

Attractors of Cartan foliations

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Abstract

The paper is focused on the existence problem of attractors for foliations. Since the existence of an attractor is a transversal property of the foliation, it is natural to consider foliations admitting transversal geometric structures. As transversal structures are chosen Cartan geometries due to their universality. The existence problem of an attractor on a complete Cartan foliation is reduced to a similar problem for the action of its structure Lie group on a certain smooth manifold. In the case of a complete Cartan foliation with a structure subordinated to a transformation group, the problem is reduced to the level of the global holonomy group of this foliation. Each countable automorphism group preserving a Cartan geometry on a manifold and admitting an attractor is realized as the global holonomy group of some Cartan foliation with an attractor. Conditions on the linear holonomy group of a leaf of a reductive Cartan foliation sufficient for the existence of an attractor (and a global attractor) which is a minimal set are found. Various examples are considered.

Keywords: foliation; attractor; minimal set; Cartan foliation; reductive Cartan foliation; global holonomy group of a foliation; linear holonomy group of a foliation

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1 Introduction

The study of the dynamical properties of foliations is an actual area. The existence of closed leaves, attractors and minimal sets gives reach information about the structure of a foliation. By this reason, the problems of the existence and the structure description for attractors and minimal sets of foliations are the central problems in the foliation theory and topological dynamics. There are several nonequivalent notions of an attractor in the theory of dynamical systems (e.g., see [5]). Some of these notions are equivalent [6]. For „typical” dynamical systems in metric sense different notions of an attractor coincide according to Palis’s hypothesis [10]. We use the most general notion of an attractor for a foliation that generalizes the notion of an attractor from [12]. Note that the attractor of a foliation may be disconnected and it may contain other attractors. This is not a case for a transitive attractor that contains a dense leaf. Examples of transitive attractors are attractors which are minimal sets.

In Section 2 we show that the property of a singular foliation to admit an attractor is transversal, i.e., it is preserved under the transversal equivalence of foliations. By this reason it is natural to investigate the influence of different kinds of the transversal structures of foliations on the existence of attractors on them. As the transversal structures we consider Cartan geometries, since they include large classes of geometries, e.g. Riemannian, Lorentzian (more generally, pseudo-Riemannian), affine, conformal, projective, transversely homogeneous, parabolic, etc.

Deroin and Kleptsyn [6] investigated attractors of foliations with conformal transversal structures on compact manifolds. The Main theorem of [6] states that for every conformal foliation on a compact manifold either there exists a transversely invariant measure, or there exists a finite number of minimal sets equipped with probability measures, which are attractors satisfying some properties.

The case of transversally similar foliations considered the second author of this work in [14, Sec. 9]. In [17], the existence problem of an attractor for foliations admitting a transversal parabolic geometry of rank one was solved. In [16], it is shown that every non-Riemannian conformal foliation (M, F) of codimension $q \geq 3$ admits an attractor which is a minimal set, and the restriction of the foliation to the basin of the attractor is a transversely conformally flat foliation. Moreover, if the foliated manifold M is compact, then (M, F) is a $(\text{Conf}(\mathbb{S}^q), \mathbb{S}^q)$ -foliation [15, Th. 4]. Every complete non-Riemannian conformal foliation (M, F) of codimension $q \geq 3$ admits a global attractor which is a minimal set, and (M, F) is covered by a locally trivial bundle over the standard q -dimensional sphere \mathbb{S}^q or the Euclidean space \mathbb{R}^q [16, Th. 5].

Note that in [15, 16] as well as in the present work we use the methods of local and global differential geometry, while Deroin and Kleptsyn [6] used the Lyapunov exponentials and invariant transversal measures, including the harmonic measures.

In Section 3 we introduce the Cartan foliations, discuss the effectivity, the completeness of the Cartan foliations, the construction of the lifted foliation, and the aureole foliation.

The Cartan geometry has an infinitesimal nature, consequently in order to describe the global structure of Cartan foliations one should use global conditions. The most important global condition for such foliations is the completeness. In Section 4, we use the associated singular aureole foliation and prove a criteria (Theorem 1) that reduces the existence problem of an attractor for a complete Cartan foliation of type (G, H) to a similar problem for the induced action of the Lie group H on a certain manifold called the basic one.

In Section 5 we study attractors of complete Cartan (Φ, N) -foliations (M, F) . We prove that problems of the existence and the structure description of attractors (resp. global attractors) for these foliations are equivalent to the similar problems for the countable automorphism groups of complete Cartan geometries on simply connected manifolds.

Next we consider the case of reductive Cartan foliations, i.e., foliations, admitting transversal reductive Cartan geometry. The class of reductive Cartan foliations includes transversally similar, Riemannian, Lorentzian (more generally, pseudo-Riemannian), reductive transversally homogeneous foliations and foliations with transversal linear connections. In Section 6 we show that a reductive Cartan foliation admits also a transversal linear connection; this simplifies the study of reductive Cartan foliations.

In Section 7 we find the conditions on the linear holonomy group of a leaf L of the foliation (M, F) that are sufficient for the closure $\mathcal{M} = \overline{L}$ to be an attractor and a minimal set of this foliation. Some other results about the linearization of the holonomy group and the geometry around leaves of foliations obtained using various methods Crainic, Struchiner, Weinstein, Zung, and also del Hoyo with Fernandes (see e.g. [7] and the references therein). We also find the sufficient conditions for the existence a global attractor which is a minimal set of (M, F) .

In Section 8 we consider several examples.

Assumptions Throughout this paper we assume for simplicity that all manifolds and maps are smooth of the class C^∞ ; in fact, the main results of the paper are valid for foliations of the class C^2 . All neighborhoods are assumed to be open and all manifolds are assumed to be Hausdorff.

Notations The algebra of smooth functions on a manifold M will be denoted by $\mathfrak{F}(M)$.

Let $\mathfrak{X}(N)$ denote the Lie algebra of smooth vector fields on a manifold N . If \mathfrak{M} is a smooth distribution on M and $f : K \rightarrow M$ is a submersion, then let $f^*\mathfrak{M}$ be the distribution on the manifold K such that $(f^*\mathfrak{M})_z = \{X \in T_zK \mid f_{*z}(X) \in \mathfrak{M}_{f(z)}\}$, where $z \in K$. Let $\mathfrak{X}_{\mathfrak{M}}(M) = \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \ \forall u \in M\}$. As usually we denote by $P(N, H)$ the principal H -bundle P over the manifold N . The symbol \cong will denote the isomorphism of objects in the corresponding category.

2 Attractors of foliations and transversality

In this section we give a definition of an attractor of a singular foliation in the sense of Stefan and Sussmann and show that the property of a foliation to admit an attractor is transversal. Most of the results of this paper are obtained for smooth foliations. We use singular foliations in the proof of Theorem 1

Definition 1. Let (M, F) be a singular foliation. A subset of a manifold M is called saturated if it is a union of leaves of this foliation. A nonempty closed saturated subset \mathcal{M} of M is called *an attractor* of (M, F) if there exists an open saturated neighbourhood $\mathcal{U} = \mathcal{U}(\mathcal{M})$ of the set \mathcal{M} such that the closure of every leaf from $\mathcal{U} \setminus \mathcal{M}$ contains the set \mathcal{M} , i.e., if $\overline{L} \supset \mathcal{M} \ \forall L \subset \mathcal{U} \setminus \mathcal{M}$. The neighbourhood \mathcal{U} is uniquely determined by this condition and it is called *the basin of this attractor*; we denote it by $\mathcal{B}(\mathcal{M})$. If in addition $\mathcal{B}(\mathcal{M}) = M$, then the attractor \mathcal{M} is called *global*.

Definition 2. Two smooth singular foliations (M_1, F_1) and (M_2, F_2) are called transversally equivalent if there exists a smooth singular foliation (\mathbb{M}, \mathbb{F}) and two submersions $p_1 : \mathbb{M} \rightarrow M_1$ and $p_2 : \mathbb{M} \rightarrow M_2$ such that

$$\mathbb{F} = \{p_1^{-1}(L_\alpha) \mid L_\alpha \in F_1\} = \{p_2^{-1}(L_\beta) \mid L_\beta \in F_2\}.$$

This notion generalizes the notion of the transversal equivalence for smooth foliation in the sense of Molino [9, Def. 2.1]. It can be checked directly that the transversal equivalence of singular foliations is an equivalence relation.

A property of singular foliations is said to be *transversal* if it is preserved under the transversal equivalence.

Proposition 1. *The existence of an attractor is a transversal property of a foliation.*

Proof of Proposition 1. Since transversally equivalent singular foliations have the common leaf space, it is sufficient to characterize the existence of an attractor in terms of the topology of the leaf space M/F of a singular foliation (M, F) .

It is well known that the projection $f : M \rightarrow M/F$ is an open map and the closure in M of any saturated subset is again a saturated subset. Due to this we observe that a singular foliation (M, F) has an attractor if and only if there exists a nonempty closed subset $\widetilde{\mathcal{M}} \subset M/F$ and its open neighbourhood $\widetilde{\mathcal{U}}$ in M/F such that the closure $\overline{\{z\}}$ of any one-point set $\{z\} \subset \widetilde{\mathcal{U}} \setminus \widetilde{\mathcal{M}}$ satisfies the inclusion $\widetilde{\mathcal{M}} \subset \overline{\{z\}}$. Note that the set $\mathcal{M} = f^{-1}(\widetilde{\mathcal{M}})$ is an attractor of the singular foliation (M, F) . This proves the proposition. \square

Definition 3. Let H be a Lie group with a smooth action on a manifold W . A nonempty closed union \mathcal{K} of orbits of H is called an attractor of the action if there exists an open invariant neighbourhood $\mathcal{V} = \mathcal{V}(\mathcal{K})$ of the set \mathcal{K} such that the closure of every orbit from $\mathcal{V} \setminus \mathcal{K}$ contains the set \mathcal{K} . If $\mathcal{V} = W$, then the attractor \mathcal{K} is called *global*.

The following proposition can be proved in the same way as Proposition 1 (for connected Lie groups this proposition follows from Proposition 1).

Proposition 2. *The existence of an attractor of a smooth action of a Lie group on a manifold is a transversal property.*

3 Cartan foliations and the associated constructions

3.1 Cartan geometries

Let us first recall the definition of a Cartan geometry [3, 4]. Let G be a Lie group and H be a closed subgroup of G . Denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of the Lie groups G and H , respectively. Let N be a smooth (not necessary connected) manifold. A *Cartan geometry* on N of type (G, H) (or $\mathfrak{g}/\mathfrak{h}$) is a principal right H -bundle $P(N, H)$ with the projection $p : P \rightarrow N$ and together with a \mathfrak{g} -valued 1-form β on P satisfying the following conditions:

- (c_1) the map $\beta_w : T_w P \rightarrow \mathfrak{g}$ is an isomorphism of the vector spaces for every $w \in P$;
- (c_2) $R_h^* \beta = \text{Ad}_G(h^{-1})\beta$ for all $h \in H$, where $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$ is the adjoint representation of the Lie group G on its Lie algebra \mathfrak{g} ;
- (c_3) $\beta(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field defined by the element A .

This Cartan geometry is denoted by $\xi = (P(N, H), \beta)$. The pair (N, ξ) is called a *Cartan manifold*.

Definition 4. Let G/H be a homogeneous space and let G act on G/H by left translations. Denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of the Lie groups G and H , respectively. If there exists an $\text{Ad}_G(H)$ -invariant vector subspace V of \mathfrak{g} such that

$$\mathfrak{g} = \mathfrak{h} \oplus V,$$

then the homogeneous space G/H is called reductive. A Cartan geometry $\xi = (P(N, H), \beta)$ of type (G, H) , where G/H is a reductive homogeneous space, is called a *reductive Cartan geometry*.

Cartan manifolds form a category, where morphisms of two Cartan manifolds $\xi = (P(N, H), \beta)$ and $\xi' = (P'(N', H), \beta')$ of the same type (G, H) are morphisms of the principle bundles $\Gamma : P(N, H) \rightarrow P'(N', H)$ satisfying the condition $\Gamma^* \beta' = \beta$.

3.2 Cartan foliations

A foliation (M, F) is said to be a *Cartan foliation* of type (G, H) if it admits a Cartan geometry of type (G, H) (or $\mathfrak{g}/\mathfrak{h}$) as a transversal structure. More precisely, this means the following. Let $\xi = (P(N, H), \beta)$ be a Cartan geometry. A Cartan foliation (M, F) may be defined using an (N, ξ) -*cocycle*, i.e. a family $\{U_i, f_i, \Gamma_{ij}\}_{i,j \in J}$ satisfying the following properties:

- (i) $\{U_i\}_{i \in J}$ is an open covering of the manifold M by connected subsets U_i , and $f_i : U_i \rightarrow V_i \subset N$ are submersions with connected leaves;
- (ii) if $U_i \cap U_j \neq \emptyset$, $i, j \in J$, then there exists an isomorphism Γ_{ij} of the Cartan geometries $\xi_{f_j(U_i \cap U_j)}$ and $\xi_{f_i(U_i \cap U_j)}$ induced on $f_j(U_i \cap U_j)$ and $f_i(U_i \cap U_j)$, respectively, such that its projection γ_{ij} satisfies the equality $f_i = \gamma_{ij} \circ f_j$;

- (iii) $\Gamma_{ij} \circ \Gamma_{jk} = \Gamma_{ik}$ for $U_i \cap U_j \cap U_k \neq \emptyset$; moreover, $\Gamma_{ii} = \text{id}_{P_{f_i(U_i)}}$.
- (iv) the cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ defines the foliation (M, F) .

One says also that the Cartan foliation (M, F) satisfying the above properties is modelled on the Cartan geometry ξ of type (G, H) .

3.3 The lifted foliation

We will use the construction of the lifted foliation $(\mathcal{R}, \mathcal{F})$ for a Cartan foliation (M, F) from [14]; it generalizes a similar construction for a Riemannian foliation from [9]. For a given Cartan foliation (M, F) of type (G, H) one may construct a principle H -bundle $\mathcal{R}(M, H)$ (called a foliated bundle) with a projection $\pi : \mathcal{R} \rightarrow M$, an H -invariant transversely parallelizable foliation $(\mathcal{R}, \mathcal{F})$ such that π is a morphism of $(\mathcal{R}, \mathcal{F})$ into (M, F) in the category of foliations; moreover, there exists a \mathfrak{g} -valued 1-form ω on \mathcal{R} having the following properties:

- (i) $\omega(A^*) = A$ for any $A \in \mathfrak{h}$, where A^* is the fundamental vector field corresponding to A ;
- (ii) $R_a^* \omega = \text{Ad}_G(a^{-1}) \omega \quad \forall a \in H$;
- (iii) for any $u \in \mathcal{R}$, the map $\omega_u : T_u \mathcal{R} \rightarrow \mathfrak{g}$ is surjective with the kernel $\ker \omega = T\mathcal{F}$, where $T\mathcal{F}$ is the tangent distribution to the foliation $(\mathcal{R}, \mathcal{F})$;
- (iv) the Lie derivative $L_X \omega$ is zero for any vector field X tangent to the leaves of $(\mathcal{R}, \mathcal{F})$.

The foliation $(\mathcal{R}, \mathcal{F})$ is called the *lifted foliation*. The restriction $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow L$ of π to a leaf \mathcal{L} of $(\mathcal{R}, \mathcal{F})$ is a covering map onto the corresponding leaf L of (M, F) . If \mathcal{R} is disconnected, then we consider a connected component of \mathcal{R} .

3.4 Effectivity of transversal Cartan geometries

Let us recall several standard definitions.

Definition 5. A pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, where \mathfrak{h} is subalgebra of \mathfrak{g} , is called effective if the maximal ideal of \mathfrak{g} belonging to \mathfrak{h} is zero.

Definition 6. A Cartan geometry of the type (G, H) is said to be *effective* if the group G acts effectively on G/H , or in other words, if the maximal normal subgroup of G belonging to H is trivial.

Note that the effectivity of a pair of Lie groups (G, H) , where H is a closed subgroup of G , implies the effectivity of the pair $(\mathfrak{g}, \mathfrak{h})$ of their Lie algebras. It is known [14, Prop. 1] that if (M, F) admits an ineffective Cartan geometry, then (M, F) admits also an effective Cartan geometry. Therefore without loss of generality we assume further in this work that all Cartan foliations are modelled on effective Cartan geometries if the contrary is not indicated.

In the case of an effective Cartan geometry, the definition of a Cartan foliation from the previous section is equivalent to the definition of a Cartan foliation by Blumenthal [1].

3.5 Completeness of Cartan foliations

Let (M, F) be a foliation. A q -dimensional smooth distribution \mathfrak{M} on the manifold M is called transversal to the foliation (M, F) if the equality $T_x M = T_x F \oplus \mathfrak{M}_x$ holds for all $x \in M$. One may identify a transversal distribution \mathfrak{M} with the vector quotient bundle TM/TF .

Let (M, F) be a Cartan foliation of codimension q and let \mathfrak{M} be a transversal q -dimensional distribution. Denote by $\widetilde{\mathfrak{M}}$ the induced distribution $\pi^* \mathfrak{M}$ on \mathcal{R} . Let ω be the \mathfrak{g} -valued 1-form on \mathcal{R} defined in the previous section. The Cartan foliation (M, F) is said to be \mathfrak{M} -complete if

every vector field $X \in \mathfrak{X}_{\mathfrak{M}}(\mathcal{R})$ satisfying the condition $\omega(X) = c$, $c = \text{const} \in \mathfrak{g}$, is complete. A Cartan foliation (M, F) is called *complete* if there exists a transversal distribution \mathfrak{M} such that (M, F) is \mathfrak{M} -complete.

3.6 The associated aureole foliation

Consider a complete Cartan foliation (M, F) and its lifted foliation $(\mathcal{R}, \mathcal{F})$. Let $\pi : \mathcal{R} \rightarrow M$ be the projection of the corresponding H -bundle. Since $(\mathcal{R}, \mathcal{F})$ is a complete transversally parallelizable foliation, the closures $\overline{\mathcal{L}}$ of its leaves \mathcal{L} form a simple foliation $(\mathcal{R}, \overline{\mathcal{F}})$ which leaves are fibres of a locally trivial bundle $\pi_B : \mathcal{R} \rightarrow W$ [9, Th.4.2]. The manifold W is called the *basic manifold associated to the foliation* (M, F) . The image $\mathcal{O}(L) = \pi(\overline{\mathcal{L}})$, where $\mathcal{L} \in \mathcal{F}$, is called the *aureole* of the leaf $L = \pi(\mathcal{L})$ of (M, F) . The aureole $\mathcal{O}(L)$ of a leaf $L = L(x)$ is also denoted $\mathcal{O}(x)$.

Theorem [14, Th. 2]. *The set of all aureoles $\mathcal{O} = \{\pi(\overline{\mathcal{L}}) \mid \mathcal{L} \in \mathcal{R}\}$ of a complete Cartan foliation (M, F) is a smooth singular foliation (M, \mathcal{O}) that has the following properties:*

- 1) *the leaf $L(y)$ of (M, F) is dense in $\mathcal{O}(x)$, $x \in M$, for every point $y \in \mathcal{O}(x)$;*
- 2) *$L(x) \subset \mathcal{O}(x) \subset \overline{L(x)} \forall x \in M$, where $\overline{L(x)}$ is the closure of $L(x)$ in M .*

The foliation (M, \mathcal{O}) defined in the above theorem is called *the aureole foliation* associated with (M, F) . The map

$$\Phi^W : W \times H \rightarrow W : (w, a) \mapsto \pi_b(R_a(u)) \quad \forall (w, a) \in W \times H, u \in \pi_b^{-1}(w), \quad (1)$$

defines an action of the Lie group H on the basic manifold $W = \mathcal{R}/\overline{\mathcal{F}}$, and the orbit space W/H is homeomorphic to the leaf space M/\mathcal{O} of the aureole foliation (M, \mathcal{O}) , i.e. $W/H \cong M/\mathcal{O}$.

4 A criteria for the existence of an attractor for a complete Cartan foliation

Theorem 1. *Let (M, F) be a complete Cartan foliation of type (G, H) . The foliation (M, F) admits an attractor (or a global attractor) if and only if the induced action of the Lie group H on the basic manifold has an attractor (or a global attractor).*

Proof of Theorem 1. Assume that a complete Cartan foliation (M, F) admits an attractor \mathcal{M} with the basin $\mathcal{B} = \mathcal{B}(\mathcal{M})$. Let us show that \mathcal{M} is an attractor for the associated singular aureole foliation (M, \mathcal{O}) with the same basin \mathcal{B} . Pick a point $x \in \mathcal{B} \setminus \mathcal{M}$. Let $L = L(x)$, then $L \cap \mathcal{M} = \emptyset$. By the definition of an attractor, we get

$$\mathcal{M} \subset \overline{L}.$$

Consider the aureole $\mathcal{O}(L)$. According to the above theorem, $\overline{L} = \overline{\mathcal{O}(L)}$, hence

$$\mathcal{M} \subset \overline{\mathcal{O}(L)}. \quad (2)$$

First we check that

$$\mathcal{M} \cap \mathcal{O}(L) = \emptyset. \quad (3)$$

Suppose that $\mathcal{M} \cap \mathcal{O}(L) \neq \emptyset$. Since both \mathcal{M} and $\mathcal{O}(L)$ are saturated sets, there exists a leaf $L_\alpha \subset \mathcal{M} \cap \mathcal{O}(L)$. Therefore we get the following chain of relations

$$\overline{\mathcal{O}(L)} = \overline{L_\alpha} \subset \overline{\mathcal{M} \cap \mathcal{O}(L)} \subset \mathcal{M} \cap \overline{\mathcal{O}(L)} \subset \mathcal{M},$$

which together with (2) implies $\overline{\mathcal{O}(L)} = \mathcal{M}$. Consequently, $L \subset \overline{L} = \overline{\mathcal{O}(L)} \subset \mathcal{M}$. This contradicts $L \cap \mathcal{M} = \emptyset$. Thus (3) holds true.

Pick a leaf $L_\beta \subset \mathcal{O}(L)$, then

$$\overline{L_\beta} = \overline{\mathcal{O}(L)} = \overline{L} \supset \mathcal{M}.$$

Therefore, $\overline{L_\beta} \supset \mathcal{M}$. Consequently, $L_\beta \cap \mathcal{B} \neq \emptyset$. Since \mathcal{B} is a saturated set, it holds $L_\beta \subset \mathcal{B}$. This and (3) imply that $\mathcal{O}(L) \subset \mathcal{B} \setminus \mathcal{M}$. Consequently \mathcal{M} is an attractor of the aureole foliation (M, \mathcal{O}) with the same basin $\mathcal{B} = \mathcal{B}(\mathcal{M})$.

From Propositions 1 and 2 it follows that the induced action of the Lie group H on the basic manifold W has the attractor $\mathcal{K} = \pi_B(\pi^{-1}(\mathcal{M}))$, and $\mathcal{B}(\mathcal{K}) = \pi_B(\pi^{-1}(\mathcal{B}(\mathcal{M})))$ is its basin.

Conversely, assume that the induced action of the Lie group H on the basic manifold W has an attractor \mathcal{K} . Since $M/\mathcal{O} \cong W/H$, by Propositions 1 and 2, the aureole foliation (M, \mathcal{O}) admits the attractor $\mathcal{M} = \pi(\pi_B^{-1}(\mathcal{K}))$. Let $\mathcal{B} = \mathcal{B}(\mathcal{M})$ be its basin. Consider any leaf $L_\beta \subset \mathcal{B} \setminus \mathcal{M}$. According to the above theorem, $\overline{L_\beta} = \overline{\mathcal{O}(L_\beta)} \supset \mathcal{M}$. Therefore \mathcal{M} is an attractor of (M, F) with the same basin $\mathcal{B}(\mathcal{M})$. Since $\mathcal{B}(\mathcal{M}) = M$ if and only if $\mathcal{B}(\mathcal{K}) = W$, \mathcal{M} is a global attractor of the foliation (M, F) if and only if \mathcal{K} is a global attractor of the group Ψ . \square

5 Attractors of (Φ, N) -foliations

5.1 (Φ, N) -manifolds and (Φ, N) -foliations

Let N be a connected manifold and let Φ be a group of diffeomorphisms of N . One says that a group Φ acts quasi-analytically on N if for any open subset U of N the condition $\phi|_U = \text{id}_U$ implies $\phi = \text{id}_N$. In this section we assume that a group Φ of diffeomorphisms of a manifold N acts quasi-analytically.

Definition 7. A foliation (M, F) defined by an N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ is called a (Φ, N) -foliation if for every $U_i \cap U_j \neq \emptyset$, $i, j \in J$, there exists an element $\phi \in \Phi$ satisfying the equality $\phi|_{f_j(U_i \cap U_j)} = \gamma_{ij}$.

Definition 8. A manifold B is called a (Φ, N) -manifold if its natural zero-dimensional foliation (B, F^0) is a (Φ, N) -foliation.

We emphasize that a group of automorphisms of a Cartan manifold (N, ξ) acts quasi-analytically on N .

Definition 9. If Φ is a subgroup of the Lie group of all automorphisms of a Cartan manifold (N, ξ) , then a (Φ, N) -foliation is called a Cartan (Φ, N) -foliation.

Theorem 2. *Let (M, F) be a complete Cartan (Φ, N) -foliation. Then*

- (i) *there exists a regular covering map $\kappa : \widehat{M} \rightarrow M$ such that the induced foliation $(\widehat{M}, \widehat{F})$, $\widehat{F} = \kappa^*F$, consists of the fibres of a locally trivial bundle $r : \widehat{M} \rightarrow B$, where (B, ξ) is a simply connected Cartan (Φ, N) -manifold with a complete Cartan geometry ξ ;*
- (ii) *an epimorphism $\chi : \pi_1(M) \rightarrow \Psi$ of the fundamental group $\pi_1(M)$ onto a subgroup Ψ of the automorphism group $\text{Aut}(B, \xi)$ of the Cartan manifold (B, ξ) is defined in such a way that Ψ is isomorphic to the deck transformation group of the covering $\kappa : \widehat{M} \rightarrow M$;*

(iii) for all points $y \in M$ and $z \in \kappa^{-1}(y)$, the restriction $\kappa|_{\widehat{L}} : \widehat{L} \rightarrow L$ to the leaf $\widehat{L} = \widehat{L}(z)$ of the foliation $(\widehat{M}, \widehat{F})$ is a regular covering map onto the leaf $L = L(y)$ of the foliation (M, F) , the group of the deck transformations is isomorphic to the stationary subgroup Ψ_v of Ψ at the point $v = r(z) \in B$, and Ψ_v is isomorphic to the holonomy group $\Gamma(L, y)$ of the leaf L .

Moreover, (M, F) has an attractor (resp., a global attractor) \mathcal{M} if and only if the group Ψ has an attractor (resp., a global attractor) \mathcal{K} , and $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$. Besides, \mathcal{M} is a minimal set of the foliation (M, F) if and only if \mathcal{K} is a minimal set of the group Ψ .

Definition 10. The group Ψ appearing in Theorem 2 is called the global holonomy group of the foliation (M, F) .

Corollary 1. The transversal structure of a global attractor of a foliation (M, F) satisfying Theorem 2 is completely determined by the structure of the corresponding global attractor of its global holonomy group Ψ .

Theorem 3. Let (B, ξ) be a simply connected Cartan manifold. Let Ψ be a countable subgroup of the Lie group $\text{Aut}(B, \xi)$ of all automorphisms of (B, ξ) . Suppose that Ψ has an attractor (resp., a global attractor). Then Ψ may be realized as the global holonomy group of a certain Cartan (Φ, N) -foliation admitting an attractor (resp., a global attractor).

Theorems 2 and 3 show that the problems of the existence and the structure description of attractors (resp. global attractors) of complete Cartan (Φ, N) -foliations are equivalent to the similar problems for countable automorphism groups of complete Cartan geometries on simply connected manifolds.

5.2 Ehresmann connection for foliations

The notion of an Ehresmann connection for foliations was introduced by Blumenthal and Hebda in [2]. We use the terminology from [14]. Let (M, F) be a smooth foliation of codimension $q \geq 1$ and \mathfrak{M} be a q -dimensional transversal distribution on M . All maps considered here are assumed to be piecewise smooth. The curves in the leaves of the foliation are called vertical; the distribution \mathfrak{M} and its integral curves are called horizontal.

A map $H : I_1 \times I_2 \rightarrow M$, where $I_1 = I_2 = [0, 1]$, is called a *vertical-horizontal homotopy* if for each fixed $t \in I_2$, the curve $H|_{I_1 \times \{t\}}$ is horizontal, and for each fixed $s \in I_1$, the curve $H|_{\{s\} \times I_2}$ is vertical. The pair of curves $(H|_{I_1 \times \{0\}}, H|_{\{0\} \times I_2})$ is called the base of H .

A pair of curves (σ, h) with a common starting point $\sigma(0) = h(0)$, where $\sigma : I_1 \rightarrow M$ is a horizontal curve, and $h : I_2 \rightarrow M$ is a vertical curve, is called admissible. If for each admissible pair of curves (σ, h) there exists a vertical-horizontal homotopy with the base (σ, h) , then the distribution \mathfrak{M} is called an Ehresmann connection for the foliation (M, F) . Note that there exists at most one vertical-horizontal homotopy with a given base. Let H be a vertical-horizontal homotopy with the base (σ, h) . We say that $\tilde{\sigma} = H|_{I_1 \times \{1\}}$ is the result of the translation of the horizontal curve σ along the vertical curve h with respect to the Ehresmann connection \mathfrak{M} . Similarly the curve $\tilde{h} = H|_{\{1\} \times I_2}$ is called the translation of the curve h along σ with respect to \mathfrak{M} . We use the denotation $\sigma \xrightarrow{h} \tilde{\sigma}$ and $h \xrightarrow{\sigma} \tilde{h}$.

5.3 Proof of Theorem 2

Let (M, F) be a complete Cartan (Φ, N) -foliation of codimension q . Then there exists a transversal q -dimensional distribution \mathfrak{M} such that (M, F) is \mathfrak{M} -complete. According to

[14, Prop. 2], \mathfrak{M} is an Ehresmann connection for (M, F) . Applying [16, Th. 2] to the (Φ, N) -foliation (M, F) , we see that there exists a regular covering $\kappa : \widehat{M} \rightarrow M$ such that the induced foliation $(\widehat{M}, \widehat{F})$, $\widehat{F} = \kappa^*F$, is made up of fibres of the locally trivial bundle $r : \widehat{M} \rightarrow B$ over a simply connected smooth manifold B . Besides, there is the induced group Ψ of diffeomorphisms of B and an epimorphism

$$\chi : \pi_1(M, x) \rightarrow \Psi$$

of the fundamental group $\pi_1(M, x)$ of M , $x \in M$, onto Ψ . Further, the group of deck transformations of the covering \widehat{M} is isomorphic to the group Ψ . Note that the foliation $(\widehat{M}, \widehat{F})$ is a Cartan foliation, and it is $\widehat{\mathfrak{M}}$ -complete with respect to the induced distribution $\widehat{\mathfrak{M}}$, where $\widehat{\mathfrak{M}} = \kappa^*\mathfrak{M}$. Observe that the transversal Cartan geometry of the foliation $(\widehat{M}, \widehat{F})$ induces a complete Cartan geometry η on B . The group Ψ is a countable subgroup of $\text{Aut}(B, \eta)$ of the Lie group of all automorphisms of (B, η) . We have proven the statements (i) and (ii) of Theorem 2. The statement (iii) of Theorem 2 follows from the similar statement [16, Th. 2].

Assume that there exists an attractor (resp., a global attractor) \mathcal{M} of the foliation (M, F) . It is easy to check that $\mathcal{K} = r(\kappa^{-1}(\mathcal{M}))$ is an attractor (resp., a global attractor) of the group Ψ . Conversely, let \mathcal{K} be an attractor (resp., a global attractor) of the group Ψ . It is easy to see that $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$ is an attractor (resp., a global attractor) of the foliation (M, F) . Finally, \mathcal{K} is a minimal set of the group Ψ if and only if $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$ is a minimal set of the foliation (M, F) . \square

5.4 The suspended foliation

Let B and T be connected smooth manifolds of dimensions $n - q$ and q , respectively. Let $\rho : \pi_1(B, b) \rightarrow \text{Diff}(T)$ be a homomorphism from the fundamental group $G = \pi_1(B, b)$ to the group of diffeomorphisms of the manifold T . We consider the universal covering space \widehat{B} of B as a right G -space. Let us define the left action of the group G on the product $\widehat{B} \times T$ by the rule

$$\Phi : G \times \widehat{B} \times T \rightarrow \widehat{B} \times T, \quad (g, (\widehat{b}, t)) \mapsto (\widehat{b}.g^{-1}, \rho(g)t),$$

where $(\widehat{b}, t) \in \widehat{B} \times T$. Then we obtain a smooth n -dimension quotient manifold $M = \widehat{B} \times_G T$ with a foliation F of codimension q . The leaves of the foliation (M, F) are images of the leaves of the foliation $F_{tr} = \{\widehat{B} \times \{t\} \mid t \in T\}$ under the quotient map $f : \widehat{B} \times T \rightarrow M$, which is a regular covering. The foliation (M, F) is called *the suspension* and it is denoted by $\text{Sus}(T, B, \rho)$. One says that (M, F) is obtained from the *suspension of the homomorphism* ρ .

The images of the leaves of the foliation $F_{tr}^t = \{\{\widehat{b}\} \times T \mid \widehat{b} \in \widehat{B}\}$ on the product manifold $\widehat{B} \times T$ form a locally trivial bundle $p : M \rightarrow B$, which is transversal to the foliation (M, F) .

5.5 Proof of Theorem 3

Let (T, η) be a simply connected q -dimensional Cartan manifold and let $\text{Aut}(T, \eta)$ be the Lie group of all automorphisms of (T, η) . Assume that Ψ is a countable subgroup of the group $\text{Aut}(T, \eta)$ and assume that Ψ admits an attractor $\mathcal{K} \subset T$.

First we suppose that Ψ has a finite set of generators $\{\psi_1, \dots, \psi_m\}$. Denote by \mathbb{S}_m^2 the 2-dimensional sphere with m handles. As it is known, the fundamental group of \mathbb{S}_m^2 may be represented in the form

$$\pi_1(\mathbb{S}_m^2) = \langle a_1, \dots, a_m, b_1, \dots, b_m \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_m b_m a_m^{-1} b_m^{-1} = 1 \rangle.$$

Let $B = \mathbb{S}_m^2$ and define the homomorphism $\rho : \pi_1(B, b) \rightarrow \text{Aut}(T, \eta)$ by the conditions

$$\rho(a_i) = \psi_i, \quad \rho(b_i) = \text{id}_T, \quad i = 1, \dots, m,$$

here id_T is the neutral element of the group Ψ . Then we consider the suspended foliation $(M, F) = \text{Sus}(T, B, \rho)$. The foliation (M, F) is a Cartan foliation of codimension q covered by the locally trivial bundle $\mathbb{R}^2 \times T \rightarrow T$, and Ψ is its global holonomy group. The manifold M is the total space of the locally trivial bundle $p : M \rightarrow B$ with the standard leaf T over the base B . From the compactness of the manifolds T and B follows the compactness of M .

Suppose now that the group $\Psi \subset \text{Aut}(T, \eta)$ has a countable set of generators $\{\psi_i \mid i \in \mathbb{N}\}$. Let T_∞ be the plain with the pitched countable subset $\{(n, 0) \mid n \in \mathbb{N}\}$. Then,

$$\pi_1(T_\infty) = \langle a_n \mid n \in \mathbb{N} \rangle.$$

The assignment

$$\rho_\infty(a_n) = \psi_n, \quad n \in \mathbb{N},$$

defines the homomorphism

$$\rho_\infty : \pi_1(B, b) \rightarrow \text{Aut}(T, \eta).$$

The suspended foliation $(M, F) = \text{Sus}(T, B, \rho_\infty)$ is a Cartan foliation of codimension q with the global holonomy group Ψ .

By the assumption, the group Ψ has an attractor $\mathcal{K} \subset T$. Let us consider the regular covering $f : \mathbb{R}^2 \times T \rightarrow M$ and the projection onto the second factor $r : \mathbb{R}^2 \times T \rightarrow T$. Then $\mathcal{M} = f(r^{-1}(\mathcal{K}))$ is an attractor of the foliation (M, F) . It is easy to see that \mathcal{M} is a global attractor (resp., a minimal set) of (M, F) if and only if \mathcal{K} is a global attractor (resp., a minimal set) of Ψ . \square

6 Reductive Cartan foliations as foliations with transversal linear connections

6.1 Foliations with transversely linear connection

Let $(N^{(i)}, \nabla^{(i)})$, $i = 1, 2$, be manifolds with linear connections $\nabla^{(i)}$. A diffeomorphism $f : N^{(1)} \rightarrow N^{(2)}$ is called an isomorphism of the connections $\nabla^{(1)}$ and $\nabla^{(2)}$ if

$$f_*(\nabla_X^{(1)} Y) = \nabla_{f_* X}^{(2)} f_* Y$$

for all vector fields $X, Y \in \mathfrak{X}(M^{(1)})$, where f_* is the differential of f .

Definition 11. Suppose that an N -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ defines the foliation (M, F) . If on the manifold N a linear connection ∇ is given such that each local diffeomorphism γ_{ij} is an isomorphism of the linear connections induced by ∇ on open subsets $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$, then (M, F) is called a *foliation with a transversely linear connection* given by the (N, ∇) -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$. It is said that (M, F) is modeled on the manifold with the linear connection (N, ∇) . We stress that the connection ∇ on N may have a nonzero torsion.

Remark 1. A linear connection ∇ on N defines an effective reductive Cartan geometry ξ on N of type (G, H) , where $H = \text{GL}(q, \mathbb{R})$ and G is the semi-direct product of the Lie groups $\text{GL}(q, \mathbb{R})$ and \mathbb{R}^q . The Lie group G is interpreted as the Lie group $\text{Aff}(\mathbb{R}^q)$ of all affine transformations of the space \mathbb{R}^q , and H is its stationary subgroup. Thus a foliation (M, F) with a transversal linear connection is a reductive Cartan foliation.

6.2 A linear connection associated with a reductive Cartan geometry

Let M be a smooth manifold and let V be a vector space. A map $\sigma : M \rightarrow V$ is called a V -valued function on M . Let $\mathfrak{F}_V(M)$ denote the space of all V -valued smooth functions on M . Denote by σ_{*x} the differential of the map σ at a point $x \in M$. The action of a vector field $X \in \mathfrak{X}(M)$ on $\mathfrak{F}_V(M)$ is defined by the equality

$$(X\sigma)(x) = \sigma_{*x}(X_x) \quad \forall \sigma \in \mathfrak{F}_V(M), \forall x \in M.$$

The map

$$X : \mathfrak{F}_V(M) \rightarrow \mathfrak{F}_V(M), \quad \sigma \mapsto X\sigma$$

is \mathbb{R} -linear.

Let (M, F) be a reductive Cartan foliation modelled on a reductive Cartan geometry $\xi = (P(N, H), \omega)$ of type (G, H) , where ξ is reductive with respect to the decomposition of \mathfrak{g} into the direct sum of vector spaces $\mathfrak{g} = \mathfrak{h} \oplus V$, here \mathfrak{g} and \mathfrak{h} are the Lie algebras of the Lie groups G and H , and V is an $\text{Ad}_G(H)$ -invariant vector subspace of \mathfrak{g} . The effectivity of the Cartan geometry ξ is not assumed.

Consider the smooth q -dimensional distribution

$$Q = \{Q_u = \omega_u^{-1}(V) \mid u \in P\}$$

on P . The $\text{Ad}_G(H)$ -invariance of the vector subspace V of \mathfrak{g} implies the H -invariance of the distribution Q . Thus Q is an H -connection on the principal H -bundle $P(N, H)$. Let $p : P \rightarrow N$ be the projection.

Recall that a \mathfrak{g} -valued function $h \in \mathfrak{F}(P)$ is called H -equivariant if it satisfies the equality

$$h(ua) = \text{Ad}(a^{-1})h(u) \quad \forall u \in P, \forall a \in H.$$

Denote by $\mathfrak{F}_{H,V}(P)$ the set of H -equivariant V -valued functions on P . Let $\mathfrak{X}_{H,Q}(P)$ be the set of all H -invariant smooth vector fields on P tangent to Q . Note that $\mathfrak{F}_{H,V}(P)$ and $\mathfrak{X}_{H,Q}(P)$ are modules over the algebra of functions $\mathfrak{F}(P)$. For each vector field $X \in \mathfrak{X}(M)$ there exists a unique vector field $Y \in \mathfrak{X}_{H,Q}(P)$ such that $p_*(Y) = X$. We denote this vector field by \widehat{X} . If $h \in \mathfrak{F}(M)$, then let $\widehat{h} = h \circ p \in \mathfrak{F}(P)$

Consider the map

$$\alpha : \mathfrak{X}_{H,Q}(P) \rightarrow \mathfrak{F}_{H,V}(P), \quad Y \mapsto \omega(Y) \quad \forall Y \in \mathfrak{X}_{H,Q}(P), \quad (4)$$

which is an isomorphism of the modules over the algebra of smooth functions $\mathfrak{F}(P)$. Every vector field $Y \in \mathfrak{X}_{H,Q}(P)$ defines the \mathbb{R} -linear map

$$D_Y : \mathfrak{F}_{H,V}(P) \rightarrow \mathfrak{F}_{H,V}(P), \quad \sigma \mapsto Y\sigma \quad \forall \sigma \in \mathfrak{F}_{H,V}(P), \quad (5)$$

which satisfies the equality

$$D_Y(f\sigma) = (Yf)\sigma + fD_Y(\sigma) \quad \forall \sigma \in \mathfrak{F}_{H,V}(P), \forall f \in \mathfrak{F}(P). \quad (6)$$

We obtain

Proposition 3. *Let $\xi = (P(N, H), \omega)$ be a reductive Cartan geometry with the projection $p : P \rightarrow N$, and $\alpha : \mathfrak{X}_{H,Q}(P) \rightarrow \mathfrak{F}_{H,V}(P)$ be the isomorphism defined by (4). Then the equality*

$$\nabla_X Y = p_*(\alpha^{-1}(D_{\widehat{X}}\alpha(\widehat{Y}))) \quad \forall X, Y \in \mathfrak{X}(N) \quad (7)$$

defines a linear connection ∇ on the manifold N .

Definition 12. The linear connection ∇ defined in Proposition 3 by a reductive Cartan geometry $\xi = (P(N, H), \omega)$ is called the linear connection associated with ξ .

Remark 2. Lotta [8] proved that the statement by Sharpe [11, Lem. 6.4] about the existence of a mutation for any reductive Cartan geometry ξ of type $\mathfrak{g}/\mathfrak{h}$ with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus V$ to a Cartan geometry ξ' of type $\mathfrak{g}'/\mathfrak{h}$ with the decomposition $\mathfrak{g}' = \mathfrak{h} \oplus \mathfrak{p}$, where \mathfrak{p} is a subalgebra in the Lie algebra \mathfrak{g}' such that $[\mathfrak{p}, \mathfrak{p}] = 0$, does not hold true generally. Consequently the structure of the reductive Cartan geometries is complicated than it is stated in [11]. According to Proposition 3 and Remark 1, any reductive Cartan geometry of type $\mathfrak{g}/\mathfrak{h}$ with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus V$ induces a reductive Cartan geometry η of type $\mathfrak{g}'/\mathfrak{h}'$ with the decomposition $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{p}$, where $\mathfrak{p} = \mathbb{R}^q$, $q = \dim(V) = \dim(\mathfrak{p})$, is a subalgebra of the Lie algebra \mathfrak{g}' , which is the Lie algebra of $\text{GL}(q, \mathbb{R})$, hence $[\mathfrak{p}, \mathfrak{p}] = 0$. In general $\mathfrak{h} \neq \mathfrak{h}'$ and η is not a mutation of ξ .

6.3 A transversal linear connection associated to a reductive Cartan foliation

Theorem 4. *Each reductive Cartan foliation is a foliation with a transversal linear connection.*

Corollary 2. *The holonomy group of each leaf of a reductive Cartan foliation is linearizable.*

Remark 3. The topology of a foliated manifold M is invariant under the change of the transversal geometric structure of the foliation (M, F) by any other geometric structure on this foliation. Hence, due to Theorem 4, any topological problem for a reductive Cartan foliation may be reduced to a similar problem for a foliation with a transversal linear connection.

Proof of Theorem 4.

Lemma 1. *Let $\xi = (P(N, H), \omega)$ and $\xi' = (P'(N', H), \omega')$ be two reductive Cartan geometries of the same type (G, H) with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus V$, having the associated linear connections ∇ and ∇' , respectively. If $\Gamma : P \rightarrow P'$ is an isomorphism of the Cartan geometries ξ and ξ' , then its projection $\gamma : N \rightarrow N'$ is an isomorphism of the associated linear connections ∇ and ∇' .*

Proof. All objects related to the geometry ξ' will be denoted with the prime. Since $\Gamma : P \rightarrow P'$ is an isomorphism of the Cartan geometries ξ and ξ' , it holds $\Gamma^*\omega' = \omega$. Hence, $\Gamma_*Q = Q'$. Consequently the isomorphisms of the vector spaces

$$\Gamma^* : \mathfrak{F}_{H,V}(P') \rightarrow \mathfrak{F}_{H,V}(P), \quad \sigma \mapsto \sigma \circ \Gamma, \quad \Gamma_* : \mathfrak{X}_{H,Q}(P) \rightarrow \mathfrak{X}_{H,Q'}(P'), \quad Y \mapsto \Gamma_*Y$$

are induced. These isomorphisms satisfy the equality

$$(\Gamma^*)^{-1} \circ \alpha = \alpha' \circ \Gamma_*,$$

where α and α' were defined above. Therefore we get

$$\widehat{\gamma_* \nabla_X Y} = \Gamma_* \alpha^{-1}(D_{\widehat{X}} \alpha(\widehat{Y})) = \alpha'^{-1} \circ (\Gamma^*)^{-1}(D_{\widehat{X}} \alpha(\widehat{Y})) = \alpha'^{-1}(D_{\Gamma_* \widehat{X}} \alpha'(\Gamma_* \widehat{Y})) = \widehat{\nabla'_{\gamma_* X} \gamma_* Y},$$

that means that,

$$\widehat{\gamma_* \nabla_X Y} = \widehat{\nabla'_{\gamma_* X} \gamma_* Y}.$$

Since the map $\mathfrak{X}(N) \rightarrow \mathfrak{X}_{H,Q}(P) : X \mapsto \widehat{X}$ is bijective, it holds

$$\gamma_* \nabla_X Y = \nabla'_{\gamma_* X} \gamma_* Y$$

for all $X, Y \in \mathfrak{X}(N)$. Thus $\gamma_* : N \rightarrow N'$ is an isomorphism of the manifolds with linear connections (N, ∇) and (N', ∇') . \square

Let (M, F) be a reductive Cartan foliation modelled on a reductive Cartan geometry $\xi = (P(N, H), \omega)$. Suppose that (M, F) is defined by an (N, ξ) -cocycle $\{U_i, f_i, \Gamma_{ij}\}_{i,j \in J}$. According to Proposition 3, the associated linear connection ∇ is defined on the manifold N . Since every transformation Γ_{ij} is a local isomorphism of the corresponding reduced Cartan geometries, by Lemma 1, its projection γ_{ij} is an isomorphism of the induced linear connections on the open subsets of N . This means that (M, F) is a foliation with a transversal linear connection given by the (N, ∇) -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$. This proves Theorem 4. \square

7 Attractors and minimal sets

7.1 Statement of the results

Recall that a *minimal set* of a foliation on a manifold M is a nonempty closed subset in M that consists of a union of leaves and has no proper subset satisfying this condition. Minimal sets for transformation groups are defined in a similar way.

Remark 4. A nonempty closed saturated subset \mathcal{M} in M is an attractor and a minimal set of the foliation (M, F) if and only if there exists an open neighbourhood $\mathcal{U} = \mathcal{U}(\mathcal{M})$ of the subset \mathcal{M} such that $\overline{L} \supset \mathcal{M}$ for any leaf $L \subset \mathcal{U}$.

Definition 13. Let Φ be a group of homeomorphisms of a topological space N . We call a point $x \in N$ a local limit point of the group Φ if there exists a neighbourhood U of the point x such that the closure of the orbit $\Phi \cdot y$ of any point $y \in U$, $y \notin \Phi \cdot x$, contains x . If, moreover, $U = N$, then x is called the limit point of the group Φ .

Remark 5. The origin $0 \in \mathbb{R}^q$ is a fixed point for any subgroup Φ of the linear group $\text{GL}(q, \mathbb{R})$. Consequently the origin is the only possible local limit point of Ψ .

Consider a reductive Cartan foliation (M, F) of codimension q and a transversal q -dimensional distribution \mathfrak{M} . Let $\mathcal{R}(M, H)$ be the principal H -bundle with the H -invariant connection Q , defined by above the transversal reductive Cartan geometry ξ and let $\mathcal{R}'(M, \text{GL}(q, \mathbb{R}))$ be the principal $\text{GL}(q, \mathbb{R})$ -bundle with the $\text{GL}(q, \mathbb{R})$ -invariant connection Q' given by the transversal geometry (N, ∇) . Note that an integral curve σ of the distribution \mathfrak{M} is a geodesic of the connection Q if and only if σ is a geodesic of the connection Q' . Such curves are called \mathfrak{M} -geodesics. We observe that the completeness of a reductive Cartan foliation is equivalent to the existence of a transversal distribution \mathfrak{M} such that every maximal \mathfrak{M} -geodesic is defined on the whole real line.

We give now sufficient conditions for the existence of an attractor (and a global attractor) that is also a minimal set.

Theorem 5. *Let (M, F) be a reductive Cartan foliation of codimension q . Suppose that there exists a leaf L such that its linear holonomy group $D\Gamma(L, x)$ at some point $x \in M$ has a limit point. Then:*

- (1) *The closure of the leaf $\mathcal{M} = \overline{L}$ is an attractor and a minimal set of the foliation (M, F) .*
- (2) *If, moreover, (M, F) is a complete Cartan foliation with respect to a transversal q -dimensional distribution \mathfrak{M} and the leaf L can be connected with every leaf L_α of (M, F) by a smooth \mathfrak{M} -geodesic, then $\mathcal{M} = \overline{L}$ is a global attractor and a minimal set of this foliation.*

Corollary 3. *Let (M, F) be a complete reductive Cartan foliation of codimension q . Suppose that the curvature and the torsion of the associated transversal linear connection are zero. If there exists a leaf L with a linear holonomy group admitting a limit point, then:*

- (1) *the closure of the leaf $\mathcal{M} = \overline{L}$ is a global attractor and a minimal set of the foliation (M, F) ;*
- (2) *there exists a regular covering map $\kappa : \widehat{M} \rightarrow M$ such that the foliation (M, F) is covered by the trivial bundle $r : \widehat{M} = L_0 \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ over the space \mathbb{R}^q , where L_0 is a manifold diffeomorphic to every leaf without holonomy;*
- (3) *the global holonomy group Ψ of (M, F) is a subgroup of the affine Lie group $\text{Aff}(\mathbb{R}^q)$, it has a global attractor \mathcal{K} , and $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$.*

Corollary 4. *Let (M, F) be a reductive Cartan foliation of codimension q . Suppose that there exists a leaf L such that its linear holonomy group contains an element defined by a matrix of the form $B \cdot D$, where $B \in O(q)$ and $D = \text{diag}(d_1, \dots, d_q)$ with $|d_i| < 1$ for $1 \leq i \leq q$. Then (M, F) has an attractor $\mathcal{M} = \overline{L}$ which is a minimal set.*

Recall that a smooth foliation (M, F) is called *proper* if each its leaf is an embedded submanifold in M . A leaf L is called *closed* if it is a closed subset in M . As it is known, (see e.g. [13]), any minimal set of a foliation is either a closed leaf, or the closure of a non-proper leaf. This and Theorem 5 imply the following statements:

Corollary 5. *Let (M, F) be a reductive Cartan foliation. If there exists a proper leaf L with a linear holonomy group admitting a limit point, then the leaf L is closed and it is an attractor of the foliation (M, F) .*

Corollary 6. *Let (M, F) be a complete reductive Cartan foliation. Suppose that the curvature and the torsion of the associated transversal linear connection are zero. If there exists a proper leaf L with a linear holonomy group admitting a limit point, then L is a unique closed leaf, and L is a global attractor of the foliation.*

7.2 The existence of an attractor which is a minimal set

Denote by $\Gamma(L, x)$ the germ holonomy group of an arbitrary leaf $L = L(x)$ at a point x ; $\Gamma(L, x)$ consists of germs of certain holomorphic diffeomorphisms ψ of a transversal q -dimensional disk D_x^q at the point [13]. Let $D\Gamma(L, x)$ be the linear holonomy group consisting of the differentials $\psi_{*x} : \mathfrak{M}_x \rightarrow \mathfrak{M}_x$, where $\mathfrak{M}_x = T_x D_x^q$.

Suppose that the linear holonomy group $D\Gamma(L, x)$ of a leaf $L = L(x)$ has a limit point. There exists a submersion $f : U \rightarrow V$ from an (N, ∇) -cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ defining the foliation (M, F) such that $x \in U$. Let $v = f(x) \in V$. Denote by \mathcal{H} the holonomy pseudogroup generated by the local automorphisms γ_{ij} , $i, j \in J$ of the transversal manifold with the linear connection (N, ∇) . Let

$$\mathcal{H}_v = \{\phi \in \mathcal{H} \mid \phi(v) = v\}.$$

We consider the linear holonomy group $D\Gamma(L, x)$ of the leaf L as the group of linear transformations

$$\mathcal{H}_{*v} = \{\phi_{*v} \mid \phi \in \mathcal{H}_v\}$$

of the tangent space $T_v N$ at the point $v \in V \subset N$.

It is well known that the group $\Gamma(L, x)$ is isomorphic to the group of germs of local automorphisms $\phi \in \mathcal{H}_v$ at the point v . Since a linear connection defines a G -structure of the first order, there exists an isomorphism

$$d_v : \Gamma(L, x) \rightarrow D\Gamma(L, x) \cong \mathcal{H}_{*v} : \{\phi\}_v \mapsto \phi_{*v} \quad \forall \phi \in \mathcal{H}_v,$$

assigning to a germ $\{\phi\}_v$ at the point v the differential ϕ_{*v} .

Let W_0 be a normal neighbourhood of the origin in the tangent space $T_v N$. The exponential map

$$\text{Exp}_v : W_0 \rightarrow W : X \mapsto \gamma_X(1)$$

defined by the linear connection ∇ is a diffeomorphism onto an open neighbourhood W of v in N . By the property of the exponential map, each transformation $\phi \in \mathcal{H}_v$ satisfies the following equality

$$\phi \circ \text{Exp}_v = \text{Exp}_v \circ \phi_{*v} \quad (8)$$

in a neighbourhood of the origin in $T_v N$, where the both sides of Equality (8) are defined. Note that the holonomy pseudogroup \mathcal{H} of a Cartan foliation is quasi-analytical, i.e., if a transformation $\phi \in \mathcal{H}_v$ equals the identity on an open subset in N , then it coincides with the identity transformation everywhere in the domain of its definition. Since the differential ϕ_{*v} of each $\phi \in \mathcal{H}_v$ is defined on the whole tangent space $T_v N$, Equality (8) allows us to extend each local automorphism $\phi \in \mathcal{H}_v$ to the whole neighbourhood W of the point v . Since \mathcal{H} is quasi-analytical, this extension is defined uniquely. Thus we assume that Equality (8) holds on W_0 .

By the assumption, the group \mathcal{H}_{*v} has a limit point, hence this point is the origin in $T_v N$. Therefore for any vector $Y \in T_v N$ there exists a sequence $\phi_k \in \mathcal{H}_v$ such that $(\phi_k)_{*v}(Y) \rightarrow 0$ as $k \rightarrow +\infty$. Without loss generality we assume that $(\phi_k)_{*v}(Y) \in W_0$ for any $Y \in W_0$. We introduce the notation $U_0 = f^{-1}(W)$ and

$$\mathcal{U} = \bigcup_{L_\alpha \in \mathcal{F}, L_\alpha \cap U_0 \neq \emptyset} L_\alpha.$$

Consider any leaf $L_\alpha \subset \mathcal{U}$. Let $x_\alpha \in L_\alpha \cap U_0$. Then $y = f(x_\alpha) \in W$. Since the map $\text{Exp}_v|_{W_0} : W_0 \rightarrow W$ is a diffeomorphism, for each $y \in W$ there exists a vector $Y = \text{Exp}_v^{-1}(y) \in W_0$. From (8) it follows that $\phi_k(y) \rightarrow v$ as $k \rightarrow +\infty$. We emphasize that the set $f^{-1}(\{\phi_k(y) \mid k \in \mathbb{N}\})$ is contained in $L_\alpha \cap U$. Consequently, $L \subset \overline{L_\alpha}$. Thus it holds

$$\mathcal{M} = \overline{L} \subset \overline{L_\alpha} \quad \forall L_\alpha \subset \mathcal{U},$$

i.e. \mathcal{M} is an attractor with the basin \mathcal{U} .

Let us show that \mathcal{M} is a minimal set of the foliation (M, F) . Let L_α be any leaf of the foliation contained in \mathcal{M} , i.e., $L_\alpha \subset \mathcal{M}$. Then the closure $\overline{L_\alpha}$ satisfies $\overline{L_\alpha} \subset \overline{L} = \mathcal{M}$. Since $L_\alpha \subset \mathcal{U}$, from the above it follows that $\overline{L_\alpha} \supset \overline{L}$. Consequently, $\overline{L_\alpha} = \overline{L} = \mathcal{M}$. This means that \mathcal{M} is a minimal set of the foliation (M, F) . Thus the statement (1) of Theorem 5 is proved.

7.3 The existence of a global attractor which is a minimal set

We use the notations from Section 3.3. Let (M, F) be a \mathfrak{M} -complete reductive Cartan foliation. Let $\widetilde{\mathfrak{M}} = \pi^*\mathfrak{M}$, where $\pi : \mathcal{R} \rightarrow M$ is the associated H -bundle. Let $g_{\mathcal{R}}$ be a Riemannian metric on \mathcal{R} . Consider an Euclidean metric d_0 on the vector space \mathfrak{g} such that \mathfrak{h} and V are orthogonal

subspaces. Let $Z = Z_{\mathcal{F}} \oplus Z_{\widetilde{\mathfrak{M}}}$ be the decomposition of a vector field $Z \in \mathfrak{X}(\mathcal{R})$ with respect to the decomposition $T_u\mathcal{R} = T_u\mathcal{F} \oplus \widetilde{\mathfrak{M}}_u$, $u \in \mathcal{R}$. The equality

$$d(X, Y) = g_{\mathcal{R}}(X_{\mathcal{F}}, Y_{\mathcal{F}}) + d_0(\omega(X), \omega(Y)), \forall X, Y \in \mathfrak{X}(\mathcal{R}),$$

defines a Riemannian metric d on \mathcal{R} , and d is transversally projectible with respect to the lifted foliation $(\mathcal{R}, \mathcal{F})$.

Let $E_i, i = 1, \dots, \dim \mathfrak{g}$ be a basis of the Lie algebra \mathfrak{g} . Denote by X_i the vector field from $\mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $\omega(X_i) = E_i$. Let $\widehat{\nabla}$ be the Levi-Civita connection of the Riemannian manifold (\mathcal{R}, d) . It can be checked directly that the equality

$$\widetilde{\nabla}_Y Z = Y(Z^i)X_i + \widehat{\nabla}_Y Z_{\mathcal{F}}, \quad (9)$$

where $Z = Z_{\mathcal{F}} \oplus Z_{\widetilde{\mathfrak{M}}}$, $Z_{\widetilde{\mathfrak{M}}} = Z^i X_i \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$, $Z_{\mathcal{F}} \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$, and $Y \in \mathfrak{X}(\mathcal{R})$, defines a linear connection $\widetilde{\nabla}$ in \mathcal{R} . The connection $\widetilde{\nabla}$ is in general not torsion free, and it holds $\widetilde{\nabla}d = 0$. Every vector field $X \in \mathfrak{X}_{\widetilde{\mathfrak{M}}}(\mathcal{R})$ such that $\omega(X) = \text{const} \in \mathfrak{g}$, is parallel with respect to $\widetilde{\nabla}$, hence its integral curves are geodesic lines of $\widetilde{\nabla}$.

Let

$$Q = \{Q_u = \omega_u^{-1}(V) \mid u \in \mathcal{R}\}$$

be the H -connection in \mathcal{R} and let

$$\mathfrak{N} = \{\mathfrak{N}_u = Q_u \cap \widetilde{\mathfrak{M}}_u \mid u \in \mathcal{R}\}.$$

Geodesics of $\widetilde{\nabla}$ that are integral curves of the distribution \mathfrak{N} are called \mathfrak{N} -geodesics. Note that for any \mathfrak{N} -geodesic γ , the curve $\pi \circ \gamma$ is an \mathfrak{M} -geodesic.

As the Cartan foliation (M, F) is \mathfrak{M} -complete, the exponential map Exp_u , $u \in \mathcal{R}$, of $\widetilde{\nabla}$ is defined, in particular, on \mathfrak{N}_u . Therefore the map $\text{Exp}_x : \mathfrak{N}_x \rightarrow M$, $x = \pi(u) \in M$, satisfying the equality $\text{Exp}_x \circ \pi_{*u} = \pi \circ \text{Exp}_u$, is defined.

Since (M, F) admits a leaf $L = L(x_0)$ such that its linear holonomy group $D\Gamma(L, x_0)$ has a limit point, according to the proved statement (1) of Theorem 5, the closure $\mathcal{M} = \overline{L}$ is an attractor. Let $\mathcal{B} = \mathcal{B}(\mathcal{M})$ be its basin. Then there exists an open star neighbourhood V_0 of zero in \mathfrak{M}_{x_0} such that $\text{Exp}_{x_0}(V_0) \subset \mathcal{B}$.

By the assumptions, for any leaf L_α , there exists an \mathfrak{M} -geodesic $\sigma : [0, 1] \rightarrow M$ such that $x_0 = \sigma(0) \in L$ and $x = \sigma(1) \in L_\alpha$. Pick $u_0 \in \pi^{-1}(x_0)$, then there exists an \mathfrak{N} -geodesic $\gamma : [0, 1] \rightarrow \mathcal{R}$ which is a \mathfrak{N} -lift of σ starting at the point u_0 , i.e. $\gamma(0) = u_0$ and $\sigma = \pi \circ \gamma$. The \mathfrak{M} -completeness of (M, F) implies the existence of a vector $Y \in \mathfrak{N}_{u_0}$ such that $\gamma(s) = \text{Exp}_{u_0}(sY)$ for any $s \in [0, 1]$. Therefore the vector $X = \pi_{u_0}(Y)$ satisfies the relation $\sigma(s) = \text{Exp}_{x_0}(sY)$ for any $s \in [0, 1]$.

Since the linear holonomy group $D\Gamma(L, x_0)$ has a limit point, there exists an element $\phi_{*x_0} \in D\Gamma(L, x_0)$ for which $Y = \phi_{*x_0}(X) \in V_0$. There is a loop $h : [0, 1] \rightarrow L$ at the point x_0 such that φ is a local holonomic diffeomorphism of a transversal q -dimension disk $D_{x_0} = \text{Exp}_{x_0}(V_0)$ along h , and $\varphi(x_0) = x_0$. Let $\mathcal{L} = \mathcal{L}(u_0)$ be a leaf of the lifted foliation $(\mathcal{R}, \mathcal{F})$. Since $\pi_{\mathcal{L}} : \mathcal{L} \rightarrow L$ is a covering map, there exists a curve $\widetilde{h} : [0, 1] \rightarrow \mathcal{L}$ starting at $\widetilde{h}(0) = u_0$ and covering h , i.e., $\pi \circ \widetilde{h} = h$.

Recall that $\widetilde{\mathfrak{M}}$ is an Ehresmann connection for the foliation $(\mathcal{R}, \mathcal{F})$. Let $\gamma \xrightarrow{\widetilde{h}} \gamma^*$ and $\widetilde{h} \xrightarrow{\gamma} \widetilde{h}^*$ be the translations with respect to the Ehresmann connection $\widetilde{\mathfrak{M}}$. Consider the translations $\sigma \xrightarrow{h} \sigma^*$ and $h \xrightarrow{\sigma} h^*$ with respect to the Ehresmann connection \mathfrak{M} . Then \widetilde{h}^* is a curve in a leaf $\widetilde{\mathcal{L}}$ of $(\mathcal{R}, \mathcal{F})$ and h^* is a curve in a leaf L_α , with $\pi(\widetilde{\mathcal{L}}) = L_\alpha$ and $\pi \circ \widetilde{h}^* = h^*$. Note that

$\sigma^*(s) = \text{Exp}_{x_0}(sY) \in \mathcal{B}$ for all $s \in [0, 1]$. Therefore, $y = h^*(1) = \sigma^*(1) \in L_\alpha \cap \mathcal{B}$. Since \mathcal{B} is a saturated set, we have $L_\alpha \subset \mathcal{B}$. Thus, taking into attention the fact that L_α is an arbitrary leaf of (M, F) , we get $M = \mathcal{B}$, i.e. \mathcal{M} is a global attractor. This completes the proof of the statement (2) of Theorem 5. Theorem 5 is proved. \square

Proof of Corollary 3. Since the curvature and the torsion of the linear connection ∇ are equal to zero, (N, ∇) is a locally affine manifold. Therefore (M, F) is a transversally affine foliation. In other words, (M, F) is an $(\text{Aff}(\mathbb{R}^q), \mathbb{R}^q)$ -foliation. Applying Theorem 2 to the complete $(\text{Aff}(\mathbb{R}^q), \mathbb{R}^q)$ -foliation (M, F) , we see that there exists a regular covering map $\kappa : \widehat{M} \rightarrow M$ such that the leaves of the induced foliation $(\widehat{M}, \widehat{F})$ are fibres of a locally trivial bundle $r : \widehat{M} \rightarrow B$ over a simply connected affine manifold B .

Let \mathfrak{M} be a transversal distribution on M with respect to which the reductive Cartan foliation (M, F) is \mathfrak{M} -complete. Then \mathfrak{M} is an Ehresmann connection for (M, F) . This implies that $\widehat{\mathfrak{M}} = \kappa^*\mathfrak{M}$ is an Ehresmann connection for the foliation $(\widehat{M}, \widehat{F})$. In this case $\widehat{\mathfrak{M}}$ is an Ehresmann connection for the submersion $r : \widehat{M} \rightarrow M$ [2, Prop. 2]. Hence any geodesic on B admits an $\widehat{\mathfrak{M}}$ -lifts in \widehat{M} . Since every such lift $\widehat{\sigma}$ of a maximal geodesic σ from B is a maximal $\widehat{\mathfrak{M}}$ -geodesic in \widehat{M} , the canonical parameter on σ is defined on the real line. This means that the affine manifold B is complete. Thus B is a simply connected complete torsion free affine manifold with zero curvature tensor. Therefore B is the affine space \mathbb{R}^q . Every locally trivial bundle over a contractible base is trivial, hence we get the trivial bundle $r : \widehat{M} = L_0 \times \mathbb{R}^q \rightarrow \mathbb{R}^q$, and the manifold L_0 is diffeomorphic to any leaf of (M, F) with the trivial holonomy group.

Assume that there exists a leaf L of (M, F) such that its linear holonomy group has a limit point. According the statement (1) of Theorem 5, the closure $\mathcal{M} = \overline{L}$ is an attractor and a minimal set of (M, F) .

Since any two points in \mathbb{R}^q may be connected by a geodesic, using the Ehresmann connection $\widehat{\mathfrak{M}}$ we able to connect any two leaves of $(\widehat{M}, \widehat{F})$ by an $\widehat{\mathfrak{M}}$ -geodesic. Therefore for every leaf L_α of (M, F) there exists a \mathfrak{M} -geodesic connecting L with L_α . From this and the statement (2) of Theorem 5 it follows that \mathcal{M} is a global attractor of the foliation (M, F) . \square

8 Examples

First of all we stress that the foliations admitting transversally projectible Riemannian metrics do not admit attractors [14]. As it is known [14, Th. 4], Cartan foliations of type $\mathfrak{g}/\mathfrak{h}$, where the Lie algebra \mathfrak{h} is compactly embedded into the Lie algebra \mathfrak{g} , are Riemannian foliations, hence they also do not admit attractors. Example 1 below provides a complete transversely affine foliation that does not admit an attractor and that is not a Riemannian foliation. Example 2 shows that in the framework of Theorem 5, the situation $\mathcal{M} = \overline{L} = M$ is possible. According to Example 3, there exist non-complete transversal affine foliations with global attractors. In Example 4 we construct a transversally affine foliation with regular global attractor that illustrates Corollary 3. Examples of global attractors of transversally similar foliations are constructed in [14].

We denote by $f_A = \langle A, a \rangle$ the element of the affine group $\text{GL}(q, \mathbb{R}) \ltimes \mathbb{R}^q \cong \text{Aff}(\mathbb{R}^q)$, where $A \in \text{GL}(q, \mathbb{R})$ and $a \in \mathbb{R}^q$. It holds $\langle A, a \rangle x = Ax + a$ for any $x \in \mathbb{R}^q$ and $\langle A, a \rangle \circ \langle B, b \rangle = \langle AB, Ab + a \rangle$ for the composition of every two elements $\langle A, a \rangle, \langle B, b \rangle \in \text{Aff}(\mathbb{R}^q)$.

Example 1

Let $f_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the affine transformation of the plain \mathbb{R}^2 given by the matrix $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$

with respect to the canonical basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $B = \mathbb{S}^1$ be the unite circle.

We define the group homomorphism $\rho : \pi_1(\mathbb{S}^1, b) \rightarrow \text{Aff}(\mathbb{R}^2)$ by setting its value on the generator $1 \in \mathbb{Z} \cong \pi_1(\mathbb{S}^1, b)$ to $\rho(1) = f_A$. Then the suspended foliation $(M, F) = \text{Sus}(\mathbb{R}^2, B, \rho)$ is a transversely affine foliation. This foliation is \mathfrak{M} -complete, where \mathfrak{M} is the tangent distribution to the transversal locally trivial bundle $p : M \rightarrow \mathbb{S}^1$.

The foliation (M, F) is covered by the trivial bundle $r : \mathbb{R}^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It is easy to see that its global holonomy group equals to $\Psi = \langle f_A \rangle \cong \mathbb{Z}$. For each $v \in M$ there exists a point $z \in r(p^{-1}(v)) \in \mathbb{R}^2$. Note that the leaf $L = L(v)$ is compact and it is diffeomorphic to the circle if and only if $z \in \{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\} \cup \{(0, 0)\}$. Consequently, the global holonomy group Ψ has no attractors. A foliation (M, F) defined as above by a matrix A is Riemannian if and only if A belongs to the orthogonal group $O(2)$. In our case $A \notin O(2)$. Thus the complete transversely affine foliation (M, F) does not admit an attractor and it is not a Riemannian foliation.

Example 2.

Let, as above, e_1, e_2 be the canonical basis of the plain \mathbb{R}^2 . Let us consider real numbers $\lambda_0, \lambda_1, \lambda_2$ such that $0 < \lambda_i < 1$, $i = 0, 1, 2$. Let $f_0(x) = \lambda_0(x)$, $f_1(x) = \lambda_1(x - e_1) + e_1$, $f_2(x) = \lambda_2(x - e_2) + e_2 \quad \forall x \in \mathbb{R}^2$.

Denote by $B = \mathbb{S}_3^2$ the sphere with three handles. For its fundamental group we have

$$\pi_1(B) = \langle a_j, b_j \mid j = 0, 1, 2; a_0 b_0 a_0^{-1} b_0^{-1} a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle .$$

Define the group homomorphism $\rho : \pi_1(B) \rightarrow \text{Diff}(\mathbb{R}^2)$ assuming that

$$\rho(b_j) = \text{id}_{\mathbb{R}^2}, \quad \rho(a_j) = f_j, \quad j = 0, 1, 2.$$

We get the suspended foliation $(M, F) = \text{Sus}(\mathbb{R}^2, \mathbb{S}_3^2, \rho)$ on the noncompact 4-dimensional manifold M , which is the total space of a locally trivial bundle over B with the standard fibre \mathbb{R}^2 .

By [14, Prop. 16], the orbit $\Psi \cdot x$ of any point $x \in \mathbb{R}^2$ is dense in \mathbb{R}^2 . Thus (M, F) is a transversely affine foliation, and M is its minimal set.

Example 3.

Denote by 0_q the origin in \mathbb{R}^q . Suppose that $q \geq 3$. Consider the submanifold $\widehat{M} = \mathbb{R}^q \setminus \{0_q\}$ of \mathbb{R}^q . Let $(\widehat{M}, \widehat{F})$ be the simple foliation defined by the submersion

$$r : \widehat{M} \rightarrow \mathbb{R}^1, \quad (x_1, \dots, x_q) \mapsto x_1.$$

Consider the homothety $\gamma = \langle \lambda E, 0 \rangle$ with the coefficient $\lambda > 1$, where E is the identity matrix of order q . We obtain the affine Hopf manifold $M = \widehat{M}/\Gamma$, where $\Gamma = \langle \gamma \rangle$ is a group of similarity transformations of \widehat{M} ; note that $M \cong \mathbb{S}^{q-1} \times \mathbb{S}^1$. Since $r \circ \gamma = \gamma \circ r$, on the manifold M the foliation F is induced such that its leaves are images of the leaves of the foliation $(\widehat{M}, \widehat{F})$ under the universal covering $\kappa : \widehat{M} \rightarrow M = \widehat{M}/\Gamma$. Consequently the foliation (M, F)

is covered by the simple foliation $(\widehat{M}, \widehat{F})$, and both these foliations are transversally affine. Note that the leaf $L_0 = \kappa(r^{-1}(0_1))$ is compact and diffeomorphic to $\mathbb{S}^{q-2} \times \mathbb{S}^1$, and it is a global attractor of the foliation (M, F) . All the other leaves of this foliation are diffeomorphic to \mathbb{R}^{q-1} .

Suppose that the foliation (M, F) admits an Ehresmann connection. Then by Proposition 2, it is covered by a locally trivial bundle $p : \widehat{M} \rightarrow \mathbb{R}^1$, whose fibres are all diffeomorphic to each other. Since not all leaves of the submersion $r : \widehat{M} \rightarrow \mathbb{R}^1$ are diffeomorphic to each other, we get a contradiction. Thus (M, F) does not admit an Ehresmann connection, consequently it is not a complete transversely affine foliation.

The linear holonomy group of the leaf L_0 has a limit point, hence the foliation (M, F) with a global attractor satisfies the conditions of the statement (1) of Theorem 5, but it does not satisfy the conditions of the statement (2) of Theorem 5.

Example 4.

Let \mathbb{R}^2 be the plane with the coordinates x, y . Consider two affine transformations of \mathbb{R}^2 : $c = \langle A, 0 \rangle$, where $A = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ and $\psi_2 = \langle C, c \rangle$, where $C = \begin{pmatrix} \mu_3 & 0 \\ 0 & \nu \end{pmatrix}$, $\mu_i, i = 1, 2, 3, \nu$ are real numbers such that $|\mu_i| < 1$ and $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The point $x_0 = \begin{pmatrix} 1 \\ \frac{1}{1-\mu_3} \end{pmatrix}$ is the only fixed point of ψ_2 . Denote by Ψ the subgroup of the affine group $\text{Aff}(\mathbb{R}^2)$ generated by ψ_1 and ψ_2 . We show that Ψ has a global attractor \mathcal{K} coinciding with the coordinate axis x . Let

$$h_n = \psi_1^n \circ \psi_2^n \circ \psi_1^{-n} \circ \psi_2^{-n}.$$

It can be shown that

$$h_n = \langle E, d_n \rangle,$$

where $d_n = \begin{pmatrix} \frac{(\mu_1^n - 1)(\mu_3^n - 1)}{\mu_3 - 1} \\ 0 \end{pmatrix}$ for every $n \in \mathbb{N}$. It can be checked directly that

$$h_{n+1} \circ h_n^{-1} = \langle E, \delta_n \rangle,$$

where $\delta_n = \begin{pmatrix} \delta_n^{(1)} \\ 0 \end{pmatrix}$, and $\delta_n^{(1)} = \frac{\mu_1^{n+1}(\mu_3^{n+1} - 1) - \mu_1^n(\mu_3^n - 1)}{\mu_3 - 1} - \mu_3^n$. Hence, $\delta_n \rightarrow 0$ as $n \rightarrow +\infty$.

Taking into account the fact that $h_{n+1} \circ h_n^{-1} \in \Psi$ is a parallel translation along the axis x , we see that the orbit $\Psi \cdot 0$ of the origin of \mathbb{R}^2 is dense in the axis x , and the closure of this orbit \mathcal{K} is a global attractor of the group Ψ .

Let $B = (\mathbb{S}^1 \times \mathbb{S}^2) \# (\mathbb{S}^1 \times \mathbb{S}^2)$ be the connected sum of two copies of the product $\mathbb{S}^1 \times \mathbb{S}^2$ of the unit circle \mathbb{S}^1 and the unite two-sphere \mathbb{S}^2 . The fundamental group $\pi_1(B) = \langle g_1, g_2 \rangle$ is a free group of rank two.

Consider the group homomorphism $\rho : \pi_1(B) \rightarrow \Psi$ defined by the conditions $\rho(g_i) = \psi_i$, $i = 1, 2$. It defines the suspended foliation $(M, F) = \text{Sus}(\mathbb{R}^2, B, \rho)$ with the global holonomy group Ψ . Let $\kappa : \widehat{M} \rightarrow M$ be the regular covering map satisfying the conditions of Theorem 2. According to Corollary 3, $\widehat{M} \cong L_0 \times \mathbb{R}^2$, where L_0 is diffeomorphic to every leaf without holonomy, and leaves of the induced foliation $(\widehat{M}, \widehat{F})$, $\widehat{F} = \kappa^*F$, are fibres of the trivial bundle $r : L_0 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Consequently $\mathcal{M} = \kappa(r^{-1}(\mathcal{K}))$ is a global attractor of the complete transversally affine foliation (M, F) , and \mathcal{M} is an embedded submanifold of M . Since M is the space of a locally

trivial bundle $p : M \rightarrow B$ with the contractible standard fibre \mathbb{R}^2 , the exact homotopy sequence of this bundle and Whitehead's theorem imply the homotopic equivalence of the manifolds M and B . In particular, the fundamental group $\pi_1(M, x)$ is a free group of rank two.

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