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# Morse–Smale Diffeomorphisms with Three Fixed Points

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**Abstract**—It is proved that the closures of separatrices for a Morse–Smale diffeomorphism with three fixed points are flatly embedded spheres if the dimension of the manifold is at least 6 and may be wildly embedded spheres if the dimension of the manifold is 4.

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## INTRODUCTION

Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism (the basic notions and facts of the theory of dynamical systems can be found in [1]–[3]) of a closed $n$-manifold $M^n$ ($n \geq 3$), and let $\sigma$ be a saddle periodic point of the diffeomorphism $f$ with $k$-dimensional ($1 \leq k \leq n-1$) stable manifold $W^{\mathrm{s}}(\sigma)$ or unstable manifold $W^{\mathrm{u}}(\sigma)$. The set $\mathrm{Sep}^\tau(\sigma) = W^\tau(\sigma) \setminus \{\sigma\}$ is called a *separatrix* ($\tau$ is either s or u; for brevity, we use the notation $\tau = \mathrm{s}$ and $\tau = \mathrm{u}$). If $\mathrm{Sep}^\tau(\sigma)$ does not intersect the separatrices of other saddle periodic points, then $\mathrm{Sep}^\tau(\sigma)$ belongs to the unstable (if $\tau = \mathrm{s}$) or stable (if $\tau = \mathrm{u}$) manifold of some nodal periodic point, say $N$. In this case, the topological closure of the separatrix $\mathrm{Sep}^\tau(\sigma)$ coincides with $W^\tau(\sigma) \cup \{N\}$ and is a $k$-sphere topologically embedded in $M^n$, provided that $k = \dim \mathrm{Sep}^\tau(\sigma) \geq 2$ [4]. The possibility of a wild embedding of such a $k$-sphere was first proved in [5] when the manifold is a 3-sphere ($M^3 = S^3$) and $k = 2$ (similar examples were constructed in [4], [6]–[10], where classification questions were also considered). More precisely, in [5], a gradient-like diffeomorphism of the 3-sphere with one saddle and three nodes was constructed (we describe the idea of the construction at the beginning of Sec. 2). If follows from results of [11] that there exist no orientable closed 3-manifolds admitting a Morse–Smale diffeomorphism with three periodic points. Since a Morse–Smale diffeomorphism of any closed manifold has at least one periodic source and one periodic sink [3], it follows that, in the case $n = 3$, the least number of periodic points for which the closure of a separatrix can be wildly embedded is four.

In [12], the existence of closed $n$-manifolds with $n \geq 4$ admitting Morse functions with precisely three critical points was proved, and such manifolds were studied. Thus, in the case $n \geq 4$, there exist Morse–Smale diffeomorphisms with precisely three periodic points. Any such diffeomorphism has precisely one saddle (see Lemma 3). Therefore, it is natural to consider the question of whether the topological closure of a separatrix of the (unique) saddle can be wildly embedded. The present paper is devoted to this question. The main result is contained in the following theorem.

**Theorem.** *Suppose that $f\colon M^n \to M^n$ is a Morse–Smale diffeomorphism of a closed manifold of dimension $n \geq 4$ and its nonwandering set consists of three fixed points, namely, a sink $\omega$, a source $\alpha$, and a saddle $s_0$. Then*

- *$M^n$ is orientable*;

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- *the separatrices of the saddle* $s_0$ *have the same dimension* (*and, therefore, the dimension* $n$ *of* $M^n$ *is even*);

- *the closures of the unstable separatrix* $\mathrm{Sep}^{\mathrm{u}}(s_0)$ *and the stable separatrix* $\mathrm{Sep}^{\mathrm{s}}(s_0)$ *are topologically embedded* $(n/2)$*-spheres, i.e.,*

$$W^{\mathrm{u}}(s_0) \cup \{\omega\} = S_\omega, \qquad W^{\mathrm{s}}(s_0) \cup \{\alpha\} = S_\alpha,$$

  *respectively.*

*Moreover,*

- *if* $n \geq 6$*, then the spheres* $S_\omega$ *and* $S_\alpha$ *are locally flat*;

- *if* $n = 4$*, then there exists an* $f\colon M^4 \to M^4$ *for which the spheres* $S_\omega$ *and* $S_\alpha$ *are wildly embedded.*

The paper is organized as follows. All the assertions of the main theorem, except the last one, are proved in Sec. 1. The last statement, which asserts the existence of an $f\colon M^4 \to M^4$ with wildly embedded closures of separatrices, is proved in Sec. 2.

## 1. CLOSURES OF SEPARATRICES FOR $n \geq 6$

### *1.1. Basic Definitions*

First, we recall some basic definitions. A diffeomorphism $f$ of a smooth manifold $M$ is called a *Morse–Smale diffeomorphism* if its nonwandering set $NW(f)$ consists of finitely many periodic points (and, therefore, $NW(f) = \mathrm{Per}(f)$), all periodic points are hyperbolic, and the invariant manifolds $W^s(x)$ and $W^{\mathrm{u}}(y)$ either are disjoint or intersect transversally for any points $x, y \in NW(f)$.

The *Kronecker–Poincaré index* is the number $\mathrm{Ind}_p(f) = (-1)^{\dim W^{\mathrm{u}}(p)}\Delta$, where $\Delta$ is $+1$ or $-1$, depending on whether or not $f|_{W^{\mathrm{u}}(p)}$ preserves orientation. By $\mathrm{tr}(f_{*k})$ we denote the trace of the (linear) map $f_{*k}\colon H_k(M, \mathbb{R})$ induced by the diffeomorphism $f$ on the $k$-dimensional homology group

$$H_k(M, \mathbb{R}) = H_k(M), \qquad 0 \leq k \leq \dim M.$$

If the fixed point set $\mathrm{Fix}(f)$ of a diffeomorphism $f$ consists of hyperbolic points, then this diffeomorphism satisfies the following relation, called the Lefschetz formula:

$$\sum_{k=0}^{\dim M} (-1)^k \,\mathrm{tr}(f_{*k}) = \sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}_p(f).$$

A wild and a locally flat embedding of a submanifold in a manifold are defined as follows. For positive integers $1 \leq m \leq n$, consider Euclidean space $\mathbb{R}^m$ embedded in $\mathbb{R}^n$ so that the last $n - m$ coordinates of points from $\mathbb{R}^m$ are equal to 0. Let $e\colon M^m \to N^n$ be an embedding of a closed $m$-manifold $M^m$ in the interior of an $n$-manifold $N^n$. Then $e(M^m)$ is said to be *locally flat at a point* $e(x)$, $x \in M^m$, if there exists a neighborhood $U(e(x)) = U$ of $e(x)$ and a homeomorphism $h\colon U \to \mathbb{R}^n$ for which

$$h(U \cap e(M^m)) = \mathbb{R}^m \subset \mathbb{R}^n.$$

Otherwise $e(M^m)$ is said to be *wildly embedded at* $e(x)$. In the case of a compact manifold $M^m$ with boundary, definitions are similar.

### *1.2. Preliminary Results*

In [4], the following assertion was proved; we state it below as a lemma for reference.

**Lemma 1.** *Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism for which a separatrix $\mathrm{Sep}^\tau(\sigma)$ of some saddle $\sigma$ does not intersect the separatrices of other saddles and $k = \dim \mathrm{Sep}^\tau(\sigma) \ge 2$. Then $\mathrm{Sep}^\tau(\sigma)$ is contained in the unstable (if $\tau = \mathrm{s}$) or the stable (if $\tau = \mathrm{u}$) manifold of some periodic sink, say $N$, the topological closure of $\mathrm{Sep}^\tau(\sigma)$ coincides with $W^\tau(\sigma) \cup \{N\}$, and $\mathrm{Sep}^\tau(\sigma)$ itself is a $k$-sphere topologically embedded in $M^n$.*

The orientability of the manifold $M^n$ is a consequence of the following lemma, which is of independent interest.

**Lemma 2.** *Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism for which there are no one-dimensional separatrices and no separatrices with heteroclinic intersections. Then the manifold $M^n$ is orientable.*

**Proof.** Suppose than $M^n$ is nonorientable. Without loss of generality, we can assume that all periodic points of the diffeomorphism $f$ are fixed (otherwise we pass to some iteration of $f$). As is well known, there exists a double covering $\widehat{\pi}\colon \widehat{M}^n \to M^n$, where $\widehat{M}^n$ is an orientable manifold. Let us show that there exists a pullback $\widehat{f}$ of the diffeomorphism $f$ by the covering $\widehat{\pi}$. We set $\widehat{f} = \mathrm{id}$ at all points $\widehat{\pi}^{-1}(\mathrm{Fix}\, f)$. Take any point

$$\widehat{x} \in \widehat{M}^n, \qquad \widehat{x} \notin \widehat{\pi}^{-1}(\mathrm{Fix}\, f).$$

Its image $\widehat{\pi}(\widehat{x})$ belongs to either the stable manifold $W^{\mathrm{s}}(\omega)$ of some sink $\omega$ or the stable separatrix $\mathrm{Sep}^{\mathrm{s}}(\sigma)$ of some saddle $\sigma$. In the former case, since $W^{\mathrm{s}}(\omega)$ is simply connected and, therefore, the preimage $\widehat{\pi}^{-1}(W^{\mathrm{s}}(\omega))$ consists of pairwise disjoint simply connected domains, it follows that there exists a unique component $\widehat{W}^{\mathrm{s}}$ of the preimage $\widehat{\pi}^{-1}(W^{\mathrm{s}}(\omega))$ containing $\widehat{x}$. Note that there also exists a unique point $\widehat{\omega} \in \widehat{\pi}^{-1}(\omega)$ belonging to the same component. We set

$$\widehat{f}(\widehat{x}) = \widehat{y} \in \widehat{\pi}^{-1}(f(\widehat{\pi}(\widehat{x}))) \cap \widehat{W}^{\mathrm{s}}.$$

In the latter case, where $\widehat{\pi}(\widehat{x}) \in \mathrm{Sep}^{\mathrm{s}}(\sigma)$, it follows by Lemma 1 that the closure of the separatrix $\mathrm{Sep}^{\mathrm{s}}(\sigma)$ is the $k$-sphere $S_0^k$. By assumption, we have $k \ge 2$. Therefore, $S_0^k$ is simply connected, and, therefore, the preimage $\widehat{\pi}^{-1}(S_0^k)$ consists of pairwise disjoint $k$-spheres, one of which, say $\widehat{S}_0^k$, contains $\widehat{x}$. We set

$$\widehat{f}(\widehat{x}) = \widehat{y} \in \widehat{\pi}^{-1}(f(\widehat{\pi}(\widehat{x}))) \cap \widehat{S}_0^k.$$

It can be verified directly that the map $\widehat{f}$ thus constructed is a Morse–Smale diffeomorphism satisfying the relation $\widehat{\pi} \circ \widehat{f} = f\widehat{\pi}$.

Clearly, $\widehat{f}$ has no one-dimensional separatrices. It was shown in [13] that any Morse–Smale diffeomorphism for which there are no one-dimensional separatrices has precisely one source and precisely one sink. Since $f$ has at least one source and at least one sink, it follows that $\widehat{f}$ must have at least two sources and two sinks. This contradiction shows that the manifold $M^n$ is orientable. $\square$

Following [14], we say that a *saddle $\sigma$ is of type $(\mu, \nu)$* if $\mu = \dim W^{\mathrm{u}}(\sigma)$ and $\nu = \dim W^{\mathrm{s}}(\sigma)$. The number $\mu$ ($\nu$) is called the *unstable* (respectively, *stable*) *Morse index*.

**Lemma 3.** *Let $f\colon M^n \to M^n$ be a Morse–Smale diffeomorphism whose nonwandering set $NW(f)$ consists of three fixed points. Then*

- *$NW(f)$ consists of a sink, a source, and a saddle; moreover, the separatrices of the saddle have the same dimension (and, therefore, the dimension $n$ of the manifold $M^n$ is even);*
- *$M^n$ is orientable.*

**Proof.** First, we recall the Morse–Smale inequalities [15]. Let $M_j$ denote the number of periodic points $p$ of $f$ for which the stable manifold has dimension $j = \dim W^{\mathrm{s}}(p)$, and let $\beta_i(M^n) = \beta_i$ be the $i$th Betti number of the manifold $M^n$, i.e., $\beta_i(M^n) = \operatorname{rank} H_i(M^n, \mathbb{Z})$. Then the following relations hold [15]:

$$M_0 \geq \beta_0, \quad M_1 - M_0 \geq \beta_1 - \beta_0, \quad \dots, \quad M_{n-1} - M_{n-1} + \dots \geq \beta_{n-1} - \beta_{n-1} + \cdots, \tag{1}$$

$$\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i \beta_i. \tag{2}$$

For a connected manifold, we have $\beta_0 = 1$; therefore, it follows from (1) that $f$ has at least one sink and at least one source. If $f$ has two sinks $\omega_1$ and $\omega_2$ and one source $\alpha$, then the connected set $M^n \setminus \{\alpha\}$ is the union of the two disjoint open sets $W^{\mathrm{s}}(\omega_1)$ and $W^{\mathrm{s}}(\omega_2)$. Similarly, $f$ cannot have two sources and one sink. Thus, $NW(f)$ consists of a sink $\omega$, a source $\alpha$, and a saddle $\sigma$. Suppose that $\sigma$ is of type $(n-k, k)$. Then $M_0 = M_n = M_k = 1$. For the diffeomorphism $f^{-1}$, we have $M_0 = M_n = M_{n-k} = 1$ and

$$M_j = 0, \qquad j \neq 0, n, k, n-k.$$

Equating the left-hand sides of (2) for $f$ and $f^{-1}$, we obtain $(-1)^k = (-1)^{n-k}$; therefore, the number $n = 2m$ is even. Moreover, $n \geq 4$.

Let us show that $k \neq 1$. Assume the contrary. Since the manifolds $W^{\mathrm{s}}(\sigma)$ and $W^{\mathrm{u}}(\sigma)$ have no heteroclinic intersections, it follows that their topological closures are

$$W^{\mathrm{s}}(\sigma) \cup \{\alpha\} \overset{\mathrm{def}}{=} S^1_\alpha, \qquad W^{\mathrm{u}}(\sigma) \cup \{\omega\} \overset{\mathrm{def}}{=} S^{n-1}_\omega;$$

these are a topologically embedded circle and a topologically embedded $(n-1)$-sphere, respectively [4]. Since $n \geq 4$ and $S^{n-1}_\omega$ is smoothly embedded, except possibly at one point, it follows that $S^{n-1}_\omega$ has a neighborhood $U_\omega$ homeomorphic to $S^{n-1}_\omega \times (-1; +1)$ [16], [17]. Moreover, $U_\omega$ can be constructed so that $f(U_\omega) \subset U_\omega$. The only intersection point of $S^{n-1}_\omega$ and $S^1_\alpha$ is $\sigma$; therefore, $S^{n-1}_\omega$ does not separate $M^n$. Hence $M^n_1 = M^n \setminus U_\omega$ is a connected manifold with two boundary components homeomorphic to $S^{n-1}_\omega$. Attaching disjoint $n$-balls to these components, we obtain a closed manifold $M^n_2$. It follows from $f(U_\omega) \subset U_\omega$ that $f$ can be extended over $M^n_2$ to a diffeomorphism with one source and two sinks. It was shown above that such a diffeomorphism does not exist. This contradiction proves the inequality $k \neq 1$. Applying this result to $f^{-1}$, we obtain $k \neq n-1$. Thus,

$$M_1 = M_{n-1} = 0.$$

For a Morse–Smale diffeomorphism, the separatrices of the same saddle do not intersect; therefore, both separatrices of the (unique) saddle of the diffeomorphism $f$ have no heteroclinic intersections. This observation and Lemma 2 imply the orientability of the manifold $M^n$.

Let us show that $k = m$. Suppose that, on the contrary, $k \neq m$. We can assume that $k > m$ (otherwise consider the diffeomorphism $f^{-1}$). According to (1), we have $\beta_1 = \dots = \beta_{n-k-1} = 0$, because $M_1 = \dots = M_{n-k-1} = 0$. Poincaré duality for orientable manifolds (see e.g., [18, p. 145]) implies $\beta_1 = \dots = \beta_{k-1} = 0$. Thus, $\beta_i = 0$ for all $i = 1, \dots, n-1$, and relation (2) takes the form $1 + (-1)^k + (-1)^n = 1 + (-1)^n$, which is impossible.

The equality $k = m$ can be proved by a different method, which does not use the orientability of $M^n$. Again, suppose that $k \neq m$; to be definite, assume that $k < m$. In this case, the codimension of the manifold $W^{\mathrm{s}}(\sigma)$ is at least 2. Hence there is a diffeomorphism $\kappa\colon M^n \to M^n$ close enough to the identity which maps the union $W^{\mathrm{s}}(\sigma) \cup \alpha$ to $\kappa(W^{\mathrm{s}}(\sigma) \cup \alpha)$ so that

$$(\kappa(W^{\mathrm{s}}(\sigma) \cup \alpha)) \cap (W^{\mathrm{s}}(\sigma) \cup \alpha) = \varnothing.$$

Moreover, we can assume that $\kappa$ is equal to the identity diffeomorphism in some neighborhood of the sink $\omega$. The diffeomorphism $\kappa^{-1} \circ f \circ \kappa = \kappa'$ is a Morse–Smale diffeomorphism for which $\omega$ is a sink and the closure of the saddle separatrix does not intersect $W^{\mathrm{s}}(\sigma) \cup \alpha$. Hence the stable manifolds of the sink $\omega$ of the diffeomorphisms $f$ and $\kappa'$ cover the entire manifold $M^n$. Since the stable manifold of the sink is homeomorphic to the open $n$-ball, it follows that $M^n$ is the $n$-sphere $S^n$ [19]. Passing, if necessary, to some iteration, we can assume that $f$ and the restriction $f|_{W^{\mathrm{u}}(\sigma)}$ preserve orientation. For the $n$-sphere $S^n$, we have

$$H_0(S^n) = H_n(S^n) = 1, \qquad H_k(S^n) = 0, \quad 1 \le k \le n-1;$$

therefore, the Lefschetz formula for a Morse–Smale diffeomorphism of the sphere $S^n$ has the form

$$1 + (-1)^n = \sum_{p \in \mathrm{Fix}(f)} \mathrm{Ind}_p(f). \tag{3}$$

Clearly, $\mathrm{Ind}_\alpha(f) = (-1)^n$ and $\mathrm{Ind}_\omega(f) = 1$. Applying (3), we obtain $\mathrm{Ind}_\sigma(f) = 0 = (-1)^{\dim W^{\mathrm{u}}(\sigma)}$, which is impossible. This contradiction proves the equality $k = m$. $\square$

### *1.3. Proof of Local Flatness for $n \ge 6$*

We have proved that if $f$ is a diffeomorphism satisfying the assumptions of the main theorem, then $n = 2k$, where $k \ge 2$, and the nonwandering set $NW(f)$ consists of a sink $\omega$, a source $\alpha$ and a saddle $s_0$ of type $(k,k)$. Lemma 1 implies the following assertion.

**Lemma 4.** *Let $f : M^{2k} \to M^{2k}$ be a Morse–Smale diffeomorphism whose nonwandering set $NW(f)$ consists of a sink $\omega$, a source $\alpha$, and a saddle $s_0$ of type $(k,k)$. Then the closure of the unstable manifold $W^{\mathrm{u}}(s_0)$ and the stable manifold $W^{\mathrm{s}}(s_0)$ are the topologically embedded $k$-spheres $W^{\mathrm{u}}(s_0) \cup \{\omega\}$ and $W^{\mathrm{s}}(s_0) \cup \{\alpha\}$, respectively.*

Set

$$S^k_\omega = W^{\mathrm{u}}(s_0) \cup \{\omega\}, \qquad S^k_\alpha = W^{\mathrm{s}}(s_0) \cup \{\alpha\}.$$

**Lemma 5.** *Let $f\colon M^{2k} \to M^{2k}$ be a Morse–Smale diffeomorphism whose nonwandering set $NW(f)$ consists of a sink $\omega$, a source $\alpha$, and a saddle $s_0$, and let $k \ge 3$. Then $S^k_\omega$ and $S^k_\alpha$ are flat $k$-spheres.*

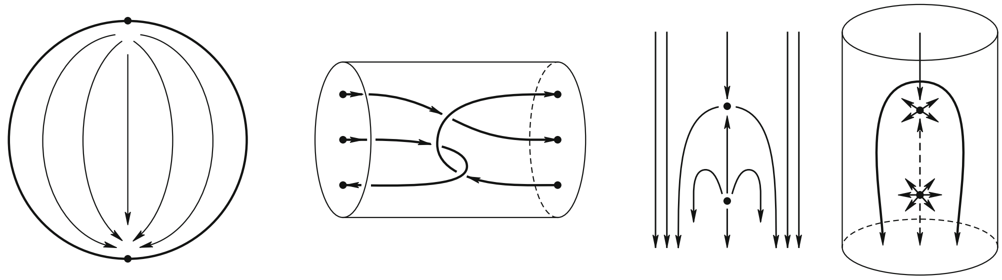
**Proof.** Let $e\colon M^k \to \mathbb{R}^n$ be an embedding of a $k$-manifold (possibly with boundary) into $\mathbb{R}^n$. It was proved in [20] (see also [21], [22]) that, if $n \ge 5$ and $k \ne n-2$, then the embedding $e$ has no isolated points of wild embedding. Since the unstable and stable manifolds are smoothly embedded submanifolds, it follows that the $k$-spheres $S^k_\omega$ and $S^k_\alpha$ can have points of wild embedding only at nodes. Applying results of [20] to a neighborhood of a node homeomorphic to $\mathbb{R}^n$, we see that $S^k_\omega$ and $S^k_\alpha$ are locally flat topologically embedded $k$-spheres. $\square$

## 2. EXAMPLE OF A WILD EMBEDDING OF THE CLOSURE OF A SEPARATRIX

### *2.1. Idea of the Construction*

We borrow the idea of the construction of similar examples from [9], [5]. For this reason, it makes sense to recall the key points of these constructions. Consider a north-south flow $f^t_{NS}$ on the 3-sphere $S^3$ which has one sink $\omega$ and one source $\alpha$ (see Fig. 1 (a)). All other orbits are wandering.

Let $f_{NS} = f^1_{NS}$ denote the shift along the orbits of the flow $f^t_{NS}$ in the time $t = 1$. Consider the Artin–Fox configuration consisting of the three arcs shown in Fig. 1 (b). As is well known, the Artin–Fox curve $l_{\mathrm{AF}}$ is obtained by shifts of this configuration. Therefore, we can embed the Artin–Fox curve in $S^3$ so that $l_{\mathrm{AF}}$ is invariant with respect to $f_{NS}$ and joins the points $\omega$ and $\alpha$, which are points of wild embedding (in Fig. 3 (b), a tubular neighborhood of the curve $l_{\mathrm{AF}}$ is shown). Let us represent a tubular neighborhood $T$ of the curve $l_{\mathrm{AF}}$ (to be more precise, of the open arc $l_{\mathrm{AF}} \setminus \{\omega, \alpha\}$) as an infinite cylindrical solid figure, on which we define a flow with one saddle and one node. This flow can be obtained by rotating a Cherry cell in a strip around the central line (see Fig. 1 (c)). It is easy to define a flow $g^t_T$ on $T$ so that the shift $g^1_T = g$ along the orbits in the time $t = 1$ on the boundary of $T$ coincides with the shift $f_{NS}$. Now, we can define a diffeomorphism $f\colon S^3 \to S^3$ by setting $f$ equal to $f_{NS}$ outside $T$ and to $g$ inside $T$. As a result, we obtain a gradient-like Morse–Smale diffeomorphism with one saddle and three



(a) (b) (c)

**Fig. 1.**

nodes for which the closure of the two-dimensional separatrix of the saddle is a wildly embedded (at one point) 2-sphere. Note that the closure of one of the one-dimensional separatrices is wildly embedded as well (at an endpoint).

To extend this construction, we represent the 4-sphere $S^4$ as the result of the application to the 3-sphere $S^3$ of a rotation $\mathcal{R}$ with precisely two fixed points, $\omega = S$ and $\alpha = N$. Then the rotation of the Artin–Fox curve yields the 2-sphere $\mathcal{R}(l_{\mathrm{AF}})$ wildly embedded at the two points $S$ and $N$. A tubular neighborhood $T_{\mathcal{R}}$ of this 2-sphere (to be more precise, of the open cylinder $\mathcal{R}(l_{\mathrm{AF}}) \setminus \{N, S\}$) is replaced by a special neighborhood $U_0$ of a saddle of type $(2,2)$. By analogy with the three-dimensional case, a diffeomorphism of the resulting 4-manifold is defined so that it has one sink, one source, and one saddle, and the two-dimensional separatrices of the saddle, together with the nodes, form two wildly embedded 2-spheres.

### *2.2. The Special Neighborhood of a Saddle of Type* $(2,2)$

In Euclidean space $\mathbb{R}^4$ with canonical coordinates $(x_1, x_2, x_3, x_4)$, consider the flow $f_{\mathrm{s}}^t$ determined by the system of differential equations

$$\dot{x}_1 = -x_1, \qquad \dot{x}_2 = -x_2, \qquad \dot{x}_3 = x_3, \qquad \dot{x}_4 = x_4. \tag{4}$$

The origin $O = (0,0,0,0)$ is a saddle for the flow $f_s^t$; it has the stable 2-manifold

$$W^{\mathrm{s}}(O) = \{(x_1, x_2, x_3, x_4) \mid x_3 = 0 = x_4\}$$

and the unstable 2-manifold

$$W^{\mathrm{u}}(O) = \{(x_1, x_2, x_3, x_4) \mid x_1 = 0 = x_2\}.$$

It can be verified directly that the function

$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2)(x_3^2 + x_4^2)$$

is an integral of system (4). The equality $F = 1$ determines a 3-manifold, which we denote by $H^3$ (see Fig. 2(a)).

This manifold separates $\mathbb{R}^4$ into two open invariant sets, one of which is a neighborhood of the saddle $O$. We denote this neighborhood by $U_0$ and call it the *special neighborhood*. Clearly, $\partial U_0 = H^3$.

The set of points whose coordinates satisfy the relations

$$x_1^2 + x_2^2 = r^2 \qquad \text{and} \qquad x_3^2 + x_4^2 = \frac{1}{r^2}$$

with fixed $r > 0$ is homeomorphic to the standard 2-torus $\mathbb{T}^2$, because it can be naturally represented as the direct product of the two circles

$$S^1_{1,2}(r) = \{(x_1, x_2, 0, 0) \mid x_1^2 + x_2^2 = r^2\}, \qquad S^1_{3,4}\left(\frac{1}{r}\right) = \left\{(0,0,x_3,x_4) \;\middle|\; x_3^2 + x_4^2 = \frac{1}{r^2}\right\}.$$

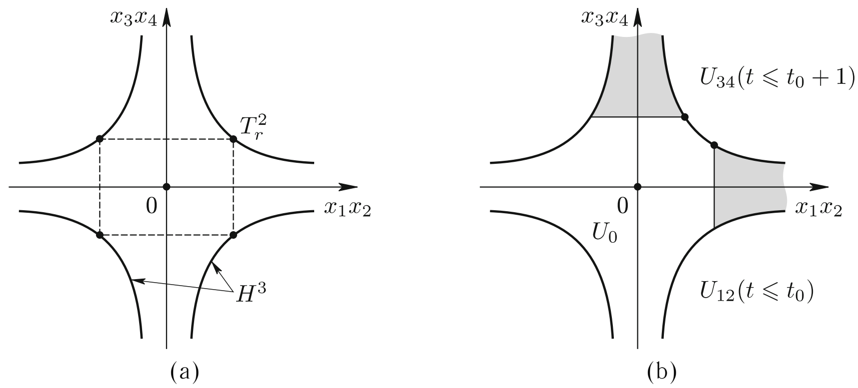



**Fig. 2.**

We denote this 2-torus by $T_r^2$. The one-parameter family $\{T_r^2\}_{r>0}$ forms a foliation of codimension one on $H^3$. Note that $T_r^2$ is the common boundary of the two solid tori

$$P^3_{1,2,r} = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 = r^2,\ x_3^2 + x_4^2 \le \frac{1}{r^2}\right\},$$
$$P^3_{3,4,r} = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 = r^2,\ x_3^2 + x_4^2 \le \frac{1}{r^2}\right\},$$

whose interiors are contained in the neighborhood $U_0$ (see Fig. 2 (b)).

Suppose that the torus $T^2$ is the boundary of a solid torus $P^3 = S^1 \times D^2$, that is, $T^2 = \partial P^3 = S^1 \times \partial D^2$. On $T^2$, there is a unique (up to isotopy) simple closed curve $\{\,\cdot\,\} \times \partial D^2$ homotopic to zero in $P^3$ (because it bounds a disk $\{\,\cdot\,\} \times D^2$) and not homotopic to zero in $T^2$. Any such curve is called a *meridian*. It is natural to refer to a simple closed curve $S^1 \times \{\,\cdot\,\}$ on $T^2$ which intersects the zero meridian at precisely one point as a *parallel*. As is well known, the identification of the boundaries of two copies of $P^3$ by means of a diffeomorphism $T^2 \to T^2$ taking meridians to parallels and vice versa yields a 3-sphere $S^3$. Such a representation of $S^3$ is called a *standard Heegaard diagram of genus* 1.

**Lemma 6.** *The union $P^3_{1,2,r} \cup P^3_{3,4,r}$ is a representation of the* 3*-sphere in the form of a standard Heegaard diagram of genus* 1 (*the boundaries of the solid tori $P^3_{1,2,r}$ and $P^3_{3,4,r}$ are identified by means of the identity map*). *Moreover, in $\mathbb{R}^4$, the* 3*-sphere*

$$S^3(r) = P^3_{1,2,r} \cup P^3_{3,4,r}$$

*bounds an open* 4*-ball $B_0^4 \subset U_0$ containing the saddle* $(0,0,0,0)$ *and separates the special neighborhood $U_0$ into three domains, $U_0$, $U^4_{1,2}(r)$, and $U^4_{3,4}(r)$, where*

$$U^4_{1,2}(r) = \left\{(x_1, x_2, x_3, x_4) \;\middle|\; x_1^2 + x_2^2 > r^2,\ x_3^2 + x_4^2 < \frac{1}{r^2}\,,\ (x_1^2 + x_2^2)(x_3^2 + x_4^2) < 1\right\},$$
$$U^4_{3,4}(r) = \{(x_1, x_2, x_3, x_4) \;\Big|\; x_1^2 + x_2^2 < r^2,\ x_3^2 + x_4^2 > \frac{1}{r^2}\,,\ (x_1^2 + x_2^2)(x_3^2 + x_4^2) < 1\Big\}.$$

**Proof.** Take any point

$$(a_1, a_2, a_3, a_4) \in T_r^2, \qquad a_1^2 + a_2^2 = r^2, \quad a_3^2 + a_4^2 = \frac{1}{r^2}\,.$$

It is easy to show that the curve

$$\{(x_1, x_2, a_3, a_4) \mid x_1^2 + x_2^2 = r^2\}$$

is a meridian of $T_r^2$ treated as the boundary of the solid torus $P^3_{3,4,r}$ and a parallel of $T_r^2$ treated as the boundary of $P^3_{1,2,r}$. Similarly, the curve

$$\{(a_1, a_2, x_3, x_4) \mid x_3^2 + x_4^2 = 1/r^2\}$$

is a parallel of $T_r^2$ treated as the boundary of the solid torus $P^3_{3,4,r}$ and a meridian of $T_r^2$ treated as the boundary of $P^3_{1,2,r}$. It follows that the union $S^3(r) = P^3_{1,2,r} \cup P^3_{3,4,r}$ is a representation of the 3-sphere in the form of a standard Heegaard diagram of genus 1. Clearly, in $\mathbb{R}^4$, $S^3(r)$ bounds a 4-ball $B_0^4 \subset U_0$ containing the saddle $(0,0,0,0)$. The remaining assertions are verified directly as well. □

**Lemma 7.** *Each orbit of the flow $f_s^t$ contained in $H^3$ intersects each 2-torus $T_r^2$ precisely once, and the intersection is quasi-transversal* (*this means that the tangent space to $T_r^2$ and the orbits of the flow intersect only in zero*).

**Proof.** It follows from the form of (4) that the projection of any trajectory $l \subset H^3$ on the plane $(x_1, x_2, 0, 0)$ is the orbit of an attracting node. Therefore, the projection of $l$ intersects $S^1_{1,2}(r)$ precisely once, and the intersection is transversal. Similarly, the projection of $l$ on the plane $(0,0,x_3,x_4)$ intersects $S^1_{3,4}(1/r)$ precisely once, and the intersection is transversal. The required assertion follows. □

Lemma 7 makes it possible to parameterize the family $\{T_r^2\}_{r=0}^{+\infty}$ by the moment of time $t$ at which the 2-tori $T_r^2$ intersect a given orbit; this parameterization is more convenient for our purposes. Let $l^t$ be the trajectory passing through the point $(1,0,1,0)$ at $t=0$. It can be verified directly that $(1,0,1,0)$ belongs to the torus $T_1^2$, and $l^t$ passes through the points $(e^{-t}, 0, e^t, 0)$ of the tori $T^2_{\exp(-t)}$ with $t \in \mathbb{R}$. As a consequence, $H^3$ is diffeomorphic to $\mathbb{R} \times \mathbb{T}^2$ under the map

$$\{t\} \times \mathbb{T}^2 \to \{t\} \times T^2_{\exp(-t)}.$$

For simplicity, we denote the torus $T^2_{\exp(-t)}$ by $\mathbb{T}_t^2$ (we can assume that we have made the change $t = -\ln r$) and the corresponding solid tori $P^3_{1,2,r}$ and $P^3_{3,4,r}$ by $P^3_{1,2,t}$ and $P^3_{3,4,t}$, respectively. We denote the sets into which the solid tori $P^3_{1,2,t_0}$ and $P^3_{3,4,t_0}$ separate $U_0$ at fixed $t = t_0$ according to Lemma 6 by $U_{12}$ (if $t \le t_0$) or $U_{34}$ (if $t \ge t_0$). The torus $\mathbb{T}^2_{t_0}$ separates $H^3$ into the sets

$$\mathbb{T}^2_{t\le t_0} = \bigcup_{t\le t_0} \mathbb{T}_t^2, \qquad \mathbb{T}^2_{t\ge t_0} = \bigcup_{t\ge t_0} \mathbb{T}_t^2,$$

for which

$$\partial U_{12}(t \le t_0) = \mathbb{T}^2_{t\le t_0}, \qquad \partial U_{34}(t \ge t_0) = \mathbb{T}^2_{t\ge t_0}.$$

On each $\mathbb{T}_t^2$, we introduce coordinates $(u, v)$, $u, v \in [0;1)$, by setting

$$x_1 = e^{-t}\cos 2\pi u, \qquad x_2 = e^{-t}\sin 2\pi u, \qquad x_3 = e^{t}\cos 2\pi v, \qquad x_4 = e^{t}\sin 2\pi v. \tag{5}$$

On $H^3 = \partial U_0$, we obtain the coordinate system $(t, u, v)$, which we denote by $(t_2, u_2, v_2)$.

For any fixed $t$, the curve $u = 0$ is the *zero meridian* of $\mathbb{T}_t^2$ treated as the boundary of $P^3_{1,2,t}$. Similarly, the curve $v = 0$ is the *zero meridian* of $\mathbb{T}_t^2$ treated as the boundary of $P^3_{3,4,t}$. We refer to the curve $u = 0$ as the *zero parallel* of $\mathbb{T}_t^2$ treated as the boundary of $P^3_{1,2,t}$ and to the curve $v = 0$ as the *zero parallel* of $\mathbb{T}_t^2$ treated as the boundary of $P^3_{3,4,t}$. The intersection point of the zero meridian with the zero parallel is said to be *marked*. On $\mathbb{T}_t^2$, the marked point has coordinates $(e^{-t}, 0, e^t, 0)$.

*2.3. The Special Neighborhood of an Artin–Fox Cylinder*

Consider the copy of the space $\mathbb{R}^4$ with canonical coordinates $(x_1, x_2, x_3, x_4)$. Suppose that the flow $f^t_{NS}$ is determined by the system of differential equations

$$\dot{x}_1 = x_1, \qquad \dot{x}_2 = x_2, \qquad \dot{x}_3 = x_3, \qquad \dot{x}_4 = x_4. \tag{6}$$

The origin $(0, 0, 0, 0)$ is a repelling node, which we denote by $N$. It is convenient to consider this copy of $\mathbb{R}^4$ as the 4-sphere $S^4$ minus the point $S$: $\mathbb{R}^4 = S^4 \setminus \{S\}$. The point $S$ is identified in an obvious sense with the ideal 3-sphere at infinity in $\mathbb{R}^4$ so that any ray starting at the origin is an arc joining the points $N$ and $S$. Clearly, $f^t_{NS}$ can be extended to the entire 4-sphere $S^4$ so that the point $S$ becomes an attracting node. The flow $f^t_{NS}$ is of type north-south, and any ray going from the origin is an orbit of $f^t_{NS}$. The diffeomorphism

$$f_{NS} = f^1_{NS} : (x_1, x_2, x_3, x_4) \to (ex_1, ex_2, ex_3, ex_4)$$

is the shift by time $t = 1$ along the orbits of the linear flow $f^t_{NS}$. Clearly, the spheres

$$S^3_m = \{(x_1, \dots, x_4) : x_2^2 + x_2^2 + x_3^2 + x_4^2 = e^{2m}\}, \qquad S^2_m = S^3_m \cap \{x_4 = 0\}$$

are invariant with respect to $f_{NS}$.

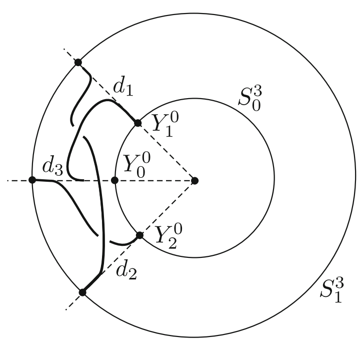
In

$$\mathbb{R}^3_+ = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_3 \geq 0,\ x_4 = 0\},$$

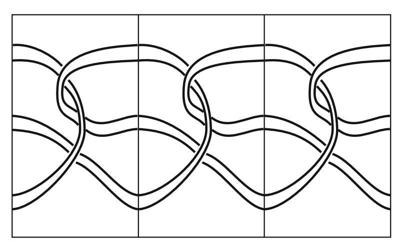
we construct an Artin–Fox curve $l_{\mathrm{AF}} = l$ as follows. On $S^2_0$, we take the three point

$$Y^0_0\left(\frac{1}{2}; 0; \frac{\sqrt{3}}{2}; 0\right), \qquad Y^0_1(0; 0; 1; 0), \qquad Y^0_2\left(\frac{\sqrt{3}}{2}; 0; \frac{1}{2}; 0\right).$$

In the annulus $K^3_{01}$ bounded by the spheres $S^3_0$ and $f_{NS}(S^3_0) = S^3_1$, we join the points $Y^0_0$ and $Y^0_1$ by an arc $d_1$ and the points $f_{NS}(Y^0_0) = Y^0_3$ and $Y^0_2$ by an arc $d_3$ (we specify the arrangement of the arcs later on; see Fig. 3(a)). We also join the points $f_{NS}(Y^0_1)$ and $f_{NS}(Y^0_2)$ by an arc $d_2$ so that the arcs $d_1$, $d_2$, and $d_3$ form an Artin–Fox configuration in the annulus $K^3_{01}$.




(a)



(b)

**Fig. 3.**

We require that the arcs $d_1$, $d_2$, and $d_3$ lie on rays issuing from the origin in neighborhoods of their endpoints. We can assume that $A = d_1 \cup f^{-1}_{NS}(d_2) \cup d_3$ is a simple arc whose endpoints $Y^0_0$ and $f_{NS}(Y^0_0)$ are identified by $f_{NS}$. Then the union

$$l^\circ \overset{\mathrm{def}}{=} \bigcup_{k \in \mathbb{Z}} f^k_{NS}(A)$$

is a simple curve joining the points $S$ and $N$ in $S^4$. The arc

$$l = \{S, N\} \bigcup_{k \in \mathbb{Z}} f^k_{NS}(A) = \{S, N\} \cup l^\circ$$

is an Artin–Fox curve [23].

As is well known (see, e.g., [19, p. 118 of the Russian original]), at each of the points $N$ and $S$, the complement $S^3 \setminus l$ does not have the homotopy type of the circle (this means that, for any sufficiently small neighborhood $U$ of, say, the point $N$, there exists a smaller neighborhood $U'$ such that the image $\pi_1(U' \setminus l)$ in $\pi_1(U \setminus l)$ under the embedding homomorphism is not an infinite cyclic group). Note that we regard the 3-sphere $S^3$ both as the space $\mathbb{R}^3$ extended by adding the point $S$ and as a part of the 4-sphere $S^4$.

The rotation $\mathcal{R}$ of the half-space $\mathbb{R}^3_+$ about the 2-plane $x_4 = 0 = x_3$ is determined by

$$\overline{x}_1 = x_1, \quad \overline{x}_2 = x_2, \quad \overline{x}_3 = x_3 \cos 2\pi v - x_4 \sin 2\pi v, \quad \overline{x}_4 = x_3 \sin 2\pi v + x_4 \cos 2\pi v, \tag{7}$$

where $v \in [0, 1]$. Thus, $\mathcal{R}(l)$ is a 2-sphere topologically embedded in $S^4$, because the points $S$ and $N$ are fixed with respect to the rotation $\mathcal{R}$, and the remaining part $l^\circ = l \setminus (\{S, N\})$ of the Artin–Fox curve is contained in the interior of the half-space $\mathbb{R}^3_+$. We sometimes endow objects obtained by applying the rotation $\mathcal{R}$ with the subscript $\mathcal{R}$; for example, $\mathcal{R}(l) = l_{\mathcal{R}}$.

**Lemma 8.** *The 2-sphere $\mathcal{R}(l)$ is wildly embedded in $S^4$ at the points $S$ and $N$.*

**Proof.** According to [24, Theorem 3], the groups $\pi_1(S^4 \setminus l_{\mathcal{R}})$ and $\pi_1(S^3 \setminus l)$ are isomorphic (to be more precise, any closed path in $S^4 \setminus l_{\mathcal{R}}$ is homotopic to a path in $S^3 \setminus l$, and vice versa). Since the complement $S^3 \setminus l$ does not have the homotopy type of the circle at each of the points $N$ and $S$, it follows that $S^4 \setminus l_{\mathcal{R}}$ does not have the homotopy type of the circle either at each of the points $N$ and $S$. By virtue of [20, Theorem 1 (c)], none of the points $N$ and $S$ is locally flat. $\square$

Let us parameterize the arc $A$ by means of any diffeomorphism $\theta_0\colon [0; 1] \to A$ for which

$$\theta(0) = Y_0^0, \qquad \theta\left(\frac{1}{3}\right) = Y_1^0, \qquad \theta\left(\frac{2}{3}\right) = Y_2^0, \qquad \theta(1) = f_{NS}(Y_0^0).$$

Clearly, we can extend $\theta_0$ to a smooth parameterization $\theta\colon \mathbb{R} \to l^\circ$ by setting

$$\theta(t) = f_{NS}^{[t]} \circ \theta_0(t \bmod 1),$$

where $[t]$ denotes the integer part of the number $t \in \mathbb{R}$. It follows by construction that the Artin–Fox curve $l$ is invariant with respect to $f_{NS}$ and perpendicularly intersects each 3-sphere $S^3_m$, $m \in \mathbb{Z}$, at the points

$$l^\circ(m), \qquad l^\circ\left(m + \frac{1}{3}\right), \qquad l^\circ\left(m + \frac{2}{3}\right)$$

corresponding to the parameters $t = m, m + 1/3$, and $m + 2/3$; moreover, the open arc $l^\circ$ is contained in the interior of the half-space $\mathbb{R}^3_+$. This implies the existence of a tubular neighborhood $T(A)$ of the arc $A$ such that $T(A)$ is diffeomorphic to the direct product $A \times D^2$ and intersects $S^3_0$ in the three 2-disks

$$D_{0,0} = \{0\} \times D^2, \qquad D_{1/3,1} = \left\{\frac{1}{3}\right\} \times D^2, \qquad D_{2/3,2} = \left\{\frac{2}{3}\right\} \times D^2,$$

where $Y_i^0 \in D_{i/3,i}$, $i = 1, 2, 3$, and the 3-sphere $S^3_1$ in the 2-disk

$$f_{NS}(D_{00}) = \{1\} \times D^2.$$

Without loss of generality, we can assume that $T(A)$ does not intersect the plane $x_3 = 0 = x_4$ (and, therefore, is contained in the interior of the half-space $\mathbb{R}^3_+$). Then the set

$$T(l^\circ) = \bigcup_{k \in \mathbb{Z}} f_{NS}^k(T(A))$$

is a tubular neighborhood of $l^\circ$ invariant with respect to $f_{NS}$. The product structure on $T(A)$ is carried over to $T(l^\circ)$ by means of iterations of $f_{NS}$, so that $T(l^\circ)$ is diffeomorphic under the identification $i_{\mathrm{AF}}$

to the product $\mathbb{R} \times D^2$. To emphasize that the product $\mathbb{R} \times D^2$ is related to $T(l^\circ)$, we denote it by $(\mathbb{R} \times D^2)_{\mathrm{AF}}$. The product structure makes it possible to naturally carry over the parameterization $\theta$ to any curve

$$\mathbb{R} \times \{z\})_{\mathrm{AF}} \subset (\mathbb{R} \times D^2)_{\mathrm{AF}}$$

with any $z \in D^2$; moreover, any curve of this form is invariant with respect to $f_{NS}$. This and Lemma 8 imply the following assertion.

**Corollary.** *The complement $S^4 \setminus (T(l^\circ)_{\mathcal{R}} \cup \{N, S\})$ does not have the homotopy type of the circle at each of the points $N$ and $S$.*

**Proof.** The curve $(\mathbb{R} \times \{z\})_{\mathrm{AF}} \cup \{N, S\}$ is an Artin–Fox curve isotopic to $l$ in $T(l^\circ) \cup \{N, S\}$, and the isotopy is fixed at the points $N$ and $S$. Therefore, $l$ is a retract of the set $T(l^\circ) \cup \{N, S\}$, and hence $l_{\mathcal{R}}$ is a retract of the set $T(l^\circ)_{\mathcal{R}} \cup \{N, S\}$. $\square$

### *2.4. Main Construction*

Clearly, under the rotation $\mathcal{R}$ defined by (7), the tubular neighborhood $(\mathbb{R} \times D^2)_{\mathrm{AF}}$ forms a neighborhood $\mathcal{R}(\mathbb{R} \times D^2)_{\mathrm{AF}}$ of the infinite 2-cylinder $\mathcal{R}(l^\circ) = l^\circ_{\mathcal{R}}$. It follows from the presence of a product structure that the neighborhood $\mathcal{R}(\mathbb{R} \times D^2)_{\mathrm{AF}}$ is diffeomorphic to $\mathbb{R} \times D^2 \times S^1$, and its boundary is homeomorphic to $\mathbb{R} \times S^1 \times S^1$. Therefore, on the boundary of the neighborhood $\mathcal{R}(\mathbb{R} \times D^2)_{\mathrm{AF}}$, there are coordinates $(t, u, v)$, where $v$ is defined according to (7). We denote these coordinates by $(t_1, u_1, v_1)$.

Let $\mathcal{I}$ be the union of the points $N$ and $S$ with the interior of $(\mathbb{R} \times D^2)_{\mathrm{AF}}$:

$$\mathcal{I} = \operatorname{int}(\mathbb{R} \times D^2)_{\mathrm{AF}} \cup \{N, S\}.$$

We set $M_1 = S^4 \setminus \mathcal{I}$. Note that, removing the interior of the set $(\mathbb{R} \times D^2)_{\mathrm{AF}}$ from $S^4$, we obtain a compact set with boundary homeomorphic to $S^2 \times S^1$. Removing also the points $N$ and $S$, we obtain a noncompact smooth 4-manifold, and its noncompactness is caused by the removal of two boundary points. Thus, the boundary $\partial M_1$ of the set $M_1$ is homeomorphic to $\mathbb{R} \times S^1 \times S^1$, that is, $\partial M_1 \simeq \mathbb{R} \times S^1 \times S^1$.

Take $M_2 = \operatorname{cl} U_0$. Since $U_0$ is the domain in $\mathbb{R}^4$ bounded by the submanifold $H^3$ of codimension one, it follows that $M_2$ is a noncompact smooth 4-manifold. Recall that, on the boundary

$$\partial M_2 = H^3 \simeq \mathbb{R} \times S^1 \times S^1$$

of the manifold $M_2$, there are coordinates $(t_2, u_2, v_2)$. Consider the map $\Xi\colon \partial M_2 \to \partial M_1$ defined by

$$t_1 = t_2, \qquad u_1 = u_2 - v_2, \qquad v_1 = v_2. \tag{8}$$

According to [25] (see also [26]), the set

$$M^4_* = M_1 \bigcup_{\Xi} M_2,$$

that is, the sets $M_1$ and $M_2$ attached to each other along boundaries by the diffeomorphism $\Xi$, is a smooth noncompact 4-manifold. As mentioned above,

$$M_1' = S^4 \setminus \operatorname{int}(\mathbb{R} \times D^2)_{\mathrm{AF}}$$

is a compact set, and $M_1 = M_1' \setminus \{N, S\}$. Since the image $\Xi(\partial M_2)$ of the attaching diffeomorphism $\Xi$ does not contain the points $N$ and $S$, it follows that $M^4_* = M_1 \cup_\Xi M_2$ can be represented as the result of attaching $M_1'$ to $M_2$ and removing the two points $N$ and $S$, i.e., as

$$M^4_* = (M_1' \cup_\Xi M_2) \setminus \{N, S\}.$$

In what follows, we shall prove that the set $M_1' \cup_\Xi M_2$ admits the structure of a smooth (closed) 4-manifold. To this end, for each of the points $N$ and $S$, we shall construct a sequence of four-dimensional annuli $K_i \simeq S^3 \times [0; 1]$ converging to $N$ and $S$ and surrounding these points in an obvious sense.

First, we prove a lemma in which the case where the Heegaard diagram of genus 1 is the 3-sphere is considered. In this lemma, the boundary of the solid torus $P^3$ is denoted by $T^2$, and on $T^2$, meridians and parallels are naturally defined. By $\mu$ and $\lambda$ we denote the generators of the homology group $H_1(T^2,\mathbb{Z})$ corresponding, respectively, to a meridian and a parallel of the torus $T^2$. As is well known, a diffeomorphism of the torus $T^2$ induces an automorphism of the group $H_1(T^2,\mathbb{Z})$, which can be represented as

$$\mu \mapsto r\mu + p\lambda, \quad \lambda \mapsto s\mu + q\lambda, \quad qr - ps = \pm 1, \qquad \text{where} \quad r, s, p, q \in \mathbb{Z}.$$

**Lemma 9.** *Suppose that the 3-sphere $S^3$ is represented as the standard Heegaard splitting $S^3 = P^3 \cup P_*$ of genus 1 of two solid tori $P^3$ and $P_*$. Consider the manifold obtained by removing the solid torus $P^3$ from $S^3$ and again attaching $P^3$ to $P_*$ by a diffeomorphism $\psi\colon T^2 \to T^2$ (the boundaries of $P^3$ to $P_*$ are naturally identified with $T^2$), which induces either the automorphism $\mu \mapsto -\mu + \lambda$, $\lambda \mapsto -\lambda$ or the automorphism $\mu \mapsto \lambda$, $\lambda \mapsto -\mu - \lambda$. Then the 3-manifold $P^3 \cup_\psi P_*$ thus obtained is a 3-sphere.*

**Proof.** As is well known, if $P^3$ and $P_*$ are attached to each other by a diffeomorphism $\psi\colon T^2 \to T^2$ inducing an automorphism of the form

$$\mu \mapsto r\mu + p\lambda, \quad \lambda \mapsto s\mu + q\lambda, \qquad qr - ps = \pm 1,$$

then $P^3 \bigcup_\psi P_*$ is the lens space $L(p,q)$. In the case of the automorphisms specified in the statement of the lemma, we obtain the lens space $L(1,-1)$. Since $L(p,q) = L(p, q \bmod p)$, it follows that

$$L(1,-1) = L(1, -1 \bmod 1) = L(1,0) = S^3. \quad \square$$

Clearly, we have

$$\mathcal{R}(S^2_m) = S^3_m \subset \mathbb{R}^4 \setminus \{N, S\}.$$

Let us denote the exterior of the sphere $S^3_m$ together with $S^3_m$ by

$$K(\geq m) = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq e^{2m}\}.$$

By $K(m_1, m_2)$ we denote the closed annulus bounded by the spheres $S^3_{m_1}$ and $S^3_{m_2}$. Finally, we denote the interior of the sphere $S^3_m$ together with $S^3_m$ punctured at $N$ by $K(\leq m)$.

According to Lemma 7, the tori $\mathbb{T}^2_0$ and $\mathbb{T}^2_1$ separate $H^3$ into the three sets $\mathbb{T}^2_{t\geq 1}$, $\mathbb{T}^2_{0\leq t\leq 1}$, and $\mathbb{T}^2_{t\leq 0}$, one of which ($\mathbb{T}^2_{0\leq t\leq 1}$) is compact and homeomorphic to the direct product of the 2-torus $T^2$ and $[0;1]$. The other two sets are homeomorphic to the direct product of $T^2$ and $[0;\infty)$. Let us denote the restrictions of the diffeomorphism $\Xi$ to the sets $\mathbb{T}^2_{t\geq 1}$, $\mathbb{T}^2_{0\leq t\leq 1}$, and $\mathbb{T}^2_{t\leq 0}$ by $\Xi_{t\geq 1}$, $\Xi_{0\leq t\leq 1}$, and $\Xi_{t\leq 0}$, respectively.

The boundary of the tubular neighborhood $(\mathbb{R} \times D^2)_{\mathrm{AF}}$ is a 2-cylinder; the circles $(\{0\} \times \partial D^2)_{\mathrm{AF}}$ and $(\{1\} \times \partial D^2)_{\mathrm{AF}}$ separate this cylinder into the compact cylinder

$$C_{0\leq t\leq 1} = ([0;1] \times S^1)_{\mathrm{AF}}$$

and the two noncompact cylinders

$$C_{t\geq 1} = ([1;+\infty) \times S^1)_{\mathrm{AF}}, \qquad C_{t\leq 0} = ((-\infty;0] \times S^1)_{\mathrm{AF}}.$$

Under the action of the rotation $\mathcal{R}$, these cylinders $C_{0\leq t\leq 1}$, $C_{t\geq 1}$, and $C_{t\leq 0}$ form the sets

$$C_{0\leq t\leq 1,\mathcal{R}} \simeq [0;1] \times S^1 \times S^1, \quad C_{t\geq 1,\mathcal{R}} \simeq [1;+\infty) \times S^1 \times S^1, \quad C_{t\leq 0,\mathcal{R}} \simeq (-\infty;0] \times S^1 \times S^1,$$

respectively. By virtue of (8), $\Xi_{t\geq 1}$ maps $\mathbb{T}^2_{t\geq 1}$ to $C_{t\geq 1,\mathcal{R}}$, $\Xi_{0\leq t\leq 1}$ maps $C_{0\leq t\leq 1,\mathcal{R}}$ to $\mathbb{T}^2_{0\leq t\leq 1}$, and $\Xi_{t\leq 0}$ maps $C_{t\leq 0,\mathcal{R}}$ to $\mathbb{T}^2_{t\leq 0}$.

Since

$$\partial U_{12}(t \leq 0) = \mathbb{T}^2_{t\leq 0} \qquad \text{and} \qquad \partial U_{34}(t \geq 1) = \mathbb{T}^2_{t\geq 1},$$

it follows that $M^4_* = M_1 \cup_\Xi M_2$ can be represented as the union of the following three (intersecting) sets:

1) $U_{12}(t \le 0) \cup_{\Xi_{t\le 0}} [K(\le 0) \setminus \mathcal{I}] \overset{\text{def}}{=} B_N$;

2) $U_{34}(t \ge 1) \cup_{\Xi_{t\ge 1}} [K(r \ge 1) \setminus \mathcal{I}] \overset{\text{def}}{=} B_S$;

3) $B^4(0 \le t \le 1) \cup_{\Xi_{0\le t\le 1}} [K(-1,1) \setminus \mathcal{I}] \overset{\text{def}}{=} B_*$.

The set $B_*$ is compact, while $B_N$ and $B_S$ are not. According to [25] (see also [26, Chap. 3]), $B_*$ has the structure of a smooth manifold induced by $\Xi$ and the structures on $B^4(0 \le t \le 1)$ and $K(-1,1)$. Let us prove that each of the sets $B_N$ and $B_S$ has a one-point compactification, and the compact set thus obtained can be endowed with the structure of a smooth manifold extending the smooth structure of the manifold $B_*$. First, consider $B_N$. The set $B_N$ contains the sequence of 3-manifolds $S_{*,-m}$, $m \in \mathbb{N}$, obtained by removing the sets $S^3_{-m} \cap \mathcal{I}$ from the 3-spheres $S^3_{-m}$. For fixed $m$, the manifold $S_{*,-m}$ is the 3-sphere $S^3_{-m}$ minus the three (disjoint) solid tori

$$D_{-m,1} \times S^1, \qquad D_{-m+1/3,2} \times S^1, \qquad D_{-m+2/3,3} \times S^1.$$

In $B_N$, to each boundary component (homeomorphic to the 2-torus) of the manifold $S_{*,-m}$ the corresponding solid tori $P^3_{1,2,-m}$, $P^3_{1,2,-m+1/3}$, and $P^3_{1,2,-m+2/3}$ are attached by the diffeomorphism $\Xi_{t\le 0}$.

Let us show that

$$S_{*,-m} \bigcup_{\Xi_{t\le 0}} (P^3_{1,2,-m} \cup P^3_{1,2,-m+1/3} \cup P^3_{1,2,-m+2/3}) \overset{\text{def}}{=} S^3_{m,*}$$

is a 3-sphere. On the boundary $\mathbb{T}_{-m}$ of the solid torus $P^3_{1,2,-m}$, there are coordinates $(-m, u_2, v_2)$ defined by (5). According to Lemma 6, the meridians of the torus $\mathbb{T}_{-m}$ bounding the solid tori $P^3_{1,2,-m}$ are determined by $v_2 = \text{const}$, and the parallels are determined by $u_2 = \text{const}$.

On the boundary $T^2_{-m,1}$ of the solid torus $D_{-m,1} \times S^1$, there exist coordinates $(-m, u_1, v_1)$ in which the meridians are given by $v_1 = \text{const}$ and parallels, by $u_1 = \text{const}$. It follows from (8) and Lemma 9 that, removing the solid torus $D_{-m,1} \times S^1$ from the 3-sphere $S^3_{-m}$ and attaching the solid torus $P^3_{1,2,-m}$, we again obtain a 3-sphere. Repeating this argument for the solid tori

$$D_{-m+1/3,2} \times S^1, \qquad P^3_{1,2,-m+1/3}, \qquad D_{-m+2/3,3} \times S^1, \qquad P^3_{1,2,-m+2/3},$$

we see that $S^3_{m,*}$ is a 3-sphere.

Let us show that, for any $m \ge 1$, the spheres $S^3_{m,*}$ and $S^3_{m+1,*}$ in $B_N$ bound an annulus homeomorphic to $S^3 \times [0;1]$. Note that the spheres $S^3_{m,*}$ and $S^3_{m+1,*}$ in $B_N$ bound a compact set $N^4$ with nonempty boundary $\partial N^4 = S^3_{m,*} \cup S^3_{m+1,*}$. Indeed, $N^4$ is obtained by attaching the three compact sets

$$U_{12}\biggl(-m + \frac{1}{3} \le t \le -m + \frac{2}{3}\biggr), \qquad U_{12}\biggl(-m - \frac{1}{3} \le t \le -m\biggr),$$
$$U_{12}\biggl(-m - 1 \le t \le -m - \frac{2}{3}\biggr)$$

to $K(-m \le -m-1) \setminus \mathcal{I}$. The boundaries of these sets consist of the corresponding solid tori contained in the boundaries of the 3-spheres $S^3_{m,*}$ and‘$S^3_{m+1,*}$ and the sets

$$\mathbb{T}^2_{-m+1/3 \le t \le -m+2/3}, \qquad \mathbb{T}^2_{-m-1/3 \le t \le -m}, \qquad \mathbb{T}^2_{-m-1 \le t \le -m-2/3},$$

in which all points become interior after the attachment.

To prove that $N^4$ and $S^3 \times [0;1]$ are homeomorphic, we show that $N^4$ is embedded in $\mathbb{R}^4$. First, on the Artin–Fox curve, we take the arc $d$ corresponding to the parameter $-m-2 \le t \le -m+1$ and consider its tubular neighborhood

$$[-m-2; -m+1] \times D^2 \subset (\mathbb{R} \times D^2)_{\text{AF}},$$

which is a part of the tubular neighborhood of the entire Artin–Fox curve and has the natural structure of a direct product. We extend $d$ over the half-space $\operatorname{int}\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \mid x_3 > 0\}$ to a smooth (open) curve $L$ going from the origin to infinity so that the following conditions hold.

**Condition 1.** There exists a smooth diffeotopy $\varphi_\alpha$, $0 \le \alpha \le 1$, on $\mathbb{R}^3_+$ such that

(a) $\varphi_0 = \mathrm{id}$;

(b) the diffeomorphism $\varphi_1$ takes $L$ to the ray

$$\mathcal{L} = \varphi_1(L) = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3 \ge 0\};$$

(c) $\varphi_\alpha = \mathrm{id}$ outside the 3-ball containing the arc $d$, and $\varphi_\alpha = \mathrm{id}$ near the plane $x_3 = 0$ for all $0 \le \alpha \le 1$.

**Condition 2.** The tubular neighborhood $[-m+1; -m-2] \times D^2$ can be extended to a tubular neighborhood $T(L) \subset \mathbb{R}^3_+$ of the curve $L$ endowed with the direct product structure: $T(L) = (\mathbb{R} \times D^2)_L$.

The arc $d$ can be deformed into the interval in space $\mathbb{R}^3_+$; hence there exists an extension to the curve $L$ and a diffeotopy $\varphi_\alpha$ with the required properties. It follows from Condition 1 that $L$ coincides with the ray $\mathcal{L}$ near the origin and outside a sufficiently large ball.

Clearly, the ray $\mathcal{L}$ transversally intersects the 3-spheres

$$S^3_r = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = r^2\}$$

of radius $r > 0$. Hence the ray $\mathcal{L}$ has a tubular neighborhood $T(\mathcal{L})$ admitting the structure of a direct product generated by the intersections of the spheres $S^3_r$ with $\mathcal{L}$, i.e., such that

$$T(\mathcal{L}) = (\mathbb{R} \times D^2)_{\mathcal{L}}, \qquad \text{where} \quad (\{r\} \times D^2)_{\mathcal{L}} = S^3_r \cap T(\mathcal{L}).$$

Passing, if necessary, to smaller tubular neighborhoods, we can assume that the diffeotopy $\varphi_\alpha$ satisfies the following additional condition.

**Condition 3.** The diffeotopy $\varphi_1$ takes $T(L)$ to $T(\mathcal{L})$ so that the direct product structure is preserved.

Let $\mathcal{R}_\tau$ denote the rotation of the half-space $\mathbb{R}^3_+$ about the 2-plane $x_4 = 0 = x_3$ which is defined by (7) with $v = \tau$. Clearly, $\mathcal{R}_\tau$ is a diffeomorphism. Therefore, if $A, B \subset \mathbb{R}^3_+$ are two sets diffeomorphic under fixed $\mathcal{R}_\tau$ and disjoint from the plane $x_4 = 0 = x_3$, then

$$\mathcal{R}(A) = \bigcup_{0 \le \tau \le 1} \mathcal{R}_\tau(A) = \mathcal{R}(B) = \bigcup_{0 \le \tau \le 1} \mathcal{R}_\tau(B).$$

It follows that $T(L)_{\mathcal{R}} = T(\mathcal{L})_{\mathcal{R}}$, and, therefore,

$$\mathbb{R}^4 \setminus T(L)_{\mathcal{R}} = \mathbb{R}^4 \setminus T(\mathcal{L})_{\mathcal{R}}. \tag{9}$$

The direct product structures on $T(L) = (\mathbb{R} \times D^2)_L$ and $T(\mathcal{L}) = (\mathbb{R} \times D^2)_{\mathcal{L}}$ can be naturally extended to direct product structures on $T(L)_{\mathcal{R}}$ and $T(\mathcal{L})_{\mathcal{R}}$, respectively, which allows us to introduce coordinates on $\partial T(L)_{\mathcal{R}}$ and $\partial T(\mathcal{L})_{\mathcal{R}}$ by analogy with the coordinates $(t_1, u_1, v_1)$ on the boundary of the neighborhood $\mathcal{R}(\mathbb{R} \times D^2)_{\mathrm{AF}}$. We use the same symbol $\Xi$ (hoping that this will not lead to confusion) to denote the diffeomorphisms

$$\Xi \colon \partial M_2 \to \partial T(L)_{\mathcal{R}}, \qquad \Xi \colon \partial M_2 \to \partial T(\mathcal{L})_{\mathcal{R}}$$

defined by (8). Identifying the boundaries by means of $\Xi$, we obtain the two sets

$$R_1 = (\mathbb{R}^4 \setminus T(L)_{\mathcal{R}}) \cup_\Xi M_2, \qquad R_2 = (\mathbb{R}^4 \setminus T(\mathcal{L})_{\mathcal{R}}) \cup_\Xi M_2.$$

Condition 3 and relation (9) imply that $R_1$ and $R_2$ are homeomorphic. Since $\mathcal{L}$ is a ray, it follows from Lemma 9 that $R_2$ is homeomorphic to the direct product of 3-spheres and a once-punctured ray, i.e., $R_2$

is the space $\mathbb{R}^4$ minus the origin. According to Condition 2, $R_1$ contains $N^4$. Therefore, $N^4$ is embedded in $\mathbb{R}^4$.

It follows from the construction that the 3-spheres $S^3_{m,*}$ and $S^3_{m+1,*}$ have no points of wildness. Locally flat 3-spheres in $\mathbb{R}^4$ bound an annulus homeomorphic to $S^3 \times [0;1]$ [17]; hence, for any $m \geq 1$, the spheres $S^3_{m,*}$ and $S^3_{m+1,*}$ in $B_N$ bound an annulus homeomorphic to $S^3 \times [0;1]$. We denote this annulus by $K_N(-m,-m-1)$. It follows from the above considerations that the noncompact part $B_N$ is the countable union of annuli adjacent to each other along boundary 3-spheres. Therefore, $B_N$ has a one-point compactification $B_{N*}$ to which the topological structure of the manifold $B_N$ can be extended.

It can also be proved in a quite similar way that $B_S$ has a one-point compactification $B_{S*}$ to which the topological structure of the manifold $B_N$ can be extended.

By virtue of (8), the diffeomorphisms

$$f_{NS}\colon M_1 \to M_1 \qquad \text{and} \qquad f_s^1\colon M_2 \to M_2$$

are compatible on the boundaries $\partial M_1$ and $\partial M_2$. Hence they induce a homeomorphism $f\colon M^4 \to M^4$ with three fixed points, namely, the sink $S$, the source $N$, and the saddle $O$. According to [26] and [25], the compact set obtained from $M^4$ by removing spherical neighborhoods $U_S$ and $U_N$ of the nodes $S$ and $N$, respectively, admits the structure of a smooth manifold. By construction, we can choose $U_S$ and $U_N$ so that

$$f(U_S) \subset U_S \qquad \text{and} \qquad f^{-1}(U_N) \subset U_N.$$

Thus, we can return the spherical neighborhoods $U_S$ and $U_N$ back so as to obtain a closed smooth 4-manifold (we denote it by the same symbol $M^4$), on which a diffeomorphism conjugate to $f$ is defined (we denote this diffeomorphism by the same symbol $f$). According to the Corollary and Theorem 1 from [20], the closures of the unstable and stable separatrices of the saddle $O$ are 2-spheres topologically embedded in $M^4$, which are not locally flat at the points $S$ and $N$, respectively. This completes the construction.

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