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Complexity of Functions from Some Classes of Three-Valued Logic

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Abstract—The problem of the realization complexity for functions of the three-valued logic taking values from the set $\{0,1\}$ by formulas over incomplete generating systems is considered. Upper and lower asymptotic estimates for the corresponding Shannon functions are obtained.

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In this paper we consider the problem of the complexity of realization for functions of the three-valued logic taking values from the set $\{0,1\}$ by formulas over finite systems. Some results in this direction were obtained in [1]. All definitions can be found in [2–6].

Let $k \geq 2$ and $n \geq 1$. Assume $E_k = \{0,1,\dots,k-1\}$. By E_k^n we denote the set of all collections $\widetilde{\alpha} = (\alpha_1,\dots,\alpha_n)$ such that $\alpha_1,\dots,\alpha_n \in E_k$. Denote the set of all functions of the k-valued logic by P_k and the set of all functions of the three-valued logic taking values from the set E_2 by $P_{3,2}$. Let $G \subseteq P_k$. Denote the closed class generated by the system G by $[G]$ and the set of all functions from G dependent on the variables $x_1,\dots,x_n$, $n \geq 1$, by $G(n)$. If $f(x_1,\dots,x_n) \in [G]$, Φ is a formula over G realizing the function f, and $F \subseteq [G]$, then we denote the number of symbols of variables and constants contained in the formula Φ (the complexity of the formula Φ) by $L(\Phi)$, the complexity of the function f by $L_G(f)$, and the Shannon function for the set F by $L_G(F(n))$. For a variable x contained in a formula Φ, denote the number of occurrences of the variable x in the formula Φ by $N(\Phi;x)$.

Lupanov [4] proved that for any complete system of Boolean functions G the relation

$$L_G(P_2(n)) \sim \frac{2^n}{\log_2 n}$$

holds (see also [2, 3]). It is known [7] that for any finite system $G \subseteq P_2$ there exists a constant $c = c(G)$ such that each function $f(x_1,\dots,x_n)$ from $[G]$ satisfies the inequality $L_G(f) \leq c^n$. In [8, 9], for some finite complete bases $G \subseteq P_k$, $k \geq 3$, the relation

$$L_G(P_k(n)) \sim \frac{k^n}{\log_k n}$$

was obtained (see also [10]). An example of a sequence of functions of the 4-valued logic whose complexity of realization in the class of formulas over some finite incomplete system has an over-exponential order of growth with respect to the number of variables was given in [11].

We use the following notation from [12] for closed classes of Boolean functions: S is the set of all self-dual functions; T_i is the set of all functions preserving the constant i, $i = 0,1$; M is the set of all monotone functions; L is the set of all linear functions; O^∞ is the set of all functions satisfying the condition $\langle 0^\infty \rangle$; I^∞ is the set of all functions satisfying the condition $\langle 1^\infty \rangle$; K is the set of all conjunctions; D is the set of all disjunctions; U is the set of all functions essentially dependent on at most one variable; C is the set of all functions having no essential variables.

Assume

$$L_i = L \cap T_i,\ M_i = M \cap T_i,\ K_i = K \cap T_i,\ D_i = D \cap T_i,\ U_i = U \cap T_i,\ C_i = C \cap T_i, \quad i = 0,1;$$

$$M_{01} = M_0 \cap M_1,\ L_{01} = L_0 \cap L_1,\ K_{01} = K_0 \cap K_1,\ D_{01} = D_0 \cap D_1,\ U_{01} = U_0 \cap U_1;$$

$$SU = S \cap U,\ MU = M \cap U,\ O_0^\infty = T_0 \cap O^\infty,\ I_1^\infty = T_1 \cap I^\infty;$$

$$MO^\infty = M \cap O^\infty,\ MI^\infty = M \cap I^\infty,\ MO_0^\infty = M \cap O_0^\infty,\ MI_1^\infty = M \cap I_1^\infty.$$

The projection of a function $f(x_1, \ldots, x_n) \in P_{3,2}$ is the Boolean function $\operatorname{pr} f(x_1, \ldots, x_n)$ whose values on an arbitrary collection $\widetilde{\alpha} \in E_2^n$ are defined by the equality $\operatorname{pr} f(\widetilde{\alpha}) = f(\widetilde{\alpha})$. The projection $\operatorname{pr} F$ of a set of functions $F \subseteq P_{3,2}$ is the set $\bigcup\{\operatorname{pr} f\}$, where the union is taken over all functions $f \in F$. It is not difficult to show that for any closed class $F \subseteq P_{3,2}$ the set $\operatorname{pr} F$ is a closed class of Boolean functions.

Let B be an arbitrary closed class of Boolean functions. Assume

$$\operatorname{pr}^{-1} B = \{f \in P_{3,2} \mid \operatorname{pr} f \in B\}.$$

It is easy to see that the set $\operatorname{pr}^{-1} B$ is a closed class and any closed class $F \subseteq P_{3,2}$ such that $\operatorname{pr} F = B$ satisfies the relation $F \subseteq \operatorname{pr}^{-1} B$. Such a class $\operatorname{pr}^{-1} B$ is called the maximal closed class. Thus, for each closed class of Boolean functions we have the corresponding maximal class of functions from $P_{3,2}$. It is known [6] that a closed class $\operatorname{pr}^{-1} B$ is finitely-generated if and only if $B \notin \{C, C_0, C_1\}$.

Denote the function from $P_{3,2}$ that is equal to 1 for $x = i$ and to 0 in the other cases by $j_i(x)$, $i \in E_3$. By $k(x)$ we denote the function from $P_{3,2}$ that is equal to 1 for $x \in E_2$ and to 0 for $x = 2$. By $x + y$ and $x \cdot y$ we denote the functions from $P_{3,2}$ such that for any $\alpha, \beta \in E_3$ the equalities $\alpha + \beta = j_1(\alpha) \oplus j_1(\beta)$ and $\alpha \cdot \beta = j_1(\alpha) \& j_1(\beta)$ holds respectively, where $\oplus$ and $\&$ are the addition and multiplication modulo 2. Let $p \in E_3$. Assume

$$\delta(x_1, x_2) = j_1(x_1) \cdot k(x_2), \quad \theta(x_1, x_2) = j_1(x_1) + j_2(x_2), \quad \rho_p(x_1, x_2, x_3) = j_1(x_1) + j_p(x_2) \cdot j_2(x_3);$$

$$\psi_p(x_1, x_2, x_3) = j_1(x_3) + j_1(x_1) \cdot j_p(x_2) \cdot j_2(x_3), \quad \zeta_p(x_1, x_2, x_3, x_4) = j_1(x_4) + j_1(x_1) \cdot j_p(x_2) \cdot j_2(x_3).$$

Note that the projections of the functions $\delta, \theta, \rho_p, \psi_p, \zeta_p$, where $p \in E_3$, belong to the set U_{01}. Assume $\mathfrak{U} = \{j_1, \delta, \theta, \rho_0, \rho_1, \rho_2, \psi_0, \psi_1, \psi_2, \zeta_0, \zeta_1, \zeta_2\}$. It is evident that $\operatorname{pr} \mathfrak{U} \subseteq U_{01}$. It is known [6] that $[\mathfrak{U}] = \operatorname{pr}^{-1} U_{01}$ and also that for any closed class of Boolean functions B different from the classes C, C_0, C_1 and for any set $A \subseteq P_{3,2}$ such that $\operatorname{pr} A = B$ the set $A \cup \mathfrak{U}$ is a generating system for the class $\operatorname{pr}^{-1} B$.

The main result of this paper is the following

Theorem 1. *Let B be an arbitrary closed class of Boolean functions such that $B \notin \{C, C_0, C_1\}$, $\mathfrak{E}(B)$ be an arbitrary finite subset of the set $P_{3,2}$ such that $[\operatorname{pr} \mathfrak{E}(B)] = B$, and $G = \mathfrak{E}(B) \cup \mathfrak{U}$. Then*

$$\frac{3^n}{\log_2 n} \lesssim L_G(\operatorname{pr}^{-1} B(n)) \lesssim \frac{3^n}{\log_2 n} + L_{\operatorname{pr} G}(B(n)).$$

Present here a sketch of the proof of Theorem 1. First, based on the method from [13], we construct a partition of the set E_3^r, $r \geq 3$, into disjoint subsets $U_0, U_1, \ldots, U_{T(r)}$ such that the cardinality of the set U_0 satisfies the inequality

$$|U_0| \leq 2^r + r \cdot 2^{r-1} + \frac{r(r-1)}{2} \cdot 2^{r-2}$$

and each of the sets U_i, $i = 1, \ldots, T(r)$, possesses the following properties:

1) U_i is a subset of some ball of radius 1;

2) there exists $l = l(i)$ such that $1 \leq l \leq r$ and the lth component of each collection from U_i is equal to 2.

Then we estimate the cardinality $T(r)$ of the given partition: we prove the inequality

$$T(r) \leq 2 \cdot \frac{3^{r+1}}{r} \cdot \ln r.$$

Further, for each function $f(x_1, \ldots, x_n)$, $n \geq 3$, from the maximal class $\operatorname{pr}^{-1} B$ we construct a certain decomposition. By $g_f(x_1, \ldots, x_n)$ we denote the function from $P_{3,2}$ whose values coincide with the values of the function f on the set E_2^n and are equal to zero on all collections from $E_3^n \setminus E_2^n$, and by $\widehat{h}_f(x_1, \ldots, x_n)$ we denote the function from $P_{3,2}$ whose values coincide with the values of the function f on the set $E_3^n \setminus E_2^n$ and are equal to zero on all collections from E_2^n. Assume $h_f(x_1, \ldots, x_n, x_{n+1}) = j_1(x_{n+1}) + \widehat{h}_f(x_1, \ldots, x_n)$. It is evident that the function h_f belongs to the class $\operatorname{pr}^{-1} U_{01}$. It is easy to see that the following equality is valid:

$$f(x_1, \ldots, x_n) = h_f(x_1, \ldots, x_n, g_f(x_1, \ldots, x_n)). \tag{1}$$

In addition, using the partition of the set E_3^r described above, we construct a representation for the function h_f which is similar to the third representation of Boolean functions from [3].

After that, we construct a formula Φ_h over the system G realizing the function h_f so that

$$L_G(\Phi_h) \lesssim \frac{3^n}{\log_2 n}, \tag{2}$$

$$N(\Phi_h; x_{n+1}) = 1. \tag{3}$$

Further, we construct a formula Φ_g over G realizing the function g_f so that

$$L(\Phi_g) \leq L_{\mathrm{pr}\, G}(B(n)) + c_1 n, \tag{4}$$

where c_1 is some constant dependent on G. Equality (1) and relations (2)–(4) imply the upper estimate for the function $L_G(\mathrm{pr}^{-1}B(n))$. The lower estimate follows from cardinality considerations (see, e.g., [2, 3]).

It follows from Theorem 1 that the problem of the behavior of the function $L_G(\mathrm{pr}^{-1}B(n))$ can be reduced in some cases to the problem of the complexity of realization of Boolean functions in incomplete bases (i.e., to the problem of the behavior of the function $L_{\mathrm{pr}\, G}(B(n))$). In particular, Theorem 1 and previously known upper estimates for the complexity of realization of Boolean functions (see, e.g., [2, 4, 14]) imply asymptotically exact estimates for Shannon functions corresponding to some maximal classes. Thus, the following assertion is valid.

Theorem 2. *Let B be a closed class of Boolean functions such that at least one of the following conditions is satisfied*:

1) $L_{01} \subseteq B$;
2) $M_{01} \subseteq B$;
3) $B \in \{O^\infty, O_0^\infty, I^\infty, I_1^\infty, MO^\infty, MO_0^\infty, MI^\infty, MI_1^\infty\}$;
4) $B \in \{D_{01}, D_0, D_1, D, K_{01}, K_0, K_1, K, U, SU, U_{01}, MU, U_0, U_1\}$.

Then there exists a finite system $G \subseteq P_{3,2}$ such that $[G] = \mathrm{pr}^{-1}B$ and

$$L_G(\mathrm{pr}^{-1}B(n)) \sim \frac{3^n}{\log_2 n}.$$

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