Formation of coalition structures as a non-cooperative game 1: theory

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Abstract

The paper defines a non-cooperative simultaneous finite game to study coalition structure formation with intra and inter-coalition externalities. The novelty of the game is that the game definition embeds a *coalition structure formation mechanism*.

This mechanism portions a set of strategies of the game into partition-specific strategy domains, what makes every partition to be a non-cooperative game with partition-specific payoffs for every player.

The mechanism includes a maximum coalition size, a set of eligible partitions with coalitions sizes no greater than this number (which

*This paper comes as a further development of the 3-rd and the 4-th chapters of my PhD Thesis "Essays on Trade and Cooperation" at Ca Foscari University, Venezia, Italy. Acknowledge: Nick Baigent, Phillip Bich, Jean-Marc Bonnisseau, Alex Boulatov, Emiliano Catonini, Giulio Codognato, Sergio Currarini, Luca Gelsomini, Izhak Gilboa, Olga Gorelkina, Piero Gottardi, Roman Gorpenko, Eran Hanany, Mark Kelbert, Ludovic Julien, Alex Kokorev, Dmitry Makarov, Francois Maniquet, Igal Miltaich, Stephane Menozzi, Roger Myerson, Miklos Pinter, Ariel Rubinstein, Marina Sandomirskaya, William Thompson, Konstantin Sonin, Simone Tonin, Dimitrios Tsomocos, Eyal Winter, Shmuel Zamir. Special thanks to Fuad Aleskerov, Shlomo Weber and Lev Gelman. Many thanks for advices and for discussion to participants of SI&GE-2015, 2016 (an earlier version of the title was "A generalized Nash equilibrium"), CORE 50 Conference, CEPET 2016 Workshop, Games 2016 Congress, Games and Applications 2016 at Lisbon, Games and Optimization at St Etienne. All possible mistakes are only mine. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
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also serves as a restriction for a maximum number of deviators) and a coalition structure formation rule. The paper defines a family of nested non-cooperative games parametrized by a size of a maximum coalition size.

Every game in the family has an equilibrium in mixed strategies. The equilibrium can generate more than one coalition and encompasses intra and inter group externalities, what makes it different from the Shapley value. Presence of individual payoff allocation makes it different from a strong Nash, coalition-proof equilibrium, and some other equilibrium concepts. The accompanying papers demonstrate applications of the proposed toolkit.

Keywords: Noncooperative Games
JEL : C72

1 Subject of the paper

The research topic of this paper was inspired by John Nash's "Equilibrium Points in n-person games "(1950). This remarkably short, but highly influential note of only 5 paragraphs established an equilibrium concept and the proof of its existence which did not require an explicit specification of a final coalition structure for a set of players. Prior to Nash's paper, the generalization of the concept of equilibrium provided by von Neumann for the case of two-players zero-sum game was done by portioning the players into two groups and regarding several players as a single player. However up to now, non of these approaches resulted into expected progress in studying intra- and inter group externalities between players.

In this paper "coalition structure", or "partition"[1] for short, is a collection of non-overlapping subsets from a set of players, which in a union make the

[1] Existing literature uses both terms.

original set. A group, or a coalition, is an element of a coalition structure or of a partition.[2]

A partition induces two types of effects on a player's payoff. The first, through actions of players of the same coalition. These effects will be addressed as *intra*-coalition externalities. The second, from all other players, who are outside coalition, and belong to different coalitions. This effects will be addressed as *inter*-group externalities.

Nash (1953) suggested that cooperation should be studied within a group and in terms of non-cooperative fundamentals. This suggestion is known now as the Nash Program. Cooperative behavior was understood as an activity inside a group with positive externalities between players. Nash did not write explicitly about multi-coalition framework or coalition structures. Coalition structures allow us to study inter-coalition externalities, along with intra-coalition, ones and to separate cooperation in payoffs and cooperation in allocation of players.

The best analogy for the difference between Nash Program and the current research is the difference between partial and general strategic equilibrium analysis[3] in economics. The former isolates a market ignoring cross-market interactions, the latter explicitly studies cross market interactions.

The research agenda of the paper is: how to construct coalition structures from actions of self-interested agents. Moreover, the paper offers a generalization of a non-cooperative game from Nash (1950) to address the problem of coalition structure formation absent in Nash (1953). The contributions of this paper are: a construction of a non-cooperative game with an embedding coalition structure formation mechanism, and a parametrization of all constructed games by a number of deviators. The three subsequent papers will demonstrate various applications of the model to different types of games, including construction of a non-cooperative stability criterium and defining

[2]the same.
[3]meaning, strategic market games of Shapley and Shubik

a cooperation.

The paper has the following structure: Section 2 presents an example, on why studying inter-coalition externalities requires including coalition structures into individual strategy sets, Section 3 presents a general model of the game, Section 4 presents an example of the game. The last section discusses the approach used and the results of the paper.

2 An example: corporate dinner game

The simple example below shows why to study coalition structures we need to incorporate them into a strategy set, what is different from Hart and Kurz (1983). They studied coalition formation in terms of choice of a coalition by every player, not of a coalition structure.

Consider a game of 4 players: A is a President; B is a senior vice-president; C_1, C_2 are two other vice-presidents. They have a corporate dinner. [4] A coalition is a group of players at *one* table. Every player may sit only at one table. A coalition structure is an allocation of all players over no more than four tables. Empty tables are not taken into account.

Individual set of strategies is a set of all coalition structures for the players, i.e. a set of all possible allocations of players over 4 non-empty tables. A player chooses one coalition structure, an element from his strategy set. A set of strategies in the game is a direct product of four individual strategy sets. The choice of all players is a point in the set of strategies of the game.

Preferences of the players are such that everyone (besides A) would like to have a dinner with A, but A only with B. Everyone wants players outside his table to eat individually, due to possible dissipation of rumors or information exchange. No one can enforce others to form or not to form coalitions.

In every coalition structure (or a partition) any coalition (i.e. a table)

[4] The game is designed to obtain pure strategies equilibrium for some of the players. Compare this game with a lunch game in the next paper, where all players are identical, and the result is a stochastic game, where states are equilibrium coalition structures.

is formed only if everybody at the table agrees to have dinner together, otherwise a non–selected player eats alone. Further we address this as a coalition structure formation rule or a rule for simplicity. The same coalition may belong to different coalition structures, but with different allocations of players beyond it, compare lines 1 and 2 in Table 1 further.

The game is simultaneous and one shot. A realization of a final partition (a coalition structure) depends on choices of coalition structures of all players. A choice of a player is an element of his/her strategy set. Example. Let player A choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player B choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$; player C_1 choose $\{\{A, C_1\}, \{B\}, \{C_2\}\}$, and player C_2 choose $\{\{A, C_2\}, \{B\}, \{C_2\}\}$. Then the final partition is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$. It is clear that a strong Nash equilibrium (Aumann, 1960), which is based on a deviation of a coalition of any size, does not discriminate between the coalition structures mentioned above.

Payoff profile of all players in the game should be defined for every final coalition structure. Table 1 presents coalition structures only with the best individual payoffs.[5] Thus only some partitions from the big set of all strategies deserves attention. The first column is a number of a strategy. The second column is an allocation of players over coalition structures, and also is a list of the best[6] final coalition structures from a set of all strategies. The third column is an individual payoff profile of all players. The fourth column is a list of values for coalitions in coalition structures if to calculate values using cooperative game theory.

The game runs as follows. Players simultaneously announce individually chosen coalition structures, what makes a point in a set of strategies of all players. Then a final coalition structure is formed according to the rule above, and payoffs are assigned for the formed coalition structure.

The way the game happens means that a set of all strategies is divided

[5] All other coalition structures have significantly lower payoffs.

[6] Payoffs in the rest coalition structures are much lower, and they are excluded from consideration.

Table 1: Strategies and payoffs in the corporate dinner game

num	Best final partitions	Non-cooperative payoff profile $(U_A, U_B, U_{C_1}, U_{C_2})$	Values of coalitions as in cooperative game theory
1	$\{\{A,B\},\{C_1\},\{C_2\}\}$	(10,10,3,3)	20_{AB}, 3_{C_1}, 3_{C_2}
2*	$\{\{A,B\},\{C_1,C_2\}\}$	(8,8,5,5)	16_{AB}, $10_{C_1,C_2}$
3	$\{\{A,C_1\},\{B,C_2\}\}$	(3,5,10,5)	13_{AC_1}, 10_{BC_2}
4	$\{\{A,C_1\},\{B\},\{C_2\}\}$	(3,3,10,3)	13_{AC_1}, 3_B, 3_{C_2}
5	$\{\{A,C_2\},\{B,C_1\}\}$	(3,5,5,10)	13_{AC_2}, 10_{BC_1}
6	$\{\{A,C_2\},\{B\},\{C_1\}\}$	(3,3,3,10)	13_{AC_1}, 3_B, 3_{C_1}
7	all other partitions	(0,0,0,0)	all payoffs are 0

by the rule of coalition structure formation into domains: every domain corresponds to exactly one coalition structure. Or in other words, every point in the set of strategies of the game is assigned: exactly one final coalition structure and payoffs for every player. A player may have different payoffs in different coalition structures. Clearly a final coalition structure may not coincide with someone's initial choice.

Final payoffs have the following interpretation. Players A and B would always like to be together. Being rational they would choose a coalition structure with the highest payoff for them, i.e. the strategy 1. By the same reason the first best choices of C_1 and C_2 would be to choose coalition structures with A. But A will never choose to be with either of them. The unavailability of the first best makes C_1 and C_2 to choose option 2.[7] By doing so they do not disallow a coalition $\{A, B\}$, but reduce payoffs for A and B. And players A and B cannot prevent this (or to insure against).

On the other hand, if players A and B choose strategy 2 they will obtain coalition $\{A, B\}$ in any case, but in a different final coalition structure. In

[7] In sociology this behavior is referred as a cooperation: players C_1 and C_2 group together against other options when they are not together and have lower payoffs, player A will never choose them. This problem will be addressed in the accompanying paper.

terms of mixed strategies this means that an equilibrium mixed strategy for A and for B is a whole probability space over two points, two coalition structures 1 and 2. For simplicity payoffs in coalition structures with coalition sizes three and four are much smaller.

From the forth column we can see that the corresponding cooperative game has an empty core.

The constructed game has a unique equilibrium. In terms of individual payoffs it is characterized as the second-best efficient for everybody. Equilibrium coalition structure contains two coalitions. This equilibrium is different from the strong Nash and a coalition proof equilibria. Both these concepts assume deviation of coalitions, but here definition of a coalition is not enough to identify a final result of the game. Additional problems appear with cores - what is a direction of a deviation, may players deviate simultaneously from different coalitions, do deviators from different coalitions interact with each other? Within cooperative game theory all these questions are left without well-defined answers. The proposed formal game in the paper targets to provide coherent answers for these questions in terms of non-cooperative game theory. And the result of the game allows to study inter / intra coalition externalities in a consistent non-cooperative way.

The equilibrium in the corporate dinner game does not need super-additivity of payoffs. There are also differences from partition approach (Yi, 1999), as, for example, there is no initial allocation of players over coalitions.

3 Formal setup of the model

Nash (1950, 1951) suggested a non-cooperative game which consists of a set of players N, with a general element i, sets of individual finite strategies $S_i, i \in N$, and payoffs, defined as a mapping from a set of all strategies into payoff profiles of all players, $\Big(U_i(s)\Big)_{i\in N}$, such that $S = \times_{i\in N} S_i \mapsto \Big(U_i(s)\Big)_{i\in N} \subset \mathbb{R}^{\#N}$, where $\Big(U_i(s)\Big)_{i\in N} < \infty, \forall s \in S$.

The suggested game modifies the mapping by preliminary portioning S into coalition structure specific domains and assigning payoffs for every point in these domains. The division of S is done with a coalition structure formation mechanism, defined further.

Let there is a set of agents N, with a general element i, a size of N is $\#N$, a finite integer, $2 \leq \#N < \infty$.

Every game has a parameter K, $K \in \{1, \ldots, \#N\}$. This parameter has two interpretations. Let for N agents there is a coalition with a maximum size K. Then no more than K agents are required to dissolve it. The reverse is also true: we need no more than K agents to form this maximum size K coalition. Closeness of construction of the object under investigation requires these two simultaneous interpretations for K be equal.

Every value of K from the set $\{1, \ldots, \#N\}$ induces a family of coalition structures (or a family of partitions) $\mathcal{P}(K)$ over the set of all players N:

$$\mathcal{P}(K) = \{P = \{g_j \colon g_j \subset N; \#g \leq K; \cup_j g_j = N; \forall j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset\}.$$

Every coalition g_j in a partition P from $\mathcal{P}(K)$ has a size (a number of members) no bigger than K, but can be less. The condition $j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset$ is interpreted as that an agent can participate only in one coalition.

If we increase value of the parameter K by one, then we need to add partitions from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes families of partitions for different K be nested: $\mathcal{P}(K=1) \subset \ldots \subset \mathcal{P}(K) \subset \ldots \subset \mathcal{P}(K=N)$. The bigger is K, the more coalition structures (or partitions) are involved into consideration.

For every partition P an agent i has a finite strategy set $S_i(P)$.[8] A set of strategies of agent i for a family of coalition structures $\mathcal{P}(K)$ is

$$S_i(K) = \Big\{ s_i(K) \colon s_i(K) \in \{S_i(P) \colon P \in \mathcal{P}(K)\} \Big\}$$

[8]Finite strategies are chosen as used in Nash (1950),

with a general element $s_i(K)$. For a given K an agent chooses $s_i(K)$ from $S_i(K)$. *A choice of $s_i(K)$ means a choice of a desirable partition and an action for this partition.*[9] If we increase the parameter K by one, then we need to construct additional strategies only for the newly available coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes strategy sets for different K be nested: $S_i(K=1) \subset \ldots \subset S_i(K) \subset \ldots \subset S_i(K=N)$.

The set of strategies for a fixed K is $S(K) = \times_{i \in N} S_i(K)$, a direct product of individual strategy sets of all players for the given K. A choice of all players $s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$ is a point in $S(K)$. For simplicity if there is no ambiguity we will write $s = \Big(s_1, \ldots, s_N\Big) \equiv s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$. It is clear that an increase in K induces nested strategy sets: $S(K=1) \subset \ldots \subset S(K=N)$.

We have seen above that a mechanism to resolve conflicts between individual choices is required. For every value of K from the set $\{1, \ldots, \#N\}$ we define a coalition structure formation mechanism (a mechanism or a rule for short) $\mathcal{R}(K)$. For every strategy profile $s = (s_1, \ldots, s_N) \in S(K)$ the mechanism assigns a final coalition structure P, $P \in \mathcal{P}(K)$, and transports the strategy profile s into the partition P. Further we will see that this makes every P to be a game, as P will have a set of players, a non-trivial set of strategies and partition specific payoffs over it's own strategy set.

Definition 1. *For every K* ***a coalition structure formation mechanism*** *$\mathcal{R}(K)$ is a set of mappings such that:*

1. *A domain of $\mathcal{R}(K)$ is a set of all strategy profiles of $S(K)$.*
2. *A range of $\mathcal{R}(K)$ is a finite number of subsets $S(P) \subset S(K)$, $P \in \mathcal{P}(K)$. $S(P)$ is a coalition structure specific strategy set .*
3. *Number of the subsets $S(P)$ is no more than a cardinality of $\mathcal{P}(K)$.*

[9] A desirable partition may not realize. A coalition structure formation mechanism resolves conflicts of partition choices between players.

4. *$\mathcal{R}(K)$ divides $S(K)$ into coalition structure specific strategy sets, $S(K) = \cup_{P \in \mathcal{P}(K)} S(P)$.*

5. *Two different coalition structures, $\bar{P}$ and $\tilde{P}$, $\bar{P} \neq \tilde{P}$, have different coalition structure strategy sets $S(\bar{P}) \cap S(\tilde{P}) = \emptyset$.*

Formally the same:

$$\mathcal{R}(K) \colon S(K) = \times_{i \in N} S_i(K) \mapsto \colon \begin{cases} \forall s = (s_1, \ldots, s_N) \in S(K) \; \exists P \in \mathcal{P}(K) \colon s \in S(P), \\ S(K) = \cup_{P \in \mathcal{P}(K)} S(P), \\ \forall \bar{P}, \tilde{P} \in \mathcal{P}(K), \bar{P} \neq \tilde{P} \Rightarrow S(\bar{P}) \cap S(\tilde{P}) = \emptyset. \end{cases} .$$

Hence there are two ways to construct $S(K)$: in terms of initial individual strategies $S(K) = \times_{i \in N} S_i(K)$, and in terms of realized partition strategies $S(K) = \cup_{P \in \mathcal{P}(K)} S(P)$. Representation of $S(K)$ in terms of coalition structure specific strategy sets may not be a direct product of sets, see an example in the next section. We assume that a mechanism is given from outside.

If K increases we need to add only a mechanism for strategy sets from $S(K+1) \setminus S(K)$. This supports consistency of coalition structure formation mechanisms for different K. The family of mechanisms is nested: : $\mathcal{R}(K = 1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K = N)$.

Payoffs in the game are defined as state-contingent payoffs (or payoffs of Arrow-Debreu securities) in finance. For every coalition structure P player i has a payoff function $U_i(P) \colon S(P) \to \mathbb{R}_+$, such that the set $U_i(P)$ is bounded, $U_i(P) < \infty$. Payoffs are considered as von Neumann-Morgenstern utilities. All payoffs of i for a game with no more than K deviators make a family: $\mathcal{U}_i(K) = \left\{ U_i(P) \colon P \in \mathcal{P}(K) \right\}$. Every coalition structure has it's own set of strategies and a corresponding set of payoffs. Thus every coalition structure is a non-cooperative game.

An increase in K also increases the number of possible partitions and the set of strategies for every player. We need to add only payoffs for the partitions in $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$ or for strategies in $S_i(K+1) \setminus S_i(K)$. Thus we obtain a nested family of payoff functions:

$$\mathcal{U}_i(K=1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K=N).$$

We can easily see that this construction of payoffs allows to obtain both intra and inter coalition (or group) externalities, as payoffs are defined directly over strategy profiles of all players and independently from allocation of players in coalition structures.

Definition 2 (**a simultaneous coalition structure formation game**). *A non-cooperative game for coalition structure formation is*

$$\Gamma(K) = \Big\langle N, \Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}, \Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N} \Big\rangle,$$

where $\Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}$ *- coalition structure formation mechanism (a social norm, a social institute),* $\Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N}$ *- properties of players in* N*, (individual strategies and payoffs), such that:*

$$\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \Big\{S(P) \colon P \in \mathcal{P}(K)\Big\} \to \Big\{(\mathcal{U}_i(K))_{i \in N}\Big\}.$$

For a example, the corporate dinner game above was a game for $K = 2$, where an equilibrium did not change with an increase in values of K from $K = 2$ to $K = 3, 4$.

Novelty of the paper is an introduction of coalition structure formation mechanism, which portions the set of all strategies into non-cooperative partition-specific games. If we omit the mechanism part of the game and eliminate restriction on coalition sizes then we obtain the traditional non-cooperative game of Nash: $\times_{i \in N} S_i(K) \to \Big(U_i(s)\Big)_{i \in N}$. Construction of a game $\Gamma(K)$ makes every partition P be an individual game.

Another novelty of the paper is an introduction of nested games.

Definition 3 (**family of games**). *A family of games is* ***nested*** *if :*

$$\Gamma = \Gamma(K=1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K=N).$$

Nested games appear as a result of parametrization of a game by a maximum coalition size (or by a maximum number of deviators, what is equivalent). All games have consistent nesting of components, for the same set of players N.

Let $\Sigma_i(K)$ be a set of all mixed strategies of i, i.e. a space of probability measures, $\Sigma_i(K) = \left\{\sigma_i(K)\colon \int_{S_i(K)} d\sigma_i(K) = 1\right\}$, with a general element $\sigma_i(K)$, where an integral is Lebegue-Stiltjes integral. Sets of mixed strategies for all other players are defined in the standard way $\Sigma_{-i}(K) = \left\{\left(\sigma_j(K)\right)_{j\neq i}\colon \forall j \neq i \int_{\left(S_j(K)\right)_{j\neq i}} \left(d\sigma_j(K)\right) = 1\right\}$.

Expected utility can be defined in terms of strategies the players choose or in terms of final partition-specific strategies. Expected utility of i in terms of individual strategies is:

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \int_{S(K)=\times_{i\in N} S_i(K)} U_i(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K)$$

or in terms of partition-specific strategies is

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \sum_{P\in\mathcal{P}(K)} \int_{S(P)} U_i(P)(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K).$$

Expected utilities are constructed in the standard way.

Definition 4 (**an equilibrium in a game** $\Gamma(K)$). *A mixed strategies profile* $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ *is an equilibrium strategy profile for a game* $\Gamma(K)$ *if for every* $\sigma_i(K) \neq \sigma_i^*(K)$ *the following inequality for every player* i

from N holds true:

$$EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}^*(K)\Big).$$

Equilibrium in the game $\Gamma(K)$ is defined in a standard way. It's existence is just an expansion of Nash theorem. However this result for non-cooperative games with coalition structure formation is different from the results of cooperative games, where an equilibrium may not exist, for example, in coalition form games with empty cores. Another outcome of the model is that there is no need to introduce additional properties of games, like axioms on a system of payoffs, super-additivity, or weights. Equilibrium existence result can be generalized for the whole family of games.

Theorem 1. *The family of games $\mathcal{G} = \{\Gamma(K), K = 1, 2, \ldots, N\}$ has an equilibrium in mixed strategies, $\sigma^*(\mathcal{G}) = (\sigma^*(K = 1), \ldots, \sigma^*(K = N))$, $\Big(\sigma^*(K)\Big)_{i \in N}$.*

Technical side off the result is obvious. The theorem expands the classic Nash theorem. An equilibrium in the game can also be characterized by equilibrium partitions.

Definition 5 (equilibrium coalition structures or partitions). *A set of partitions $\{P^*\}(K)$, $\{P^*\}(K) \subset \mathcal{P}(K)$, of a game $\Gamma(K)$, is a set of equilibrium partitions, if it is induced by an equilibrium strategy profile $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$.*

In the same way we can define equilibrium payoffs for the whole family of games. This consideration is important for construction of a non-cooperative stability criterium in the next paper.

In the example below we will see that there can be more than one equilibrium partition, and equilibrium partitions may change with an increase in the number of deviators. These issues are addressed in the accompanying paper on applications of the suggested game.

4 An example of a game of two players

The example serves to demonstrate two points. The constructed game has the standard property of games: an equilibrium and efficiency concepts may not coincide, what is different from strong Nash concept. The second is to demonstrate that the concept of cooperation requires additional investigation (done in the next paper). The example below is based on a generalization of the Prisoner's Dilemma.

There are 2 players. They can form two types of partitions. If $K = 1$ then there is only one final partition, $\mathcal{P}(K = 1) \equiv P_{separ} = \{\{1\}, \{2\}\}$. If $K = 2$ there are two final partitions, which make a family of partitions $\mathcal{P}(K = 2) = \Big\{\{\{1\}, \{2\}\}, \{1, 2\}\Big\}$. Further we will use the notation $P_{joint} = \{1, 2\}$. Clearly, partition structures $\mathcal{P}(K = 1)$ and $\mathcal{P}(K = 2)$ are nested.

Every player in every partition has two strategies: H(igh) and L(ow). Player i for a game with $K = 1$ has a strategy set $S_i(K = 1) = \{L_{i,P_{separ}}, H_{i,P_{separ}}\}$. Player i in the game with $K = 2$ has the strategy set $S_i(K = 2) = \{L_{i,P_{separ}}, H_{i,P_{separ}}, L_{i,P_{joint}}, H_{i,P_{joint}}\}$. Clearly, strategy sets $S_i(K = 1)$ and $S_i(K = 2)$ are nested.

Set of strategies for the game with $K = 1$ is $S(K = 1) = \{L_{1,P_{separ}}, H_{1,P_{separ}}\} \times \{L_{2,P_{separ}}, H_{2,P_{separ}}\}$. Payoffs are in corresponding top-left cells of Table 2. Every cell contains a payoff profile and a *final* coalition structure. The payoffs in these cells are identical to the payoffs of the Prisoner's Dilemma.

Set of strategies for the game with $K = 2$ is

$$S(2) = \{L_{1,P_{separ}}, H_{1,P_{separ}}, L_{1,P_{joint}}, H_{1,P_{joint}}\} \times \{L_{2,P_{separ}}, H_{2,P_{separ}}, L_{2,P_{joint}}, H_{2,P_{joint}}\}.$$

Strategy sets of games with $K = 1$ and $K = 2$ are nested, i.e. $S(K = 1) \subset S(K = 2)$. Additional payoffs of the game for $K = 1$ are replicated from the game with $K = 1$, see Table 2.

For $K = 1$ there are four outcomes, for $K = 2$ there are sixteen outcomes, twelve new in comparison to $K = 1$.

A coalition structure formation mechanisms for different K are $\mathcal{R}(K = 1), \mathcal{R}(K = 2)$. The coalition structure formation mechanism is: the partition $P_{joint} = \{1, 2\}$ can be formed only from a unanimous agreement of both players. It is clear that for $K = 1$ only P_{separ} can be formed. The grand coalition, or a partition P_{joint}, can be formed only over the strategy set $(L_{1,P_{joint}}, H_{1,P_{joint}}) \times (L_{2,P_{joint}}, H_{2,P_{joint}})$. Otherwise the partition P_{separ} is formed. Thus a set of strategies for the partition P_{separ} is not a direct product of some sets. It is also clear that an increase in K results in nested mechanisms: $\mathcal{R}(K = 1) \subset \mathcal{R}(K = 2)$.

From Table 2 we can see that the whole strategy set of the game is partitioned into *coalition structure specific domains*. Every coalition structure (or a partition) is a non-cooperative game with it's own strategy set and payoff profiles. Final partition for a player may not coincide with an individual choice. A set of strategies for the partition P_{separ} is not a product of sets. Finally the games for $K = 1$ and $K = 2$ are nested.

Consider the game with a maximum coalition size $K = 2$ as described by the Table 2. If both players choose strategies only for the partition P_{separ}, then the game is the standard Prisoner's Dilemma game. However the same coalition structure can be formed if some of the players does not choose any strategy for $P_{joint} = \{1, 2\}$. Thus for the *final* partition $P_{separ} = \{\{1\}, \{2\}\}$ there are three equilibria, and every equilibrium is inefficient. There are also three efficient and desirable outcomes in the same partition.

The partition $P_{joint} = \{1, 2\}$ can be formed only if both players choose it. Within this partition there is one inefficient equilibrium and one efficient outcome.

Compare efficient payoff profiles for the partitions P_{separ} and P_{joint}. They have equal payoff profiles $(0; 0)$, but their strategy profiles belong to different final partitions. In other words, from observing only the payoff profile $(0; 0)$ we can not make a conclusion, which coalition structure is formed either P_{separ} or P_{joint}. Another interpretation is that cooperation may take place as a cooperation in terms of positive externalities and in terms of allocation of players in one coalition. We can see that they are different.

This means there are two kinds of cooperation - a cooperation in payoffs and a cooperation in allocation of players over coalitions. The example demonstrates that they may not coincide. More on that in the next paper.

Using the same game we can demonstrate appearance of intra- and inter-coalitions externalities. If partition $P_{joint} = \{1, 2\}$ is formed, then an individual payoff of a player depends on a strategy of another in the *same* coalition (presence of intra-coalition or intra-group externality). If partition $P_{separ} = \{\{1\}, \{2\}\}$, is formed then an individual payoff of a player depends on a strategy of another player in *the different* coalition (presence of inter-coalition or inter-group externality).

Multiplicity of equilibria makes both of these externalities co-exist in equilibria, but in different final coalition structures. Thus the game is able to present both intra and inter-coalition externalities, what is impossible in cooperative game theory.

Table 2: Payoff for the family of games with unanimous formation rules. Different partitions have payoff-equal efficient outcomes.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$ $\{\{1\}, \{2\}\}$	(-5;3) $\{\{1\}, \{2\}\}$	$\underline{(0;0)}$ $\{\{1\}, \{2\}\}$	(-5;3) $\{\{1\}, \{2\}\}$
$H_{1,P_{separ}}$	(3;-5) $\{\{1\}, \{2\}\}$	**(-2;-2)** $\{\{1\}, \{2\}\}$	(3;-5) $\{\{1\}, \{2\}\}$	**(-2;-2)** $\{\{1\}, \{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$ $\{\{1\}, \{2\}\}$	(-5;3) $\{\{1\}, \{2\}\}$	$\underline{(0; 0)}$ $\{1, 2\}$	$(-5; 3)$ $\{1, 2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\}, \{2\}\}$	**(-2;-2)** $\{\{1\}, \{2\}\}$	$(3; -5)$ $\{1, 2\}$	**(-2;-2)** $\{1, 2\}$

We can reinstall uniqueness of an equilibrium, what is done in Table 3.

Table 3: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition P_{joint}, when it is realized. Uniqueness of an equilibrium is reinstalled.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5) $\{\{1\},\{2\}\}$	**(-2;-2)** $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	**(-2;-2)** $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$(0+\epsilon;0+\epsilon)$ $\{1,2\}$	$(-5+\epsilon;3+\epsilon)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3+\epsilon;-5+\epsilon)$ $\{1,2\}$	**$(-2+\epsilon;-2+\epsilon)$** $\{1,2\}$

If both players are extroverts and prefer be together,[10] then every individual payoff increases by $\epsilon > 0$, if the grand coalition is realized. This means that a change of the game from $\Gamma(1)$ to $\Gamma(2)$ changes the equilibrium in terms of both strategies and the partitions.

If both players are introverts, $\epsilon < 0$, then the expansion of the game will not change initial equilibrium in terms of both strategies and the partition. This case is not presented here and is left for future.

5 Discussion

Insufficiency of cooperative game theory to study coalitions and coalition structures was earlier reported by many authors. Maskin (2011) wrote that "features of cooperative theory are problematic because most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form". My-

[10] what is equivalent to preferences over coalition structures

erson (p.370, 1991) noted that "we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory". Thus there is a demand for a specially designed non-cooperative game to study coalition structures formation along with an adequate equilibrium concept for this game.

There is a voluminous literature on the topic, a list of authors is far from complete: Aumann, Hart, Holt, Maschler, Maskin, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders and many others.

A popular approach to use a "threat"as a basic concept for coalition formation analysis was suggested by Nash (1953). Consider a strategy profile from a subset of players. Let this profile be a threat to someone, beyond this subset. The threatening player**s** may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat? At the same time there may be some other player beyond the subset of players who may obtain a bonanza from this threat. But this beneficiant may not join the group due to expected intra-group negative externalities for members or from members of this group. Thus a concept of a threat can not serve as an elementary concept.

The justification of a chosen tool, a non-cooperative game, comes from Maskin (2011) and a remark of Serrano (2014), that for studying coalition formation "it may be worth to use strategic-form games, as proposed in the Nash program".

The difference of the research agenda in this paper from the Nash program (Serrano 2004) is studying non-cooperative formation of coalition structures, but not only formation of one coalition. The best analogy for the difference is the difference between partial and general strategic equilibrium analysis in economics. The former isolates a market ignoring cross-market interactions, the latter explicitly studies cross market interactions.

The constructed finite non-cooperative game allows to study what can be a cooperative behavior, when the individuals "rationally further their individual interests" (Olson, 1971).

Nash (1950, 1951) suggested to construct a non-cooperative game as a mapping of a set of strategies into a profile of payoffs, $\times_{i\in N} S_i \to (U_i)_{i\in N}$.

This paper has two contributions in comparison to his paper: construction of a non-cooperative game with an embedding coalition structure formation mechanism, and parametrization of all constructed games by a number of deviators: $\times_{i\in N} S_i(K) \stackrel{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i\in N}\}$, where $K \in \{1, \ldots, \#N\}$. The game suggested by Nash becomes a partial case for these games.

Every game in a family has an equilibrium, may be in mixed strategies. This differs from results of cooperative game theory, where games may have no equilibrium, like games with empty cores, etc.

The introduced equilibrium concept differs from the strong Nash, coalition-proof and k equilibrium concepts. The differences are: an explicit allocation of payoffs and a combined presence of intra- and inter- coalition (or group) externalities (the list of differences is not complete). Differences from the core approach of Aumann (1960) are clear: a presence of externalities, no restrictions that only one group deviates, no restrictions on the direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players. There is no need to assume transferable utilities for players. The approach allows to study coalition structures, which differ from the grand coalition as in Shapley value. Finally the introduced concepts enables to offer a non-cooperative necessary stability criterion based only on an equilibrium of a game. This is impossible for any other equilibrium concept.

The suggested approach is different in a role for a central planner offered by Nash, who "argued that cooperative actions are the result of some process of bargaining" Myerson (p.370, 1991). The central planner offers a predefined coalition structure formation mechanism, that includes a maximum number of deviators, family of eligible partitions and a family of rules to construct these partitions from individual strategies of players.

The accompanying papers will demonstrate application of the suggested model for studying stochastic Bayesian games, cooperation, self-enforcement properties of an equilibrium (Aumann, 1990), non-cooperative criterion of partition stability, focal points, application of the same mechanism to study network games and repeated games.

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