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Weyl n-algebras and the Kontsevich integral of the unknot

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ABSTRACT

Given a Lie algebra with a scalar product, one may consider the latter as a symplectic structure on a *dg*-scheme, which is the spectrum of the Chevalley–Eilenberg algebra. In Sec. 1 we explicitly calculate the first-order deformation of the differential on the Hochschild complex of the Chevalley–Eilenberg algebra. The answer contains the Duflo character. This calculation is used in the last section. There we sketch the definition of the Wilson loop invariant of knots, which is, hopefully, equal to the Kontsevich integral, and show that for unknot they coincide. As a byproduct, we get a new proof of the Duflo isomorphism for a Lie algebra with a scalar product.

Keywords: Kontsevich integral; factorization homology; Duflo character; knot; link.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

In [18] we built perturbative Chern–Simons invariants by means of the factorization complex of Weyl n-algebras. In this paper, we continue this line and introduce the Wilson loop invariant. This invariant is supposed to be equal to the Bott–Taubes invariant and the Kontsevich integral. In fact, we are only interested in one question here: calculating the Wilson loop invariant of unknot in S^3. This problem appears to be connected with the Duflo isomorphism.

We consider the Duflo isomorphism for Lie algebras with a scalar product, which is much simpler to prove than the general statement from [8]. There are (at least) two proofs of the Duflo isomorphism for a Lie algebra with a scalar product. In [1] the authors use a quantization of the Weil algebra. In [4] the Kontsevich integral of knots and link is used. Our sketch of a proof (see remark before Proposition 4.1) is related to the both. The work [13] also connects these two approaches and it would be very interesting to compare it with our arguments.

Section 1 is not strongly connected with the rest of the paper, but is of independent interest. Here we make a very concrete calculation of the first-order deformation of the Hochschild complex for the Chevalley–Eilenberg algebra of a Lie algebra. The deformation is given by the scalar product. This calculation is closely connected with [17] and may be rephrased in the style of this paper, see Remark 2.9.

In Sec. 2, we give a very short survey of results about e_n-algebras and the factorization complex we need. For basics we refer the reader to [16] and for a much more detailed survey than ours we refer to [10]. At the end of the section we describe a construction, which plays a crucial role in the next section.

In Sec. 3, we apply this construction to the quantum Chevalley–Eilenberg algebra, the role of which for perturbative Chern–Simons invariants is explained in [18, Appendix]. The central result here is Proposition 4.1. The calculation we make here strongly reminds the one from Sec. 1. I would like to understand better reasons of this similarity. This section must be considered as an announcement. It contains no proofs.

Everything is over a field $\Bbbk$ of characteristic 0.

2. Quantization of the Chevalley–Eilenberg Complex

2.1. *Hochschild homology of the Chevalley–Eilenberg complex*

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The Chevalley–Eilenberg algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ is a super-commutative *dg*-algebra $S^*(\mathfrak{g}^\vee[1])$ generated by the dual space $\mathfrak{g}^\vee$ placed in degree 1. The differential is a derivation of this free super-commutative algebra defined on the generators by the tensor $\mathfrak{g}^\vee \to \mathfrak{g}^\vee \wedge \mathfrak{g}^\vee$ dual to the bracket. The Jacobi identity guarantees that this is indeed, a differential. In terms of [2] the Chevalley–Eilenberg algebra may be thought of as the function ring of a Q-manifold.

With any $\mathfrak{g}$-module E one may associate the module $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ over $\mathrm{Ch}^\bullet(\mathfrak{g})$ as follows. As a $S^*(\mathfrak{g}^\vee[1])$-module it is freely generated by E and the differential is defined by its value on $E \otimes 1$ given by the tensor $E \to E \otimes \mathfrak{g}^\vee$ of the $\mathfrak{g}$-action. As a complex, $\mathrm{Ch}^\bullet(\mathfrak{g}, E)$ calculates the cohomology of $\mathfrak{g}$ with coefficients in E.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$ corresponding to the adjoint $\mathfrak{g}$-module may be thought of as a cotangent complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$. The de Rham differential $d_{dR}\colon \mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee_{ad})$, which is a derivation of $\mathrm{Ch}^\bullet(\mathfrak{g})$-modules, is tautologically defined on the generators. Define the $\mathrm{Ch}^\bullet(\mathfrak{g})$-module of differential forms of $\mathrm{Ch}^\bullet(\mathfrak{g})$ as $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. It is a super-commutative algebra and the de Rham differential acts on it in the usual way, it is a derivation.

For a unital *dg*-algebra A define the reduced (or normalized) Hochschild complex $C_*(A)$ (see e.g. [15, Chap. 1.1]) as the total complex of the bi-complex with the $(-i)$th term

$$\prod_{i\geq 0}(A \otimes \underbrace{A/\Bbbk \otimes \cdots \otimes A/\Bbbk}_{i}), \tag{2.1}$$

the first differential coming from A and the second differential given by

$$\begin{aligned} & a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i \\ & \quad \mapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_i - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_i \\ & \qquad + \cdots + (-1)^{i+\deg a_i(\deg a_0 + \cdots \deg a_{i-1})} a_i a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}. \end{aligned} \tag{2.2}$$

Here one have to choose representatives of quotients $A/\Bbbk$, then apply formula and take quotients again, the result does not depend on choices. Note, that the usual definition uses direct sums instead of products, but we need the one we gave. In other words, we shall consider unbounded chains, that is the graded completion ([7, Definition A.25]) of $\sum_{i\geq 0}(A \otimes \underbrace{A/\Bbbk \otimes \cdots \otimes A/\Bbbk}_{i})$ with respect to the grading given by the grading on A. For an ungraded algebra the reduced Hochschild complex calculates $\mathrm{Tor}_*^{A\otimes A^o}(A, A)$.

The following proposition is a variant of the Hochschild–Kostant–Rosenberg isomorphism.

Proposition 2.1. *The formula*

$$a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0\, d_{dR} a_1 \cdots d_{dR} a_i \tag{2.3}$$

defines a morphism from the reduced Hochschild complex $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ of the Chevalley–Eilenberg algebra to its differential forms $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. This morphism is a quasi-isomorphism.

Proof. Direct calculation shows that this is a morphism. The proof of Proposition 2.2 implies that this is a quasi-isomorphism. □

Equip $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with a descending filtration F: the subcomplex $F_k C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ is spanned by chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_0 \geq k$.

Proposition 2.2. *The spectral sequence associated with the filtration F on $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ degenerates at the second sheet. The complex $E_1^{p,0}$ is isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ and $E_1^{p,>0} = 0$.*

Proof. The associated graded object to the filtration F is the tensor product of $S^*(\mathfrak{g}^\vee[1])$ and the normalized standard complex, which calculates the homology of algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ with coefficients in the augmentation module. More precisely, the latter complex is the total complex of the bicomplex, which is the direct product $\prod_i (\mathrm{Ch}^\bullet(\mathfrak{g})/\Bbbk)^{\otimes i}$, and with the second differential defined on $a_1 \otimes \cdots \otimes a_i$ by

$$a_1 \cdot a_2 \otimes \cdots \otimes a_i - a_1 \otimes a_2 \cdot a_3 \otimes \cdots \otimes a_i + \cdots \pm a_1 \otimes \cdots \otimes a_{i-1} \cdot a_i, \tag{2.4}$$

where a_i are elements of the augmentation ideal, which is identified with $\mathrm{Ch}^\bullet(\mathfrak{g})/\Bbbk$. To compute its cohomology consider the spectral sequence associated with the

above mentioned bicomplex with the first differential (2.4). It degenerates at the first sheet for trivial reasons and equals $\Bbbk[[\mathfrak{g}^\vee]]$ sitting in degree 0.

Equip $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ with the stupid filtration (e.g. [9, III.7.5]) and consider the map (2.3) of filtered complexes. In the light of the above, the associated map of spectral sequences gives an isomorphism on the first sheet. It follows that the first differentials also coincide. Thus the first differential of our spectral sequence is as stated and the higher differentials vanish for dimensional reasons. □

Note, that $F_iC_i(\mathrm{Ch}^\bullet(\mathfrak{g}))$ is spanned by chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ such that $\deg a_{>0} = 1$. Taking into account Proposition 2.2 we get the following.

Corollary 2.3. *Every cycle in $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ may be presented by a sum of chains $a_0 \otimes a_1 \otimes \cdots \otimes a_i$ with $\deg a_{>0} = 1$.*

Finding an explicit formula for these cycles seems to be an interesting question.

2.2. *Invariant vector fields*

Along with the Hochschild complex as above one may consider the Hochschild complex $C_*(A, M)$ of a *dg*-algebra A with coefficients in a A-bimodule M (see e.g. [15, Chap. 1.1]). It is given by the same formulas (2.1) and (2.2), but a_0 now is an element of M. For an ungraded algebra the reduced Hochschild complex calculates $\mathrm{Tor}_*^{A\otimes A^o}(A, M)$.

The $\mathrm{Ch}^\bullet(\mathfrak{g})$-module of 1-forms $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee)$ is a bimodule as well, because the algebra is supercommutative. Introduce the Hochschild complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ with coefficients in this bimodule $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$.

Proposition 2.4. *The formulas*

$$\begin{aligned} &a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0\, d_{dR} a_1 \otimes a_2 \otimes \cdots \otimes a_i, \\ &a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto \pm a_0\, d_{dR} a_i \otimes a_1 \otimes \cdots \otimes a_{i-1}, \end{aligned} \tag{2.5}$$

where the sign is defined by the Koszul rule, define morphisms from the Hochschild complex $C_*(\mathfrak{g})$ to the Hochschild complex with coefficients $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ of degree 1.

Proof. This is a direct calculation. □

The following proposition describes these morphisms in terms of the quasi-isomorphism (2.1).

Recall some basic facts from Lie group theory. For a finite-dimensional Lie algebra $\mathfrak{g}$ denote by $U_\mathfrak{g}$ its enveloping algebra. This is a Hopf algebra which is dual to the Hopf algebra of formal functions $F(G)$ on the formal group associated with $\mathfrak{g}$. The Poincaré–Birkhoff–Witt map from the symmetric power of $\mathfrak{g}$ to its universal enveloping $i_{PBW}\colon S^*\mathfrak{g} \to U_\mathfrak{g}$ provides an isomorphism between them as adjoint

$\mathfrak{g}$-modules. It is dual to the exponential coordinate map $\exp^*\colon F(G) \to \Bbbk[[\mathfrak{g}^\vee]]$. Maps

$$\mathcal{L}_L\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \quad \text{and} \quad \mathcal{L}_R\colon F(G) \to F(G) \otimes \mathfrak{g}^\vee \tag{2.6}$$

dual to the multiplications

$$U_\mathfrak{g} \otimes \mathfrak{g} \to U_\mathfrak{g} \quad \text{and} \quad \mathfrak{g} \otimes U_\mathfrak{g} \to U_\mathfrak{g}$$

respectively. After identifying G and $\mathfrak{g}$ by the exponential map, the maps (2.6) are given by elements of $\mathrm{Vect}(\mathfrak{g}) \otimes \mathfrak{g}^\vee$. Corresponding maps from $\mathfrak{g}$ to $\mathrm{Vect}(\mathfrak{g})$ are given by left and right invariant vector fields on G. Applying the constant trivialization of the tangent bundle to $\mathfrak{g}$ one may identify such a tensor with a section of the trivial vector bundle with fiber $\mathrm{End}(\mathfrak{g})$ over $\mathfrak{g}$. In other words, this section is the transformation matrix between the constant basis of the tangent bundle and the one given by left (right) invariant vector fields. By e.g. [21, Chap. 3.4] they are given by formulas

$$\mathrm{id} \pm \frac{1}{2}\mathrm{Ad} + \sum_{n\geq 1} \frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n} \tag{2.7}$$

("+" for the first and "−" for the second tensor), where Ad is the structure tensor of the $\mathfrak{g}$ considered as linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ and B_n are Bernoulli numbers:

$$\sum_{n\geq 0} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \tag{2.8}$$

Recall that Proposition 2.1 identifies $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ with the complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. In the same way, one can build a quasi-isomorphism between $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}), \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee))$ and $\mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \Bbbk[[\mathfrak{g}^\vee]]^{ad})$.

Proposition 2.5. *Under the quasi-isomorphism as above, maps* (2.5)

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \mathfrak{g}^\vee \otimes \Bbbk[[\mathfrak{g}^\vee]]^{ad})$$

are induced by (2.6), *where* $\Bbbk[[\mathfrak{g}^\vee]]$ *is identified with* $F(G)$ *by the exponential map; that is,* (2.5) *are given by formulas* (2.7).

Proof. Recall that in the proof of Proposition 2.2 we considered the direct product of terms of the standard complex calculating $\mathrm{Tor}_*^{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk, \Bbbk)$ and identified it with $\Bbbk[[\mathfrak{g}^\vee]]$. Consider also the complex calculating $\mathrm{Ext}^*_{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk, \Bbbk)$, where we take direct sum rather than direct product. The former complex is dual to the latter one. As in the proof of Proposition 2.2, the spectral sequence argument shows, that the cohomology of the latter complex is isomorphic to $S^*(\mathfrak{g})$. The Yoneda product endows it with multiplication which, as it easy to check, gives it the structure of the universal enveloping algebra of $\mathfrak{g}$. As the unbounded version of $\mathrm{Tor}_*^{\mathrm{Ch}^\bullet(\mathfrak{g})}(\Bbbk, \Bbbk)$ is dual to it, this is formal functions on the group. The quasi-isomorphism (2.3) is

dual to the PBW isomorphism; that is, it is given by the exponential coordinates. Formulas (2.5) define the left and right actions of the Lie algebra on the functions on the group. This proves the statement. □

Remark 2.6. Maps (2.5) may be thought as the Atiyah class of the diagonal of the *dg*-manifold which is a spectrum of $\mathrm{Ch}^\bullet(\mathfrak{g})$. Analogous maps and formulas for a usual complex manifold play a crucial role in [17].

2.3. *Quantization*

Let now $\mathfrak{g}$ be an finite-dimensional Lie algebra with a non-degenerate invariant scalar product $\langle\cdot\,,\cdot\rangle$. The scalar product may be thought of as a constant symplectic structure of degree -2 on the *dg*-manifold (or Q-manifold), which is the spectrum of $\mathrm{Ch}^\bullet(\mathfrak{g})$. That is, we define a Poisson bracket on $\mathrm{Ch}^\bullet(\mathfrak{g})$ on the generators by $\{x,y\}=\langle x,y\rangle$ and extend it to the whole algebra by the Leibnitz rule. In terms of [2] we get a QP-manifold.

A symplectic structure gives a first-order deformation of the product of functions on a manifold and thus deforms the Hochschild complex. Our aim is to calculate it in our case.

More precisely, consider the ring $\Bbbk[\varepsilon]$, where $\deg\varepsilon=2$ and $\varepsilon^2=0$ and the Chevalley–Eilenberg complex $\mathrm{Ch}^\bullet(\mathfrak{g})\otimes\Bbbk[\varepsilon]$ over $\Bbbk[\varepsilon]$ with the differential as before, with the product given by $x\cdot y=x\wedge y+\frac{1}{2}\varepsilon\langle x,y\rangle$. Take the Hochschild complex of $\Bbbk[\varepsilon]$-algebra $\mathrm{Ch}^\bullet(\mathfrak{g})\otimes\Bbbk[\varepsilon]$, that is, all tensor products are taken over $\Bbbk[\varepsilon]$. It is a module over $\Bbbk[\varepsilon]$. Multiplication by ε defines a 2-step filtration on it. Consider the spectral sequence associated with this filtration. The zeroth sheet is $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))\otimes\Bbbk[\varepsilon]$. The following proposition describes d_0 of this spectral sequence, which is the first-order deformation of the differential in the Hochschild complex.

Proposition 2.7. *Contract tensors* (2.6) *from* $\mathrm{Vect}(\mathfrak{g})\otimes\mathfrak{g}^\vee$ *with the pairing* $\langle\cdot\,,\cdot\rangle$ *and consider the resulting element of* $\mathrm{Vect}(\mathfrak{g})\otimes\mathfrak{g}$ *as a differential operator on* $\mathrm{Ch}^\bullet(\mathfrak{g},\Bbbk[[\mathfrak{g}^\vee]]^{ad})$ *of the second-order, where term* $\cdot\otimes\mathfrak{g}$ *differentiates* $\mathrm{Ch}^\bullet(\mathfrak{g})$ *and term* $\mathrm{Vect}(\mathfrak{g})\otimes\cdot$ *differentiates* $\Bbbk[[\mathfrak{g}^\vee]]$. *Under quasi-isomorphism* (2.3) *differential* d_0 *of the above-mentioned spectral sequence is given by half-sum of these operators on the complex* $\mathrm{Ch}^\bullet(\mathfrak{g},\Bbbk[[\mathfrak{g}^\vee]]^{ad})$. *By* (2.7), *the matrix of this differential operator is given by*

$$\mathrm{id}+\sum_{n\geq 1}\frac{B_{2n}}{(2n)!}\mathrm{Ad}^{2n}, \tag{2.9}$$

B_n *are Bernoulli numbers*, Ad *is the structure tensor of the* $\mathfrak{g}$, *being considered as linear function on* $\mathfrak{g}$ *taking values in* $\mathrm{End}(\mathfrak{g})$.

Proof. By the very definition, the derivative of the differential of the Hochschild complex along the first-order deformation given by a symplectic form is presented

by the formula

$$\begin{aligned} d_0(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \\ &= \frac{1}{2}\{a_0, a_1\} \otimes a_2 \otimes \cdots \otimes a_n - \frac{1}{2} a_0 \otimes \{a_1, a_2\} \otimes \cdots \otimes a_n \\ &\quad + \cdots \pm \frac{1}{2}\{a_n, a_0\} \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned} \tag{2.10}$$

where $\{\ ,\ \}$ is the Poisson bracket, associated with the symplectic form. Apply it to the Chevalley–Eilenberg complex. By Corollary 2.3, any class in $C_*(\mathrm{Ch}^\bullet(\mathfrak{g}))$ may be represented by a cycle with degree one elements as entries with nonzero indexes. As the Hochschild complex is reduced, it follows that in (2.10) only the first and the last terms do not vanish. These terms are given by the maps (2.5). Applying Proposition 2.5 we complete the proof. □

Proposition 2.7 defines, therefore, on the algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ a differential operator of order 2 and of cohomological degree -1. On this algebra another differential operator of the same order and degree is defined, in terms of the above proposition it is given by the unit matrix. Call it the Brylinski differential after [5] and denote it by d_{Br}. They are not chain homotopic, but by the following proposition they become such after conjugation by an automorphism of complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. This automorphism equals to multiplication by the Duflo character.

Given a Lie group G, equip it with the left invariant volume form (which is the right invariant as well, due to the invariant scalar product). Equip its Lie algebra $\mathfrak{g}$ with the constant volume form and denote by $j \in \Bbbk[[\mathfrak{g}^\vee]]$ the Jacobian of the exponential map. The Duflo character is the power series on $\mathfrak{g}$ which is the square root of the Jacobian and is given by

$$j^{\frac{1}{2}} = \exp \sum_{n=1}^{\infty} \frac{B_{2n}}{4n(2n)!} \operatorname{Tr}(\mathrm{Ad}^{2n}), \tag{2.11}$$

where B_n are the Bernoulli numbers from (2.8) and Ad is the linear function on $\mathfrak{g}$ taking values in $\mathrm{End}(\mathfrak{g})$ as above.

Proposition 2.8. *Under the quasi-isomorphism* (2.3), *the differential* d_0 *on* $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ *is chain homotopic to* $j^{-\frac{1}{2}} \circ d_{Br} \circ j^{\frac{1}{2}}$, *where* $j^{\frac{1}{2}}$ *is the operator of the multiplication of* $\Bbbk[[\mathfrak{g}^\vee]]$ *by the Duflo character and* $j^{-\frac{1}{2}}$ *is the inverse operator.*

Proof. We will use the differential operator notation for endomorphisms of complex $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ and the Einstein summation convention. For example, $d_{Br} = g_{ij}\partial/\partial x^i \partial/\partial\, d_{dR} x^j$, where g_{ij} is the scalar product, x_i is a basis in $\mathfrak{g}^\vee$ and d_{dR} is the de Rham differential (we think of $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ as of differential forms on

$\mathrm{Ch}^\bullet(\mathfrak{g})$ as in Sec. 1). By Proposition 2.3,

$$d_0 - d_{Br} = \sum_{n\geq 1} \frac{B_{2n}}{(2n)!} (\mathrm{Ad}^{2n})^i_j g_{ik} \partial/\partial x^k \partial/\partial\, d_{dR} x^j, \tag{2.12}$$

where g_{ij} is the scalar product and Ad^* is the element of $\Bbbk[[\mathfrak{g}^\vee]] \otimes \mathrm{End}(\mathfrak{g})$. Consider the differential operators of order 2 given by

$$H_{2n-1} = (\mathrm{Ad}^{2n-1})^i_j g_{ik} \partial/\partial x^k \wedge \partial/\partial x^j.$$

We leave to the reader to check that

$$[d_{CE}, H_{2n-1}] = 2(\mathrm{Ad}^{2n})^i_j g_{ik} \partial/\partial x^k \partial/\partial\, d_{dR} x^j - \frac{1}{2n}[d_{Br}, \mathrm{Tr}(\mathrm{Ad}^{2n})],$$

where d_{CE} is the differential in the Chevalley–Eilenberg complex; all other terms vanish due to the Jacobi identity. Comparing it with (2.12) we see, that $d_0 - d_{Br}$ is chain homotopic to $[d_{Br}, \ln j^{\frac{1}{2}}]$. This implies the statement. □

Remark 2.9. The above proposition can be stated and proved in a coordinate-free manner for any QP-manifold in terms of [2]. In the setting of [17] (see Remark 2.6) it describes the differential on the differential forms on a complex symplectic manifold, that is, on the Hochschild homology of the structure sheaf, coming from the first-order deformation of the structure sheaf along the symplectic structure. It seems that when applied to the cotangent bundle of a complex manifold, it gives an alternative way of calculating the Todd class of this manifold.

Remark 2.10. Proposition 2.8 was inspired by the proof of the Duflo isomorphism for a Lie algebra with an invariant scalar product from [1]. As we will see below, the calculation above is connected with another proof of the Duflo isomorphism, the one from [4].

3. e_n-algebras

3.1. e_n-algebras

The main character in what follows is a unital algebra over the operad e_n, the operad of rational chains of the little discs operad. Recall that this dg-operad and its cohomology for $n > 1$ is the shifted Poisson operad, which is generated by an associative commutative product $\cdot$ of degree 0 and a Lie bracket $\{\ ,\ \}$ of degree $1-n$, they subject to the Leibnitz rule. A e_∞-algebra is a unital homotopy commutative algebra and e_0-algebra is a complex with a chosen cocycle.

The embedding of spaces of little discs induces the map of operads $e_k \to e_n$ for $k < n$. It induces a functor from e_n-algebras to e_k-algebras which we denote by obl^n_k. In particular, functor obl^∞_n produces an e_n-algebra from any commutative (that is, e_∞-) algebra.

For our purpose it will be more convenient to consider the operad of rational chains of the Fulton–MacPherson operad, see [18] and references therein for details.

The latter operad is homotopy equivalent to e_n and below we will make no difference between them; by saying an e_n-algebra we shall mostly mean an algebra over the Fulton–MacPherson operad.

The operations of the operad of little discs are spaces of n-balls embedded in a radius one n-ball. The group $SO(n)$ acts by rotations on the big ball. In order to take this action into the account one may consider $SO(n)$ as an operad with 1-ary only operations and take the semi-direct product of this operad and the little discs operad. The result is called the framed little discs operad, see [22]. We denote the dg-operad of chains of this operad by fe_n.

An alternative and better way to take into account the $SO(n)$-action is to consider equivariant chains. It gives us a dg-operad colored by $BSO(n)$, see e.g. [18]. Modules over this operad are $SO(n)$-equivariant complexes. Call these modules equivariant e_n-algebras. In general, the category of such algebras is not the same as the one of fe_n-algebras. However, for $n = 2$ commutativity of the group simplifies things and these categories are essentially the same.

Consider the latter case in some detail. The cohomology of fe_2 is known as the Batalin–Vilkovisky (BV) operad, see e.g. [22]. It is generated by the product $\cdot$ and the bracket $\{\,,\,\}$ obeying the same relations as those in e_2 and an additional 1-ary operation Δ of degree -1 obeying the relations

$$\Delta^2 = 0, \quad \{a, b\} = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b).$$

3.2. *The factorization complex*

Given a framed n-manifold (that is, a manifold with the tangent bundle trivialized) M and a e_n-algebra, the factorization complex $\int_M A$ is defined as in [18] and in the references therein. The idea of the definition is straightforward: discs embedded in M define a right module over e_n and the factorization complex is the tensor product over e_n of this right module with the left module given by A.

In order to extend the above definition to unframed manifolds, one needs the algebra A to be equivariant. Locally, one may choose a framing on M and apply the definition and then use the equivariance to identify results for different framings.

One important property of the factorization complex is its behavior with respect to gluing, see e.g. [10] and references therein. Let M_1 and M_2 be two manifolds with isomorphic boundaries B. Then for a e_n-algebra A there is a map of complexes

$$\int_{M_1} A \otimes \int_{M_2} A \to \int_{M_1 \cup_B M_2} A.$$

It follows that for $k < n$, a k-manifold M^k and a e_n-algebra A, the complex $\int_{M^k \times I^{n-k}} A$ is a e_k-algebra, and it is equivariant, if A is. In particular, for an n-manifold M with boundary B the complex $\int_{B \times I} A$ is a (homotopy) algebra, and the map above equips $\int_M A$ with a module structure over it. In terms of this action,

the gluing rule may be written as

$$\int_{M_1 \cup_B M_2} A = \int_{M_1} A \underset{\int_{B\times I} A}{\otimes} \int_{M_2} A. \tag{3.1}$$

Another important property of the factorization complex is a kind of homotopy invariance:

$$\int_{M^k \times I^{n-k}} A = \int_{M^k} \mathrm{obl}_k^n A.$$

Below we will make no difference between the two sides of this equality and will denote them simply by $\int_{M^k} A$. In particular, the factorization complex on a disk is quasi-isomorphic, as a complex, to the algebra itself.

Example 3.1. Let A be an equivariant e_2-algebra. Then its factorization complex on the disc $\int_{D^2} A$, which is A itself, is a module over $\int_{S^1\times I^1} A = \int_{S^1} \mathrm{obl}_1^2 A$, which is the Hochschild homology complex of $\mathrm{obl}_1^2 A$. The equivariance of A is essential here: without it, the Hochschild complex of e_2-algebra A does not act on A, and, if an equivariance structure is chosen, the action depends on this choice. In order to see it, note that $S^1 \times I^1$ is a framed manifold, that is why we do not need equivariance to take its factorization complex for any, not only equivariant algebra. However, this framing, which comes from the constant framing on the square after gluing together two opposite edges, can not be extended to the whole disc obtained from the annulus $S^1 \times I^1$ by gluing one of its boundary circles with the disc. Hence, in order to construct the desired action by gluing the annulus with the disc one need to identify factorization complexes with different framings, and here one needs the equivariance.

3.3. *Weyl n-algebras*

The type of equivariant e_n-algebras we need are the Weyl n-algebras; we refer to [6, 18] for the definition. In order to build such an algebra one needs a super-vector space V with a super-skew-symmetric non-degenerate bilinear form on it. The e_n-algebra associated with such data is denoted by $\mathcal{W}^n(V)$. In analogy with the usual Weyl algebra, it is the deformation of the polynomial algebra generated by V in the direction given by pairing. In fact, this is an algebra over the field of Laurent formal series in the quantization parameter h; this, however, must be ignored, assuming, loosely speaking, that $h = 1$.

There are some important properties we need. Firstly, considered as an e_k-algebra, where $k < n$, it is commutative. In other words, $\mathrm{obl}_k^n \mathcal{W}^n(V) = \mathrm{obl}_k^\infty \Bbbk[V]$ for any $k < n$, where $\Bbbk[V]$ is the polynomial algebra.

The following property is crucial for the construction of the perturbative invariants in [18]: for any n-manifold M the complex $\int_M \mathcal{W}^n(V)$ has one dimensional cohomology ([18, Proposition 11]). I conjecture that, for any $k < n$, the factorization complex $\int_{N^k\times I^{n-k}} \mathcal{W}^n(V)$ is again a Weyl algebra for any k-dimensional manifold N^k.

Example 3.2. Let V be a vector space. Equip $V \oplus V^\vee[-1]$ with the standard form of degree -1. Then $\mathcal{W}^2(V \oplus V^\vee[-1])$ is the space of polyvector fields on $V^\vee$ and standard operations on it — the Gerstenhaber bracket and the cup product — are the operations of the cohomology of e_2.

As any Weyl algebra, $\mathcal{W}^2(V \oplus V^\vee[-1])$ is equivariant. Thus it is acted on by the operad fe_2 and by its cohomology, which is the BV operad. The operation Δ is equal to the de Rham differential, where the polyvector fields are identified with the differential forms by means of the constant volume form. Another choice of the volume form leads to another fe_2-structure with the same underlying e_2-structure.

3.4. *The action*

For associative (or e_1-) algebras the notion of modules plays the central role. The higher generalization of this notion is a e_n-algebra acting on a e_{n-1}-algebra, for the definition and the discussion see e.g. [10] and references therein. Constructively, it may be defined by means of the Swiss cheese operad, which is especially convenient for algebras over the operad of chains of the Fulton–MacPherson operad. In the same way as the operations of the little discs operad are given by the configuration spaces of $\mathbb{R}^n$, the operations of the Swiss cheese operad are given by the spaces of distinct points in $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. There are points of two types: those on the boundary and those in the interior. This gives a colored operad with two colors. If an e_n-algebra B acts on an e_{n-1}-algebra A, then elements of B sit on the interior points and elements of A on boundary points. For further details we refer the reader to [25].

Note that the action of the Swiss cheese operad may be formulated in terms of factorization sheaves; for the definition of the latter see e.g. [10] and references therein. Namely, such an action is equivalent to a factorization sheaf on the half-space such that its restriction to the boundary and to the interior is constant factorization sheaves, corresponding to the e_{n-1}-algebra A and the e_n-algebra B.

It is known that for any e_n-algebra there exists a universal e_{n+1} algebra $\mathrm{End}(A)$ acting on it ([16]). In other words, an action of an e_{n+1} algebra B on A is the same as a morphism of e_{n+1}-algebras $B \to \mathrm{End}(A)$. For an associative (or e_1-) algebra the End-object is its Hochschild cohomology complex.

Let V be a vector space. Equip $V \oplus V^\vee[1-n]$ with the standard form of degree $(1-n)$. Then $\mathcal{W}^n(V \oplus V^\vee[1-n])$ is $\mathrm{End}(\Bbbk[V])$, where $\Bbbk[V]$ is the polynomial algebra. In order to see it, one may construct an action of $\mathcal{W}^n(V \oplus V^\vee[1-n])$ on $\Bbbk[V]$ directly by using the Swiss cheese operad and the Fulton–MacPherson compactification. Then one need to check that the resulting map $\mathcal{W}^n(V \oplus V^\vee[1-n]) \to \mathrm{End}(\Bbbk[V])$ is a quasi-isomorphism.

This action commutes with taking the factorization complex. That is, if an equivariant e_{n+1}-algebra B acts on an equivariant e_n-algebra A, then for a k-manifold N the e_{n-k+1}-algebra $\int_{N^k \times I^{n-k+1}} B$ acts on e_{n-k}-algebra $\int_{N^k \times I^{n-k}} A$. It follows immediately from definitions of the Swiss cheese operad and of

the factorization complex. It seems plausible that under appropriate conditions $\int_{N^k\times I^{n-k+1}} \mathrm{End}(A) = \mathrm{End}(\int_{N^k\times I^{n-k}} A)$.

Example 3.3. Consider the polynomial algebra $A = \Bbbk[V]$ as an associative algebra. Its Hochschild cohomology complex $C^*(A, A)$ (which, as it was mentioned above, is $\mathcal{W}^2(V \oplus V^\vee[-1])$) acts on it. It follows, that $\int_{S^1} C^*(A, A)$, which is a e_1-algebra, acts on $\int_{S^1} A$. The latter complex is the Hochschild homology complex of A, which is known to be quasi-isomorphic to the direct sum of shifted differential forms (see e.g. [15]). It is shown in [19] that the first complex is quasi-isomorphic to the differential operators on differential forms, and this is in good agreement with the speculation preceding the present example.

Recall, that in Example 3.1 for any an equivariant e_2-algebra A we construct action of e_1-algebra $\int_{S^1} A$ on the underlying complex of A. In the same way for any equivariant e_n-algebra A the e_1-algebra $\int_{S^{n-1}} A$ acts on the underlying complex of A: the action is given by gluing a n-ball and $S^{n-1} \times I$. It may be generalized even further. The factorization complex $\int_{S^k} A$, which is a e_{n-k}-algebra, analogously acts on e_{n-k-1}-algebra $\mathrm{obl}^n_{n-k-1} A$. As this action plays a crucial role in the next section, let us phrase it below as the construction.

The construction. Let A be an equivariant e_n-algebra. Then, for any $k < n$, the e_{n-k}-algebra $\int_{S^k} A$ naturally acts on $\mathrm{obl}^n_{n-k-1} A$. The corresponding action of the Swiss cheese operad is defined as follows. Embed $\mathbb{R}^{\geq 0}\times\mathbb{R}^{n-k-1}$ linearly into $\mathbb{R}^n$. Put at any point of this half-space the factorization complex of A on the k-sphere lying into the $k+1$ space perpendicular to the half-space, with its center on $0 \times \mathbb{R}^{n-k-1}$ and passing through this point. In particular, for points on $0 \times \mathbb{R}^{n-k-1}$ we get the sphere of zero diameter, that is a point and the factorization complex is A itself.

In other words, consider a map $\mathbb{R}^n \to \mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$ which sends a point to the pair which consists of the distance from the point to the subspace $\{0\} \times \mathbb{R}^{n-k-1}$ and the orthogonal projection on $\mathbb{R}^{n-k-1}$. Then the direct image of the factorization sheaf on $\mathbb{R}^n$ corresponding to A is the desired factorization sheaf on $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-k-1}$.

4. Wilson Loop

4.1. *Quantum Chevalley–Eilenberg algebra*

Given a Lie algebra $\mathfrak{g}$ with an invariant scalar product, in [18, Appendix] (see also [6, 3.6.2]) a e_3-dg-algebra $\mathrm{Ch}^\bullet_h(\mathfrak{g})$ is defined as follows. Take the Weyl 3-algebra given by the space $\mathfrak{g}^\vee[1]$ with the scalar product and equip it with a differential $\frac{1}{h}\{\cdot, q\}$, where $\{\ ,\ \}$ is the image of the Lie bracket under the map $L_\infty \to e_3$ (see e.g. [18, Proposition 2]) and q is the degree 3 element, which is the composition of the Lie bracket on g and the scalar product. Call this e_3-algebra the quantum Chevalley–Eilenberg algebra.

Consider the Hochschild complex $C_*(\mathrm{Ch}_h^\bullet(\mathfrak{g}))$. Here and in what follows we will consider unbounded Hochschild chains, that is, the Hochschild complex which is the direct product of its terms.

This Hochschild complex is the factorization complex $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. As $\mathrm{Ch}_h^\bullet(\mathfrak{g})$ is e_3-algebra, the Hochschild complex is an e_2-algebra. Consider it as an e_1-algebra, that is take $\mathrm{obl}_1^2 \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. By the very definition it is equal to $\int_{S^1} \mathrm{obl}_2^3 \mathrm{Ch}_h^\bullet(\mathfrak{g})$. We mentioned above an important property of Weyl algebras: $\mathrm{obl}_k^n \mathcal{W}^n(V) = \mathrm{obl}_k^\infty \Bbbk[V]$ for any $k < n$. It follows, that $\mathrm{obl}_2^3 \mathrm{Ch}_h^\bullet(\mathfrak{g}) = \mathrm{obl}_2^\infty \mathrm{Ch}^\bullet(\mathfrak{g})$. Thus $\mathrm{obl}_2^3 \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is just the super-commutative Chevalley–Eilenberg algebra. Its Hochschild complex is again a super-commutative algebra quasi-isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ by Proposition 2.1.To recap, $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as e_1-algebra, that is $\mathrm{obl}_1^2 \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$.

Now, let us apply the construction from the previous section to $A = \mathrm{Ch}_h^\bullet(\mathfrak{g})$, $n = 3$ and $k = 1$. It gives an action of the e_2-algebra $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ on $\mathrm{obl}_1^3 \mathrm{Ch}_h^\bullet(\mathfrak{g})$, which is $\mathrm{obl}_1^\infty \mathrm{Ch}^\bullet(\mathfrak{g})$. That is we get a map from the e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ to the Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ by the universal property, which is easily seen to be a quasi-isomorphism. The Hochschild cohomology complex of $\mathrm{Ch}^\bullet(\mathfrak{g})$ is known to be equal to $\mathrm{Ch}^\bullet(\mathfrak{g}, U_\mathfrak{g}^{ad})$, where $U_\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

To be more precise, in this way we get a map from $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ to $\mathrm{Ch}^\bullet(\mathfrak{g}, U_\mathfrak{g}^{ad}) \otimes \Bbbk[[h]]$. The e_1-structure on this complex comes from the one on the universal enveloping algebra. On the other hand, as it is shown in the previous paragraph, $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as e_1-algebra isomorphic to $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$. Thus, an explicit form of this map, which is supplied by the proposition below, implies the Duflo isomorphism.

Proposition 4.1. *The map of complexes*

$$\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad}) = \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U_\mathfrak{g}^{ad}) \otimes \Bbbk[[h]] \tag{4.1}$$

as above is chain homotopic to the map induced by the composition

$$\Bbbk[[\mathfrak{g}^\vee]] \overset{\exp(h(\cdot,\cdot))}{\to} S^*\mathfrak{g} \otimes \Bbbk[[h]] \overset{j^{\frac{1}{2}}}{\to} S^*\mathfrak{g} \otimes \Bbbk[[h]] \overset{PBW}{\to} U_\mathfrak{g} \otimes \Bbbk[[h]], \tag{4.2}$$

where the first arrow is given by the scalar product multiplied by h, the second is the contraction with the Duflo character (2.11) *and the third one is the PBW map.*

Proof. (Sketch of proof) As it was mentioned above, $\mathrm{Ch}_h^\bullet(\mathfrak{g})$ as an e_2-algebra is isomorphic to the commutative algebra $\mathrm{Ch}^\bullet(\mathfrak{g})$. It follows that the map induced by the unit embedding $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ is a morphism of e_2-algebras and in composition with (4.1) it gives the standard map $\mathrm{Ch}^\bullet(\mathfrak{g}) \to \mathrm{Ch}^\bullet(\mathfrak{g}, U_\mathfrak{g}^{ad})$. Thus we know the image of the subalgebra $\mathrm{Ch}^\bullet(\mathfrak{g})$ under (4.1). One may see that the whole map (4.1) may be uniquely determined from it as the unique extension

compatible with the Lie bracket coming from the e_2-structure. To see this one may use the faithful action of $\int_{S^1\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ on $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ as in the sketch of the proof of Proposition 4.3.

So our immediate purpose is to calculate the bracket on $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]])$, which is $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. As we will see below, it is enough to calculate the bracket with an element which is image of $a \in \mathrm{Ch}^\bullet(\mathfrak{g})$ under the embedding map as above. Given an element $b \in \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$, the bracket $\{a, b\}$ may be interpreted geometrically as follows. Consider the solid torus $D^2 \times S^1$ and two circles in it: $C = (0, S^1)$, call it the big one, and $c = (\{x \in D^2 \mid |x| = 1/2\}, *)$, call it the small one. The cycle in the factorization complex of the solid torus, which is $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$, representing $\{a, b\}$ equals $C_b \otimes ([c] \otimes a)$, where by C_b we denote the image of b in $\int_{D^2\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ under the embedding $C \hookrightarrow D^2 \times S^1$. One may see that cycle $[c] \otimes a$ is equal to $c_{d_{dR}a}$, where d_{dR} is the de Rham differential. If $a = x_1 \wedge \cdots \wedge x_i$, then $d_{dR}x = \sum \pm d_{dR}x_i\, x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_n$.

Let us now start pulling the small circle to unlink it from the big one. That is, consider a family of cycles c^t where c^t is a family of circles in the solid torus such that c^0 is the small circle, c^1 is a circle unlinked with the big circle and only one circle in the family intersects the big one. Until the circles do not intersect, nothing happens and the cycle $C_b \otimes c_a^t$ remains in the same class. But, as soon as they intersect each other, this class is changed by the class which is a derivation of b. The calculation shows that for $b = d_{dR}x_0\, x_1 \wedge \cdots \wedge x_n$ it is given by the sum of maps (2.5) contracted with x_0 and multiplied by $x_1 \wedge \cdots \wedge x_n$. The reasoning is analogous to Proposition 2.7: unlinking influences only around the intersection point. When the small circle is unlinked from the big one, $C_b \otimes c_a^1$ vanishes, because $c_a^1 = [c^1] \otimes a$ is a boundary.

Note, that the e_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ is, in fact, a fe_2-algebra. Thus, instead of the Lie bracket, one may calculate the operator Δ corresponding to the rotation. Given an element $x = \sum a_i b_i \in \mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$, where a_i are in the odd part and b_i in the even part, one may show, that

$$\Delta x = \sum \{a_i, b_i\}.$$

Apply the calculations from the previous paragraph to it. Comparing it with Proposition 2.7 we see, that the operator Δ on $\mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$ coincides with the operator d_0 from there. Proposition 2.8 implies that the Duflo character gives an isomorphism between this operator and d_{Br}. In order to complete the proof, one has to verify that d_{Br} is the operator Δ for the fe_2-algebra $\mathrm{Ch}^\bullet(\mathfrak{g}, U_{\mathfrak{g}}^{ad})$. □

While proving the proposition we found that the operator Δ on the fe_2-algebra $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is equal to the first-order deformation of the Hochschild differential of $\mathrm{Ch}^\bullet(\mathfrak{g})$ that we discussed in Sec. 1. I have no explanation for this coincidence.

4.2. *Invariants of knots*

In [18] we constructed invariants of manifolds using Weyl n-algebras. Below we develop this idea for manifolds with embedded links. Let us restrict ourselves to a 3-sphere with a knot in it.

As it was observed in [18], the cohomology of the factorization complex of the Weyl n-algebra $\mathcal{W}^n(V)$ on a closed n-manifold is one-dimensional. If V lies in degree 1 and the manifold is a 3-sphere (or a homology sphere), then the generator of this cohomology is given by the class $[p] \otimes S^{\mathrm{top}}V$, where p is a point in the manifold. As it was explained in [18, Appendix], the factorization complex $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is isomorphic to the complex of the underlying Weyl 3-algebra. Since the Chevalley–Eilenberg differential is inner, one needs to consider here unbounded chains that is, take direct product rather than the direct sum. It is easy to see that the generator in the cohomology of $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is given by $[p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee$. Call it the standard cycle. The idea of invariants we construct is to produce another cycle and compare it with the standard one.

Given a knot $K\colon S^1 \hookrightarrow S^3$ and a class $f \in \int_{S^1} \mathrm{Ch}^\bullet(\mathfrak{g}) = \mathrm{Ch}^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$, denote by K_f the direct image of this class under K. The class we are interested in is $([p] \otimes S^{\mathrm{top}}\mathfrak{g}^\vee) \otimes K_f$. For dimensional reasons, only f of degree 0 are interesting, in fact, $f \in \Bbbk[[\mathfrak{g}^\vee]]^{\mathrm{inv}}$. Thus we get the following definition.

Definition 4.2. For a knot K in $\mathbb{R}^3$ the Wilson loop invariant is the function on $\Bbbk[[\mathfrak{g}^\vee]]^{\mathrm{inv}}$ given by

$$f \mapsto ([\infty] \otimes S^{\mathrm{top}}(\mathfrak{g}^\vee[1])) \otimes K_f \in \int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g}),$$

where we identify $\int_{S^3} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ with $\Bbbk[[h]]$ using the standard cycle as the generator.

In [18] it is shown that invariants constructed there are described by formulas similar to formulas for the Axelrod–Singer invariants. Following the same line, we see that the Wilson loop invariants are connected with Bott–Taubes invariants; for a survey of the latter see e.g. [24]. There is another invariant of knots — the Kontsevich integral, see [7, Part 3]. In principle, it should coincide with the Bott–Taubes invariants, see [12]. As far as I know, this point is not clear, for discussion see [14]. One may hope that the definition above will help to elucidate this.

Our construction of the Wilson loop invariant depends on the choice of a Lie algebra with a scalar product. One may give a more complicated, but universal definition of these invariants with values in the graph complex, which is the Chevalley–Eilenberg complex of Hamiltonian vector fields, in the same way as it is outlined in [18, Appendix].

An interesting property of the Kontsevich integral is its value on the unknot: it is equal to the Duflo character and this allows to prove the Duflo isomorphism, see [4; 7, Chap. 11]. The following proposition states that the Wilson loop invariant shares this property.

Proposition 4.3. *The Wilson loop invariant of the unknot is equal to the composition*

$$\Bbbk[[\mathfrak{g}^\vee]]^{\mathrm{inv}} \hookrightarrow \Bbbk[[\mathfrak{g}^\vee]] \to U_{\mathfrak{g}} \otimes \Bbbk[[h]] \to \Bbbk[[h]],$$

where the second arrow is given by (4.2) *and the third one is the standard augmentation.*

Proof. (Sketch of proof) As it was discussed in Subsec. 4.1, the e_2-algebra $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ acts on the e_1-algebra $\mathrm{Ch}_h^\bullet(\mathfrak{g})$. In Proposition 4.1 it is shown that this action is not "naive", the morphism to the End-object is the composition of the pairing and the Duflo character. As it was mentioned above, this action is compatible with taking the factorization complex: as $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ acts $\mathrm{Ch}_h^\bullet(\mathfrak{g})$ so $\int_{S^1\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ acts on $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. The e_1-algebra $\int_{S^1\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is the algebra of differential operators on $\mathrm{Ch}_h^\bullet(\mathfrak{g}, \Bbbk[[\mathfrak{g}^\vee]]^{ad})$, see also Example 3.3. The complex $\int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is a kind of a holonomic module over these differential operators. But, again, it is not "naive", this action is twisted by the Duflo character.

Cut S^3 in two solid tori in the standard way, being the infinity point inside one of them and the unknot is the middle circle of the other. Now apply (3.1) to calculate the Wilson loop invariant of the unknot. As it was mentioned, $\int_{S^1\times S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is the algebra of differential operators and the factorization complexes of solid tori are kind of holonomic modules with transversal characteristic varieties. Now, the calculation of the Wilson invariant is reduced to taking the derived tensor product of these modules and comparing cycles in the result given by different $f \in \Bbbk[[\mathfrak{g}^\vee]]^{\mathrm{inv}}$. Taking into account the Duflo twisting we get the result. □

There is another natural approach to the knot invariants mentioned in [3]. Given a knot in a closed manifold, one may cut out a small solid torus around it to get a manifold with boundary. Then factorization complex for a fe_3-algebra of this manifold is a module over the factorization complex of the boundary torus, which is an invariant of the knot.

The proof of the above proposition makes clear what happens when the fe_3-algebra is $\mathrm{Ch}_h^\bullet(\mathfrak{g})$. In this case, the pair consisting of an algebra and a module itself does not depend on the knot; they are, essentially, the algebra of differential operators and the standard holonomic module over it. But this module contains a chosen element (it is a e_0-algebra!), which is the image of the unit. And the module together with this element is the invariant of the knot. Reasoning analogous to the proof of Proposition 4.3 shows that this invariant is, essentially, equivalent to the Wilson loop invariant.

4.3. *Skein algebra*

It [23] for a Riemann surface S a skein algebra was introduced. It is generated by non-self-intersecting loops on S. We claim that there is a map from this algebra

to $\int_{S\times I} \mathrm{Ch}_h^\bullet(\mathfrak{g})$. An element corresponding to a loop L maps to L_η, where $\eta \in \int_{S^1} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ is the canonical element, which is the preimage of $\exp(hc)$ under (4.2), where c is the Casimir element given by the scalar product.

The reason to propose it is the following. The skein algebra is the quantization of a Poisson algebra. The latter appears in [11, 26] as a subalgebra of the Poisson algebra of functions on the moduli space of G-local systems on S. But $\int_{S\times I} \mathrm{Ch}_h^\bullet(\mathfrak{g})$ must be thought of as the quantization of the latter Poisson algebra, see [6].

Elements L_η play an important role since they are generating functions of Dehn twists. In other words, the cobordism corresponding to the Dehn twist gives a bimodule over $\int_{S\times I} \mathrm{Ch}_h^\bullet(\mathfrak{g})$, according to speculations in the end of the previous subsection. Then the element L_η, corresponding to the Dehn twist, gives us the characteristic function of this module. This allows us to reduce the calculation of the perturbative quantum invariants of manifolds to the Wilson loop invariant of links similarly as it was done e.g. in [20].

We hope to elaborate all of this elsewhere.

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