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SHORT-WAVE TRANSVERSE INSTABILITIES OF LINE SOLITONS OF THE TWO-DIMENSIONAL HYPERBOLIC NONLINEAR SCHRÖDINGER EQUATION

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We prove that line solitons of the two-dimensional hyperbolic nonlinear Schrödinger equation are unstable under transverse perturbations of arbitrarily small periods, i.e., short waves. The analysis is based on the construction of Jost functions for the continuous spectrum of Schrödinger operators, the Sommerfeld radiation conditions, and the Lyapunov–Schmidt decomposition. We derive precise asymptotic expressions for the instability growth rate in the limit of short periods.

Keywords: nonlinear Schrödinger equation, soliton, transverse instability, Lyapunov–Schmidt decomposition, Fermi's golden rule

1. Introduction

Transverse instabilities of line solitons have been studied in many nonlinear evolution equations (see the pioneering work [1] and also [2]). In particular, this problem was studied in the context of the hyperbolic nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0, \tag{1}$$

which models oceanic wave packets in deep water. Solitary waves of the one-dimensional (y-independent) NLS equation exist in closed form. If all parameters of a solitary wave are eliminated by using the translational and scaling invariance, then we can consider the one-dimensional trivial-phase solitary wave in the simple form $\psi = e^{it}\operatorname{sech}(x)$. Adding a small perturbation $e^{i\rho y+\lambda t+it}(U(x)+iV(x))$ to the one-dimensional solitary wave and linearizing the underlying equations (see our previous work [3]), we obtain the coupled spectral stability problem

$$(L_+ - \rho^2)U = -\lambda V, \qquad (L_- - \rho^2)V = \lambda U, \tag{2}$$

where λ is the spectral parameter, ρ is the transverse wave number of the small perturbation, and $L_\pm$ are the Schrödinger operators

$$L_+ = -\partial_x^2 + 1 - 6\operatorname{sech}^2(x), \qquad L_- = -\partial_x^2 + 1 - 2\operatorname{sech}^2(x).$$

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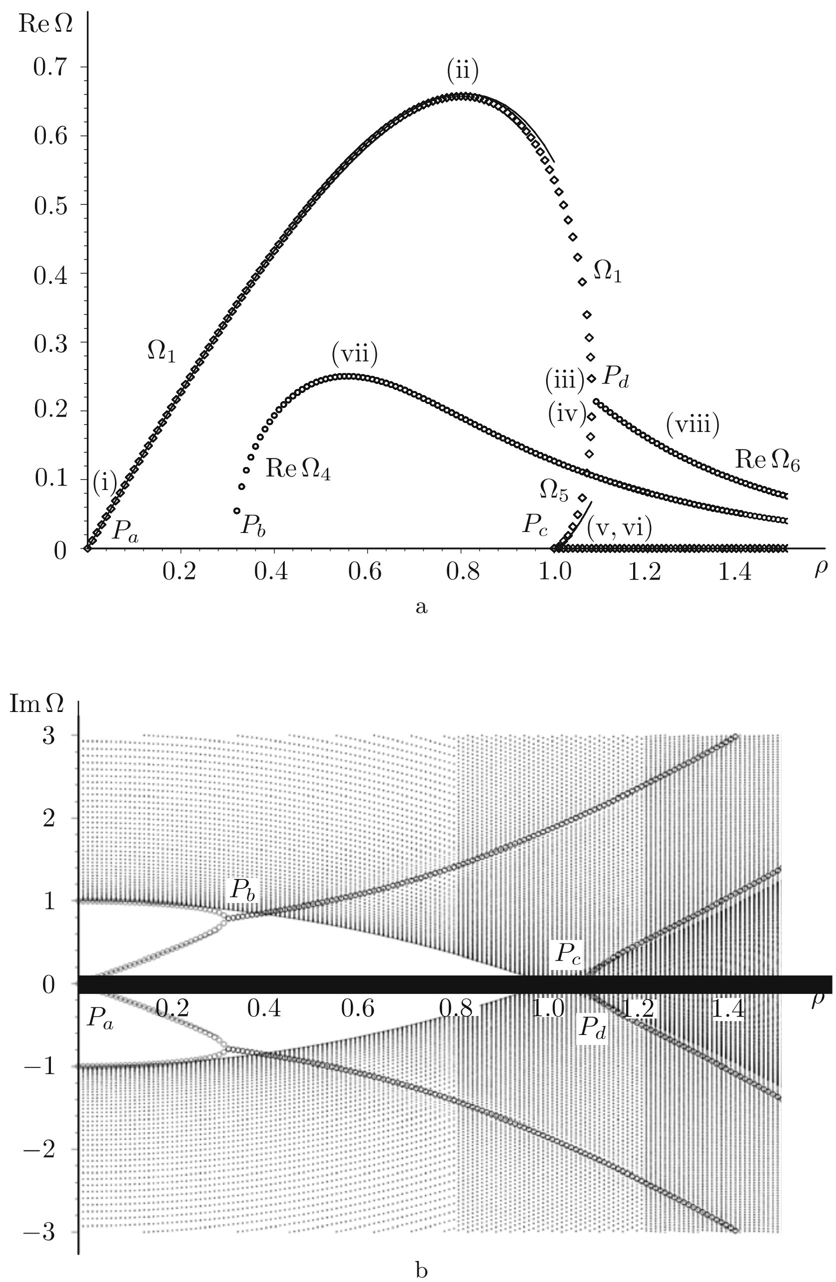

Fig. 1. Numerical computations of the (a) real and (b) imaginary parts of the isolated eigenvalues and the continuous spectrum of spectral stability problem (2) versus the transverse wave number ρ. The figure is reprinted from [3].

We note that a small ρ corresponds to long-wave perturbations in the transverse directions and a large ρ corresponds to short-wave transverse perturbations.

Numerical approximations of unstable eigenvalues (positive real part) of spectral stability problem (2) were computed in [3] and recently reproduced by independent numerical computations in [4] (see Fig. 5.27) and [5] (see Fig. 2). We reprint Fig. 2 in [3] here as Fig. 1. The figure illustrates various bifurcations and also the behavior of eigenvalues and the continuous spectrum in spectral stability problem (2) as a function of the transverse wave number ρ.

An asymptotic argument for the presence of a real unstable eigenvalue bifurcating at P_a for small values of ρ was given in [1]. The Hamiltonian Hopf bifurcation of a complex quartet at P_b for $\rho \approx 0.31$ was explained in [3] based on the negative index theory. The bifurcation of a new unstable real eigenvalue at P_c for $\rho > 1$ was also proved in [3] using Evans function methods. What remains in this puzzle is an argument for the existence of unstable eigenvalues for arbitrarily large values of ρ. We address this problem in this paper.

The motivation to develop a proof of the existence of unstable eigenvalues for large values of ρ originates from different physical experiments (both old and new). First, Ablowitz and Segur [6] predicted that there are no instabilities in the limit of large ρ and referred to water wave experiments done in narrow wave tanks by J. Hammack at the University of Florida in 1979, which showed good agreement with the dynamics of the one-dimensional NLS equation. Observation of one-dimensional NLS solitons in this limit seems to exclude transverse instabilities of line solitons. Second, experimental observations of transverse instabilities are quite robust in the context of nonlinear laser optics via a four-wave mixing interaction. Gorza et al. [7] observed the primary snake-type instability of line solitons at P_a for small values of ρ and also the persistence of the instabilities for large values of ρ. Recently, Gorza et al. [8] experimentally demonstrated the presence of the secondary neck-type instability that bifurcates at P_b near $\rho \approx 0.31$.

In a different physical context of solitary waves in PT-symmetric waveguides, results on the transverse instability of line solitons were rediscovered by Alexeeva et al. [5]. (The authors of [5] did not notice that their mathematical problem is identical to the problem for transverse instability of line solitons in the hyperbolic NLS equation.) Appendix B in [5] contains asymptotic results suggesting that if there are unstable eigenvalues of spectral problem (2) in the limit of large ρ, then the instability growth rate is exponentially small in terms of the large parameter ρ. No evidence that these eigenvalues have a *nonzero* instability growth rate was reported in [5].

Finally, similar instabilities of line solitons in hyperbolic NLS equation (1) were observed numerically in the context of the discrete NLS equation away from the anticontinuum limit [9].

This article is organized as follows. We present our main results in Sec. 2 and prove the main theorem analytically in Sec. 3. We devote Sec. 4 to computing the precise asymptotic formula for the unstable eigenvalues of spectral stability problem (2) in the limit of large values of ρ. In Sec. 5, we summarize our obtained results and discuss further problems to be solved in the future.

2. Main results

To study the transverse instability of line solitons in the limit of large ρ, we cast spectral stability problem (2) in the semiclassical form by using the transformation

$$\rho^2 = 1 + \frac{1}{\epsilon^2}, \qquad \lambda = \frac{i\omega}{\epsilon^2},$$

where ϵ is a small parameter. We rewrite spectral problem (2) as

$$\begin{aligned} \left(-\epsilon^2\partial_x^2 - 1 - 6\epsilon^2\operatorname{sech}^2(x)\right)U &= -i\omega V, \\ \left(-\epsilon^2\partial_x^2 - 1 - 2\epsilon^2\operatorname{sech}^2(x)\right)V &= i\omega U. \end{aligned} \tag{3}$$

We note that we are especially interested in the spectrum of this problem as $\epsilon \to 0$, which corresponds to $\rho \to \infty$ in the original problem. Also, the real part of λ, which determines the instability growth rate, corresponds to the imaginary part of ω up to a factor of ϵ^2.

We next introduce new dependent variables $\varphi = U + iV$ and $\psi = U - iV$, which are more suitable for working with the continuous spectrum for real values of ω. We note that φ and ψ are generally not complex conjugates of each other: because spectral problem (3) is not self-adjoint, U and V can be complex valued. We rewrite spectral problem (3) as

$$\begin{aligned} &\big(-\epsilon^2\partial_x^2+\omega-1-4\epsilon^2\operatorname{sech}^2(x)\big)\varphi-2\epsilon^2\operatorname{sech}^2(x)\psi=0,\\ &\big(-\epsilon^2\partial_x^2-\omega-1-4\epsilon^2\operatorname{sech}^2(x)\big)\psi-2\epsilon^2\operatorname{sech}^2(x)\varphi=0. \end{aligned} \tag{4}$$

We note that the Schrödinger operator

$$L_0=-\partial_x^2-4\operatorname{sech}^2(x) \tag{5}$$

admits exactly two eigenvalues of the discrete spectrum located at $-E_0$ and $-E_1$ [10], where

$$E_0=\left(\frac{\sqrt{17}-1}{2}\right)^2, \qquad E_1=\left(\frac{\sqrt{17}-3}{2}\right)^2. \tag{6}$$

The associated eigenfunctions are

$$\varphi_0=\operatorname{sech}^{\sqrt{E_0}}(x), \qquad \varphi_1=\tanh(x)\operatorname{sech}^{\sqrt{E_1}}(x). \tag{7}$$

In the neighborhood of each of these eigenvalues, a perturbation expansion can be constructed for exponentially decaying eigenfunction pairs (φ,ψ) and a quartet of complex eigenvalues ω of the original spectral problem (4). This idea already appeared in Appendix B in [5], where formal perturbation expansions were developed in powers of ϵ.

We note that the perturbation expansion for spectral stability problem (4) is not a standard application of the Lyapunov–Schmidt reduction method [11], because the eigenvalues of the limit problem given by the operator L_0 are embedded into a branch of the continuous spectrum. Therefore, to justify the perturbation expansions and to derive the main result, we need a perturbation theory that involves Fermi's golden rule [12]. An alternative version of this perturbation theory can use the analytic continuation of the Evans function across the continuous spectrum, similar to the one in [3]. In addition, semiclassical methods like WKB theory can be considered as suitable for applications to this problem [13].

Our main results in this paper are as follows. To formulate the statements, we use the notation $|a|\lesssim\epsilon$ to indicate that for sufficiently small positive values of ϵ, there is an ϵ-independent positive constant C such that $|a|\le C\epsilon$. Also, $H^2(\mathbb{R})$ denotes the standard Sobolev space of distributions whose derivatives up to order two are square integrable.

Theorem 1. *For sufficiently small $\epsilon>0$, there exist two quartets of complex eigenvalues $\{\omega,\bar\omega,-\omega,-\bar\omega\}$ in spectral problem (4) associated with components of eigenvectors $\varphi,\psi\in H^2(\mathbb{R})$.*

Let $-E_0$ and φ_0 be one of the two eigenvalue–eigenvector pairs of the operator L_0 in (5). There exists an $\epsilon_0>0$ such that for all $\epsilon\in(0,\epsilon_0)$, the complex eigenvalue ω of problem (4) in the first quadrant and its associated eigenfunction satisfy

$$|\omega-1-\epsilon^2E_0|\lesssim\epsilon^3, \qquad \|\varphi-\varphi_0\|_{L^2}\lesssim\epsilon, \qquad \|\psi\|_{L^\infty}\lesssim\epsilon, \tag{8}$$

while the positive value of $\operatorname{Im}\omega>0$ is exponentially small in ϵ.

Proposition 1. *No eigenvalues of spectral problem (4) except the two quartets of complex eigenvalues in Theorem 1 exist for sufficiently small $\epsilon>0$.*

Proposition 2. *The instability growth rates for the two complex quartets of eigenvalues in Theorem* 1 *as* $\epsilon \to 0$ *are explicitly given by*

$$\begin{aligned} \mathrm{Re}\,\lambda &= \frac{\mathrm{Im}\,\omega}{\epsilon^2} \sim \frac{2^{p+3/2}\pi^2}{\Gamma^2(p)}\epsilon^{3-2p}e^{-\sqrt{2}\pi/\epsilon}, \\ \mathrm{Re}\,\lambda &= \frac{\mathrm{Im}\,\omega}{\epsilon^2} \sim \frac{2^{q+5/2}\pi^2}{q^2\Gamma^2(q)}\epsilon^{1-2q}e^{-\sqrt{2}\pi/\epsilon}, \end{aligned} \tag{9}$$

where $p = 2 + \sqrt{E_0}$ *and* $q = 2 + \sqrt{E_1}$.

We note that the result of Theorem 1 guarantees that the two quartets of complex eigenvalues seen in Fig. 1 remain unstable for all large values of the transverse wave number ρ in spectral stability problem (2).

3. Proof of Theorem 1

By the symmetry of problem (4), we only need to prove Theorem 1 for one eigenvalue of each complex quartet, for example, for ω in the first quadrant of the complex plane. We let $\omega = 1 + \epsilon^2 E$ and rewrite spectral problem (4) in the equivalent form

$$\begin{aligned} &\left(-\partial_x^2 - 4\,\mathrm{sech}^2(x)\right)\varphi - 2\,\mathrm{sech}^2(x)\psi = -E\varphi, \\ &-2\psi - \epsilon^2\left(\partial_x^2 + E + 4\,\mathrm{sech}^2(x)\right)\psi = 2\epsilon^2\,\mathrm{sech}^2(x)\varphi. \end{aligned} \tag{10}$$

In the leading order, the first equation in system (10) has exponentially decaying eigenfunctions (7) for $E = E_0$ and $E = E_1$ in (6). But the second equation in system (10) does not admit exponentially decaying eigenfunctions for these values of E because the operator

$$L_\epsilon(E) = -2 - \epsilon^2\left(\partial_x^2 + E + 4\,\mathrm{sech}^2(x)\right)$$

is not invertible for these values of E. The scattering problem for the Jost functions associated with the continuous spectrum of $L_\epsilon(E)$ admits solutions that behave at infinity as

$$\psi(x) \sim e^{ikx}, \quad \text{where } k^2 = E + \frac{2}{\epsilon^2}.$$

If $\mathrm{Im}\,E > 0$, then $\mathrm{Re}\,k \cdot \mathrm{Im}\,k > 0$. The Sommerfeld radiation conditions $\psi(x) \sim e^{\pm ikx}$ as $x \to \pm\infty$ correspond to solutions $\psi(x)$ that decay exponentially in x when k is extended from real positive values for $\mathrm{Im}\,E = 0$ to complex values with $\mathrm{Im}\,E > 0$. We therefore impose Sommerfeld boundary conditions for the component ψ satisfying spectral problem (10):

$$\psi(x) \to a \begin{cases} e^{ikx}, & x \to \infty, \\ \sigma e^{-ikx}, & x \to -\infty, \end{cases} \qquad k = \frac{1}{\epsilon}\sqrt{2 + \epsilon^2 E}, \tag{11}$$

where a is the radiation tail amplitude to be determined and $\sigma = \pm 1$ depends on whether ψ is even or odd in x. To compute a, we note the following elementary result.

Lemma 1. *We consider bounded (in* $L^\infty(\mathbb{R})$*) solutions* $\psi(x)$ *of the second-order differential equation*

$$\psi'' + k^2\psi = f, \tag{12}$$

where $k \in \mathbb{C}$ with $\operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0$ and $f \in L^1(\mathbb{R})$ is a given function, either even or odd. Then

$$\psi(x) = \frac{1}{2ik} \int_{-\infty}^{x} e^{ik(x-y)} f(y)\, dy + \frac{1}{2ik} \int_{x}^{+\infty} e^{-ik(x-y)} f(y)\, dy \tag{13}$$

is the unique solution of differential equation (12) *with the same parity as f that satisfies the Sommerfeld radiation conditions* (11) *with*

$$a = \frac{1}{2ik} \int_{-\infty}^{+\infty} f(y) e^{-iky}\, dy. \tag{14}$$

Proof. Solving (12) using variation of parameters, we obtain

$$\psi(x) = e^{ikx}\left(u(0) + \frac{1}{2ik}\int_0^x f(y)e^{-iky}\, dy\right) + e^{-ikx}\left(v(0) - \frac{1}{2ik}\int_0^x f(y)e^{iky}\, dy\right),$$

where $u(0)$ and $v(0)$ are arbitrary constants. We fix these constants using Sommerfeld radiation conditions (11), which yields

$$u(0) = \frac{1}{2ik}\int_{-\infty}^{0} f(y)e^{-iky}\, dy, \qquad v(0) = \frac{1}{2ik}\int_0^{+\infty} f(y)e^{iky}\, dy.$$

Using these expressions and the definition $a = \lim_{x\to\infty} \psi(x)e^{-ikx}$, we obtain (13) and (14). It is easily verified that ψ has the same parity as f. The lemma is proved.

To prove Theorem 1, we select one of the two eigenvalue–eigenvector pairs E_0 and φ_0 of the operator L_0 in (5) and proceed with the Lyapunov–Schmidt decomposition

$$E = E_0 + \mathcal{E}, \qquad \varphi = \varphi_0 + \phi,$$

where ϕ is orthogonal to φ_0 (in the sense of the scalar product in $L^2(\mathbb{R})$). To simplify calculations, we assume that φ_0 is normalized to unity in the $L^2(\mathbb{R})$ norm. Moreover, we assume that $\phi \in L^2(\mathbb{R})$ in the decomposition.

We rewrite spectral problem (10) as

$$\begin{aligned} &(L_0 + E_0)\phi - 2\operatorname{sech}^2(x)\psi = -\mathcal{E}(\varphi_0 + \phi), \\ &L_\epsilon(E_0 + \mathcal{E})\psi = 2\epsilon^2 \operatorname{sech}^2(x)(\varphi_0 + \phi). \end{aligned} \tag{15}$$

Because ϕ is orthogonal to φ_0, the correction term $\mathcal{E}$ is uniquely determined by projecting the first equation in system (15) onto φ_0:

$$\mathcal{E} = 2\int_{-\infty}^{\infty} \operatorname{sech}^2(x)\varphi_0(x)\psi(x)\, dx. \tag{16}$$

If $\psi \in L^\infty(\mathbb{R})$, then $|\mathcal{E}| = O(\|\psi\|_{L^\infty})$. Let P be the orthogonal projection from $L^2(\mathbb{R})$ to the range of $L_0 + E_0$. Then ϕ is uniquely determined from the linear inhomogeneous equation

$$P(L_0 + E_0 + \mathcal{E})P\phi = 2\operatorname{sech}^2(x)\psi - 2\varphi_0 \int_{-\infty}^{\infty} \operatorname{sech}^2(x)\varphi_0(x)\psi(x)\, dx, \tag{17}$$

where $P(L_0 + E_0)P$ is invertible with a bounded inverse and we assume $\psi \in L^\infty(\mathbb{R})$. On the other hand, $\psi \in L^\infty(\mathbb{R})$ is uniquely found using the linear inhomogeneous equation

$$\psi'' + k^2\psi = f, \quad \text{where } f = -2\operatorname{sech}^2(x)(\varphi_0 + \phi + 2\psi), \tag{18}$$

subject to Sommerfeld radiation condition (11). We note that ψ is not real, because of Sommerfeld radiation condition (11), and depends on ϵ because of the ϵ-dependence of k in

$$k = \frac{1}{\epsilon}\sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}. \tag{19}$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. The function f in the right-hand-side of (18) decays exponentially as $|x| \to \infty$ if $\phi, \psi \in L^\infty(\mathbb{R})$. From solution (13), we rewrite the equation in the integral form

$$\psi(x) = \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}} \int_{-\infty}^{x} e^{ik(x-y)} \operatorname{sech}^2(y)(\varphi_0 + \phi + 2\psi)(y)\, dy + \\ + \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}} \int_{x}^{+\infty} e^{-ik(x-y)} \operatorname{sech}^2(y)(\varphi_0 + \phi + 2\psi)(y)\, dy. \tag{20}$$

The operator acting on $\psi \in L^\infty(\mathbb{R})$ in the right-hand side is a contraction for small values of ϵ if $\phi \in L^\infty(\mathbb{R})$ and $\mathcal{E} \in \mathbb{C}$ are bounded as $\epsilon \to 0$ and for $\operatorname{Im} \mathcal{E} \geq 0$ (yielding $\operatorname{Im} k \geq 0$). By the fixed point theorem [11], we have a unique solution $\psi \in L^\infty(\mathbb{R})$ of integral equation (20) for small values of ϵ such that $\|\psi\|_{L^\infty} = O(\epsilon)$ as $\epsilon \to 0$. This solution can be substituted in inhomogeneous equation (17).

Because $|\mathcal{E}| = O(\|\psi\|_{L^\infty}) = O(\epsilon)$ as $\epsilon \to 0$ and the operator $P(L_0 + E_0)P$ is invertible with a bounded inverse, we apply the implicit function theorem and obtain a unique solution $\phi \in H^2(\mathbb{R})$ of inhomogeneous equation (17) for small values of ϵ such that $\|\phi\|_{H^2} = O(\epsilon)$ as $\epsilon \to 0$. We note that by the Sobolev embedding of $H^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$, the a priori assumption $\phi \in L^\infty(\mathbb{R})$ for finding $\psi \in L^\infty(\mathbb{R})$ in (18) is consistent with the solution $\phi \in H^2(\mathbb{R})$ obtained by the implicit function theorem.

This proves bounds (8). It remains to show that $\operatorname{Im} \mathcal{E} > 0$ for small nonzero values of ϵ. If so, then the real eigenvalue $1+\epsilon^2 E_0$ bifurcates to the first complex quadrant and yields the eigenvalue $\omega = 1+\epsilon^2 E_0+\epsilon^2 \mathcal{E}$ of spectral problem (4). The persistence of such an isolated eigenvalue with respect to small values of ϵ follows from regular perturbation theory. Also, the eigenfunction ψ in (20) decays exponentially in x at infinity if $\operatorname{Im} \mathcal{E} > 0$. As a result, the eigenvector (ϕ, ψ) is defined in $H^2(\mathbb{R})$ for small nonzero values of ϵ, although $\|\psi\|_{H^2}$ diverges as $\epsilon \to 0$.

To prove that $\operatorname{Im} \mathcal{E} > 0$ for small but nonzero values of ϵ, we use (11) and (18), integrate by parts, and obtain the relation

$$2\int_{-\infty}^{\infty} \operatorname{sech}^2(x)(\varphi_0 + \phi)\psi(x)\, dx = \int_{-\infty}^{\infty} \bar{\psi}(x)\big(-\partial_x^2 - k^2 - 4\operatorname{sech}^2(x)\big)\psi(x)\, dx = \\ = (-\bar{\psi}\psi_x + \bar{\psi}_x\psi)\Big|_{x\to-\infty}^{x\to+\infty} + \int_{-\infty}^{\infty} \psi(x)\big(-\partial_x^2 - k^2 - 4\operatorname{sech}^2(x)\big)\bar{\psi}(x)\, dx = \\ = 4ik|a(\epsilon)|^2 + 2\int_{-\infty}^{\infty} \operatorname{sech}^2(x)(\varphi_0 + \bar{\phi})\bar{\psi}(x)\, dx,$$

where the parameter $k = \sqrt{2}\epsilon^{-1}(1 + O(\epsilon^2))$ is real in the leading order. Hence,

$$\operatorname{Im} \int_{-\infty}^{\infty} \operatorname{sech}^2(x)(\varphi_0 + \phi)\psi(x)\, dx = k|a(\epsilon)|^2.$$

We here recall that φ_0 is real-valued (see (7)).

Using bounds (8), definition (14), and projection (16), we obtain

$$\begin{aligned}\operatorname{Im}\mathcal{E} &= 2\operatorname{Im}\int_{-\infty}^{\infty}\operatorname{sech}^2(x)\varphi_0(x)\psi(x)\,dx = 2k|a(\epsilon)|^2\big(1+O(\epsilon)\big) = \\ &= \frac{2}{k}\left|\int_{-\infty}^{+\infty}\operatorname{sech}^2(x)\varphi_0(x)e^{-ikx}\,dx\right|^2\big(1+O(\epsilon)\big).\end{aligned} \tag{21}$$

The quantity $\operatorname{Im}\mathcal{E}$ is therefore strictly positive. We note that this expression (21) is called Fermi's golden rule in quantum mechanics [12]. Because $k = O(\epsilon^{-1})$ as $\epsilon \to 0$, the Fourier transform of $\operatorname{sech}^2(x)\varphi_0(x)$ at this k is exponentially small in ϵ. Therefore, $\operatorname{Im}\omega > 0$ is exponentially small in ϵ. The theorem is proved.

4. Proofs of Propositions 1 and 2

To prove Proposition 1, we fix E_c to be independent of ϵ and different from E_0 and E_1 in (6). We write $E = E_c + \mathcal{E}$ for some small ϵ-dependent values of $\mathcal{E}$. We rewrite spectral problem (10) as

$$\begin{aligned}&(L_0 + E_c)\varphi - 2\operatorname{sech}^2(x)\psi = -\mathcal{E}\varphi,\\ &L_\epsilon(E_c + \mathcal{E})\psi = 2\epsilon^2\operatorname{sech}^2(x)\varphi.\end{aligned} \tag{22}$$

Proof of Proposition 1. If E_c is real and negative, then system (22) has only oscillatory solutions, and exponentially decaying eigenfunctions hence do not exist for values of E near E_c. Furthermore, we note that the Schrödinger operator L_0 in (5) has no endpoint resonances. Therefore, no bifurcation of isolated eigenvalues can occur if $E_c = 0$. We therefore consider positive values of E_c if E_c is real and values with $E_c > 0$ if E_c is complex.

By Lemma 1, we rewrite the second equation in system (22) in the integral form

$$\begin{aligned}\psi(x) &= \frac{i\epsilon}{\sqrt{2+\epsilon^2E_c+\epsilon^2\mathcal{E}}}\int_{-\infty}^{x} e^{ik(x-y)}\operatorname{sech}^2(y)(\varphi+2\psi)(y)\,dy + \\ &\quad + \frac{i\epsilon}{\sqrt{2+\epsilon^2E_c+\epsilon^2\mathcal{E}}}\int_{x}^{+\infty} e^{-ik(x-y)}\operatorname{sech}^2(y)(\varphi+2\psi)(y)\,dy.\end{aligned} \tag{23}$$

Again, the operator on $\psi \in L^\infty(\mathbb{R})$ in the right-hand side is a contraction for small values of ϵ if $\varphi \in L^\infty(\mathbb{R})$ and $\mathcal{E} \in \mathbb{C}$ are bounded as $\epsilon \to 0$ with $\operatorname{Im}(E_c + \mathcal{E}) \ge 0$ (yielding $\operatorname{Im} k \ge 0$). By the fixed point theorem, under these conditions, we have a unique solution $\psi \in L^\infty(\mathbb{R})$ of integral equation (23) for small values of ϵ such that $\|\psi\|_{L^\infty} = O(\epsilon)$ as $\epsilon \to 0$. Moreover, $\psi = 0$ if $\varphi = 0$. This solution can be substituted in the first equation in system (22).

The operator $L_0 + E_c$ is invertible with a bounded inverse if E_c is complex or if E_c is real and positive but different from E_0 and E_1. There exists a solution $\varphi = 0$ of the first equation in system (22) because $\psi = 0$ if $\varphi = 0$. By the implicit function theorem, this solution $\varphi = 0$ is unique for small values of ϵ and for any value of $\mathcal{E}$ as long as $\mathcal{E}$ is small as $\epsilon \to 0$ (because E_c is fixed independently of ϵ). Hence, $E = E_c + \mathcal{E}$ is not an eigenvalue of spectral problem (10). The proposition is proved.

To prove Proposition 2, we compute $\operatorname{Im}\omega$ in Theorem 1 explicitly in the asymptotic limit $\epsilon \to 0$. It follows from (19) and (21) that

$$\operatorname{Im}\omega = \sqrt{2}\epsilon^3\left|\int_{-\infty}^{+\infty}\operatorname{sech}^2(x)\varphi_0(x)e^{-ikx}dx\right|^2\big(1+O(\epsilon)\big),$$

where $k = \sqrt{2}\epsilon^{-1}(1+O(\epsilon^2))$.

Proof of Proposition 2. We consider the first eigenfunction φ_0 in (7) for the lowest eigenvalue in (6). Using integral 3.985 in [14], we obtain

$$I_0 = \int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\varphi_0(x)e^{-ikx}\,dx = 2\int_0^{\infty} \operatorname{sech}^p(x)\cos(kx)\,dx = \frac{2^{p-1}}{\Gamma(p)}\left|\Gamma\left(\frac{p+ik}{2}\right)\right|^2,$$

where $p = 2 + \sqrt{E_0} = (\sqrt{17}+3)/2$. Because $k = O(\epsilon^{-1})$ and $\epsilon \to 0$, we can use asymptotic limit 8.328 in [14],

$$\lim_{|y|\to\infty} |\Gamma(x+iy)|e^{\pi|y|/2}|y|^{1/2-x} = \sqrt{2\pi}, \tag{24}$$

whence we establish the asymptotic equivalence

$$I_0 = \frac{2^{p-1}}{\Gamma(p)}\left|\Gamma\left(\frac{p+ik}{2}\right)\right|^2 \sim \frac{2\pi}{\Gamma(p)k^{1-p}}e^{-\pi k/2} \sim \frac{2^{(p+1)/2}\pi}{\Gamma(p)}\epsilon^{1-p}e^{-\pi/\sqrt{2}\epsilon}.$$

Therefore, the leading asymptotic order for $\operatorname{Im}\omega$ is given by

$$\operatorname{Im}\omega \sim \frac{2^{p+3/2}\pi^2}{\Gamma^2(p)}\epsilon^{5-2p}e^{-\sqrt{2}\pi/\epsilon}. \tag{25}$$

We next consider the second eigenfunction φ_1 in (7) for the second eigenvalue in (6). Using integral 3.985 in [14] and integrating by parts, we obtain

$$I_1 = \int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\varphi_1(x)e^{-ikx}\,dx =$$

$$= -\frac{2ik}{q}\int_0^{\infty} \operatorname{sech}^q(x)\cos(kx)\,dx = -i\frac{k2^{q-1}}{q\Gamma(q)}\left|\Gamma\left(\frac{q+ik}{2}\right)\right|^2,$$

where $q = 2 + \sqrt{E_1} = (\sqrt{17}+1)/2$. Using limit (24), we obtain

$$I_1 = -\frac{ik2^{q-1}}{q\Gamma(q)}\left|\Gamma\left(\frac{q+ik}{2}\right)\right|^2 \sim -\frac{2\pi ik}{q\Gamma(q)k^{1-q}}e^{-\pi k/2} \sim -i\frac{2^{(q+2)/2}\pi}{q\Gamma(q)}\epsilon^{-q}e^{-\pi/\sqrt{2}\epsilon}.$$

Therefore, the leading asymptotic order for $\operatorname{Im}\omega$ is given by

$$\operatorname{Im}\omega \sim \frac{2^{q+5/2}\pi^2}{q^2\Gamma^2(q)}\epsilon^{3-2q}e^{-\sqrt{2}\pi/\epsilon}. \tag{26}$$

In both cases (25) and (26), the expressions for $\operatorname{Im}\omega$ have the algebraically large prefactor in ϵ with the exponents $5-2p = 2-\sqrt{17} < 0$ and $3-2q = 2-\sqrt{17} < 0$. Nevertheless, $\operatorname{Im}\omega$ is exponentially small as $\epsilon \to 0$.

5. Conclusion

We have proved that spectral stability problem (2) has exactly two quartets of complex unstable eigenvalues in the asymptotic limit of large transverse wave numbers. We obtained precise asymptotic expressions for the instability growth rate in the same limit.

It would be interesting to verify the validity of our asymptotic results numerically. The numerical approximation of eigenvalues in this asymptotic limit is a delicate problem of numerical analysis because of the high-frequency oscillations of the eigenfunctions for large values of λ, i.e., for small values of ϵ, as discussed in [3]. As can be seen in Fig. 1, the existing numerical results do not allow comparing with the asymptotic results obtained here. This numerical problem is left for further studies.

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