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Nonlinear Standing Waves on the Surface of a Viscous Fluid

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Abstract—Standing surface waves in a viscous infinite-depth fluid are studied. The solution of the problem is obtained in the linear and quadratic approximations. The case of long, as compared with the boundary layer thickness, waves is analyzed in detail. The trajectories of fluid particles are determined and an expression for the vorticity is derived.

Keywords: water waves, viscosity, Lagrangian coordinates.

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The asymptotic theory of water waves is traditionally constructed in the inviscid-fluid approximation [1]. Taking fluid viscosity into account requires the development of fundamentally new methods of an analysis. In analytically studying the propagation of nonlinear gravity waves there arises a difficulty connected with the satisfaction of the boundary conditions on the free surface. Near the free surface the boundary layer thickness can be considerably smaller than the wave amplitude; for this reason, to formulate the boundary condition on a horizontal surface, as in the case of waves in an ideal fluid, would generally be incorrect. In this sense, the results of [2] need an additional validation. Transferring the boundary condition onto the horizontal surface level is possible for a low-viscosity fluid. Thus, in [3] an example of the calculations of the Faraday high-frequency ripple is presented, while in [4] the Stokes wave parameters are calculated in the cubic approximation. In the case of an arbitrary viscosity, in the quadratic approximation in the small wave-steepness parameter this difficulty can be overcome by means of going over to curvilinear coordinates that give the parameteric representation of quasistationary linear waves [5–7]. For progressive waves this approach is ineffective in the higher approximations, while for the standing waves it is inapplicable owing to the unsteadiness of the free surface shape.

However, this problem does not arise when using the Lagrangian variables. In the Lagrange description the vertical coordinate corresponding to the free surface is assumed to be zero and the boundary condition is formulated in the conventional fashion [8]. In the case of progressive waves the Lagrangian approach turns out to be very convenient in calculating drift flows [9–11] in the quadratic approximation. Below it is shown that it makes it possible to analyze the dynamics of a weakly-nonlinear standing wave as well.

1. FORMULATION OF THE PROBLEM

We will consider plane flows of an incompressible, viscous, infinitely deep fluid with a free surface. For a viscous fluid the equations of two-dimensional fluid dynamics in the Lagrange form can be written as follows [8]:

$$\begin{gathered}[X,\,Y] = \frac{D(X,\,Y)}{D(a,\,b)} = 1,\\ X_{tt} = -\rho^{-1}[p,\,Y] + \nu\{[X,\,[X,\,X_t]] + [Y,\,[Y,\,X_t]]\},\\ Y_{tt} = -g - \rho^{-1}[X,\,p] + \nu\{[X,\,[X,\,Y_t]] + [Y,\,[Y,\,Y_t]]\}.\end{gathered} \tag{1.1}$$

Here, X and Y are the coordinates of a fluid particle trajectory, a and b are its Lagrangian coordinates, t is time, ρ is the density, p is the pressure, ν is viscosity, and g is the gravity acceleration; the brackets mean the operation of taking the Jacobian with respect to the variables a and b. The boundary conditions are those of impermeability at the bottom ($Y_t = 0$ at $b = -\infty$) and of the absence of viscous stresses on the free surface [8]

$$T_{ik}n_k = -p_0 n_i, \quad \mathbf{n}\{n_x,\, n_y\} = \mathbf{n}\left\{-\frac{Y_a}{\sqrt{X_a^2 + Y_a^2}},\, \frac{X_a}{\sqrt{X_a^2 + Y_a^2}}\right\}, \quad b = 0,$$
$$T_{xx} = -p + 2\nu\rho[X_t,\, Y], \quad T_{yy} = -p - 2\nu\rho[Y_t,\, X], \quad T_{xy} = \nu\rho([Y_t,\, Y] - [X_t,\, X]), \tag{1.2}$$

where T_{ik} is the viscous stress tensor, p_0 is the constant external pressure, and $\mathbf{n}$ is the outward normal to the free surface.

We will represent all the unknown functions in the form of a series in the small parameter of wave steepness $\varepsilon = kA$, where k is the wavenumber and A is the wave amplitude

$$X = a + \varepsilon\xi_1 + \varepsilon^2\xi_2 + O(\varepsilon^3),$$
$$Y = b + \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3), \tag{1.3}$$
$$p = p_0 - \rho g b + \varepsilon p_1 + \varepsilon^2 p_2 + O(\varepsilon^3).$$

Substituting relations (1.3) in Eqs. (1.1) and (1.2) yields the equations for the unknown functions in the corresponding order of perturbation theory.

2. LINEAR APPROXIMATION

In the first order of perturbation theory Eqs. (1.1) take the form:

$$\xi_{1a} + \eta_{1b} = 0, \qquad \xi_{1tt} + \rho^{-1}p_{1a} + g\eta_{1a} - \nu\Delta_L\xi_{1t} = 0,$$
$$\eta_{1tt} + \rho^{-1}p_{1b} - g\xi_{1a} - \nu\mu_0\Delta_L\eta_{1t} = 0, \tag{2.1}$$

while the boundary conditions on the free surface are written as follows:

$$\eta_{1a} + \xi_{1b} = 0, \qquad -p_1 + 2\nu\rho\eta_{1tb} = 0, \qquad b = 0. \tag{2.2}$$

In Eq. (2.1) the Laplacian is taken with respect to the Lagrangian variables, while the functions ξ_1, η_1, and ρ_1 are assumed to be space-periodic.

For the convenience of calculations we will represent the deviations from the initial positions of the particles and the hydrostatic pressure as the real parts of complex functions ξ_i^k, η_i^k, and p_i^k letting

$$\xi_i = \frac{1}{2}\left(\xi_i^k + \bar{\xi}_i^k\right), \qquad \eta_i = \frac{1}{2}\left(\eta_i^k + \bar{\eta}_i^k\right), \qquad p_i = \frac{1}{2}\left(p_i^k + \bar{p}_i^k\right).$$

Here, the bars refer to complex-conjugate variables. We will also introduce a new independent variable $\tau = \mu t$, where $\mu = \mu_0 + \varepsilon^2\mu_2$. The value of μ_1 is chosen to be zero, as for the potential standing waves [12]. The value of μ_0 is determined in the process of calculations.

We will seek the solution of system (2.1), (2.1) in the form:

$$\xi_1^k = A(b)e^{\tau}\sin ka, \quad \eta_1^k = B(b)e^{\tau}\cos ka, \quad p_1^k = C(b)e^{\tau}\cos ka, \quad \mathrm{Re}\,\tau < 0. \tag{2.3}$$

Here, the quantity k is real and the functions A, B, C, and τ and the constant μ_0 are complex. The real parts of expressions (2.3) have the physical meaning. After the substitution of expressions (2.3) in system (2.1) and some algebra we arrive to the following equation

$$A^{\mathrm{IV}}_{bbbb} - \left(2k^2 + \frac{\mu_0}{\nu}\right)A''_{bb} + k^2\left(k^2 + \frac{\mu_0}{\nu}\right)A = 0. \tag{2.4}$$

Letting $A = e^{ib}$ in Eq. (2.4) we obtain a biquadratic equation in l, whose solution is given by the relations

$$(l^2)_1 = k^2, \qquad (l^2)_2 = k^2 + \frac{\mu_0}{\nu} = m^2. \tag{2.5}$$

Wave disturbances must decrease with the depth (as $b \to -\infty$); for this reason, the function A should be taken in the form:

$$A = \alpha e^{kb} + \beta e^{mb}, \qquad \mathrm{Re}\, m > 0. \tag{2.6}$$

The functions B and C are determined by the equalities

$$B = -\left(\alpha e^{kb} + \frac{k}{m}\beta e^{mb}\right), \qquad C = \frac{\rho}{k}\left[(\mu_0^2 + kg)\alpha e^{kb} + \frac{k^2}{m} g\beta e^{mb}\right]. \tag{2.7}$$

In view of Eqs. (2.6) and (2.7), the solution of system (2.1) takes the form:

$$\xi_1^k = (\alpha e^{kb} + \beta e^{mb})e^{\tau}\sin ka,$$

$$\eta_1^k = -\left(\alpha e^{kb} + \frac{k}{m}\beta e^{mb}\right)e^{\tau}\cos ka,$$

$$p_1^k = \frac{\rho}{k}\left[(\mu_0^2 + kg)\alpha e^{kb} + \frac{k^2}{m}\beta e^{mb}\right]e^{\tau}\cos ka, \qquad \mathrm{Re}\,\tau < 0.$$

Substituting this solution in conditions (2.2) yields

$$2k\nu m\alpha + \beta(2\nu k^2 + \mu_0) = 0, \quad (\mu_0^2 + 2\nu k^2\mu_0 + kg)\alpha + \left(\frac{k^2}{m}g + 2\nu\mu_0 k^2\right)\beta = 0.$$

The condition of compatibility of these two equations can be written as follows:

$$(\mu_0 + 2\nu k^2)^2 + kg = 4\nu^2 k^3 m. \tag{2.8}$$

For the sake of brevity, we will use the designations $\omega^2 = gk$, $\nu k^2/\omega = \theta$, and $\mu_0 + 2\nu k^2 = s\omega$, where ω is the frequency of the propagation of linear gravity waves. When raised to the square, in the new designations Eq. (2.8) takes the form:

$$(s^2 + 1)^2 = 16\theta^3(s - \theta). \tag{2.9}$$

This equation is precisely the same as that which appears in considering the linear waves traveling over a viscous fluid surface. Of four its roots only two satisfy the condition $\mathrm{Re}\, m > 0$ [13]. The real part of a root determines the decrement value and the imaginary part determines the wave oscillation frequency. One of two quantities α and β can be arbitrarily chosen. Let, for example, α be given; then the expression for β can be written as follows:

$$\beta = -\frac{2k\nu m}{\mu_0 + 2\nu k^2}\alpha = \frac{2km}{m^2 + k^2}\alpha. \tag{2.10}$$

The quantity α specifies the initial wave amplitude.

In this approximation, the wave motion vorticity is determined by the expression

$$\Omega_1 = -\mathrm{Re}\,(\mu_0^2\beta/\nu m)e^{mb+\tau}\sin ka.$$

It is concentrated in a surface layer, $2\pi(\mathrm{Re}\, m)^{-1}$ in thickness.

By way of illustration, we will consider the decay of sufficiently long waves for which the wave length $\lambda = 2\pi k^{-1}$ satisfies the condition $\theta = 4\pi^2\nu/\omega\lambda^2 \ll 1$. In this case, the right side of Eq. (2.9) can be neglected. Then its solution is given by the values $s = \pm i$, or, in the dimensional quantities, $\mu_0 = -2\nu k^2 \pm i\omega$. Hence we conclude that the decrement value is $-2\nu k^2$ and ω is the wave oscillation frequency whose sign determines the oscillation phase and can be arbitrary; for the sake of definiteness, we will choose the plus sign. As follows from Eq. (2.5), the value of m is approximately $m = (1 + i)\Delta$, where $\Delta = \sqrt{\omega/2\nu}$, and Eq. (2.10) takes the form $\beta/\alpha = -(1 - i)k/\Delta$. This quantity is in absolute value always less than unity; for this reason, in the approximation adopted the quantity β can be neglected as compared with α. Because of this, the expression for the free surface elevation η_1 can be written as follows:

$$\eta_1 = \alpha_0 e^{-2\nu k^2 t + kb} \cos ka \sin \omega t. \tag{2.11}$$

Here, it is chosen $\alpha = i\alpha_0$, where α_0 is real, in order for at zero viscosity expression (2.11) have the form similar to the solution for the potential waves [12]; the quantity $\varepsilon\alpha_0$ is equal to the wave amplitude. The horizontal displacement of fluid particles is written in the form:

$$\xi_1 = -\alpha_0 e^{-2\nu k^2 t + kb} \sin ka \sin \omega t, \tag{2.12}$$

so that the fluid particle trajectories satisfy the equation of a straight line

$$Y - b = -(X - a) \cot ka. \tag{2.13}$$

As in the potential wave, the fluid particles move relative to its equilibrium position $X_0 = a$, $Y_0 = b$ along a line inclined by the angle $-\cot ka$ but, due to the viscosity effect, the amplitude of their oscillations exponentially decays with time. The particle with the Lagrangian coordinates $ka = \pi/2 \pm \pi n$, where n is an integer, correspond to nodes and move in the horizontal direction. In antinodes the particles move in the vertical direction (for these $ka = \pi/2$).

For fairly long waves the viscosity effect reduces to the exponential decay of the oscillation amplitude with time. In the surface layer the flow is vortical, the vorticity varying in accordance with the law $\Omega_1 = -2k\omega\alpha_0 e^{-2\nu k^2 t + \Delta b} \sin ka \cos(\Delta b + \omega t)$. In this case, the standing waves furnish a rare example of the analytical representation of the flow with a time-dependent vorticity distribution.

3. QUADRATIC APPROXIMATION

The second-order equations are as follows:

$$\begin{aligned}
&\xi^k_{2a} + \eta^k_{2b} = \frac{k}{4}\left[(F_1(b) + F_2(b)\cos 2ka)e^{2\tau} + (H_1(b) + H_2(b)\cos 2ka)e^{\tau+\bar{\tau}}\right],\\
&\xi^k_{2tt} + g\eta^k_{2a} + \rho^{-1}p^k_{2a} - \nu\Delta_L\xi^k_{2t} = \frac{k}{4}\left[F_3(b)e^{2\tau} + H_3(b)e^{\tau+\bar{\tau}}\right]\sin 2ka,\\
&\eta^k_{2tt} - g\xi^k_{2a} + \rho^{-1}p^k_{2b} - \nu\Delta_L\eta^k_{2t}\\
&= \frac{k}{4}\left[(F_4(b) + F_5(b)\cos 2ka)e^{2\tau} + (H_4(b) + H_5(b)\cos 2ka)e^{\tau+\bar{\tau}}\right].
\end{aligned} \tag{3.1}$$

The complex functions F_i and H_i are determined by the solution of the linear approximation in the following functions

$$F_1 = -AB' - A'B, \qquad F_2 = A'B - AB', \qquad H_1 = -A\bar{B}' - A'\bar{B},$$

$$H_2 = A'' - A\bar{B}', \qquad F_3 = \rho^{-1}(B'C - BC') + \nu\mu_0(AA'' - A'^2 + 3k(A'B - AB')),$$

$$H_3 = \rho^{-1}(\bar{B}'C - \bar{B}C') + \nu\mu_0(2\bar{A}A'' - A\bar{A}'' - A'\bar{A}' + 3kA' - 3kA\bar{B}'),$$

$$F_4 = -\rho^{-1}(A'C + AC') + \nu\mu_0(2AB'' + A''B + 3A'B' - 2kBB'), \tag{3.2}$$

$$F_5 = \rho^{-1}(A'C - AC') + \nu\mu_0(2AB'' - A''B - A'B'),$$

$$H_4 = -\rho^{-1}(\bar{A}''C + \bar{A}C') + \nu\mu_0(2\bar{A}B'' + \bar{A}''B + 3A'B' - k(\bar{B}B' + \bar{B}'B)),$$

$$H_5 = \rho^{-1}(\bar{A}'C - \bar{A}C') + \nu\mu_0(2\bar{A}B'' - \bar{A}''B - \bar{A}'B' + 3k(\bar{B}B' - \bar{B}'B)).$$

System (3.1), (3.2) should be supplemented with the boundary conditions at the free surface $b = 0$

$$\begin{aligned}
&\xi^k_{2tb} + \eta^k_{2ta} = \frac{1}{4}(F_6(b)e^{2\tau} + H_6(b)e^{\tau+\bar{\tau}})\sin ka,\\
&F_6 = k[(\nu\rho)^{-1}BC - 2k\mu_0 AB],\\
&H_6 = k[(\nu\rho)^{-1}\bar{B}C + \mu_0(B\bar{B}' - \bar{B}B' - 2kA\bar{B} + A\bar{A}' - A'\bar{A})],\\
&(\nu\rho)^{-1}p^k_2 - 2\eta^k_{2tb}\\
&= \frac{1}{4}\left[(F_7(b) + F_8(b)\cos 2ka)e^{2\tau} + (H_7(b) + H_8(b)\cos 2ka)e^{\tau+\bar{\tau}}\right],\\
&F_{7,8} = k\mu_0(2AB' \pm 3A'B \mp kB^2), \quad H_{7,8} = k\mu_0(2\bar{A}B' \pm 2\bar{A}'B \mp k\bar{B}B \pm A'\bar{B}).
\end{aligned} \tag{3.3}$$

In the two last relations the upper signs relate to the first subscript.

We will seek the solution of system (3.1), (3.2) in the form:

$$\begin{aligned}
&\xi^k_2 = [e^{2\tau}f_1(b) + e^{\tau+\bar{\tau}}h_1(b)]\sin 2ka,\\
&\eta^k_2 = e^{2\tau}(f_2(b) + f_3(b)\cos 2ka) + e^{\tau+\bar{\tau}}(h_2(b) + h_3(b)\cos 2ka),\\
&p^k_2 = e^{2\tau}(f_4(b) + f_5(b)\cos 2ka) + e^{\tau+\bar{\tau}}(h_4(b) + h_5(b)\cos 2ka).
\end{aligned} \tag{3.4}$$

The functions $f_i(b)$ and $h_i(b)$ are complex. Thus, the problem of describing the waves is reduced to the determination of ten unknown complex functions in Eq. (3.4) satisfying the boundary conditions (3.3).

We note the basic properties of the disturbances in the quadratic approximation. Their spatial oscillation scale is equal to half the main wavelength. For $\mu_1 = 0$ the particle oscillation frequency and decrement are $2\mathrm{Im}\,\mu_0$ and $2\mathrm{Re}\,\mu_0$, respectively. The presence of the term independent of the horizontal coordinate a in the expression for the vertical displacement of the particles η_2 means that the mean level of the free surface may, generally, oscillate relative to the horizon $Y = 0$, gradually approaching it. It lies strictly in this horizontal plane if the relation

$$\int\limits_{-\pi/k}^{\pi/k} Y\,dX\bigg|_{b=0} = \varepsilon^2 \int\limits_{-\pi/k}^{\pi/k} (\eta_1\xi_{1a} + \eta_2)\,da\bigg|_{b=0} + O(\varepsilon^3) = 0$$

is fulfilled.

In the quadratic approximation this relation is fulfilled if $f_2 = f_3 = -kAB/4$ and $h_2 = h_3 = -kA\bar{B}/4$ at $b = 0$.

If all these relations hold for the functions f_3 and h_3 but are not fulfilled for, at least, one of the functions f_2 and h_2, then the mean level simply monotonically decreases with time, down to the $Y = 0$ horizon.

In our designations, the solution for the potential standing waves in the quadratic approximation is written as follows [12]:

$$\xi_{2\text{pot}} = 0, \quad \eta_{2\text{pot}} = \frac{k\alpha_0^2}{4} e^{2kb}(1 - \cos 2\omega t), \quad p_{2\text{pot}} = \frac{\rho\omega^2\alpha_0^2}{2}(1 - e^{2kb})\cos 2\omega t. \tag{3.5}$$

The expressions for the disturbances are independent of the coordinate a; because of this, the particles with the same value of b move in a similar manner, oscillating in the vertical direction. In a viscous fluid in this approximation the motion of the particles is considerably more complicated. Not only the dependence on the horizontal Lagrangian coordinate is added but also the formulas for the quadratic disturbances of the fluid particle coordinates involve time multipliers of two types. Apart from the typical multiplier determining the exponential decay, one of these includes oscillation with a double frequency and the other does not contain them (zero frequency harmonic).

4. SECOND FREQUENCY HARMONIC

After substitution of relations (3.4) in system (3.1) and fairly cumbersome algebra the following equation for the function $f_3(b)$ can be obtained

$$f_3^{\text{IV}} - \left(8k^2 + \frac{2\mu_0}{\nu}\right)f_3'' + 4k^2\left(4k^2 + \frac{2\mu_0}{\nu}\right)f_3 = \frac{k^2(k-m)^3(k+m)^2\alpha^2}{2(k^2+m^2)}e^{(k+m)b}.$$

The expression for the function $f_3(b)$ is as follows:

$$f_3 = C_1 e^{2kb} + C_2 e^{\sqrt{2(m^2+k^2)}b} + J_3 e^{(k+m)b}, \qquad J_3 = \frac{k^2(k+m)^2\alpha^2}{2(3k+m)(k^2+m^2)}.$$

Here, C_1 and C_2 are some constants.

In the general solution of the homogeneous equation only those terms are retained that decrease, as $b \to -\infty$ (two of the four free constants are taken to be zero). In the second term the exponent is chosen from the condition $\mathrm{Re}\sqrt{2(k^2+m^2)} > 0$. The formulas for the other functions f_i are as follows:

$$f_1 = -C_1 e^{2kb} - \frac{\sqrt{2(m^2+k^2)}}{2k} C_2 e^{\sqrt{2(m^2+k^2)}b} + J_1 e^{(k+m)b}, \quad J_1 = \frac{k^2(k^2 - 4km - m^2)\alpha^2}{2(3k+m)(k^2+m^2},$$

$$f_2 = \frac{k\alpha^2}{4}\left[e^{2kb} + \frac{4k^3 m}{(m^2+k^2)^2}e^{2mb} - \frac{2k(k+m)}{m^2+k^2}e^{(k+m)b} + C_3\right],$$

$$f_4 = \rho\nu^2\alpha^2(J_2 e^{2kb} + J_4 e^{2mb} + J_5 e^{(k+m)b}) - \rho(4\mu_0^3 b C_3 - \nu C_4),$$

$$J_2 = \frac{k-m}{2}[2k^3 - (k+m)(3k^2 - m^2)], \qquad J_5 = \frac{k^2(k^2-m^2)(k^2-m^2+2mk)}{k^2+m^2},$$

$$J_4 = \frac{k(k-m)}{(k^2+m^2)^2}[(k^4+m^4)(k+2m) - 2k^2m^2(3k+2m)],$$

$$f_5 = \rho\nu^2(J_6 C_1 e^{2kb} + J_7 C_2 e^{\sqrt{2(k^2+m^2)}b} + J_8 e^{(k+m)b},$$

$$J_6 = \frac{1}{k}(k - m)[m^2(k + m) - k^2(k + 5m)], \qquad J_7 = \frac{k - m}{k}[k^2(k - 3m) - m^2(k + m)],$$

$$J_8 = \frac{\alpha^2 k^2 (k + m)^2 (k - m)}{(3k + m)(k^2 + m^2)}\left[3k^2 + m^2 - 6km\right].$$

The constant C_3 should be let zero for the impermeability condition be fulfilled at the bottom. The three other constants are determined from the boundary conditions (3.3). They can be represented as follows:

$$C_1 = -\frac{3k^2 + m^2}{4k^2}C_2 + \frac{k + m}{4k}J_1 - \frac{1}{2}J_3 - \frac{1}{32k\nu(m^2 - k^2)}F_0(0),$$

$$C_2 = \frac{(4\nu)^{-1}F_8(0) - C_1[J_0 - 8(m^2 - k^2)k] - J_8 + 4(m^2 - k^2)(k + m)J_3}{J_7 - 4(m^2 - k^2)\sqrt{2(k^2 + m^2)}},$$

$$C_4 = \alpha^2(m - k)\left\{\frac{k^2(k + m)}{m^2 + k^2}\left[\frac{2km(k - m)^2}{m^2 + k^2} + k^2 + 3m^2 + 2km\right] + \frac{k^3 + m^3 + km(m - 3k)}{2}\right\}.$$

The first two quantities enter in the expressions for fluid particle displacements and the last determines the pressure.

5. ZERO FREQUENCY HARMONIC

Substituting relations (3.4) in system (3.1) yields an equation for the function $h_3(b)$

$$\begin{gathered} h_3^{\mathrm{IV}} - \left(8k^2 + \frac{\mu_0 + \bar{\mu}_0}{\nu}\right)h_3'' + 4k^2\left(4k^2 + \frac{\mu_0 + \bar{\mu}_0}{\nu}\right)h_3 \\ = J_1^* e^{(\bar{m}+k)b} + J_2^* e^{(m+k)b} + J_3^* e^{(\bar{m}+m)b}. \end{gathered} \tag{5.1}$$

The coefficients of the exponents on the right side of the equation are functions of k, m, and $\bar{m}$

$$J_1^* = \frac{\alpha\bar{\beta}k(k - \bar{m})}{\bar{m}(2k^2 - m^2 - \bar{m}^2)}\{4k[4k^3 m - (k^2 + m^2)^2] + (k + \bar{m})(k^2 + m^2 - 2k\bar{m})$$

$$+ (2k^2 - m^2 - \bar{m}^2) + 4k(k - m)[2k^2 m(\bar{m} - k) - \bar{m}(k + m)(3k^2 - m^2 - 2k\bar{m})]\},$$

$$J_2^* = -\frac{\alpha\beta(k - m}{2k^2 - m^2 - \bar{m})}\{4k[4k^3 m - (k^2 + m^2)^2] + (k + m)(k^2 + \bar{m}^2 - 2km)$$

$$+ (2k^2 - m^2 - \bar{m}) + \frac{2k(k - m)}{m}[k^2(k + m)(3k - 5m) - m^2(k + m)(3k + m)]\},$$

$$J_3^* = \frac{|\beta|^2 k(m - \bar{m})}{\bar{m}(2k^2 - m^2 - \bar{m}^2)}\{4k(4k^3 m - (m^2 + k^2)^2) + 2(m + \bar{m})(2k^2 - m^2 - \bar{m}^2)$$

$$+ (k^2 - \bar{m}m) + \frac{2(k - m)(m + \bar{m})}{m}[k^3(k - 3m) - m(k + m)(3k^2 - \bar{m}^2 + km - 2m\bar{m})]$$

$$- \frac{4k(m - \bar{m})}{m}[k^2\bar{m}(k - 3m) - (k + m)(3k^3 + m^2\bar{m} - 2km\bar{m} - k\bar{m}^2)]\}.$$

The general solution of Eq. (5.1) satisfying the condition of the wave disturbance decay down to zero at the bottom can be written in the form:

$$h_3 = C_1^* e^{2kb} + C_2^* e^{\sqrt{m^2+\bar{m}^2+2k^2}b} + \frac{J_1^* e^{(k+\bar{m})b}}{(k^2 + m^2 - 2\bar{m}k)(3k^2 - \bar{m}^2 - 2\bar{m}k)}$$

$$+ \frac{J_2^* e^{(k+m)b}}{(k^2 + \bar{m}^2 - 2mk)(3k^2 - m^2 - 2mk)} + \frac{J_3^* e^{(\bar{m}+m)b}}{2(k^2 - m\bar{m})(4k^2 - m^2 - \bar{m}^2 - 2m\bar{m})}.$$

Here, C_1^* and C_2^* are constants.

In the general solution of the homogeneous equation only those terms are retained that decay, as $b \to -\infty$ (two of four free constants are taken to be zero). In the second term the exponent is chosen from the condition $\mathrm{Re}\sqrt{m^2 + \bar{m}^2 + 2k^2} > 0$. From the known h_3 the functions h_i, $i = 1,\ 2,\ 4,\ 5$ are determined from the relations

$$h_1 = \frac{H_2}{8} - \frac{1}{2k}h_3', \quad h_2' = \frac{k}{4}H_1, \quad h_4' = \rho\left(\frac{k}{4}H_4 - (\mu_0 + \bar{\mu}_0)^2 h_2 + \nu(\mu_0 + \bar{\mu}_0)h_2''\right),$$

$$h_5 = \frac{\rho}{2k}\left\{\left[(\mu_0 + \bar{\mu}_0)^2 + 4\nu(\mu_0 + \bar{\mu}_0)k^2\right]h_1 - \nu(\mu_0 + \bar{\mu}_0)h_1'' - 2gkh_3 - \frac{k}{4}H_3\right\}.$$

The values of the constants are determined from the boundary conditions

$$4(\mu_0 + \bar{\mu}_0)(h_1' - 2kh_3)\Big|_{b=0} = H_6(0),$$

$$4\left[(\nu\rho)^{-1}h_4 - 2(\mu_0 + \bar{\mu}_0)h_2'\right]\Big|_{b=0} = H_7(0),$$

$$4\left[(\nu\rho)^{-1}h_5 - 2(\mu_0 + \bar{\mu}_0)h_3'\right]\Big|_{b=0} = H_8(0).$$

All algebraic calculations are performed in the same fashion as those of the previous section. They are extremely cumbersome. For this reason, the complete expressions for functions (3.4) are not presented. Actually, the boundary layer thickness Δ is almost always considerably smaller than the wavelength λ; because of this, it seems natural to present the form of solution (3.4) in the long-wave limit.

Following the solution in the linear approximation, we let $m = (1 + i)\Delta$ and expand the polynomial expression in a series in the small parameter k/Δ, restricting ourselves to the terms linear in this parameter. Ultimately, the formulas for the fluid particle coordinates are written as follows:

$$\xi_2 = \left[\frac{\Delta e^{\Delta b}}{k\sqrt{2}}\sin\left(\Delta b - \frac{1}{4}\pi\right) - \frac{3e^{2\Delta b}}{4} - \frac{ke^{\sqrt{2}\Delta b}}{2\Delta}\sin\left(\sqrt{2}\Delta b + 2\omega t + \frac{1}{4}\pi\right)\right]k\alpha_0^2 e^{-4\nu k^2 t}\sin 2ka,$$

$$\eta_2 = \left[(1 - \cos 2\omega t)e^{2kb} + \frac{ke^{\Delta b}}{\Delta}\left[\frac{\sqrt{2}}{2}\sin\left(\Delta b + \omega t + \frac{1}{4}\pi\right) - \sqrt{2}\sin\left(\Delta b + \frac{1}{4}\pi\right)\right]\right] \tag{5.2}$$

$$+ \left[4e^{\Delta b}\cos\Delta b + \frac{3ke^{2\Delta b}}{\Delta}\right]\cos 2ka\Bigg\}\frac{k\alpha_0^2}{4}e^{-4\nu k^2 t}.$$

When viscosity vanishes and Δ increases without bound, expressions (5.2) go over into formulas (3.5) determining the potential standing waves. With increase in the depth ($\Delta|b| \gg 1$, $\nu \neq 0$) these formulas take the form:

$$\xi_2^{\text{int}} = 0, \qquad \eta_2^{\text{int}} = \frac{k\alpha_0^2}{4}(1 - \cos 2\omega t)e^{-4\nu k^2 t}\cos 2ka.$$

Rectilinear motions of fluid particles in the linear wave are superimposed by vertical oscillations, inhomogeneous in the horizontal direction. In this case, the fluid particle trajectories are governed (in view of Eqs. (2.11)–(2.13)) by the equation

$$Y - b = -(X - a)\cot ka + \frac{k(X - a)^2}{2}(\cot^2 ka - 1),$$

that is, now the particles move along parabolic sections; in antinodes they oscillate in the vertical direction.

Within the framework of the long-wave approximation the terms of the order of k/Δ should, generally be neglected in expressions (5.2). Then the solutions of the quadratic approximation are rewritten as follows:

$$\xi_2^{\text{bound}} = k\alpha_0^2\left(\frac{\Delta e^{\Delta b}}{k\sqrt{2}}\sin\left(\Delta b - \frac{1}{4}\pi\right) - \frac{3e^{2\Delta b}}{4}\right)e^{-4\nu k^2 t}\sin 2ka,$$

$$\eta_2^{\text{bound}} = \frac{k\alpha_0^2}{4}[(1 - 2\cos 2\omega t)e^{2kb} + 4e^{\Delta b}\cos\Delta b\cos 2ka]e^{-4\nu k^2 t}.$$

The boundary layer effect consists in the appearance of additional terms in the expressions for the displacements, which depend on the Lagrangian coordinates but are independent of the oscillation frequency ω. Qualitatively, their effect can be represented as nonlinear deformation of the flow structure with the depth; fluid particles oscillate along curves somewhat more complicated than parabolic sections.

For the terms of the order of k/Δ in expressions (5.2) the frequency dependence already exists. Hence follows that near the free boundary for the shorter waves the fluid particle motion becomes more complicated than the motion of the particles within the fluid depth. In the situation in which the derivation of analytical representations for the boundary layer flows is fairly difficult problem expressions (5.2) can be used for approximately describing the fluid motion also for finite values of the parameter k/Δ. Obviously that in this case it makes sense to speak of only qualitative features of the flow.

In the quadratic approximation the flow vorticity is given by the expression

$$\Omega_2 = [\xi_{1t}, \xi_1] + [\eta_{1t}, \eta_1] + \xi_{2tb} - \eta_{2ta}.$$

Substituting the first approximation and relations (5.2) in the above relation yields the following representation

$$\Omega_2 = \omega k^2\alpha_0^2\left[(2\cos\Delta b - \sin\Delta b)e^{\Delta b} + 2e^{\sqrt{2}\Delta b}\cos(\sqrt{2}\Delta b + 2\omega t)\right]e^{-4\nu k^2 t}\sin 2ka + o\left(\frac{k}{\Delta}\right).$$

The vorticity is the sum of two fields, namely, the quasistationary field ($\nu k^2 \ll 1$) and that oscillating at double frequency. The vortex layer thickness is by a factor of $\sqrt{2}$ greater than in the linear waves.

Summary. Within the framework of the Lagrangian approach the asymptotic theory of weakly-nonlinear standing waves in a viscous fluid is developed. For the case of an infinitely deep fluid the complete solution of the problem is given in the quadratic approximation. This is the first example of the complete analytical description for the nonlinear wave motion of a viscous fluid.

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