff}_{\rm loc} := Z^{\rm aff}_{\rm loc} \otimes_{Z^{\rm aff}} \mathcal{R}^{\rm aff} \otimes_{Z^{\rm aff}} Z^{\rm aff}_{\rm loc}$. Then it follows from the main theorem of [2] that $\mathcal{R}^{\rm aff}_{\rm loc} \simeq Z^{\rm aff}_{\rm loc} \otimes_{\mathbb{Q}} \operatorname{Mat}_{N!}(\mathbb{Q}) \otimes_{\mathbb{Q}} Z^{\rm aff}_{\rm loc}$.

2.6. Geometric interpretation

It is well known that $\mathcal{H}^{\rm aff}$ is the Grothendieck ring (with respect to convolution) of the derived constructible $\mathbf{I}$-equivariant category of Tate Weil $\overline{\mathbb{Q}}_l$-sheaves on $\mathbf{Fl}$, and multiplication by $\mathbf{v}$ corresponds to the twist by $\overline{\mathbb{Q}}_l(-1/2)$ (so that $\mathbf{v}$ has weight 1). In particular, H_w is the class of the shriek extension of $\overline{\mathbb{Q}}_l[\ell(w)](\ell(w)/2)$ from the corresponding orbit $\mathbf{Fl}_w$, and $\underline{\tilde{H}}_w$ is the selfdual class of the Goresky–MacPherson extension of $\overline{\mathbb{Q}}_l[\ell(w)](\ell(w)/2)$ from this orbit. We will interpret $\mathcal{R}^{\rm aff}$ in a similar vein, as the Grothendieck group of the derived constructible $\mathbf{I}$-equivariant category of Tate Weil $\overline{\mathbb{Q}}_l$-sheaves on $\mathbf{Fl} \times \mathring{\mathbf{V}}$.

To be more precise, we view $\mathbf{V}$ as an indscheme (of ind-infinite type), the union of schemes (of infinite type) $\mathbf{V}_i := t^{-i}\mathsf{k}[[t]] \otimes V$, $i \in \mathbb{Z}$. Here $\mathbf{V}_i$ is the projective limit of the finite-dimensional affine spaces $\mathbf{V}_i/\mathbf{V}_j$, $j < i$. Note that $\mathbf{I}$ acts on $\mathbf{V}_i$ linearly (over k), and it acts on any quotient $\mathbf{V}_i/\mathbf{V}_j$ through a finite-dimensional quotient group. Thus we have the derived constructible $\mathbf{I}$-equivariant category of Weil $\overline{\mathbb{Q}}_l$-sheaves on $\mathbf{Fl}\times\mathbf{V}_i/\mathbf{V}_j$, to be denoted by $D_{\mathbf{I}}(\mathbf{Fl}\times\mathbf{V}_i/\mathbf{V}_j)$. For $j' < j$ we have the inverse image functor from $D_{\mathbf{I}}(\mathbf{Fl}\times\mathbf{V}_i/\mathbf{V}_j)$ to $D_{\mathbf{I}}(\mathbf{Fl}\times\mathbf{V}_i/\mathbf{V}_{j'})$, and we denote by $D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}}_i)$ the 2-limit of this system. Now for $i' > i$ we have the direct image functor from $D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}}_i)$ to $D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}}_{i'})$, and we denote by $D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}})$ the 2-limit of this system.

Clearly, $D_{\mathbf{I}}(\mathbf{Fl})$ acts by convolution both on the left and on the right on $D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}})$.

The $\mathbf{I}$-orbits in $\mathbf{Fl} \times \mathring{\mathbf{V}}$ are numbered by $RB^{\rm aff}$; for $\tilde{w} \in RB^{\rm aff}$, the locally closed embedding of the orbit $\Omega_{\tilde{w}} \hookrightarrow \mathbf{Fl} \times \mathring{\mathbf{V}}$ is denoted by $j^{\tilde{w}}$.

Proposition 4. *The Goresky–MacPherson sheaf $j^{\tilde{w}}_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w})](\ell(\tilde{w})/2)$ is Tate for any $\tilde{w} \in RB^{\rm aff}$.*

Proof. Repeats word for word the proof of Proposition 4 of [14]. For the base of induction, we use the fact that the orbit closure $\bar{\Omega}_{\tilde{w}_{i,j}}$ (see 2.5) is smooth. For the induction step we use the Demazure type resolutions as in [14]. □

2.7. The completed bimodule $\hat{\mathcal{R}}^{\mathrm{aff}}$

Let $D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl}) \subset D_{\mathbf{I}}(\mathbf{Fl})$ (resp. $D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl} \times \mathring{\mathbf{V}}) \subset D_{\mathbf{I}}(\mathbf{Fl} \times \mathring{\mathbf{V}})$) stand for the full subcategory of Tate sheaves. Then $D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl})$ is closed under convolution, and its K-ring is isomorphic to $\mathcal{H}^{\mathrm{aff}}$. The proof of Proposition 4 implies that $D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl}\times\mathring{\mathbf{V}})$ is closed under both left and right convolution with $D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl})$. It follows that $K(D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl} \times \mathring{\mathbf{V}}))$ forms an $\mathcal{H}^{\mathrm{aff}}$-bimodule. This bimodule is isomorphic to a completion $\hat{\mathcal{R}}^{\mathrm{aff}}$ of $\mathcal{R}^{\mathrm{aff}}$ we presently describe.

Recall that for an $\mathbf{O}$-sublattice $F \subset \mathbf{V}$ its *virtual dimension* is $\dim(F) := \dim(F/(F\cap(\mathbf{O}\otimes V))) - \dim((\mathbf{O}\otimes V)/(F\cap(\mathbf{O}\otimes V)))$. Recall that $\mathbf{I}$ is the stabilizer of the flag $F_\bullet^1$, where $F_k^1 = \langle e_k, e_{k-1}, e_{k-2}, \ldots\rangle$. The connected components of $\mathbf{G_F}/\mathbf{I} = \mathbf{Fl}$ are numbered by $\mathbb{Z}$: a flag $F_\bullet$ lies in the component $\mathbf{Fl}_i$ where $i = \dim(F_N)$. For the same reason, the connected components of $\mathbf{Fl}\times\mathring{\mathbf{V}}$ are numbered by $\mathbb{Z}$: a pair $(F_\bullet, v)$ lies in the connected component $(\mathbf{Fl}\times\mathring{\mathbf{V}})_i$ where $i = \dim(F_N)$. We will write $\tilde{w} \in RB_i^{\mathrm{aff}}$ iff $\Omega_{\tilde{w}} \subset (\mathbf{Fl} \times \mathring{\mathbf{V}})_i$. Note that for any $i, k \in \mathbb{Z}$ there are only finitely many $\tilde{w} \in RB^{\mathrm{aff}}$ such that $\tilde{w} \in RB_i^{\mathrm{aff}}$ and $\ell(\tilde{w}) = k$.

We define $\hat{\mathcal{R}}^{\mathrm{aff}}$ to be the direct sum $\bigoplus_{i\in\mathbb{Z}} \hat{\mathcal{R}}_i^{\mathrm{aff}}$ where $\hat{\mathcal{R}}_i^{\mathrm{aff}}$ is formed by all the formal sums $\sum_{\tilde{w}\in RB_i^{\mathrm{aff}}} a_{\tilde{w}} H_{\tilde{w}}$ with $a_{\tilde{w}} \in \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ and $a_{\tilde{w}} = 0$ for $\ell(\tilde{w}) \gg 0$. So we have $K(D_{\mathbf{I}}^{\mathrm{Tate}}(\mathbf{Fl} \times \mathring{\mathbf{V}})) \simeq \hat{\mathcal{R}}^{\mathrm{aff}}$ as an $\mathcal{H}^{\mathrm{aff}}$-bimodule, and the isomorphism takes the class $[j_!^{\tilde{w}}\underline{\overline{\mathbb{Q}}}_l[\ell(\tilde{w})](\ell(\tilde{w})/2)]$ to $H_{\tilde{w}}$.

2.8. Bruhat order

Following Ehresmann and Magyar (see [9]) we will define a partial order $\tilde{w}'' \le \tilde{w}'$ on a connected component RB_i^{aff}. Let $(F_\bullet^1, F_\bullet', v')$ (resp. $F_\bullet^1, F_\bullet'', v''$) be a triple in the relative position $\tilde{w}'$ (resp. $\tilde{w}''$). For any $k, j \in \mathbb{Z}$ we define $r_{jk}(\tilde{w}') := \dim(F_j^1 \cap F_k')$. We also define $\delta(j, k, \tilde{w}')$ to be 1 iff $v' \in F_j^1 + F_k'$, and 0 iff $v' \notin F_j^1 + F_k'$; we set $r_{\langle jk\rangle}(\tilde{w}') := r_{jk}(\tilde{w}') + \delta(j, k, \tilde{w}')$. Finally, we define $\tilde{w}'' \le \tilde{w}'$ iff $r_{jk}(\tilde{w}'') \ge r_{jk}(\tilde{w}')$, and $r_{\langle jk\rangle}(\tilde{w}'') \ge r_{\langle jk\rangle}(\tilde{w}')$ for all $j, k \in \mathbb{Z}$.

The following proposition is proved similarly to the Rank Theorem 2.2 of [9].

Proposition 5. *For $\tilde{w}', \tilde{w}'' \in RB_i^{\mathrm{aff}}$ the orbit $\Omega_{\tilde{w}''}$ lies in the orbit closure $\bar{\Omega}_{\tilde{w}'}$ iff $\tilde{w}'' \le \tilde{w}'$.*

2.9. Duality and the Kazhdan–Lusztig basis of $\hat{\mathcal{R}}^{\mathrm{aff}}$

Recall that the Grothendieck–Verdier duality on $\mathbf{Fl}$ induces an involution (denoted by $h \mapsto \overline{h}$) of $\mathcal{H}^{\mathrm{aff}}$ which takes $\mathbf{v}$ to $\mathbf{v}^{-1}$ and $\underline{\tilde{H}}_w$ to $\underline{\tilde{H}}_w$. We will describe the involution on $\hat{\mathcal{R}}^{\mathrm{aff}}$ induced by the Grothendieck–Verdier duality on $\mathbf{Fl} \times \mathring{\mathbf{V}}$. Recall the elements $\tilde{w}_{i,j}$ introduced in 2.5. We set $\underline{\tilde{H}}_{\tilde{w}_{i,j}} := \sum_{k\le i} (-\mathbf{v})^{k-i} H_{\tilde{w}_{k,j}}$. This is the class of the selfdual (geometrically constant) IC-sheaf on the closure of the orbit $\Omega_{\tilde{w}_{i,j}}$. The following proposition is proved exactly as Proposition 5 of [14].

Proposition 6. (a) *There exists a unique involution* $r \mapsto \overline{r}$ *on* $\hat{\mathcal{R}}^{\mathrm{aff}}$ *such that* $\overline{\underline{\tilde{H}}_{\tilde{w}_{i,j}}} = \underline{\tilde{H}}_{\tilde{w}_{i,j}}$ *for any* $i, j \in \mathbb{Z}$, *and* $\overline{hr} = \overline{h}\overline{r}$ *and* $\overline{rh} = \overline{r}\overline{h}$ *for any* $h \in \mathcal{H}^{\mathrm{aff}}$ *and* $r \in \hat{\mathcal{R}}^{\mathrm{aff}}$.

(b) *The involution in* (a) *is induced by the Grothendieck–Verdier duality on* $\mathbf{Fl} \times \mathring{\mathbf{V}}$.

The following proposition is proved exactly as Proposition 6 of [14].

Proposition 7. (a) *For each* $\tilde{w} \in RB^{\mathrm{aff}}$ *there exists a unique element* $\underline{\tilde{H}}_{\tilde{w}} \in \hat{\mathcal{R}}^{\mathrm{aff}}$ *such that* $\overline{\underline{\tilde{H}}_{\tilde{w}}} = \underline{\tilde{H}}_{\tilde{w}}$ *and* $\underline{\tilde{H}}_{\tilde{w}} \in H_{\tilde{w}} + \sum_{\tilde{y} < \tilde{w}} \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]H_{\tilde{y}}$.

(b) *For each* $\tilde{w} \in RB^{\mathrm{aff}}$ *the element* $\underline{\tilde{H}}_{\tilde{w}}$ *is the class of the selfdual* $\mathbf{I}$*-equivariant IC-sheaf with support* $\bar{\Omega}_{\tilde{w}}$. *In particular, for* $\tilde{w} = \tilde{w}_{i,j}$, *the element* $\underline{\tilde{H}}_{\tilde{w}_{i,j}}$ *is consistent with the notation introduced before Proposition* 6.

We conjecture that the sheaves $j_{!*}\overline{\mathbb{Q}}_l[\ell(\tilde{w})](\ell(\tilde{w})/2)$ are pointwise pure. The parity vanishing of their stalks, and the positivity properties of the coefficients of the transition matrix from $\{H_{\tilde{w}}\}$ to $\{\underline{\tilde{H}}_{\tilde{w}}\}$, would then follow.

3. Mirabolic affine Grassmannian

3.1. $\mathbf{G_F}$-orbits in $\mathbf{Gr} \times \mathbf{Gr} \times \mathbf{P}$

We consider the spherical counterpart of the objects of the previous section. First, recall that the $\mathbf{G_F}$-orbits in $\mathbf{Gr} \times \mathbf{Gr}$ are numbered by the set $\mathfrak{S}_N^{\mathrm{sph}}$ formed by all the nonincreasing N-tuples of integers $\nu = (\nu_1 \geq \ldots \geq \nu_N)$. Namely, for such ν, the following pair (L^1, L^2) of $\mathbf{O}$-sublattices in $\mathbf{V}$ lies in the orbit $\mathbb{O}_\nu$:

$$L^1 = \mathbf{O}\langle e_1, \ldots, e_N\rangle, \qquad L^2 = \mathbf{O}\langle t^{-\nu_1}e_1, \ldots, t^{-\nu_N}e_N\rangle. \tag{4}$$

We define RB^{sph} as $\mathfrak{S}_N^{\mathrm{sph}} \times \mathfrak{S}_N^{\mathrm{sph}}$. We have the addition map $RB^{\mathrm{sph}} \to \mathfrak{S}_N^{\mathrm{sph}} : (\lambda, \mu) \mapsto \nu = \lambda + \mu$ where $\nu_i = \lambda_i + \mu_i$, $i = 1, \ldots, N$.

Proposition 8. *There is a one-to-one correspondence between the set of* $\mathbf{G_F}$*-orbits in* $\mathbf{Gr} \times \mathbf{Gr} \times \mathring{\mathbf{V}}$ (*equivalently, in* $\mathbf{Gr} \times \mathbf{Gr} \times \mathbf{P}$) *and* RB^{sph}.

Proof. The argument is entirely similar to the proof of Proposition 1. We only mention that a representative of an orbit $\mathbb{O}_{(\lambda,\mu)}$ corresponding to (λ, μ) with $\lambda + \mu = \nu$ is given by (L^1, L^2, v) where (L^1, L^2) are as in (4), and $v = \sum_{i=1}^N t^{-\lambda_i}e_i$. □

Proposition 9. *There is a one-to-one correspondence between the set of* $\mathbf{G_F}$*-orbits in* $\mathbf{Gr}\times\mathbf{Fl}\times\mathring{\mathbf{V}}$ (*equivalently, in* $\mathbf{Gr}\times\mathbf{Fl}\times\mathbf{P}$) *and the set of pairs of integer sequences* $(\{b_1, \ldots, b_N\}, \{c_1, \ldots, c_N\})$ *such that if* $b_i - i/N < b_j - j/N$ *then* $c_i \leq c_j$. *Namely, a representative of the orbit corresponding to* $(\{b_1, \ldots, b_N\}, \{c_1, \ldots, c_N\})$ *is given by* (L, F, v) *where* $L = \mathbf{O}\langle t^{b_1+c_1}e_1, \ldots, t^{b_N+c_N}e_N\rangle$, $F_k = \langle e_k, e_{k-1}, e_{k-2}, \ldots\rangle$, $v = \sum_{i=1}^N t^{b_i}e_i$. □

3.2. The spherical mirabolic bimodule

Let $\mathsf{k} = \mathbb{F}_q$. Then the *spherical affine Hecke algebra* H^{sph} of G is the endomorphism algebra of the induced module $\mathrm{End}_{\mathbf{G_F}}(\mathrm{Ind}_{\mathbf{G_O}}^{\mathbf{G_F}} \mathbb{Z})$. It coincides with the convolution ring of $\mathbf{G_F}$-invariant functions on $\mathbf{Gr} \times \mathbf{Gr}$. It has the standard basis $\{U_\nu \mid \nu \in \mathfrak{S}_N^{\mathrm{sph}}\}$ of characteristic functions of $\mathbf{G_F}$-orbits in $\mathbf{Gr} \times \mathbf{Gr}$, and the structure constants are polynomial in q (Hall polynomials), so we may and will view $H^{\mathrm{sph}} = \mathrm{End}_{\mathbf{G_F}}(\mathrm{Ind}_{\mathbf{G_O}}^{\mathbf{G_F}} \mathbb{Z})$ as a specialization of the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra $\mathbf{H}^{\mathrm{sph}}$ under $\mathbf{q} \mapsto q$.

The algebra H^{sph} acts by right and left convolution on the bimodule R^{sph} of $\mathbf{G_F}$-invariant functions on $\mathbf{Gr} \times \mathbf{Gr} \times \mathring{\mathbf{V}}$. For $(\lambda, \mu) \in RB^{\mathrm{sph}}$ let $U_{(\lambda,\mu)}$ stand for the characteristic function of the corresponding orbit in $\mathbf{Gr} \times \mathbf{Gr} \times \mathring{\mathbf{V}}$. We are going to describe the right and left action of H^{sph} on the bimodule in the basis $\{U_{(\lambda,\mu)} \mid (\lambda, \mu) \in RB^{\mathrm{sph}}\}$. To this end recall that H^{sph} is a commutative algebra freely generated by $U_{(1,0,\ldots,0)}, U_{(1,1,0,\ldots,0)}, \ldots, U_{(1,1,\ldots,1,0)}$, and $U^{\pm 1}$ where $U^{\pm 1}$ is the characteristic function of the orbit of $(L^1, t^{\mp 1} L^1)$. We will denote $\nu = (1, \ldots, 1, 0, \ldots, 0)$ (with r 1's and $N - r$ 0's) by (1^r).

Note that the assignment $\phi_{i,j} : (L_1, L_2, v) \mapsto (L_1, t^{-i-j} L_2, t^{-i} v)$ is a $\mathbf{G_F}$-equivariant automorphism of $\mathbf{Gr} \times \mathbf{Gr} \times \mathring{\mathbf{V}}$ sending an orbit $\mathbb{O}_{(\lambda,\mu)}$ to $\mathbb{O}_{(\lambda+i^N, \mu+j^N)}$. We will denote the corresponding automorphism of the bimodule R^{sph} by $\phi_{i,j}$ as well: $\phi_{i,j}(U_{\lambda,\mu)}) = U_{(\lambda+i^N,\mu+j^N)}$. Furthermore, an automorphism $(L_1, L_2) \mapsto (L_2, L_1)$ of $\mathbf{Gr} \times \mathbf{Gr}$ induces an (anti)automorphism ϱ of the (commutative) algebra H^{sph}, $\varrho(U^{\pm 1}) = U^{\mp 1}$, $\varrho(U_\nu) = U_{\nu^*}$ where for $\nu = (\nu_1, \ldots, \nu_N)$ we set $\nu^* = (-\nu_N, -\nu_{N-1}, \ldots, -\nu_1)$. Similarly, the automorphism $(L_1, L_2, v) \mapsto (L_2, L_1, v)$ of $\mathbf{Gr} \times \mathbf{Gr} \times \mathring{\mathbf{V}}$ induces an antiautomorphism ϱ of the bimodule R^{sph} such that $\varrho(U_{(\lambda,\mu)}) = U_{(\mu^*,\lambda^*)}$, and $\varrho(hm) = \varrho(m)\varrho(h)$ for any $h \in H^{\mathrm{sph}}$, $m \in R^{\mathrm{sph}}$. Clearly, $U^{\pm 1} U_{(\lambda,\mu)} = U_{(\lambda \pm 1^N, \mu)}$ and $U_{(\lambda,\mu)} U^{\pm 1} = U_{(\lambda, \mu \pm 1^N)}$.

3.3. Structure constants

In this subsection we will compute the structure constants $G_{(1^r)(\lambda',\mu')}^{(\lambda,\mu)}$ such that $U_{(1^r)} U_{(\lambda',\mu')} = \sum_{(\lambda,\mu) \in RB^{\mathrm{sph}}} G_{(1^r)(\lambda',\mu')}^{(\lambda,\mu)} U_{(\lambda,\mu)}$ (see Proposition 10 below). Due to the existence of the automorphisms $\phi_{i,j}$ of R^{sph}, it suffices to compute $G_{(1^r)(\lambda',\mu')}^{(\lambda,\mu)}$ for $\lambda', \mu' \in \mathbb{N}^N$. In this case λ, μ necessarily lie in $\mathbb{N}^N$ as well, that is, all the four $\lambda', \mu', \lambda, \mu$ are partitions (with N parts). We have $\lambda = (\lambda_1, \ldots, \lambda_N)$; we may and will assume that $\lambda_1 > 0$. We set $n := |\lambda| + |\mu|$, and let $D = \mathsf{k}^n$. We fix a nilpotent endomorphism u of D and a vector $v \in D$ such that the type of $\mathrm{GL}(D)$-orbit of the pair (u, v) is (λ, μ) (see [14, Theorem 1]). By the definition of the structure constants in the spherical mirabolic bimodule, $G_{(1^r)(\lambda',\mu')}^{(\lambda,\mu)}$ is the number of r-dimensional vector subspaces $W \subset \mathrm{Ker}(u)$ such that the type of the pair $(u|_{D/W}, v \pmod{W})$ is (λ', μ').

To formulate the answer we need to introduce certain auxiliary data in $\mathrm{Ker}(u)$. First of all, $u^{\lambda_1 - 1} v$ is a nonzero vector in $\mathrm{Ker}(u)$. We consider the pair of partitions $(\nu,\theta)=\Upsilon(\lambda,\mu)$ (notations introduced before Corollary 1 of [14]), so that $\nu=\lambda+\mu$ is the Jordan type of u. We consider the dual partitions $\tilde\nu,\tilde\theta$. We consider the following flag of subspaces of $\mathrm{Ker}(u)$:

$$F^{\tilde\nu_{\nu_1}}:=\mathrm{Ker}(u)\cap\mathrm{Im}(u^{\nu_1-1})\subset F^{\tilde\nu_{\nu_2}}:=\mathrm{Ker}(u)\cap\mathrm{Im}(u^{\nu_2-1})\subset\cdots$$
$$\subset F^{\tilde\nu_2}:=\mathrm{Ker}(u)\cap\mathrm{Im}(u^{\nu_{\tilde\nu_2}-1})\subset F^{\tilde\nu_1}:=\mathrm{Ker}(u).$$

It is an (incomplete, in general) flag of intersections of $\mathrm{Ker}(u)$ with the images of $u,u^2,u^3,\dots$. More precisely, for any $k=0,1,\dots,\nu_1$ we have

$$F_k:=\mathrm{Ker}(u)\cap\mathrm{Im}(u^k)=F^{\tilde\nu_{k+1}}\quad\text{and}\quad\dim(F^{\tilde\nu_{k+1}})=\tilde\nu_{k+1}.$$

There is a unique k_0 such that $u^{\lambda_1-1}v\in F_{k_0}$ but $u^{\lambda_1-1}v\notin F_{k_0+1}$; namely, we choose the maximal i such that $\lambda_i=\lambda_1$, and then $k_0=\nu_i-1$.

Let $Q\subset\mathrm{GL}(\mathrm{Ker}(u))$ be the stabilizer of the flag $F_\bullet$, a parabolic subgroup of $\mathrm{GL}(\mathrm{Ker}(u))$; and let $Q'\subset Q$ be the stabilizer of the vector $u^{\lambda_1-1}v$. Both Q and Q' have finitely many orbits in the Grassmannian Gr of r-dimensional subspaces in $\mathrm{Ker}(u)$. The orbits of Q are numbered by the compositions $\rho=(\rho_1,\dots,\rho_{\nu_1})$ such that $|\rho|=r$ and $0\le\rho_k\le\tilde\nu_k-\tilde\nu_{k+1}$. Namely, $W\in\mathrm{Gr}$ lies in the orbit $\mathbb{O}_\rho$ iff $\dim(W\cap F_k)=\rho_{k+1}+\cdots+\rho_{\nu_1}$; equivalently, $\dim(W+F_k)=\tilde\nu_{k+1}+\rho_1+\cdots+\rho_k$. If we extend the flag $F_\bullet$ to a complete flag in $\mathrm{Ker}(u)$, then the stabilizer of the extended flag is a Borel subgroup $B\subset Q$. The orbit $\mathbb{O}_\rho$ is a union of certain B-orbits in Gr, that is, Schubert cells. So the cardinality of $\mathbb{O}_\rho$ is a sum of powers of q given by the well known formula for the dimension of the Schubert cells (see e.g. Appendix to Chapter II of [8]). We will denote this cardinality by P_ρ. Note that the Jordan type of $u|_{D/W}$ for $W\in\mathbb{O}_\rho$ is $\nu':=\rho(\nu)$ where $\rho(\nu)$ is defined as the partition dual to $\tilde\nu'=(\tilde\nu'_1,\tilde\nu'_2,\dots)$ where $\tilde\nu'_k:=\tilde\nu_{k+1}+\dim(W+F_{k-1})-\dim(W+F_k)=\tilde\nu_k-\rho_k$.

Now each Q-orbit $\mathbb{O}_\rho$ in Gr splits as a union $\bigsqcup_{0\le j\le\nu_1}\mathbb{O}_{\rho,j}$ of Q'-orbits. Namely, $W\in\mathbb{O}_\rho$ lies in $\mathbb{O}_{\rho,j}$ iff $u^{\lambda_1-1}v\in W+F_j$ but $u^{\lambda_1-1}v\notin W+F_{j+1}$ (so that for some j, e.g. $j<k_0$, $\mathbb{O}_{\rho,j}$ may be empty). The type of $(u|_{D/W},v\pmod W)$ for $W\in\mathbb{O}_{\rho,j}$ is $(\nu',\theta'):=(\rho,j)(\nu,\theta)$ where $\nu'=\rho(\nu)$ and θ' is defined as the partition dual to $\tilde\theta'=(\tilde\theta'_1,\tilde\theta'_2,\dots)$ where $\tilde\theta'_k:=\tilde\theta_{k+1}+\dim(W+F_{k-1}+\mathsf{k}u^{\lambda_1-1}v)-\dim(W+F_k+\mathsf{k}u^{\lambda_1-1}v)$. Finally, note that

$$\dim(W+F_{k-1}+\mathsf{k}u^{\lambda_1-1}v)-\dim(W+F_k+\mathsf{k}u^{\lambda_1-1}v)$$
$$=\begin{cases}\dim(W+F_{k-1})-\dim(W+F_k)=\tilde\nu_k-\tilde\nu_{k+1}-\rho_k & \text{if } j\neq k-1,\\ \dim(W+F_{k-1})-\dim(W+F_k)-1=\tilde\nu_k-\tilde\nu_{k+1}-\rho_k-1 & \text{if } j=k-1.\end{cases}$$

It remains to find the cardinality $P_{\rho,j}$ of $\mathbb{O}_{\rho,j}$. Let us denote $u^{\lambda_1-1}v$ by v' for short. Then $v'\in F_{k_0}$, $v'\in W+F_j$, $v'\notin F_{k_0+1}$, $v'\notin W+F_{j+1}$, thus

$$v'\in A:=\{(W+F_j)\cap F_{k_0}\}\setminus(\{(W+F_j)\cap F_{k_0+1}\}\cup\{(W+F_{j+1})\cap F_{k_0}\}).$$

The cardinality of A equals

$$P_A := q^{\dim(W+F_j)\cap F_{k_0}} - q^{\dim(W+F_j)\cap F_{k_0+1}} - q^{\dim(W+F_{j+1})\cap F_{k_0}} + q^{\dim(W+F_{j+1})\cap F_{k_0+1}},$$

while for any $i > l$ we have

$$\dim(W+F_i)\cap F_l = \dim(W+F_i) + \dim F_l - \dim(W+F_l) = \tilde{\nu}_{i+1} + \rho_{l+1} + \cdots + \rho_i.$$

Now we can count the set of pairs (W, v') in a relative position (ρ, j) with respect to $F_\bullet$ in two ways. First all v' in $F_{k_0} \setminus F_{k_0+1}$ ($q^{\tilde{\nu}_{k_0+1}} - q^{\tilde{\nu}_{k_0+2}}$ choices altogether), and then for each v' all W in $\mathbb{O}_{\rho,j}$ ($P_{\rho,j}$ choices). Second, all W in $\mathbb{O}_\rho$ (P_ρ choices), and then for each W all v' in A (P_A choices). We find

$$P_{\rho,j} = P_\rho \cdot P_A/(q^{\tilde{\nu}_{k_0+1}} - q^{\tilde{\nu}_{k_0+2}}). \tag{5}$$

Note that $P_{\rho,j}$ is a polynomial in q. We conclude that this polynomial computes the desired structure constant

$$G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')} = P_{\rho,j} \tag{6}$$

where $(\lambda', \mu') = \Xi(\nu', \theta')$ (notations introduced before Corollary 1 of [14]), and $(\nu', \theta') = (\rho, j)(\nu, \theta)$ where as before we have $(\nu, \theta) = \Upsilon(\lambda, \mu)$.

Clearly, for any $i, j \geq 0$ we have $G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')} = G^{(\lambda+i^N,\mu+j^N)}_{(1^r)(\lambda'+i^N,\mu'+j^N)}$. Hence for any $(\lambda, \mu), (\lambda', \mu') \in RB^{\mathrm{sph}}$ we can set

$$G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')} := G^{(\lambda+i^N,\mu+j^N)}_{(1^r)(\lambda'+i^N,\mu'+j^N)} \qquad \text{for any } i, j \gg 0.$$

Also, we set

$$G^{(\lambda,\mu)}_{(\lambda',\mu')(1^r)} := G^{(\mu^*-1^N,\lambda^*)}_{(1^{N-r})(\mu'^*,\lambda'^*)}. \tag{7}$$

Thus we have proved the following proposition (the second statement is equivalent to the first one via the antiautomorphism ϱ).

Proposition 10. *Let $(\lambda', \mu') \in RB^{\mathrm{sph}}$ and $1 \leq r \leq N-1$. Then*

$$\begin{aligned} U_{(1^r)} U_{(\lambda',\mu')} &= \sum_{(\lambda,\mu)\in RB^{\mathrm{sph}}} G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')} U_{(\lambda,\mu)}, \\ U_{(\lambda',\mu')} U_{(1^r)} &= \sum_{(\lambda,\mu)\in RB^{\mathrm{sph}}} G^{(\lambda,\mu)}_{(\lambda',\mu')(1^r)} U_{(\lambda,\mu)}. \end{aligned} \tag{8}$$

3.4. Modified bases and generators

The formulas (8) being polynomial in q, we may and will view the H^{sph}-bimodule R^{sph} of $\mathbf{G_F}$-invariant functions on $\mathbf{Gr}\times\mathbf{Gr}\times\mathring{\mathbf{V}}$ as the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-bimodule $\mathbf{R}^{\mathrm{sph}}$ over the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra $\mathbf{H}^{\mathrm{sph}}$. We extend the scalars to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$:

$$\mathcal{H}^{\mathrm{sph}} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]} \mathbf{H}^{\mathrm{sph}}, \qquad \mathcal{R}^{\mathrm{sph}} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]} \mathbf{R}^{\mathrm{sph}}.$$

Recall the selfdual basis C_λ of $\mathcal{H}^{\mathrm{sph}}$ (see e.g. [5]). In particular, for $1 \leq r \leq N-1$, $C_{(1^r)} = (-\mathbf{v})^{-r(N-r)} U_{(1^r)}$. For $(\lambda, \mu) \in RB^{\mathrm{sph}}$ with $\nu = \lambda+\mu$, we denote by

$\ell(\lambda,\mu)$ the sum $d(\nu)+|\lambda|$ with $|\lambda| := \lambda_1+\cdots+\lambda_N$ and $d(\nu) := |\nu|(N-1)-2n(\nu)$ where $n(\nu)=\sum_{i=1}^N (i-1)\nu_i$.

We introduce a new basis $\{H_{(\lambda,\mu)} := (-\mathbf{v})^{-\ell(\lambda,\mu)}U_{(\lambda,\mu)}\}$ of $\mathcal{R}^{\mathrm{sph}}$. We consider the elements $(i^N, j^N) = ((i,\dots,i),(j,\dots,j)) \in RB^{\mathrm{sph}}$ for any $i,j\in\mathbb{Z}$. The following lemma is proved the same way as Lemma 1.

Lemma 2. *$\mathcal{R}^{\mathrm{sph}}$ is generated by $\{H_{(i^N,j^N)} \mid i,j\in\mathbb{Z}\}$ as an $\mathcal{H}^{\mathrm{sph}}$-bimodule.*

3.5. Geometric interpretation and the completed bimodule $\hat{\mathcal{R}}^{\mathrm{sph}}$

Following the pattern of Subsection 2.6 we define the category $D_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}})$ on which $D_{\mathbf{G_O}}(\mathbf{Gr})$ acts by convolution (both on the left and on the right). Similarly to Proposition 4, we have (in obvious notations):

Proposition 11. *The Goresky–MacPherson sheaf $j_{!*}^{(\lambda,\mu)}\underline{\overline{\mathbb{Q}}_l}[\ell(\lambda,\mu)](\ell(\lambda,\mu)/2)$ is Tate for any $(\lambda,\mu)\in RB^{\mathrm{sph}}$.*

We also have the full subcategories of Tate sheaves $D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr})\subset D_{\mathbf{G_O}}(\mathbf{Gr})$ and $D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}})\subset D_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}})$. Furthermore, $D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr})$ is closed under convolution, and $D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}})$ is closed under both right and left convolution with $D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr})$. The K-ring $K(D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr}))$ is isomorphic to $\mathcal{H}^{\mathrm{sph}}$, and this isomorphism sends the class of the selfdual Goresky–MacPherson sheaf on the orbit closure $\mathbf{Gr}_\lambda$ to C_λ. The K-group $K(D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}}))$ forms an $\mathcal{H}^{\mathrm{sph}}$-bimodule isomorphic to a completion $\hat{\mathcal{R}}^{\mathrm{sph}}$ of $\mathcal{R}^{\mathrm{sph}}$ we presently describe.

The connected components of $\mathbf{Gr}\times\mathring{\mathbf{V}}$ are numbered by $\mathbb{Z}$: a pair (L,v) lies in the connected component $(\mathbf{Gr}\times\mathring{\mathbf{V}})_i$ where $i=\dim(L)$. We will say that $(\lambda,\mu)\in RB^{\mathrm{sph}}_i$ if the corresponding orbit lies in $(\mathbf{Gr}\times\mathring{\mathbf{V}})_i$; equivalently, $\sum_{j=1}^N\lambda_j+\sum_{j=1}^N\mu_j=i$. Note that for any $i,k\in\mathbb{Z}$ there are only finitely many $(\lambda,\mu)\in RB^{\mathrm{sph}}$ such that $(\lambda,\mu)\in RB^{\mathrm{sph}}_i$ and $\ell(\lambda,\mu)=k$.

We define $\hat{\mathcal{R}}^{\mathrm{sph}}$ to be the direct sum $\bigoplus_{i\in\mathbb{Z}}\hat{\mathcal{R}}^{\mathrm{sph}}_i$ where $\hat{\mathcal{R}}^{\mathrm{sph}}_i$ is formed by all the formal sums $\sum_{(\lambda,\mu)\in RB^{\mathrm{sph}}_i} a_{(\lambda,\mu)}H_{(\lambda,\mu)}$ with $a_{(\lambda,\mu)}\in\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]$ and $a_{(\lambda,\mu)}=0$ for $\ell(\lambda,\mu)\gg 0$. So we have $K(D^{\mathrm{Tate}}_{\mathbf{G_O}}(\mathbf{Gr}\times\mathring{\mathbf{V}}))\simeq\hat{\mathcal{R}}^{\mathrm{sph}}$ as $\mathcal{H}^{\mathrm{sph}}$-bimodules, and the isomorphism takes the class $[j_!^{(\lambda,\mu)}\underline{\overline{\mathbb{Q}}_l}[\ell(\lambda,\mu)](\ell(\lambda,\mu)/2)]$ to $H_{(\lambda,\mu)}$.

3.6. Bruhat order, duality and the Kazhdan–Lusztig basis

Following Achar and Henderson [1], we define a partial order $(\lambda,\mu)\le(\lambda',\mu')$ on a connected component RB^{sph}_i: we say $(\lambda,\mu)\le(\lambda',\mu')$ iff $\lambda_1\le\lambda'_1$, $\lambda_1+\mu_1\le\lambda'_1+\mu'_1$, $\lambda_1+\mu_1+\lambda_2\le\lambda'_1+\mu'_1+\lambda'_2$, $\lambda_1+\mu_1+\lambda_2+\mu_2\le\lambda'_1+\mu'_1+\lambda'_2+\mu'_2,\dots$ (in the end we have $\sum_{k=1}^N\lambda_k+\sum_{k=1}^N\mu_k=\sum_{k=1}^N\lambda'_k+\sum_{k=1}^N\mu'_k=i$). The following proposition is due to Achar and Henderson (Theorem 3.9 of [1]):

Proposition 12. *For $(\lambda,\mu),(\lambda',\mu')\in RB^{\mathrm{sph}}_i$ the $\mathbf{G_O}$-orbit $\Omega_{(\lambda,\mu)}\subset\mathbf{Gr}\times\mathring{\mathbf{V}}$ lies in the orbit closure $\bar{\Omega}_{(\lambda',\mu')}$ iff $(\lambda,\mu)\le(\lambda',\mu')$.*

Now we will describe the involution on $\hat{\mathfrak{R}}^{\mathrm{sph}}$ induced by the Grothendieck–Verdier duality on $\mathbf{Gr} \times \mathring{\mathbf{V}}$. Recall the elements (i^N, j^N) introduced in 3.4. We set $\underline{\tilde{H}}_{(i^N,j^N)} := \sum_{k\le 0}(-\mathbf{v})^{Nk} H_{((i-k)^N,(j+k)^N)}$. This is the class of the selfdual (geometrically constant) IC-sheaf on the closure of the orbit $\Omega_{(i^N,j^N)}$. The following propositions are proved exactly as Propositions 6 and 7:

Proposition 13. (a) *There exists a unique involution $r \mapsto \overline{r}$ on $\hat{\mathfrak{R}}^{\mathrm{sph}}$ such that $\overline{\underline{\tilde{H}}_{(i^N,j^N)}} = \underline{\tilde{H}}_{(i^N,j^N)}$ for any $i, j \in \mathbb{Z}$, and $\overline{hr} = \overline{h}\overline{r}$ and $\overline{rh} = \overline{r}\overline{h}$ for any $h \in \mathcal{H}^{\mathrm{sph}}$ and $r \in \hat{\mathfrak{R}}^{\mathrm{sph}}$.*

(b) *The involution in* (a) *is induced by the Grothendieck–Verdier duality on $\mathbf{Gr} \times \mathring{\mathbf{V}}$.*

Proposition 14. (a) *For each $(\lambda, \mu) \in RB^{\mathrm{sph}}$ there exists a unique element $\underline{\tilde{H}}_{(\lambda,\mu)} \in \hat{\mathfrak{R}}^{\mathrm{sph}}$ such that $\overline{\underline{\tilde{H}}_{(\lambda,\mu)}} = \underline{\tilde{H}}_{(\lambda,\mu)}$ and $\underline{\tilde{H}}_{(\lambda,\mu)} \in H_{(\lambda,\mu)} + \sum_{(\lambda',\mu')<(\lambda,\mu)} \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]H_{(\lambda',\mu')}$.*

(b) *For each $(\lambda, \mu) \in RB^{\mathrm{sph}}$ the element $\underline{\tilde{H}}_{(\lambda,\mu)}$ is the class of the selfdual $\mathbf{G_O}$-equivariant IC-sheaf with support $\bar{\Omega}_{(\lambda,\mu)}$. In particular, for $(\lambda, \mu) = (i^N, j^N)$, the element $\underline{\tilde{H}}_{(i^N,j^N)}$ is consistent with the notation introduced before Proposition* 13.

We will write

$$\underline{\tilde{H}}_{(\lambda,\mu)} = \sum_{(\lambda',\mu')\le(\lambda,\mu)} \Pi_{(\lambda',\mu'),(\lambda,\mu)} H_{(\lambda',\mu')}. \tag{9}$$

The coefficients $\Pi_{(\lambda',\mu'),(\lambda,\mu)}$ are polynomials in $\mathbf{v}^{-1}$. As we will see in Subsection 4.2 below, they coincide with a generalization of Kostka–Foulkes polynomials introduced by Shoji in [12].

We define a sub-bimodule $\tilde{\mathfrak{R}}^{\mathrm{sph}} \subset \hat{\mathfrak{R}}^{\mathrm{sph}}$ generated (*not* topologically) by the set $\{\underline{\tilde{H}}_{(\lambda,\mu)} \mid (\lambda, \mu) \in RB^{\mathrm{sph}}\}$. It turns out to be a free $\mathcal{H}^{\mathrm{sph}}$-bimodule of rank one:

Theorem 1. $C_\lambda \underline{\tilde{H}}_{(0^N,0^N)} C_\mu = \underline{\tilde{H}}_{(\lambda,\mu)}$.

The proof will be given in Subsection 3.9 after we introduce the necessary ingredients in 3.7 and 3.8.

3.7. Lusztig's construction

Following Lusztig (see [5, Section 2]) we will prove that the G-orbit closures in $\mathcal{N} \times V$ are equisingular to (certain open pieces of) the $\mathbf{G_O}$-orbit closures in $\mathbf{Gr} \times \mathring{\mathbf{V}}$. So we set $E = V \oplus \cdots \oplus V$ (N copies), and let $t : E \to E$ be defined by $t(v_1, \dots, v_N) = (0, v_1, \dots, v_{N-1})$. Let $\mathcal{Y}$ be the variety of all pairs (E', e) where $E' \subset E$ is an N-dimensional t-stable subspace, and $e \in E'$. Let $\mathcal{Y}_0$ be the open subvariety of $\mathcal{Y}$ consisting of those pairs (E', e) in which E' is transversal to $V \oplus \cdots \oplus V \oplus 0$. According to [5], $\mathcal{Y}_0$ is isomorphic to $\mathcal{N} \times V$, the isomorphism sending (u, v) to $(E' = (u^{N-1}w, u^{N-2}w, \dots, uw, w)_{w\in V}, e = (u^{N-1}v, u^{N-2}v, \dots, uv, v))$. Now E is naturally isomorphic to $(t^{-N}\mathsf{k}[[t]]/\mathsf{k}[[t]]) \otimes V$ (together with the action of t), and the assignment $(E', e) \mapsto (L := E' \oplus \mathsf{k}[[t]] \otimes V, e)$ embeds $\mathcal{Y}$ into $\mathbf{Gr}_{(N,0,\dots,0)} \times \mathbf{V}$. We will denote the composite embedding $\mathcal{N} \times V \hookrightarrow \mathbf{Gr} \times \mathbf{V}$ by $\psi : (u, v) \mapsto (L(u, v), e(u, v))$. There is an open subset $\mathcal{W} \subset \mathsf{k}[[t]] \otimes V$ with the property that for any $w \in \mathcal{W}$ and any $(u, v) \in (\mathcal{N} \times V)_{(\lambda,\mu)}$ (a G-orbit, see [14, Theorem 1]), we have $(L(u, v), e(u, v) + w) \in \Omega_{(\lambda,\mu)}$ (the corresponding $\mathbf{G_O}$-orbit in $\mathbf{Gr} \times \mathring{\mathbf{V}}$). Moreover, the resulting embedding $\mathcal{W} \times (\mathcal{N} \times V)_{(\lambda,\mu)} \hookrightarrow \Omega_{(\lambda,\mu)}$ is an open embedding. Also, the embedding $\mathcal{W} \times \overline{(\mathcal{N} \times V)}_{(\lambda,\mu)} \hookrightarrow \bar{\Omega}_{(\lambda,\mu)}$ of the orbit closures is open as well. Hence the Frobenius action on the IC stalks of $\overline{(\mathcal{N} \times V)}_{(\lambda,\mu)}$ is encoded in the polynomials $\Pi_{(\lambda',\mu'),(\lambda,\mu)}$ introduced after Proposition 14.

3.8. Mirković–Vybornov construction

The $\mathbf{G_O}$-orbits $\Omega_{(\lambda,\mu)} \subset \mathbf{Gr} \times \mathring{\mathbf{V}}$ considered in Subsection 3.7 are rather special: all the components λ_k, μ_k are nonnegative integers, and $\sum_{k=1}^N \lambda_k + \sum_{k=1}^N \mu_k = N$. To relate the singularities of more general orbit closures $\bar{\Omega}_{(\lambda',\mu')}$ to the singularities of orbits in the enhanced nilpotent cones (for different groups GL_n, $n \neq N$) we need a certain generalization of Lusztig's construction, due to Mirković and Vybornov [11].

To begin, note that the assignment $\phi_{i,j} : (L, v) \mapsto (t^{-i-j}L, t^{-i}v)$ is a $\mathbf{G_O}$-equivariant automorphism of $\mathbf{Gr} \times \mathring{\mathbf{V}}$ sending $\Omega_{(\lambda,\mu)}$ to $\Omega_{(\lambda+i^N,\mu+j^N)}$. Thus we may restrict ourselves to the study of orbits $\Omega_{(\lambda,\mu)}$ with $\lambda, \mu \in \mathbb{N}^N$ without restricting generality. Geometrically, this means to study the pairs (L, v) such that $L \supset L^1 = \mathbf{O}\langle e_1, \dots, e_N\rangle$ and $L \ni v \notin L^1$.

Let $n = rN$ for $r \in \mathbb{N}$. We consider an n-dimensional k-vector space D with a basis $\{e_{k,i} \mid 1 \le k \le r, 1 \le i \le N\}$ and a nilpotent endomorphism $x : e_{k,i} \mapsto e_{k-1,i},\ e_{1,i} \mapsto 0$. The *Mirković–Vybornov transversal slice* is defined as $T_x := \{x + f \mid f \in \mathrm{End}(D),\ f^{l,j}_{k,i} = 0 \text{ if } k \neq r\}$. Its intersection with the nilpotent cone of $\mathrm{End}(D)$ is $T_x \cap \mathcal{N}_n$.

Let $L^2 \in \mathbf{Gr}$ be given as $L^2 = t^{-r}L^1$. It lies in the orbit closure $\mathbf{Gr}_{(n,0,\dots,0)}$, and we will describe an open neighbourhood $\mathcal{U}$ of L^2 in $\mathbf{Gr}_{(n,0,\dots,0)}$ isomorphic to $T_x \cap \mathcal{N}_n$. We choose a direct complement to L^2 in $\mathbf{V}$ so that $L_2 := t^{-r-1}\mathsf{k}[t^{-1}] \otimes V$. Then $\mathcal{U}$ is formed by all the lattices whose projection along L_2 is an isomorphism onto L^2. Any $L \in \mathcal{U}$ is of the form $(1+g)L^2$ where $g : L^2 \to L_2$ is a linear map with kernel containing L^1, i.e. $g : L^2/L^1 \to L_2$. Now we use the natural identification of L^2/L^1 with D (so that the action of t corresponds to the action of x). Furthermore, we identify $t^{-r}V$ with a subset of $L^2/L^1 = D$. Hence we may view g as a sum $\sum_{k=1}^\infty t^{-k} g_k$ where $g_k : D \to t^{-r}V$ are linear maps. Composing with $t^{-r}V \hookrightarrow D$ we may view g_k as an endomorphism of D. Then L being a lattice is equivalent to the condition that $g_k = g_1(t + g_1)^{k-1}$ and $t + g_1$ is nilpotent. In other words, the desired isomorphism $T_x \cap \mathcal{N}_n \xrightarrow{\sim} \mathcal{U}$ is of the form

$$T_x \cap \mathcal{N}_n \ni x + f \mapsto L = L(x + f) := \Big(1 + \sum_{k=1}^{\infty} t^{-k} f(t + f)^{k-1}\Big)L^2.$$

Now we identify D with $t^{-1}V \oplus \cdots \oplus t^{-r}V \subset L^2$. Given a vector $v \in D$ we consider its image $v \in L^2$ under the above embedding, and define $e(x+f, v) \in L(x+f)$ as the preimage of v under the isomorphism $L \xrightarrow{\sim} L^2$ (projection along L_2). Thus we have constructed an embedding $\psi : (T_x \cap \mathcal{N}_n) \times D \hookrightarrow \mathbf{Gr} \times \mathbf{V}$, $(x+f, v) \mapsto (L(x+f), e(x+f, v))$. Note that the Jordan type of any nilpotent $x+f$ is given by a partition ν with the number of parts less than or equal to N. There is an open subset $\mathcal{W} \subset \mathsf{k}[[t]] \otimes V$ with the property that for any $w \in \mathcal{W}$ and any $(x+f, v) \in ((T_x \cap \mathcal{N}_n) \times V)_{(\lambda,\mu)}$ (the intersection with a GL_n-orbit), we have $(L(x+f), e(x+f, v) + w) \in \Omega_{(\lambda,\mu)}$ (the corresponding $\mathbf{G_O}$-orbit in $\mathbf{Gr} \times \mathring{\mathbf{V}}$). Moreover, the resulting embedding $\mathcal{W} \times ((T_x \cap \mathcal{N}_n) \times D)_{(\lambda,\mu)} \hookrightarrow \Omega_{(\lambda,\mu)}$ is an open embedding. Also, the embedding $\mathcal{W} \times \overline{((T_x \cap \mathcal{N}_n) \times V)}_{(\lambda,\mu)} \hookrightarrow \bar{\Omega}_{(\lambda,\mu)}$ of the intersection with the orbit closure is an open embedding as well.

We conclude that the orbit closures $\bar{\Omega}_{(\lambda,\mu)}$ with $\sum_{k=1}^N \lambda_k + \sum_{k=1}^N \mu_k$ divisible by N are equisingular to certain GL_n-orbit closures in $\mathcal{N}_n \times D$ for some n divisible by N.

3.9. Semismallness of convolution

We are ready for the proof of Theorem 1. Let us denote the selfdual Goresky–MacPherson sheaf on the orbit $\mathbf{Gr}_\lambda$ (whose class is C_λ) by IC_λ for short. Then the convolution power $IC_{(1,0,\dots,0)}^{*l}$ is isomorphic to $\bigoplus_{|\lambda|=l} K_\lambda \otimes IC_\lambda$ for certain vector spaces K_λ (equal to the multiplicities of irreducible GL_N-modules in $V^{\otimes l}$). We stress that K_λ is concentrated in degree 0, that is, the convolution morphism is stratified semismall. Thus it suffices to prove

$$IC_{(1,0,\dots,0)}^{*l} * IC_{(0^N,0^N)} * IC_{(1,0,\dots,0)}^{*m} \simeq \bigoplus_{\substack{|\mu|=m \\ |\lambda|=l}} K_\mu \otimes K_\lambda \otimes IC_{(\lambda,\mu)}. \tag{10}$$

Moreover, it suffices to prove (10) for m, l divisible by N. In fact, this would imply that the convolution morphism $\mathbf{Gr}_{(1,0,\dots,0)}^{*l} * \bar{\Omega}_{(0^N,0^N)} * \mathbf{Gr}_{(1,0,\dots,0)}^{*m} \to \mathbf{Gr} \times \mathring{\mathbf{V}}$ is stratified semismall for *any* $m, l \geq 0$. Indeed, if the direct image of the constant IC-sheaf under the above morphism involved some summands with nontrivial shifts in the derived category, the further convolution with $IC_{(1,0,\dots,0)}$ could not possibly kill the nontrivially shifted summands (due to selfduality and decomposition theorem), and so they would persist for some larger m, l divisible by N.

Having established the semismallness for arbitrary $m, l \geq 0$, we see that the semisimple abelian category formed by direct sums of $IC_{(\lambda,\mu)}$, $(\lambda,\mu) \in RB^{\mathrm{sph}}$, is a bimodule category over the tensor category formed by direct sums of IC_λ, $\lambda \in \mathfrak{S}_N^{\mathrm{sph}}$ (equivalent by Satake isomorphism to $\mathrm{Rep}(\mathrm{GL}_N)$). To specify such a bimodule category it suffices to specify the left and right action of the generator $IC_{(1,0,\dots,0)}$, and there is only one action satisfying (10) with m, l divisible by N: it necessarily satisfies (10) for any m, l.

We set $n = m + l$. The advantage of having n divisible by N is that according to 3.8, the (open part of the) orbit closure is equisingular to a certain slice of the

GL_n-orbit closure in $\mathcal{N}_n \times D$. To describe the convolution diagram in terms of GL_n we need to recall a Springer type construction of [3, 5.4].

So $\tilde{\mathfrak{Y}}_{n,m}$ is the smooth variety of triples $(u, F_\bullet, v)$ where $F_\bullet$ is a complete flag in the n-dimensional vector space D, and u is a nilpotent endomorphism of D such that $uF_k \subset F_{k-1}$, and $v \in F_{n-m}$. We have a proper morphism $\pi_{n,m} : \tilde{\mathfrak{Y}}_{n,m} \to \mathcal{N}_n \times D$ with the image $\mathfrak{Y}_{n,m} \subset \mathcal{N}_n \times D$ formed by all the pairs (u, v) such that $\dim\langle v, uv, u^2v, \dots\rangle \le n-m$. It follows from the proof of [3, Proposition 5.4.1] that $\pi_{n,m}$ is a semismall resolution of singularities, and

$$(\pi_{n,m})_* IC(\tilde{\mathfrak{Y}}_{n,m}) \simeq \bigoplus_{\substack{|\mu|=m \\ |\lambda|=n-m}} \mathrm{L}_\mu \otimes \mathrm{L}_\lambda \otimes IC_{(\lambda,\mu)} \tag{11}$$

where L_μ (resp. L_λ) is the irreducible representation of $\mathfrak{S}_m$ (resp. $\mathfrak{S}_{n-m}$) corresponding to the partition μ (resp. λ); furthermore, $IC_{(\lambda,\mu)}$ is the IC-sheaf of the GL_n-orbit closure $\overline{(\mathcal{N}_n \times D)}_{(\lambda,\mu)}$ (cf. Theorem 4.5 of [1]).

Recall the nilpotent element $x \in \mathcal{N}_m$ introduced in 3.8, and the slice $T_x \cap \mathcal{N}_n$. We will denote $\pi_{n,m}^{-1}((T_x \cap \mathcal{N}_n) \times D)$ by $T\tilde{\mathfrak{Y}}_{n,m} \subset \tilde{\mathfrak{Y}}_{n,m}$. Recall the open embedding $\varphi : \mathcal{W} \times ((T_x \cap \mathcal{N}_n) \times D) \hookrightarrow \bar{\Omega}_{(n,0,\dots,0),(0^N)}$ of 3.8. Let us denote the convolution diagram $\mathbf{Gr}^{*l}_{(1,0,\dots,0)} * \bar{\Omega}_{(0^N,0^N)} * \mathbf{Gr}^{*m}_{(1,0,\dots,0)}$ by $\tilde{\Omega}_{(l,0,\dots,0),(m,0,\dots,0)}$ for short; let us denote its morphism to $\bar{\Omega}_{(n,0,\dots,0),(0^N)}$ (with the image $\bar{\Omega}_{(l,0,\dots,0),(m,0,\dots,0)}$) by $\varpi_{n,m}$. Finally, let us denote the preimage under $\varpi_{n,m}$ of $\varphi(\mathcal{W} \times ((T_x \cap \mathcal{N}_n) \times D))$ by $T\tilde{\Omega}_{(l,0,\dots,0),(m,0,\dots,0)}$. The next lemma follows by comparison of definitions:

Lemma 3. *We have a commutative diagram*

$$\begin{array}{ccc}
\mathcal{W} \times T\tilde{\mathfrak{Y}}_{n,m} & \xrightarrow{\ \sim\ } & T\tilde{\Omega}_{(l,0,\dots,0),(m,0,\dots,0)} \\
\Big\downarrow {\scriptstyle \mathrm{id} \times \pi_{n,m}} & & {\scriptstyle \varpi_{n,m}} \Big\downarrow \\
\mathcal{W} \times ((T_x \cap \mathcal{N}_n) \times D) & \xrightarrow{\ \varphi\ } & \bar{\Omega}_{(n,0,\dots,0),(0^N)}
\end{array}$$

Since $\mathrm{L}_\lambda = K_\lambda$ by Schur–Weyl duality, the proof of the theorem is finished. □

Remark 2. Due to Lusztig's construction of 3.7, Theorem 1 implies Proposition 4.6 of [1].

4. Mirabolic Hall bimodule

4.1. Recollections

The field k is still $\mathbb{F}_q$. The *Hall algebra* Hall $=$ Hall_N of all finite $\mathsf{k}[[t]]$-modules which are direct sums of $\le N$ indecomposable modules is defined as in [8, II.2]. It is a quotient algebra of the "universal" Hall algebra $H(\mathsf{k}[[t]])$ of [8]. It has a basis $\{\mathfrak{u}_\lambda\}$ where λ runs through the set ${}^+\mathfrak{S}_N^{\mathrm{sph}}$ of partitions with $\le N$ parts. It is a free polynomial algebra with generators $\{\mathfrak{u}_{(1^r)} \mid 1 \le r \le N-1\}$. The structure constants $G^\lambda_{\mu\nu}$ being polynomial in q, we may and will view Hall as the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra **Hall**. Extending scalars to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ we obtain a $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}a\ell\ell$.

Let $\Lambda = \Lambda_N$ denote the ring of symmetric polynomials in the variables $X = (X_1, \dots, X_N)$ over $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$. There is an isomorphism $\Psi : \mathcal{H}a\ell\ell \xrightarrow{\sim} \Lambda$ sending $\mathfrak{u}_{(1^r)}$ to $\mathbf{v}^{-r(r-1)}e_r$ (elementary symmetric polynomial), and $\mathfrak{u}_\lambda$ to $\mathbf{v}^{-2n(\lambda)}P_\lambda(X, \mathbf{v}^{-2})$ ([8, Chapter III]) where $P_\lambda(X, \mathbf{v}^{-2})$ is the Hall–Littlewood polynomial and $n(\lambda) = \sum_{i=1}^N (i-1)\lambda_i$. Let us denote by ${}^+\mathcal{H}^{\mathrm{sph}}$ the subalgebra of $\mathcal{H}^{\mathrm{sph}}$ spanned by $\{U_\lambda \mid \lambda \in {}^+\mathfrak{S}_N^{\mathrm{sph}}\}$. Then we have a natural identification of ${}^+\mathcal{H}^{\mathrm{sph}}$ with $\mathcal{H}a\ell\ell$ sending U_λ to $\mathfrak{u}_\lambda$, and C_λ to $\mathfrak{c}_\lambda$. Furthermore, $\Psi(\mathfrak{c}_\lambda) = (-\mathbf{v})^{-(N-1)|\lambda|}s_\lambda$ (Schur polynomial).

4.2. The Mirabolic Hall bimodule

A finite $\mathsf{k}[[t]]$-module which is a direct sum of $\leq N$ indecomposable modules is the same as a k-vector space D with a nilpotent operator u with $\leq N$ Jordan blocks. The isomorphism classes of pairs (u, v) (where $v \in D$) are numbered by the set ${}^+RB^{\mathrm{sph}}$ of pairs of partitions (λ, μ) with $\leq N$ parts in λ and $\leq N$ parts in μ. We define the structure constants $G^{(\lambda,\mu)}_{(\lambda',\mu')\nu}$ and $G^{(\lambda,\mu)}_{\nu(\lambda',\mu')}$ as follows.[1] $G^{(\lambda,\mu)}_{\nu(\lambda',\mu')}$ is the number of u-invariant subspaces $D'' \subset D$ such that the isomorphism type of $u|_{D''}$ is given by ν, and the isomorphism type of $(u|_{D/D''}, v \pmod{D''})$ is given by (λ', μ'). Furthermore, $G^{(\lambda,\mu)}_{(\lambda',\mu')\nu}$ is the number of u-invariant subspaces $D' \subset D$ containing v such that the isomorphism type of $(u|_{D'}, v)$ is given by (λ', μ'), and the isomorphism type of $u|_{D/D'}$ is given by ν. Note that some similar quantities were introduced in Proposition 5.8 of [1]: in notations there, we have $g^{\lambda;\mu}_{\theta;\nu} = \sum_{\lambda'+\mu'=\theta} G^{(\lambda,\mu)}_{(\lambda',\mu')\nu}$.

Lemma 4. *For any ${}^+RB^{\mathrm{sph}} \ni (\lambda, \mu), (\lambda', \mu')$, $1 \leq r \leq N-1$, the structure constants $G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')}$ and $G^{(\lambda,\mu)}_{(\lambda',\mu')(1^r)}$ are given by the formulas* (6) *and* (7).

Proof. Was given in Subsection 3.3. □

We define the *mirabolic Hall bimodule* Mall over Hall to have a $\mathbb{Z}$-basis $\{\mathfrak{u}_{(\lambda,\mu)} \mid (\lambda, \mu) \in {}^+RB^{\mathrm{sph}}\}$ and the structure constants

$$\mathfrak{u}_\nu \mathfrak{u}_{(\lambda',\mu')} = \sum_{(\lambda,\mu)\in {}^+RB^{\mathrm{sph}}} G^{(\lambda,\mu)}_{\nu(\lambda',\mu')}\mathfrak{u}_{(\lambda,\mu)}, \ \mathfrak{u}_{(\lambda',\mu')}\mathfrak{u}_\nu = \sum_{(\lambda,\mu)\in {}^+RB^{\mathrm{sph}}} G^{(\lambda,\mu)}_{(\lambda',\mu')\nu}\mathfrak{u}_{(\lambda,\mu)}.$$

The structure constants $G^{(\lambda,\mu)}_{(\lambda',\mu')(1^r)}$ and $G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')}$ for the generators of Hall being polynomial in q, we may and will view Mall as the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-bimodule **Mall** over the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra **Hall**. Extending scalars to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ we obtain a $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$-bimodule $\mathcal{M}a\ell\ell$ over the $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}a\ell\ell$. Let us denote by ${}^+\mathcal{R}^{\mathrm{sph}}$ the ${}^+\mathcal{H}^{\mathrm{sph}}$-subbimodule of $\mathcal{R}^{\mathrm{sph}}$ spanned by

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[1] The notation $G^{(\lambda,\mu)}_{(\lambda',\mu')(1^r)}$ and $G^{(\lambda,\mu)}_{(1^r)(\lambda',\mu')}$ introduced in Subsection 3.3 is just a particular case of the present one for $\nu = (1^r)$ as we will see in Lemma 4.

$\{U_{(\lambda,\mu)} \mid (\lambda,\mu) \in {}^+RB^{\mathrm{sph}}\}$. Then we have a natural identification of ${}^+\mathcal{R}^{\mathrm{sph}}$ with $\mathcal{M}a\ell\ell$ sending $U_{(\lambda,\mu)}$ to $\mathfrak{u}_{(\lambda,\mu)}$. For $(\lambda,\mu) \in {}^+RB^{\mathrm{sph}}$ we set

$$ {}^+C_{(\lambda,\mu)} := \sum_{{}^+RB^{\mathrm{sph}} \ni (\lambda',\mu') \le (\lambda,\mu)} \Pi_{(\lambda',\mu'),(\lambda,\mu)} H_{(\lambda',\mu')} $$

(notation introduced after Proposition 14). We define $\mathfrak{c}_{(\lambda,\mu)} \in \mathcal{M}a\ell\ell$ as the element corresponding to ${}^+C_{(\lambda,\mu)}$ under the above identification.

Theorem 1 has the following corollary:

Corollary 1. *For any $\lambda, \mu \in {}^+\mathfrak{S}_N^{\mathrm{sph}}$ we have $\mathfrak{c}_\lambda \mathfrak{c}_{(0^N,0^N)} \mathfrak{c}_\mu = \mathfrak{c}_{(\lambda,\mu)}$.*

Hence there is a unique isomorphism $\Psi : \mathcal{M}a\ell\ell \xrightarrow{\sim} \Lambda \otimes \Lambda$ of $\mathcal{H}a\ell\ell \simeq \Lambda$-bimodules sending $\mathfrak{c}_{(\lambda,\mu)}$ to $(-\mathbf{v})^{-(N-1)(|\lambda|+|\mu|)} s_\lambda \otimes s_\mu$. We define

$$ \Lambda \otimes \Lambda \ni P_{(\lambda,\mu)}(X, Y, \mathbf{v}^{-1}) := (-\mathbf{v})^{2n(\lambda)+2n(\mu)+|\mu|} \Psi(\mathfrak{u}_{(\lambda,\mu)}) $$

(*mirabolic Hall–Littlewood polynomials*).

Thus the polynomials $\Pi_{(\lambda',\mu'),(\lambda,\mu)}$ are the matrix coefficients of the transition matrix from the basis $\{P_{(\lambda,\mu)}(X, Y, \mathbf{v}^{-1})\}$ to the basis $\{s_\lambda(X)s_\mu(Y)\}$ of $\Lambda \otimes \Lambda$. It follows from Theorem 5.2 of [1] that the mirabolic Hall–Littlewood polynomial $P_{(\lambda,\mu)}(X, Y, \mathbf{v}^{-1})$ coincides with Shoji's Hall–Littlewood function $P^{\pm}_{(\lambda,\mu)}(X, Y, \mathbf{v}^{-1})$ (see Section 2.5 and Theorem 2.8 of [12]).

5. Frobenius traces in mirabolic character sheaves

5.1. Unipotent mirabolic character sheaves

Recall the construction of certain mirabolic character sheaves in [3, 5.4]. So $\tilde{\mathfrak{X}}_{n,m}$ is the smooth variety of triples $(g, F_\bullet, v)$ where $F_\bullet$ is a complete flag in an n-dimensional vector space D, and $v \in F_m$, and g is an invertible linear transformation of D preserving $F_\bullet$. We have a proper morphism $\pi_{n,m} : \tilde{\mathfrak{X}}_{n,m} \to \mathrm{GL}_n \times D$ with the image $\mathfrak{X}_{n,m} \subset \mathrm{GL}_n \times D$ formed by all the pairs (g, v) such that $\dim\langle v, gv, g^2v, \dots\rangle \le n - m$. According to [3, Corollary 5.4.2], we have

$$ (\pi_{n,m})_* IC(\tilde{\mathfrak{X}}_{n,m}) \simeq \bigoplus_{\substack{|\mu|=m \\ |\lambda|=n-m}} \mathrm{L}_\mu \otimes \mathrm{L}_\lambda \otimes \mathcal{F}_{\lambda,\mu} \tag{12} $$

for certain irreducible perverse mirabolic character sheaves $\mathcal{F}_{\lambda,\mu}$ on $\mathrm{GL}_n \times D$.

Following [AH], we set $b(\lambda,\mu) := 2n(\lambda) + 2n(\mu) + |\mu|$, so that $b(\lambda',\mu') - b(\lambda,\mu)$ equals the codimension of $\Omega_{(\lambda',\mu')}$ in $\bar{\Omega}_{(\lambda,\mu)}$, and the codimension of $(\mathcal{N}_n \times D)_{(\lambda',\mu')}$ in $\overline{(\mathcal{N}_n \times D)}_{(\lambda,\mu)}$.

Theorem 2. *Let $(u, v) \in (\mathcal{N}_n \times D)_{(\lambda',\mu')}(\mathbb{F}_q)$. The trace of the Frobenius automorphism of the stalk of $\mathcal{F}_{\lambda,\mu}$ at (u, v) equals $\sqrt{q}^{\,b(\lambda',\mu')-b(\lambda,\mu)} \Pi_{(\lambda',\mu'),(\lambda,\mu)}(\sqrt{q})$ (see (9)).*

Proof. We identify the nilpotent cone $\mathcal{N}_n$ and the variety of unipotent elements of GL_n by adding the identity matrix, so that we may view $\mathcal{N}_n \subset \mathrm{GL}_n$. Then $\mathfrak{X}_{n,m} \cap (\mathcal{N}_n \times D) = \mathfrak{Y}_{n,m}$, and $\pi_{n,m}^{-1}(\mathfrak{X}_{n,m} \cap (\mathcal{N}_n \times D)) = \tilde{\mathfrak{Y}}_{n,m}$ (notations of the proof of Theorem 1). Comparing (12) with (11), we see that $\mathcal{F}_{\lambda,\mu}|_{\mathcal{N}_n \times D} \simeq IC_{(\lambda,\mu)}$. Hence the trace of Frobenius in the stalk of $\mathcal{F}_{\lambda,\mu}$ at (u,v) equals the trace of Frobenius in the stalk of $IC_{(\lambda,\mu)}$ at (u,v). The latter is equal to the matrix coefficient of the transition matrix from the basis $\{j_! \underline{\overline{\mathbb{Q}}_l}_{(\mathcal{N}_n \times D)_{(\lambda',\mu')}}[n^2 - b(\lambda',\mu')]\}$ to the basis $\{j_{!*} \underline{\overline{\mathbb{Q}}_l}_{(\mathcal{N}_n \times D)_{(\lambda,\mu)}}[n^2 - b(\lambda,\mu)]\}$. And the latter by construction, up to the factor of $\sqrt{q}^{b(\lambda',\mu') - b(\lambda,\mu)}$, is equal to $\Pi_{(\lambda',\mu'),(\lambda,\mu)}(\sqrt{q})$. □

5.2. $\mathbb{G}_m$-equivariant mirabolic character sheaves

More generally, we recall the notion [4] of mirabolic character sheaves equivariant with respect to the dilation action of $\mathbb{G}_m$ on D. Let $\mathcal{B}$ be the flag variety of $\mathrm{GL}(D)$, let $\widetilde{\mathcal{B}}$ be the base affine space of $\mathrm{GL}(D)$, so that $\widetilde{\mathcal{B}} \to \mathcal{B}$ is a $\mathrm{GL}(D)$-equivariant H-torsor, where H is the abstract Cartan torus of $\mathrm{GL}(D)$. Let $\mathcal{Y}$ be the quotient of $\widetilde{\mathcal{B}} \times \widetilde{\mathcal{B}}$ modulo the diagonal action of H; it is called the *horocycle space* of $\mathrm{GL}(D)$. Clearly, $\mathcal{Y}$ is an H-torsor over $\mathcal{B} \times \mathcal{B}$ with respect to the right action $(\tilde{x}_1, \tilde{x}_2) \cdot h := (\tilde{x}_1 \cdot h, \tilde{x}_2)$. We consider the following diagram of $\mathrm{GL}(D)$-varieties and $\mathrm{GL}(D) \times \mathbb{G}_m$-equivariant maps:

$$\mathrm{GL}(D) \times D \xleftarrow{\mathrm{pr}} \mathrm{GL}(D) \times \mathcal{B} \times D \xrightarrow{f} \mathcal{Y} \times D.$$

In this diagram, the map pr is given by $\mathrm{pr}(g,x,v) := (g,v)$. To define the map f, we think of $\mathcal{B}$ as $\widetilde{\mathcal{B}}/H$, and for a representative $\tilde{x} \in \widetilde{\mathcal{B}}$ of $x \in \mathcal{B}$ we set $f(g,x,v) := (g\tilde{x}, \tilde{x}, gv)$. The group $\mathrm{GL}(D)$ acts diagonally on all the product spaces in the above diagram, and acts on itself by conjugation. The group $\mathbb{G}_m$ acts by dilations on D.

The functor CH from the constructible derived category of l-adic sheaves on $\mathcal{Y} \times D$ to the constructible derived category of l-adic sheaves on $\mathrm{GL}(D) \times D$ is defined as $\mathsf{CH} := \mathrm{pr}_* f^![-\dim \mathcal{B}]$. Now let $\mathcal{F}$ be a $\mathrm{GL}(D) \times \mathbb{G}_m$-equivariant, H-monodromic perverse sheaf on $\mathcal{Y} \times D$. The irreducible perverse constituents of $\mathsf{CH}\mathcal{F}$ are called $\mathbb{G}_m$-*equivariant mirabolic character sheaves* on $\mathrm{GL}(D) \times D$. We define a $\mathbb{G}_m$-equivariant mirabolic character sheaf as a direct sum of the above constituents for various $\mathcal{F}$ as above. The semisimple abelian category of $\mathbb{G}_m$-equivariant mirabolic character sheaves will be denoted $\mathcal{MC}(\mathrm{GL}(D) \times D)$. Clearly, this is a direct analogue of Lusztig's definition of character sheaves. The semisimple abelian category of character sheaves on $\mathrm{GL}(D)$ will be denoted $\mathcal{C}(\mathrm{GL}(D))$.

5.3. Left and right induction

Following Lusztig's construction of *induction* of character sheaves, we define the left and right action of Lusztig's character sheaves on the mirabolic character sheaves (for varying D). To this end it will be notationally more convenient to consider $\mathcal{MC}(\mathrm{GL}(D) \times D)$ (resp. $\mathcal{C}(\mathrm{GL}(D))$) as a category of perverse sheaves on the quotient stack $\mathrm{GL}(D)\backslash(\mathrm{GL}(D) \times D)$ (resp. $\mathrm{GL}(D)\backslash\mathrm{GL}(D)$). Let $m \leq n = \dim(D)$, let V be an $n-m$-dimensional k-vector space, and let W be an m-dimensional k-vector space. We have the following diagrams:

$$\mathrm{GL}(D)\backslash(\mathrm{GL}(D)\times D) \xleftarrow{p} A \xrightarrow{q} \mathrm{GL}(V)\backslash\mathrm{GL}(V)\times\mathrm{GL}(W)\backslash(\mathrm{GL}(W)\times W), \quad (13)$$

$$\mathrm{GL}(D)\backslash(\mathrm{GL}(D)\times D) \xleftarrow{d} B \xrightarrow{b} \mathrm{GL}(V)\backslash(\mathrm{GL}(V)\times V)\times\mathrm{GL}(W)\backslash\mathrm{GL}(W). \quad (14)$$

Here A is the quotient stack of $\tilde{A}$ by the action of $\mathrm{GL}(D)$, where

$$\tilde{A} := \{(g\in\mathrm{GL}(D), F\subset D, v\in D) \mid \dim F = n-m \text{ and } gF=F\},$$

and p forgets F, while q sends (g,F,v) to $(g|_F; (g|_{D/F}, v \pmod F))$ (under an arbitrary identification $V\simeq F$, $W\simeq D/F$). Note that p is proper, and q is smooth of relative dimension $n-m$.

Furthermore, B is the quotient stack of $\tilde{B}$ by the action of $\mathrm{GL}(D)$, where

$$\tilde{B} := \{(g\in\mathrm{GL}(D), F\subset D, v\in F) \mid \dim F = n-m \text{ and } gF=F\},$$

and d forgets F, while b sends (g,F,v) to $((g|_F, v); g|_{D/F})$ (under an arbitrary identification $V\simeq F$, $W\simeq D/F$). Note that d is proper, and b is smooth of relative dimension 0.

Finally, for a character sheaf $\mathcal{G}\in\mathcal{C}(\mathrm{GL}(V))$ and a mirabolic character sheaf $\mathcal{F}\in\mathcal{MC}(\mathrm{GL}(W)\times W)$ we define the *left* convolution $\mathcal{G}*\mathcal{F} := p_!q^*(\mathcal{G}\boxtimes\mathcal{F})[n-m]$. Similarly, for a character sheaf $\mathcal{G}'\in\mathcal{C}(\mathrm{GL}(W))$ and a mirabolic character sheaf $\mathcal{F}'\in\mathcal{MC}(\mathrm{GL}(V)\times V)$ we define the *right* convolution $\mathcal{F}'*\mathcal{G}' := d_!b^*(\mathcal{F}'\boxtimes\mathcal{G}')$.

Note that the definition of convolution works in the extreme cases $m=0$ or $n-m=0$ as well: if $\dim V=0$, then $\mathrm{GL}(V)$ is just the trivial group. The following proposition is proved like Proposition 4.8.b) in [7].

Proposition 15. *Both $\mathcal{G}*\mathcal{F}$ and $\mathcal{F}'*\mathcal{G}'$ are $\mathbb{G}_m$-equivariant mirabolic character sheaves on* $\mathrm{GL}(D)\times D$.

We denote by $\underline{\overline{\mathbb{Q}}_l}$ the unique $\mathbb{G}_m$-equivariant mirabolic character sheaf on $\mathrm{GL}(D)\times D$ for $\dim(D)=0$.

Proposition 16. *Let $\mathcal{G}\in\mathcal{C}(\mathrm{GL}(V))$ and $\mathcal{G}'\in\mathcal{C}(\mathrm{GL}(W))$ be* irreducible *character sheaves. Then $\mathcal{G}*\underline{\overline{\mathbb{Q}}_l}*\mathcal{G}'$ is irreducible.*

Proof. Let $\dim(D)=n$, $\dim(W)=m$, $\dim(V)=n-m$. Recall the diagram (14), and denote by $r:\mathrm{GL}(V)\backslash(\mathrm{GL}(V)\times V)\to\mathrm{GL}(V)\backslash\mathrm{GL}(V)$ the natural projection (forgetting the vector v). Then

$$\mathcal{G}*\underline{\overline{\mathbb{Q}}_l}*\mathcal{G}' = d_!b^*(r^*\mathcal{G}\boxtimes\mathcal{G}'[n-m]).$$

The sheaf $b^*(r^*\mathcal{G}\boxtimes\mathcal{G}'[n-m])$ is irreducible perverse on B; more precisely, it is the intermediate extension of a local system on an open part of B. The image of the proper morphism d coincides with $\mathrm{GL}(D)\backslash\mathfrak{X}_{n,m}$ (notations of 5.1), and $d: B\to\mathrm{GL}(D)\backslash\mathfrak{X}_{n,m}$ is generically an isomorphism: F is reconstructed as $F=\langle v, gv, g^2v,\ldots\rangle$. Finally, the arguments absolutely similar to the proof of Proposition 4.5 of [6] prove that d is stratified small. This implies that $d_!b^*(r^*\mathcal{G}\boxtimes\mathcal{G}'[n-m])$ is irreducible. □

Conjecture 1. *Any irreducible* $\mathbb{G}_m$*-equivariant mirabolic character sheaf on* $\mathrm{GL}(D)\times D$ *is isomorphic to* $\mathcal{G}*\overline{\mathbb{Q}}_l*\mathcal{G}'$ *for some* $\mathcal{G}\in\mathcal{C}(\mathrm{GL}(V))$ *and* $\mathcal{G}'\in\mathcal{C}(\mathrm{GL}(W))$ *where* $\dim(V)+\dim(W)=\dim(D)$.

5.4. Mirabolic Green bimodule

Once again $\mathsf{k}=\mathbb{F}_q$. We will freely use the notation of Chapter IV of [8]. In particular, Φ is the set of Frobenius orbits in $\overline{\mathbb{F}}_q^\times$, or equivalently, the set of irreducible monic polynomials in $\mathbb{F}_q[t]$ with the exception of $f=t$. We consider the set of isomorphism classes (D,g,v) where D is a k-vector space, $v\in D$, and g is an invertible linear operator $D\to D$. Similarly to [8, Section 2] we identify this set with the set of finitely supported functions $(\boldsymbol{\lambda},\boldsymbol{\mu}):\Phi\to\mathcal{P}\times\mathcal{P}$ to the set of pairs of partitions. Note that $\dim(D)=|(\boldsymbol{\lambda},\boldsymbol{\mu})|:=\sum_{f\in\Phi}\deg(f)(|\boldsymbol{\lambda}(f)|+|\boldsymbol{\mu}(f)|)$. Let $c_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\subset\mathrm{GL}(D)\times D$ be the corresponding $\mathrm{GL}(D)$-orbit, and let $\boldsymbol{\pi}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}$ be its characteristic function. Let $\mathcal{MA}$ be the $\overline{\mathbb{Q}}_l$-vector space with the basis $\{\boldsymbol{\pi}_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\}$. It is evidently isomorphic to $\bigoplus_{n\geq 0}\overline{\mathbb{Q}}_l(\mathrm{GL}(\mathsf{k}^n)\times\mathsf{k}^n)^{\mathrm{GL}(\mathsf{k}^n)}$.

Recall the Green algebra $\mathcal{A}=\bigoplus_{n\geq 0}\mathcal{A}_n$ of class functions on the groups $\mathrm{GL}_n(\mathbb{F}_q)$ (see [8, Section 3]; multiplication is given by parabolic induction) with the basis $\{\boldsymbol{\pi}_{\boldsymbol{\mu}}\}$ of characteristic functions of conjugacy classes. The construction of 5.3 equips $\mathcal{MA}$ with the structure of an $\mathcal{A}$-bimodule. It is easily seen to be a free bimodule of rank 1 with a generator $\boldsymbol{\pi}_{(\mathbf{0},\mathbf{0})}$ given by the zero function (taking the value of zero bipartition on any $f\in\Phi$). The structure constants are as follows (the proof is similar to [8, (3.1)]).

$$\boldsymbol{\pi}_{\boldsymbol{\nu}}\boldsymbol{\pi}_{(\boldsymbol{\lambda}',\boldsymbol{\mu}')}=\sum_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\mathrm{g}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}_{\boldsymbol{\nu}(\boldsymbol{\lambda}',\boldsymbol{\mu}')}\boldsymbol{\pi}_{(\boldsymbol{\lambda},\boldsymbol{\mu})},\qquad \boldsymbol{\pi}_{(\boldsymbol{\lambda}',\boldsymbol{\mu}')}\boldsymbol{\pi}_{\boldsymbol{\nu}}=\sum_{(\boldsymbol{\lambda},\boldsymbol{\mu})}\mathrm{g}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}_{(\boldsymbol{\lambda}',\boldsymbol{\mu}')\boldsymbol{\nu}}\boldsymbol{\pi}_{(\boldsymbol{\lambda},\boldsymbol{\mu})},\tag{15}$$

where

$$\begin{aligned}\mathrm{g}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}_{\boldsymbol{\nu}(\boldsymbol{\lambda}',\boldsymbol{\mu}')}&=\prod_{f\in\Phi}G^{(\boldsymbol{\lambda}(f),\boldsymbol{\mu}(f))}_{\boldsymbol{\nu}(f)(\boldsymbol{\lambda}'(f),\boldsymbol{\mu}'(f))}(q^{\deg(f)}),\\ \mathrm{g}^{(\boldsymbol{\lambda},\boldsymbol{\mu})}_{(\boldsymbol{\lambda}',\boldsymbol{\mu}')\boldsymbol{\nu}}&=\prod_{f\in\Phi}G^{(\boldsymbol{\lambda}(f),\boldsymbol{\mu}(f))}_{(\boldsymbol{\lambda}'(f),\boldsymbol{\mu}'(f))\boldsymbol{\nu}(f)}(q^{\deg(f)}).\end{aligned}\tag{16}$$

Now recall another basis $\{S_{\boldsymbol{\eta}}\}$ of $\mathcal{A}$ (see [8, Section 4], numbered by the finitely supported functions from Θ to $\mathcal{P}$. Here Θ is the set of Frobenius orbits on the direct limit L of the character groups $(\mathbb{F}_{q^n}^\times)^\vee$. This is the basis of irreducible characters. According to Lusztig, for $|\boldsymbol{\eta}|=m$, the function $S_{\boldsymbol{\eta}}$ is the Frobenius trace function of an irreducible Weil character sheaf $\mathcal{S}_{\boldsymbol{\eta}}$ on GL_m. Due to Proposition 16, for $|\boldsymbol{\eta}|+|\boldsymbol{\eta}'|=n$, the function $S_{\boldsymbol{\eta}'}\boldsymbol{\pi}_{(\mathbf{0},\mathbf{0})}S_{\boldsymbol{\eta}}$ is the Frobenius trace function of an irreducible $\mathbb{G}_m$-equivariant Weil mirabolic character sheaf $\mathcal{S}_{\boldsymbol{\eta}'}*\overline{\mathbb{Q}}_l*\mathcal{S}_{\boldsymbol{\eta}}$ on $\mathrm{GL}(D)\times D$, $\dim(D)=n$. We know that the set of functions $\{S_{\boldsymbol{\eta}'}\boldsymbol{\pi}_{(\mathbf{0},\mathbf{0})}S_{\boldsymbol{\eta}}\}$ forms a basis of the mirabolic Green bimodule $\mathcal{MA}$. Hence, if Conjecture 1 holds true, then the set of Frobenius trace functions of irreducible $\mathbb{G}_m$-equivariant Weil mirabolic character sheaves forms a basis of $\mathcal{MA}$. This would be a positive answer to a question of G. Lusztig.

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