Low frequency estimation of continuous-time moving average Lévy processes [1]

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Abstract

In this paper we study the problem of statistical inference for a continuous time moving average Lévy process Z observed at low-frequency. We construct a consistent estimator for the Lévy triplet of Z, derive its convergence rates and prove their optimality. The performance of our estimation procedure is illustrated by numerical example.

Keywords: moving average, Mellin transform, low-frequency estimation

1. Introduction

Continuous-time Lévy driven moving average processes of the form

$$Z_t = \int_{-\infty}^{\infty} K(s,t) dL_s,$$

where K denotes a deterministic kernel function and L a Lévy process build a large class of stochastic processes including semimartingales and non- semimartingales, cf. Basse and Pedersen [1], Basse-O'Connor and Rosinsky [2],

[1] The study has been funded by the Russian Academic Excellence Project "5-100". The first and the third author acknowledge the financial support from the Deutsche Forschungsgemeinschaft through the SFB 823 "Statistical modelling of nonlinear dynamic processes".

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Bender, Lindner and Schicks [3], as well as long memory processes. Starting point was the paper by Rajput and Rosinski [4] providing conditions on the interplay between K and L such that Z is well defined. Continuous-time Lévy driven moving average processes provide a unifying approach to many popular stochastic processes like Lévy driven Ornstein-Uhlenbeck processes, fractional Lévy processes and CARMA processes. Furthermore, they are the building blocks of more involved models such as Lévy semistationary processes and ambit processes, which are popular in turbulence and finance, cf. Barndorff-Nielsen, Benth and Veraart [5].

Statistical inference for Ornstein-Uhlenbeck processes and CARMA processes is already well-established due to the special structure of the processes, for an overview see Brockwell and Lindner [6], whereas for general continuous-time Lévy driven moving average processes so far only partial results are established mainly concerning parameters which enter the kernel function, cf. Cohen and Lindner [7] for an approach via empirical moments or Zhang, Lin and Zhang [8] for a least squares approach. Further results concern limit theorems for the power variation, cf. Glaser [9], Basse-O'Connor, Lachieze-Rey and Podolskyij [10], which may be used for statistical inference based on high-frequency data.

We now consider the special case of stationary continuous-time Lévy driven moving average processes of the form $Z_t = \int_{-\infty}^{\infty} \mathcal{K}(s-t)dL_s$ and aim to infer the unknown quantities of the driving Lévy process under low frequency observations. Our setting especially includes the Gamma-kernel $\mathcal{K}(t) = t^\alpha e^{-\lambda t} 1_{[0,\infty)}(t)$ with $\lambda > 0$ and $\alpha > -1/2$, which serves as a popular kernel for applications in finance and turbulence, cf. Barndorff-Nielsen and Schmiegel [11].

In fact, the resulting statistical problem is rather challenging for several reasons. On the one hand, the set of parameters. i.e. the Lévy triplet of the driving Lévy process contains, in general, an infinite dimensional object, a Lévy measure making the statistical problem nonparametric. On the other hand, the relation between the parameters of the underlying Lévy process and those of the resulting moving average process is rather nonlinear and implicit, pointing out to a nonlinear ill-posed statistical problem. It turns out that in Fourier domain this relation becomes more clear and has a form of a multiplicative convolution. This observation underlies our estimation procedure, which basically consists of three steps. First, we estimate the marginal characteristic function of the Lévy driven moving average process. Then we estimate the Mellin transform of the second derivative of the log-transform of characteristic function. Finally, an inverse Mellin transform technique is used to reconstruct the Lévy density of the underlying Lévy process.

The paper is organized as follows. In the next session, we explain our setup and discuss the correctness of our model. In Section 3, we present the estimation procedure. Our main theoretical results related to the rates of convergence of the estimates are given in Section 4. Next, in Section 5, we provide a numerical example, which shows the performance of our procedure. All proofs are collected in the appendix.

2. Setup

In this paper we study a stationary continuous-time moving average (MA) Lévy process $(Z_t)_{t\in\mathbb{R}}$ of the form:

$$Z_t = \int_{-\infty}^{\infty} \mathcal{K}(t-s)\, dL_s, \quad t \in \mathbb{R}, \tag{1}$$

where $\mathcal{K} : \mathbb{R} \to \mathbb{R}_+$ is a symmetric measurable function and $(L_t)_{t\in\mathbb{R}}$ is a two-sided Lévy process with the triplet $\mathcal{T} = (\gamma, \sigma^2, \nu)$. As shown in [4], under the conditions

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}\setminus\{0\}} \left(|\mathcal{K}(s)x|^2 \wedge 1 \right) \nu(dx)\, ds < \infty, \tag{2}$$

$$\sigma^2 \int_{\mathbb{R}_+} \mathcal{K}^2(s)\, ds < \infty, \tag{3}$$

$$\int_{\mathbb{R}_+} \left| \mathcal{K}(s) \left(\gamma + \int_{\mathbb{R}} x \left(1_{\{|x\mathcal{K}(s)|\le 1\}} - 1_{\{|x|\le 1\}} \right) \nu(dx) \right) \right| ds < \infty \tag{4}$$

the stochastic integral in (1) exists. Note that conditions (2) - (4) are fulfilled for any $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and any Lévy process L with finite second moment. In fact, (3) is trivial in this case; condition (2) directly follows from the inequality

$$\begin{aligned}
\int_{\mathbb{R}_+} \int_{\mathbb{R}\setminus\{0\}} \left(|\mathcal{K}(s)x|^2 \wedge 1 \right) \nu(dx)\, ds &\le \int_{\mathbb{R}_+} \int_{\mathbb{R}\setminus\{0\}} |\mathcal{K}(s)x|^2\, \nu(dx)\, ds \\
&= \int_{\mathbb{R}_+} (\mathcal{K}(s))^2\, ds \cdot \int_{\mathbb{R}\setminus\{0\}} x^2 \nu(dx)\, ds.
\end{aligned}$$

As to the condition (4), it holds

$$\int_{\mathbb{R}_+} \left| \mathcal{K}(s) \left(\gamma - \int_{\mathbb{R}} x 1_{\{|x| \leq 1\}} \nu(dx) \right) + \int_{\mathbb{R}} x \mathcal{K}(s) 1_{\{|x\mathcal{K}(s)| \leq 1\}} \nu(dx) \right| ds$$
$$= \int_{\mathbb{R}_+} \left| \mathcal{K}(s) \mathbb{E}\left[L_1\right] - \int_{\mathbb{R}} x \mathcal{K}(s) 1_{\{|x\mathcal{K}(s)| > 1\}} \nu(dx) \right| ds$$
$$\leq |\mathbb{E}[L_1]| \int_{\mathbb{R}_+} \mathcal{K}(s)\, ds + \int_{\mathbb{R}_+} \int_{\mathbb{R}} x^2 \left(\mathcal{K}(s)\right)^2 \nu(dx)\, ds.$$

In the sequel we assume that $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\int x^2 \nu(dx) < \infty. \tag{5}$$

Under above assumptions, the process $(Z_t)_{t\in\mathbb{R}}$ is strictly stationary and fulfils

$$\Phi(u) \doteq \mathrm{E}\left[e^{\mathrm{i}uZ_t}\right] = \exp\left(2 \int_0^\infty \psi(u\mathcal{K}(s))\, ds\right) \tag{6}$$

with

$$\psi(u) \doteq \mathrm{i}u\gamma - \sigma^2 u^2/2 + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux 1_{\{|x| \leq 1\}}\right) \nu(dx).$$

Our main goal is the estimation of the parameters of the Lévy process L from low-frequency observations of the process Z given that the function $\mathcal{K}$ is known.

3. Mellin transform approach

3.1. Main idea

Assume, for the clarity of presentation, that the Lévy process L is a subordinator with a Lévy triplet $(\mu, 0, \nu)$, where ν is a absolutely continuous w.r.t. to Lebesgue measure on $\mathbb{R}_+$ and satisfies (5). Denote by $\nu(x)$ the density of ν and set $\overline{\nu}(x) \doteq x^2 \nu(x)$. It follows then

$$\Psi''(u) = \int_{\mathbb{R}} \psi''\left(u\mathcal{K}(x)\right) \cdot \mathcal{K}^2(x)\, dx = -\int_{\mathbb{R}} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\right) \cdot \mathcal{K}^2(x)\, dx$$

with $\Psi(u) \doteq \log(\Phi(u))$, $u \in \mathbb{R}$. Next, let us compute the Mellin transform of Ψ':

$$\begin{aligned}
\mathcal{M}\left[\Psi''\right](z) &= -\int_{\mathbb{R}_+}\left[\int_{\mathbb{R}} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\right)\cdot\mathcal{K}^2(x)\,dx\right]u^{z-1}du \\
&= -\int_{\mathbb{R}}\left[\int_{\mathbb{R}_+} \mathcal{F}[\overline{\nu}]\left(u\mathcal{K}(x)\,\right)\cdot u^{z-1}du\right]\mathcal{K}^2(x)\,dx \\
&= -\left[\int_{\mathbb{R}_+} \mathcal{F}[\overline{\nu}]\,(v)\cdot v^{z-1}dv\right]\cdot\left[\int_{\mathbb{R}} \left(\mathcal{K}(x)\right)^{2-z}dx\right],
\end{aligned} \tag{7}$$

for all z such that $\int_{\mathbb{R}}(\mathcal{K}(x))^{2-\mathrm{Re}(z)}\,dx < \infty$ and $\int_{\mathbb{R}_+}|\mathcal{F}[\overline{\nu}]\,(v)|\cdot v^{\mathrm{Re}(z)-1}dv < \infty$. Since $\overline{\nu} \in L_1(\mathbb{R}_+)$, it holds

$$\begin{aligned}
\mathcal{M}[\mathcal{F}[\overline{\nu}]](z) &= \int_0^\infty v^{z-1}\left[\int_0^\infty e^{\mathrm{i}xv}\overline{\nu}(x)dx\right]dv \\
&= \mathcal{M}\left[e^{\mathrm{i}\cdot}\right](z)\cdot\mathcal{M}\left[\overline{\nu}\right](1-z).
\end{aligned}$$

Note that the Mellin transform $\mathcal{M}\left[\overline{\nu}\right](1-z)$ is defined for all z with $\mathrm{Re}(z) \in (0,1)$, provided $\overline{\nu}$ is bounded at 0. Next, using the fact that

$$\mathcal{M}[e^{\mathrm{i}\cdot}](z) = \Gamma(z)\left[\cos(\pi z/2) + \mathrm{i}\sin(\pi z/2)\right] = \Gamma(z)e^{\mathrm{i}\pi z/2}$$

for all z with $\mathrm{Re}(z) \in (0,1)$ (see [12], 5.1-5.2), we get

$$\mathcal{M}[\Psi''](z) = Q(z)\cdot\mathcal{M}[\overline{\nu}](1-z), \quad \mathrm{Re}(z) \in (0,1),$$

where $Q(z) \doteq -\Gamma(z)e^{\mathrm{i}\pi z/2}\int_{\mathbb{R}}\left(\mathcal{K}(x)\right)^{2-z}dx$. Finally, we apply the inverse Mellin transform to get

$$\begin{aligned}
\overline{\nu}(x) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathcal{M}[\overline{\nu}](z)x^{-z}dz \\
&= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathcal{M}[\Psi''](1-z)}{Q(1-z)}x^{-z}dz
\end{aligned} \tag{8}$$

for $c \in (0,1)$. The formula (8) connects the weighted Levy density $\overline{\nu}$ to the characteristic exponent Ψ of the process Z and forms basis for our estimation procedure.

Let us remark on the general case where L is not necessary a subordinator, but a Lévy process without Gaussian part. In this case one can show that

$$\frac{\mathcal{M}\left[\Psi''(-\cdot)\right](u)+\mathcal{M}\left[\Psi''(\cdot)\right](u)}{2} = -\left\{\mathcal{M}\big[\overline{\nu}_+\big](1-z)+\mathcal{M}\big[\overline{\nu}_-\big](1-z)\right\} \cdot \cos\left(\frac{\pi z}{2}\right)\Gamma(z)\cdot\int_{\mathbb{R}}\left(\mathcal{K}(x)\right)^{2-z}dx$$

and

$$\frac{\mathcal{M}\left[\Psi''(\cdot)\right](u)-\mathcal{M}\left[\Psi''(-\cdot)\right](u)}{2\mathrm{i}} = -\left\{\mathcal{M}\big[\overline{\nu}_+\big](1-z)-\mathcal{M}\big[\overline{\nu}_-\big](1-z)\right\} \cdot \sin\left(\frac{\pi z}{2}\right)\Gamma(z)\cdot\int_{\mathbb{R}}\left(\mathcal{K}(x)\right)^{2-z}dx,$$

where $\overline{\nu}_+(x)=\nu(x)\cdot 1(x\geq 0)$ and $\overline{\nu}_-(x)=\nu(-x)\cdot 1(x\geq 0)$. Using the above formulas one can find $\mathcal{M}\big[\overline{\nu}_-\big]$, $\mathcal{M}\big[\overline{\nu}_+\big]$ and apply the Mellin inversion formula to reconstruct $\overline{\nu}_-$ and $\overline{\nu}_+$.

3.2. Estimation procedure

Assume that the process Z is observed on the equidistant grid $\{\Delta, 2\Delta, \ldots, n\Delta\}$. Our aim is to estimate the Lévy density ν of the process L. First we approximate the Mellin transform of the function

$$\Psi''(u)=\Big(\log\left(\Phi(u)\right)\Big)''=\frac{\Phi''(u)}{\Phi(u)}-\left(\frac{\Phi'(u)}{\Phi(u)}\right)^2$$

via

$$\mathcal{M}_n[\Psi''](1-z)\doteq\int_0^{U_n}\left[\frac{\Phi''_n(u)}{\Phi_n(u)}-\left(\frac{\Phi'_n(u)}{\Phi_n(u)}\right)^2\right]u^{-z}\,du, \tag{9}$$

where

$$\Phi_n(u)\doteq\frac{1}{n}\sum_{k=1}^{n}\exp\{\mathrm{i}Z_{k\Delta}u\}$$

and a sequence $U_n\to\infty$ as $n\to\infty$. Second, by regularising the inverse Mellin transform, we define

$$\overline{\nu}_n(x)\doteq\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n}\frac{\mathcal{M}_n[\Psi''](1-z)}{Q(1-z)}x^{-z}dz \tag{10}$$

for some $c \in (0,1)$ and some sequence $V_n \to \infty$, which will be specified later. In the next section we study the properties of the estimate $\overline{\nu}_n(x)$. In particular we show that $\overline{\nu}_n(x)$ converges to $\overline{\nu}(x)$ and derive the corresponding convergence rates.

4. Convergence

Assume that the following conditions hold.

(AN) For some $A \in \mathbb{R}_+$ and $\alpha, \beta \in (0,1], \gamma > 0, c \in (0,1)$ the Levy density ν fulfills

$$\int_{\mathbb{R}_+} (1+y)^{\alpha} \left|\mathcal{F}[\overline{\nu}](y)\right| dy \leq A, \tag{11}$$

$$\int_{\mathbb{R}} e^{\gamma|u|} \left|\mathcal{M}[\overline{\nu}](c+\mathrm{i}u)\right| du \leq A, \tag{12}$$

$$\int_{\mathbb{R}_+} (x^{\beta} \vee x^2)\nu(x)\, dx \leq A. \tag{13}$$

Theorem 1. *Suppose that (AN) holds, $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\int (\mathcal{K}(x))^{\beta}\, dx \leq B$. Set*

$$\varepsilon_n \doteq \sqrt{\frac{\log(n)}{n}} \cdot \exp\left(\frac{1}{2}\pi^{2-\beta} A U_n^{\beta} \int (\mathcal{K}(x))^{\beta}\, dx\right)$$

and $D_j(u) \doteq (\Phi_n^{(j)}(u) - \Phi^{(j)}(u))/\Phi(u)$, $j = 0,1,2$. Consider the event

$$\mathcal{A}_K := \left\{ \max_{j=0,1,2} \|D_j\|_{U_n} \geq K\varepsilon_n \right\},$$

for some $K > 0$, where for any real valued function f on $\mathbb{R}$, $\|f\|_{U_n}$ stands for $\sup_{u\in[-U_n,U_n]} |f(u)|$. Let U_n fullfil

$$K\varepsilon_n \left(1 + \|\Psi'\|_{U_n}\right) \leq 1/2.$$

Choosing U_n in such a way is always possible, since $\Psi'(0) = \psi'(0)\int \mathcal{K}(s)\, ds$ is finite. On the set $\overline{\mathcal{A}}_K$, the estimate $\overline{\nu}_n(x)$ given by (10) with the same $c \in (0,1)$ as in (12) satisfies

$$\sup_{x\in\mathbb{R}_+} \left\{x^c \left|\overline{\nu}_n(x) - \overline{\nu}(x)\right|\right\} \;\leq\; \frac{1}{2\pi} \int_{\{|v|\leq V_n\}} \frac{\Omega_n}{|Q(1-c-iv)|}\, dv + \frac{A}{2\pi} e^{-\gamma V_n},$$

where

$$\Omega_n = 2K\varepsilon_n U_n^{1-c}\left(2 + \|\Psi''\|_{U_n} + \|\Psi'\|_{U_n}^2 + 3\|\Psi'\|_{U_n}\right) + 2\left(A + \frac{2^\alpha A}{1-c}\right)\int_{\mathbb{R}_+}\left[\mathcal{K}(x)\right]^{c+1}\left[1 + U_n\mathcal{K}(x)\right]^{-\alpha} dx.$$

Remark 1. *Note that in case of subordinators the sum* $2+\|\Psi''\|_{U_n}+\|\Psi'\|_{U_n}^2+3\|\Psi'\|_{U_n}$, *which is involved in the expression for* Ω_n, *can be bounded uniformly by* n. *In fact,*

$$\left|\psi'(u)\right| = \left|i\mu + \int_{\mathbb{R}_+} ixe^{iux}\nu(x)dx\right| \le \mu + \int_{\mathbb{R}_+} x\nu(x)dx \le \mu + 2A,$$

because $\left(\int_0^1 + \int_1^\infty\right) x\nu(x)dx \le \int_0^1 x^\beta\nu(x)dx + \int_1^\infty x^2\nu(x)dx \le 2A$ *by* (13). *Analogously,*

$$\left|\psi''(u)\right| = \left|\int_{\mathbb{R}_+} x^2 e^{iux}\nu(x)dx\right| \le \int_{\mathbb{R}_+} x^2\nu(x)dx \le A.$$

Therefore

$$\begin{aligned}\left\|\Psi'\right\|_{U_n} &= \left\|\int_{\mathbb{R}} \psi'(u\mathcal{K}(x))\mathcal{K}(x)dx\right\|_{U_n} \le (\mu+2A)\int_{\mathbb{R}}\mathcal{K}(x)dx,\\ \left\|\Psi'\right\|_{U_n} &= \left\|\int_{\mathbb{R}} \psi''(u\mathcal{K}(x))\mathcal{K}^2(x)dx\right\|_{U_n} \le A\int_{\mathbb{R}}\mathcal{K}^2(x)dx,\end{aligned}$$

where the integrals in the right-hand sides are bounded due to the assumption $\mathcal{K} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Example 1. *Consider a tempered stable Levy process* (L_t) *with*

$$\nu(x) = x^{-\eta-1}\cdot e^{-\lambda x}, \quad x \ge 0, \quad \beta \in (0,1). \tag{14}$$

Since

$$\mathcal{M}[\overline{\nu}](z) = \lambda^{\eta-z-1}\Gamma(z-\eta+1), \quad \mathsf{Re}(z) > \eta - 1,$$

we derive that (12) *holds for all* $0 < \gamma < \pi/2$ *and* $\alpha > 0$ *due to the asymptotic properties of the Gamma function. Moreover,*

$$\mathcal{F}[\overline{\nu}](u) = (iu-\lambda)^{-(2-\eta)}\Gamma(2-\eta)$$

and hence (11) *holds for any* $0 < \alpha < 2 - \eta$. *Moreover,* ν *satisfies* (13) *for any* $\beta \in (0, \eta)$.

Corollary 1. *Consider a class of kernels of the form*

$$\mathcal{K}(x) = |x|^r e^{-\rho|x|},$$

where r *is a nonnegative integer and* $\rho > 0$*, and assume that the Lévy measure* ν *satisfies the set of assumptions (AN). Then*

$$\Omega_n \lesssim K\varepsilon_n U_n^{1-c} + U_n^{-\alpha}, \quad n \to \infty$$

and

$$\int_{\{|v|\leq V_n\}} \frac{1}{|Q(1-c-iv)|}\, dv \lesssim \begin{cases} V_n^{c+3/2}, & r = 0, \\ V_n^{c+1}, & r \geq 1. \end{cases}$$

As a result we have on $\overline{\mathcal{A}}_K$

$$\sup_{x\in\mathbb{R}_+} \{x^c\, |\overline{\nu}_n(x) - \overline{\nu}(x)|\} \lesssim V_n^{\zeta} \left(\varepsilon_n U_n^{(1-c)} + U_n^{-\alpha}\right) + e^{-\gamma V_n}$$

with $\zeta = c + 1 + 1(r = 0)/2$*. By taking* $V_n = \varkappa \log(U_n)$ *with* $\varkappa > \alpha/\gamma$ *and* $U_n = \theta \log^{1/\beta}(n)$ *for any* $\theta < \left(\pi^{2-\beta} A \int (\mathcal{K}(x))^{\beta}\, dx\right)^{-1/\beta}$,

$$\sup_{x\in\mathbb{R}_+} \{x^c\, |\overline{\nu}_n(x) - \overline{\nu}(x)|\} \lesssim \log^{-\alpha/\beta}(n), \quad n \to \infty.$$

Let us now estimate the probability of the event $\mathcal{A}_K$. The following result holds.

Theorem 2. *Suppose that the kernel* $\mathcal{K}$ *satisfies*

$$\sum_{j=-\infty}^{\infty} \left| \mathcal{F}[\mathcal{K}]\left(2\pi \frac{j}{\Delta}\right)\right| < K^* \tag{15}$$

and

$$(\mathcal{K} \star \mathcal{K})(\Delta j) \leq \kappa_0\, |j|^{\kappa_1} e^{-\kappa_2 |j|}, \quad \forall\, j \in \mathbb{Z} \tag{16}$$

for some positive constants K^*,κ_0, κ_1 *and* κ_2*. Let* $\lambda_n(t)$ *denote the minimal eigenvalue of a positive-semidefinite matrix*

$$\left(\mathcal{K}(t - j\Delta)\mathcal{K}(t - l\Delta)\right)_{l,j=1}^{n},$$

and assume that there exists a constant $\lambda_\star$ not depending on n, such that

$$\int \lambda^{\beta/2}(t)\,dt > \lambda_\star > 0, \qquad \forall\, n \in \mathbb{N}. \tag{17}$$

If

$$\int_{|x|>1} e^{Rx}\nu(dx) < A_R, \quad \int_{|x|\le\varepsilon} x^2 e^{Rx}\nu(dx) \ge B_R \varepsilon^{2-\beta},$$

for some $\beta \in (0,1)$,$R > 0$, $A_R > 0$ and $B_R > 0$, then for any $K > 0$ it holds,

$$\mathbb{P}\Big(\mathcal{A}_K\Big) \le \frac{C_1}{\sqrt{K}} \frac{\sqrt{U_n} n^{(1/4)-C_2 K}}{\log(n)},$$

where the positive constants C_1, C_2 may depend on K^*, A_R, B_R and κ_i, $i = 1, 2$.

Example 2. *Consider again the class of kernels of the form*

$$\mathcal{K}(x) = |x|^r e^{-\rho|x|},$$

where r is a nonnegative integer and $\rho > 0$. Let us check the assumptions of Theorem 2. We have

$$\mathcal{F}[\mathcal{K}](u) = \frac{1}{(iu-1)^{r+1}} + \frac{1}{(-iu-1)^{r+1}}$$

and (15) holds. Assumption (16) is proven in Lemma 3. Next, a well-known Gershgorin circle theorem implies that

$$\lambda(t) \ge K^2(t - j\Delta) - 2\sum_{l<j} K(t - j\Delta)K(t - l\Delta)$$

for some $j \in \{1, \ldots, n\}$ and we derive

$$\begin{aligned} \int \lambda^{\beta/2}(t)\,dt &\ge \int \Big(K(t) - 2\sum_{k\ge1} K(t + k\Delta)\Big)^{\beta/2} dt \\ &= \Big(1 - 2\sum_{k\ge1} k^r e^{-\rho k}\Big) \int |t|^{\beta r/2} e^{-\rho\beta|t|/2}\,dt \end{aligned}$$

Note that

$$\sum_{k\ge1} k^r e^{-\rho k} = (-1)^r \frac{d^r}{d\rho^r}\left(\frac{1}{1 - e^{-\rho}}\right),$$

Hence (17) holds, provided ρ is large enough.

5. Numerical example

Simulation. Consider the integral $Z_t := \int_{\mathbb{R}} \mathcal{K}(t-s)\, dL_s$ with the kernel $\mathcal{K}(x) = e^{-|x|}$ and the Lévy process

$$L_t = L_t^{(1)} I\!I\{t>0\} - L_{-t}^{(2)} I\!I\{t<0\},$$

constructed from independent compound Poisson processes

$$L_t^{(1)} \stackrel{d}{=} L_t^{(2)} \stackrel{d}{=} \sum_{k=1}^{N_t} \xi_k,$$

where N_t is a Poisson process with intensity λ, and $\xi_1, \xi_2, ...$ are independent r.v.'s with standard exponential distribution. Note the the Lévy density of the process $L_t^{(1)}$ is equal to $\nu(x) = \lambda e^{-x}$.

For $k = 1, 2$, denote the moments of jumps of $L_t^{(k)}$ by $s_1^{(k)}, s_2^{(k)}, ...$ and the corresponding sizes of jumps by $\xi_1^{(k)}, \xi_2^{(k)}, ...$ Then

$$Z_t = \sum_{j=0}^{\infty} \mathcal{K}(t - s_j^{(1)})\xi_j^{(1)} - \sum_{j=0}^{\infty} \mathcal{K}(t + s_j^{(2)})\xi_j^{(2)}.$$

In practice, we truncate both series in the last representation by finding a value $x_{max} := \max_{x\in\mathbb{R}_+}\{\mathcal{K}(x) > \alpha\}$ for a given level α. Let

$$\begin{aligned} \tilde{Z}_t &= \sum_{k\in K^{(1)}} \mathcal{K}(t - s_j^{(1)})\xi_j^{(1)} - \sum_{k\in K^{(2)}} \mathcal{K}(t + s_j^{(2)})\xi_j^{(2)}, \qquad \text{where} \\ K^{(1)} &:= \left\{k : \max(0, t - x_{max}) < s_k^{(1)} < t + x_{max}\right\}, \\ K^{(2)} &:= \left\{k : 0 < s_k^{(2)} < \max(0, -t + x_{max})\right\}. \end{aligned}$$

For simulation study, we take $\lambda = 1$, $\alpha = 0.01$ (and therefore $x_{\max} = 6.908$). Typical trajectory is presented on Figure 1.

General idea of the estimation procedure. In practice the estimation procedure described in Section 3.1 can be slightly simplified by applying the Laplace transform $\mathcal{L}[Z_t](u) = \mathbb{E}\left[e^{-uZ_t}\right]$ instead of the Fourier transform. Note also that if the drift is equal to zero (as in this simulation study), one can consider the first derivative of the function Ψ instead of the second.

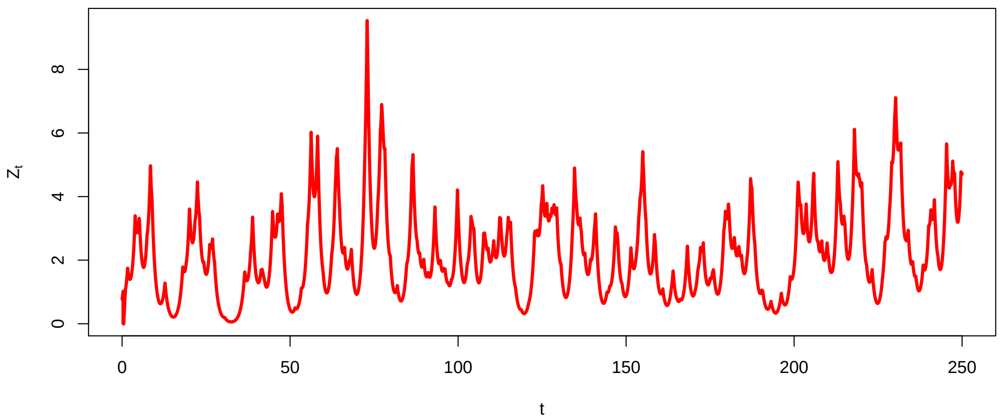

Figure 1: Typical trajectory of the process Z_t constructed from the compound Poisson process with positive jumps.

Applying these simplifications, we get by analogue to (10) the following estimate of the Lévy measure ν:

$$\widetilde{\nu}_n(x) := \frac{1}{2\pi \mathrm{i} x} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \frac{\mathcal{M}_n[\widetilde{\Psi}'](1-z)}{\widetilde{Q}(1-z)} x^{-z} dz, \tag{18}$$

where $\widetilde{Q}(z) = -\Gamma(z) \int_{\mathbb{R}} (K(x))^{1-z} \, dx$ and

$$\mathcal{M}_n[\widetilde{\Psi}'](1-z) := -\int_0^{U_n} \frac{\mathrm{mean}(Z_{\Delta k} e^{-uZ_{\Delta k}})}{\mathrm{mean}(e^{-uZ_{\Delta k}})} u^{-z} \, du \tag{19}$$

is the estimate of the the Mellin transform of the function $\widetilde{\Psi}(u) = \log(\mathcal{L}[Z_t](u))$.

The advantage of using the Laplace transform is that the proposed estimate $\widetilde{\nu}_n(x)$ is real-valued. In fact, since

$$\mathcal{M}[\widetilde{\Psi}'](1-\bar{z}) = \overline{\mathcal{M}[\widetilde{\Psi}'](1-z)}, \qquad \widetilde{Q}(1-\bar{z}) = \overline{\widetilde{Q}(1-z)},$$

we get

$$\widetilde{\nu}_n(x) = \frac{1}{\pi x} \int_0^{V_n} \mathrm{Re} \left\{ \frac{\mathcal{M}_n[\widetilde{\Psi}'](1-c-\mathrm{i}v)}{\widetilde{Q}(1-c-\mathrm{i}v)} u^{-c-\mathrm{i}v} \right\} dv.$$

Estimation of the Mellin transform of $\widetilde{\Psi}'(\cdot)$. In order to improve the numerical rates of convergence of the integral involved in (19), we slightly modify this estimate:

$$\mathcal{M}_n^\star[\widetilde{\Psi}'](1-z) := \int_0^{U_n} \Big[-\frac{\mathrm{mean}(Z_{\Delta k} e^{-uZ_{\Delta k}})}{\mathrm{mean}(e^{-uZ_{\Delta k}})} + \mathrm{mean}(Z) e^{-u} \Big] e^{-z\log(u)} du - 2\lambda\Gamma(1-z).$$

Note that $\mathcal{M}_n^\star[\widetilde{\Psi}'](1-z)$ is also a consistent estimate of $\mathcal{M}[\widetilde{\Psi}'](1-z)$ (since $\mathrm{mean}(Z) \to 2\lambda$), but involves the integral with better convergence properties.

Note that theoretically $\mathcal{L}(u) = (1+u)^{-2\lambda}$, and therefore the Mellin transform of the function $\widetilde{\Psi}'(\cdot)$ can be explicitly computed:

$$\begin{aligned} \mathcal{M}[\widetilde{\Psi}'](1-z) &= \int_{\mathbb{R}_+} \frac{\mathcal{L}'(u)}{\mathcal{L}(u)} u^{-z} du = -2\lambda \int_{\mathbb{R}_+} \frac{u^{-z}}{u+1} du \\ &= -2\lambda \frac{\Gamma(1-z)\Gamma(1+z)}{z}, \end{aligned}$$

We estimate $\mathcal{M}[\widetilde{\Psi}'](1-z)$ for $z = c + \mathrm{i}v$, where c is fixed and v is taken from the equidistant grid from 0 to V_n with step δ_V. Typical behavior of the the Mellin transform $\mathcal{M}[\widetilde{\Psi}'](1-z)$ and its estimate $\mathcal{M}_n^\star[\Psi'](1-z)$ is illustrated by Figure 2.

Estimation of $\nu(x)$. Finally, we estimate the Lévy density $\nu(x)$ by

$$\tilde{\nu}_n^\star(x) := \frac{\delta_V}{x\pi} \sum_{k=1}^{Q} \mathrm{Re} \left\{ \frac{\mathcal{M}_n^\star[\widetilde{\Psi}'](1-c-\mathrm{i}v_k)}{\widetilde{Q}(1-c-\mathrm{i}v)} \cdot e^{-(c+\mathrm{i}v_k)*\log(x)} \right\},$$

and measure the quality of this estimate by the L^2-norm on the interval $[1,3]$:

$$\mathcal{R}(\tilde{\nu}_n^\star) = \int_1^3 \left(\tilde{\nu}_n^\star(x) - \nu(x)\right)^2 dx$$

Typical behavior of the estimator $\tilde{\nu}_n^\star(x)$ in compression to the true Lévy density $\nu(x) = \lambda e^{-x}$ is depicted on Figure 3.

To show the convergence of this estimate, we made simulations with different values of n. The parameters U_n and V_n are chosen by numerical optimization of $\mathcal{R}(\tilde{\nu}_n^\star)$. The results of this optimization, for different values of n, as well as the means and variances of the estimate $\tilde{\nu}_n$ based on 20 simulation runs, are given in the next table.

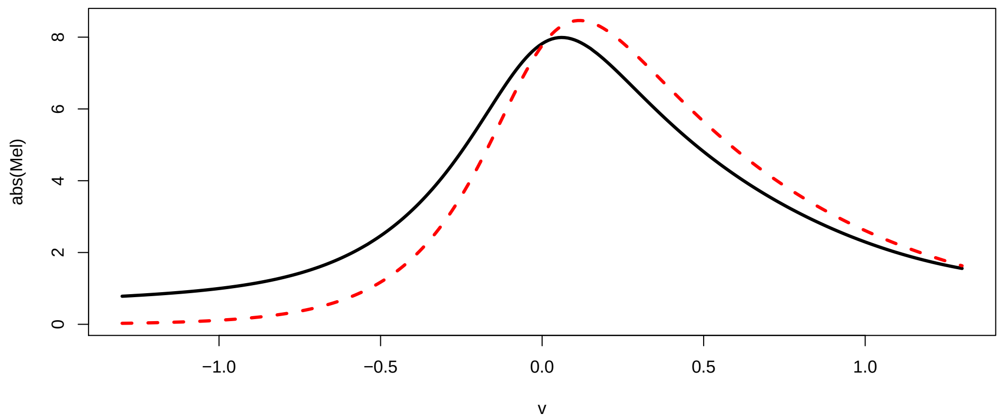

Figure 2: Left picture: graphs on the complex plain of the empirical (black solid) and theoretical (red dashed) Mellin transforms of the function $\Psi'(\cdot)$. Right picture: absolute values of the empirical and theoretical Mellin transforms depending on the imaginary part of the argument.

n	U_n	V_n	$\operatorname{mean}(\mathcal{R}(\tilde{\nu}_n^\star))$	$\operatorname{Var}(\mathcal{R}(\tilde{\nu}_n^\star))$
10000	2.5	1.2	0.0015	3.7034e-06
25000	2	1.2	0.0006	8.2935e-07
50000	2.4	1.2	0.0004	5.6539e-08
75000	3	1.2	0.0003	7.1480e-08

The boxplots of this estimate based on 20 simulation runs are presented on Figure 4.

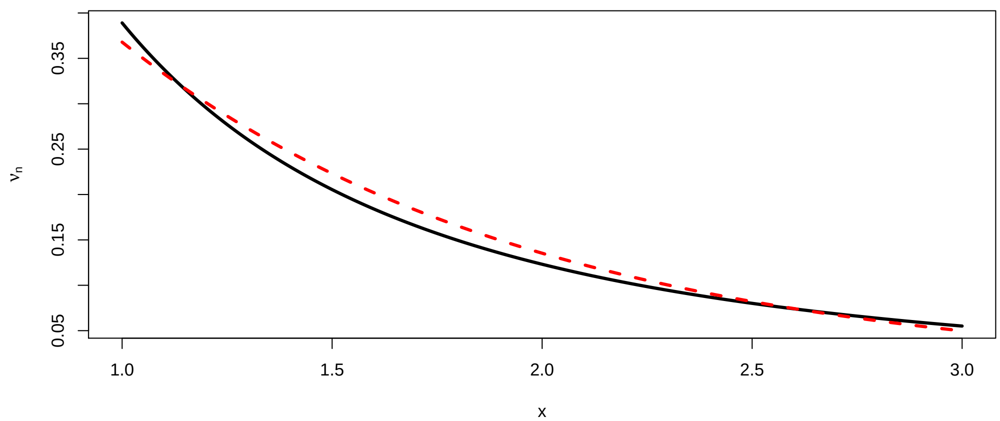

Figure 3: Plots of the Lévy density (red dashed) and its estimate (black solid) on the interval [1,3].

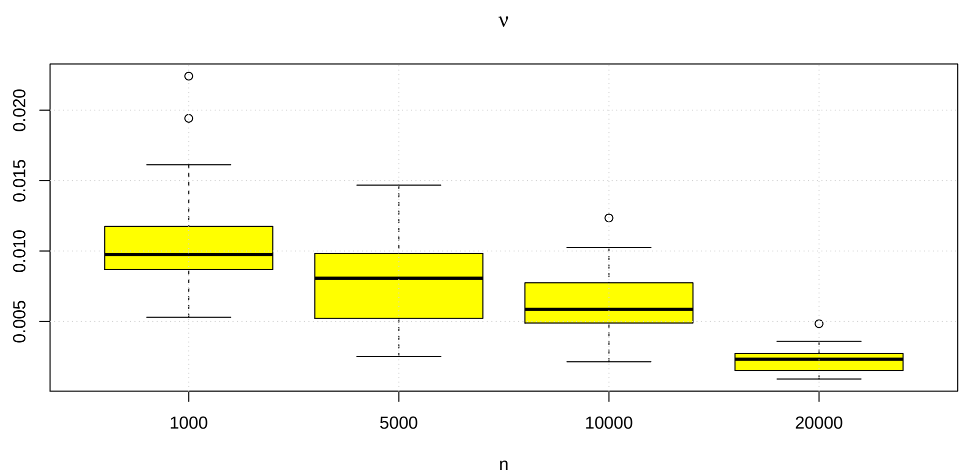

Figure 4: Boxplot of the estimate $\mathcal{R}(\tilde{\nu}_n^\star)$ based on 20 simulation runs.

Appendix A. Proof of Theorem 1

Denote $G_j(u) = \Psi_n^{(j)}(u) - \Psi^{(j)}(u)$, $j = 1, 2$, where $\Psi_n(u) = \log \Phi_n(u)$. Then

$$G_1(u) = \frac{D_1(u) - D_0(u)\Psi'(u)}{1 + D_0(u)}, \tag{A.1}$$

$$\begin{aligned} G_2(u) &= \frac{\left(\Psi''(u) + (\Psi'(u))^2 + \Psi'(u)G_1(u)\right) D_0(u)}{1 + D_0(u)} \\ &\quad - \frac{(2\Psi'(u) + G_1(u))\, D_1(u)}{1 + D_0(u)} + \frac{D_2(u)}{1 + D_0(u)}. \end{aligned} \tag{A.2}$$

We have

$$\begin{aligned} \overline{\nu}_n(x) - \overline{\nu}(x) &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}V_n}^{c+\mathrm{i}V_n} \left[\frac{\mathcal{M}_n[\Psi''](1-z) - \mathcal{M}[\Psi''](1-z)}{Q(1-z)} \right] x^{-z}\, dz \\ &\quad - \frac{1}{2\pi x} \int_{\{|v| \geq V_n\}} \mathcal{M}[\overline{\nu}](c + \mathrm{i}v)\, x^{-(c+\mathrm{i}v)}\, dv \end{aligned}$$

and

$$\begin{aligned} x^c \left(\overline{\nu}_n(x) - \overline{\nu}(x)\right) &= \frac{1}{2\pi} \int_{\{|v| \leq V_n\}} \frac{R_1(v) + R_2(v)}{Q(1 - c - \mathrm{i}v)} x^{-\mathrm{i}v}\, dv \\ &\quad - \frac{1}{2\pi} \int_{\{|v| \geq V_n\}} \mathcal{M}[\overline{\nu}](c + \mathrm{i}v)\, x^{-\mathrm{i}v}\, dv, \end{aligned} \tag{A.3}$$

where

$$R_1(v) = \int_0^{U_n} G_2(u) u^{-c-\mathrm{i}v}\, du$$

and

$$R_2(v) = -\int_{U_n}^{\infty} \Psi''(u)\, u^{-c-\mathrm{i}v}\, du.$$

We have on $\overline{\mathcal{A}_K}$ under the assumption $K\varepsilon_n(1 + \|\Psi'\|_{U_n}) \leq 1/2$, that the denominator of the fractions in G_1 and G_2 can be lower bounded as follows:

$$\min_{u \in [-U_n, U_n]} |1 + D_0(u)| \geq 1 - \max_{u \in [-U_n, U_n]} |D_0(u)| \geq 1 - K\varepsilon_n \geq 1/2.$$

Therefore,

$$\begin{aligned}\|G_1\|_{U_n} &\le 2K\varepsilon_n\left(1+\|\Psi'\|_{U_n}\right)\le 1\\ \|G_2\|_{U_n} &\le 2K\varepsilon_n\Big(1+\|\Psi''\|_{U_n}+\left\|(\Psi')^2\right\|_{U_n}\\ &\quad+\left(1+\|\Psi'\|_{U_n}\right)\|G_1\|_{U_n}+2\|\Psi'\|_{U_n}\Big),\end{aligned}$$

Thus

$$|R_1(v)|\le 2KU_n^{1-c}\varepsilon_n\Big(2+\|\Psi''\|_{U_n}+\|\Psi'\|^2_{U_n}+3\|\Psi'\|_{U_n}\Big).$$

Since

$$\Psi''(u)=-2\int_0^\infty \mathcal{K}^2(x)\cdot\mathcal{F}[\overline{\nu}](u\mathcal{K}(x))\,dx,$$

it holds for any $z\in\mathbb{C}$

$$\begin{aligned}\int_{U_n}^\infty \Psi''(u)u^{-z}\,dy &= -2\int_0^\infty \mathcal{K}^2(x)\left[\int_{U_n}^\infty \mathcal{F}[\overline{\nu}](u\mathcal{K}(x))u^{-z}\,du\right]dx\\ &= -2\int_0^\infty [\mathcal{K}(x)]^{z+1}\left[\int_{U_n\mathcal{K}(x)}^\infty \mathcal{F}[\overline{\nu}](v)v^{-z}\,dv\right]dx.\end{aligned}$$

Next, for any fixed $x\in\mathbb{R}$ we can upper bound the internal integral in the right-hand side of the last formula:

$$\begin{aligned}&\left|\int_{U_n\mathcal{K}(x)}^\infty \mathcal{F}[\overline{\nu}](v)v^{-z}\,dv\right|\\ &\qquad\le (1+U_n\mathcal{K}(x))^{-\alpha}\cdot\int_0^\infty v^{-\mathrm{Re}(z)}(1+v)^\alpha\,|\mathcal{F}[\overline{\nu}](v)|\,dv.\end{aligned}$$

Due to (11) we get we get that for any z with $\mathrm{Re}(z)\in(0,1)$ it holds

$$\int_0^\infty v^{-\mathrm{Re}(z)}(1+v)^\alpha\,|\mathcal{F}[\overline{\nu}](v)|\,dv<\frac{\bar\delta}{1-\mathrm{Re}(z)}+A$$

with $\bar\delta=2^\alpha\int_{\mathbb{R}_+}x^2\nu(x)dx\le 2^\alpha A$, see (13). Finally, we conclude that

$$\begin{aligned}|R_2(v)|:=\left|\int_{U_n}^\infty \Psi''(y)y^{-c-\mathrm{i}v}\,dy\right| &\le 2\left(\frac{\delta}{1-c}+A\right)\\ &\quad\times\int_{\mathbb{R}_+}[\mathcal{K}(x)]^{c+1}\,(1+U_n\mathcal{K}(x))^{-\alpha}\,dx.\end{aligned}$$

Upper bound for the last term in (A.3) follows from the assumption on the Mellin transform of the function $\bar{\nu}$. Since (12) is assumed, it holds

$$\left|\int_{\{|u|\geq V_n\}} \mathcal{M}[\overline{\nu}](c+\mathrm{i}u)\, x^{-\mathrm{i}u}\, du\right|$$
$$\leq e^{-\gamma V_n} \int_{\{|u|\geq V_n\}} e^{\gamma V_n}\, |\mathcal{M}[\overline{\nu}](c+\mathrm{i}u)|\, du \leq A e^{-\gamma V_n}.$$

Due to the formula (6), we get

$$|\Phi(u)| = \exp\left\{\int_{\mathbb{R}} \mathrm{Re}[\psi(u\mathcal{K}(x))]\, dx\right\}.$$

Next, we show that there exists a constant $c_0 > 0$ such that

$$1 - \cos(w) \leq c_0 |w|^\beta$$

for any $w \in \mathbb{R}$. In fact, the function $g(w) := (1-\cos(x))/x^\beta, x > 0$, attains its minimal value (equal to 0) at $x = 4\pi k,\ k = 0, 1, 2, ...$, and its points of local maximum are the solutions of the equation $\tan(x/2) = x/\beta$. The global maximum is attained at the smallest positive solution of the aforementioned equation, and therefore it is less than π. Then

$$\frac{1-\cos(x)}{x^\beta} \leq \frac{1}{2} x^{2-\beta} \leq \frac{1}{2}\pi^{2-\beta} =: c_0.$$

We derive

$$\begin{aligned} \mathrm{Re}\,\psi(v) &= -\int_{\mathbb{R}_+} (1-\cos(vy))\, \nu(y)\, dy \\ &\geq -c_0 |v|^\beta \int_{\mathbb{R}_+} y^\beta \nu(y)\, dy \end{aligned}$$

for any $v > 0$. Therefore

$$\int_{\mathbb{R}_+} \mathrm{Re}[\psi(u\mathcal{K}(x))]\, dx \geq -c_1 |u|^\beta \int_{\mathbb{R}} (\mathcal{K}(x))^\beta\, dx$$

with $c_1 = c_0 \cdot \int_{\mathbb{R}_+} y^\beta \nu(y)\, dy \leq A\pi^{2-\beta}/2$, and

$$\min_{u\in[-U_n,U_n]} |\Phi(u)| \geq \exp\left\{-c_1 U_n^\beta \int_{\mathbb{R}} (\mathcal{K}(x))^\beta dx\right\}. \tag{A.4}$$

Appendix B. Proof of Corollary 1

For the sake of simplicity we consider the case $\rho = 1$. We divide the proof into several steps.

1. *Upper bound for* $\Lambda_n := \int_{\mathbb{R}_+} \left[\mathcal{K}(x)\right]^{c+1} \left[1 + U_n\mathcal{K}(x)\right]^{-\alpha} dx$**.** Note that the function $\mathcal{K}(x) = x^r e^{-x}$ has two intervals of monotonicity on $\mathbb{R}_+$: $[0, r]$ and $[r, \infty)$. Denote the corresponding inverse functions by $g_1 : [0, r^r e^{-r}] \to [0, r]$ and $g_2 : [0, r^r e^{-r}] \to [r, \infty)$. Then

$$\begin{aligned}
\Lambda_n &= \left(\int_0^r + \int_r^\infty\right) [\mathcal{K}(x)]^{c+1} \left[1 + U_n\mathcal{K}(x)\right]^{-\alpha} dx \\
&= \int_0^{r^r e^{-r}} w^{c+1} \left(1 + U_n w\right)^{-\alpha} g_1'(w) dw \\
&\qquad + \int_{r^r e^{-r}}^0 w^{c+1} \left(1 + U_n w\right)^{-\alpha} g_2'(w) dw \\
&= \int_0^{r^r e^{-r}} w^{c+1} \left(1 + U_n w\right)^{-\alpha} G(w) dw \\
&= U_n^{-c-2} \left(\int_0^1 + \int_1^{r^r e^{-r} U_n}\right) y^{c+1} \left(1 + y\right)^{-\alpha} \cdot G(y/U_n)\, dy \\
&\qquad =: J_1 + J_2,
\end{aligned}$$

where $G(\cdot) = g_1'(\cdot) - g_2'(\cdot)$. In what follows, we separately analyze the summands J_1 and J_2.

1a. *Upper bound for* J_1**.** Clearly, the behavior of the function $G(\cdot)$ at zero is crucial for the analysis of J_1. Since $\mathcal{K}(g_1(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we get $g_1(0) = 0$ and moreover as $y \to 0$,

$$g_1'(y) = \frac{1}{\mathcal{K}'(g_1(y))} = \frac{1}{[g_1(y)]^{r-1} e^{-g_1(y)} \left(r - g_1(y)\right)} \asymp \frac{1}{r[g_1(y)]^{r-1}}.$$

Analogously, due to $\mathcal{K}(g_2(y)) = y$ for any $y \in [0, r^r e^{-r}]$, we conclude that $\lim_{y\to 0} g_2(y) = +\infty$, and as $y \to 0$

$$\begin{aligned}
g_2'(y) &= \frac{1}{[g_2(y)]^{r-1} e^{-g_2(y)} \left(r - g_2(y)\right)} \\
&\qquad \asymp \frac{-1}{[g_2(y)]^r e^{-g_2(y)}} = \frac{-1}{\mathcal{K}(g_2(y))} = \frac{-1}{y}.
\end{aligned}$$

For further analysis of the asymptotic behaviour of $g_1(\cdot)$ we apply the asymptotic iteration method. We are interested in the behaviour of the solution $g_1(y)$ of the equation

$$f(x) := x^r e^{-x} - y = 0$$

as $y \to 0$. Note that the distinction between the solutions is in the asymptotic behaviour as $y \to 0$: $g_1(y) \to 0$, $g_2(y) \to \infty$. Let us iteratively apply the recursion

$$\varphi_{n+1} = \varphi_n - \frac{f(\varphi_n)}{f'(\varphi_n)} = \varphi_n - \frac{\varphi_n^r e^{-\varphi_n} - y}{\varphi_n^{r-1} e^{-\varphi_n} (r - \varphi_n)}, \qquad n = 1, 2, ...$$

Motivated by the power series expansion of the function e^{-x} at zero,

$$x^r e^{-x} = x^r - x^{r+1} + \frac{1}{2} x^{r+2} + o(x^{r+2}),$$

we take for the initial approximation of $g_1(y)$, the function $\varphi_0 = y^{1/r}$. Then

$$\begin{aligned} \varphi_1(y) &= y^{1/r} - \frac{y e^{-y^{1/r}} - y}{y^{(r-1)/r} e^{-y^{1/r}} (r - y^{1/r})} \\ &= y^{1/r} \left(1 - \frac{e^{-y^{1/r}} - 1}{e^{-y^{1/r}} (r - y^{1/r})} \right) \\ &= y^{1/r} + O(y^{2/r}). \end{aligned}$$

Finally, we conclude that as $y \to 0$,

$$G(y) = \frac{1}{r y^{(r-1)/r}} (1 + o(1)) + \frac{1}{y} (1 + o(1)) = \frac{1}{y} (1 + o(1)).$$

Therefore J_1 can be upper bounded as follows:

$$J_1 \leq C_3 U_n^{-c-1} \int_0^1 y^c (1 + y)^{-\alpha} (1 + o(1)) \, dy.$$

The integral in the right-hand side converges iff $\int_0^1 y^c dy < \infty$. Since $c \in (0, 1)$, we get $J_1 \lesssim U_n^{-c-1}$.

1b. *Asymptotic behaviour of* J_2. Analogously, the asymptotic behavior of J_2 crucially depends on the behavior of $G(y)$ at the point $y = r^r e^{-r}$. Note that as $y \to r^r e^{-r}$,

$$g_k'(y) = \frac{1}{\mathcal{K}'(g_k(y))} = \frac{1}{[g_k(y)]^{r-1} e^{-g_k(y)} (r - g_k(y))} \asymp \frac{C}{r - g_k(y)}$$

for $k=1,2$. Taking logarithms of both parts of the equation $x^r e^{-x} = y$ and changing the variables $u = x - r$ and $\delta = r^r e^{-r} - y$, we arrive at the equality

$$u = r \log\left(1 + \frac{u}{r}\right) - \log\left(1 - \frac{\delta}{r^r e^{-r}}\right).$$

Consider this equality as $u \to 0$ and $\delta \to 0+$, we get

$$u = r\left(\frac{u}{r} - \frac{1}{2}\frac{u^2}{r^2}\right) + \frac{\delta}{r^r e^{-r}} + O\left(\delta^2\right) + O(u^3),$$

and therefore

$$u = \pm\sqrt{2r^{1-r}e^r} \cdot \sqrt{\delta} + O\left(\delta\right) + O(u^{3/2})$$

corresponding to the functions g_1 and g_2. Finally, we conclude

$$|G(y)| \asymp \frac{C\sqrt{2}}{\sqrt{r^{1-r}e^r}} \frac{1}{\sqrt{r^r e^{-r} - y}}, \qquad y \to r^r e^{-r},$$

and therefore

$$J_2 \;\sim\; U_n^{-c-3/2} \int_1^{r^r e^{-r} U_n} y^{c+1}\left(1+y\right)^{-\alpha} \cdot \frac{1}{\sqrt{r^r e^{-r} U_n - y}}\, dy.$$

We change the variable in the last integral:

$$z = \sqrt{\frac{r^r e^{-r} U_n - 1}{r^r e^{-r} U_n - y}}, \qquad y = r^r e^{-r} U_n + \frac{1 - r^r e^{-r} U_n}{z^2},$$

and get with $\widetilde{U}_n = r^r e^{-r} U_n$

$$J_2 \asymp U_n^{-c-3/2} \int_1^{\infty} \left(\widetilde{U}_n + \frac{1-\widetilde{U}_n}{z^2}\right)^{c+1} \cdot \left(1 + \widetilde{U}_n + \frac{1-\widetilde{U}_n}{z^2}\right)^{-\alpha} \cdot \frac{z}{\sqrt{\widetilde{U}_n - 1}} \frac{2(\widetilde{U}_n - 1)}{z^3} dz.$$

Therefore,

$$J_2 \asymp C_4 U_n^{-c-3/2} \widetilde{U}_n^{c+1} \left(\widetilde{U}_n + 1\right)^{-\alpha} \sqrt{\widetilde{U}_n - 1}, \qquad n \to \infty,$$

with some constant $C_4 > 0$ and we conclude that $J_2 \asymp C_5 U_n^{-\alpha}$ as $n \to \infty$. To sum up, $\Lambda_n \lesssim U_n^{-\min(\alpha,c+1)} = U_n^{-\alpha}$ as $n \to \infty$.

2. *Upper bound for* $H_n := \int_{\{|v|\le V_n\}} |Q(1-c-\mathbf{i}v)|^{-1}\, dv$**.** Recall that

$$H_n = \int_{\{|v|\le V_n\}} \frac{e^{-\pi v/2}}{\left|\Gamma(1-c-\mathrm{i}v)\right| \cdot \left|\int_{\mathbb{R}} (\mathcal{K}(x))^{c+1+\mathrm{i}v}\, dx\right|}\, dv$$

Note that for our choice of the function $\mathcal{K}(\cdot)$, it holds for any $z \in \mathbb{C}$

$$\int_{\mathbb{R}} (K(x))^z\, dx = 2\int_{\mathbb{R}_+} (x^r e^{-x})^z dx = 2\left[\lim_{R\to+\infty} \int_{\gamma_R(z)} u^{rz} e^{-u} du\right] \cdot z^{-(rz+1)},$$

where $\gamma_R(z)$ is the part of the complex line $\{(x\,\mathrm{Re}(z), x\,\mathrm{Im}(z)),\ x \in [0,R]\}$. Note that due to the Cauchy theorem, for any z with positive real part

$$\int_{\mathbb{R}_+} u^{rz} e^{-\rho u} du = \lim_{R\to+\infty} \int_{\gamma_R(z)} u^{rz} e^{-u} du + \lim_{R\to+\infty} \int_{c_R} u^{rz} e^{-u} du \qquad \text{(B.1)}$$

with $c_R := \{(R\cos(\theta), R\sin(\theta)), \theta \in (0, \arctan(\mathrm{Im}(z)/\mathrm{Re}(z))\}$. Since the last limit in (B.1) is equal to 0, we conclude that

$$\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx = 2\,\Gamma\Big(r(c+1) + 1 + \mathrm{i}vr\Big) \cdot e^{-(r(c+1)+1+\mathrm{i}vr)\cdot \log(c+1+\mathrm{i}v)}.$$

Next, using the fact that there exists a constant $\bar{C} > 0$ such that $|\Gamma(\alpha + \mathrm{i}\beta)| \ge \bar{C}|\beta|^{\alpha-1/2} e^{-|\beta|\pi/2}$ for any $\alpha \ge -2, |\beta| \ge 2$ (see Corollary 7.3 from [13]), we get that

$$\frac{e^{-\pi v/2}}{|\Gamma(1-c-\mathrm{i}v)|} \le v^{c-1/2},$$

and moreover

$$\left|\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx\right| \;=\; 2\frac{|\Gamma(r(c+1)+1+\mathrm{i}vr)|}{((c+1)^2+v^2)^{(r(c+1)+1)/2}\, e^{-vr\arctan(v/(c+1))}}.$$

The asymptotic behavior of the last expression depends on the value r. More precisely,

$$\left|\int_{\mathbb{R}} (K(x))^{c+1+\mathrm{i}v}\, dx\right| \sim \begin{cases} 2\frac{c(vr)^{r(c+1)+1/2} e^{-vr\pi/2}}{((c+1)^2+v^2)^{(r(c+1)+1)/2} e^{-vr\arctan(v/(c+1))}} \sim v^{-1/2}, \\ \qquad\qquad \text{if } r = 1,2,..., \\ v^{-1}, \qquad\qquad \text{if } r = 0. \end{cases}$$

as $v \to +\infty$. Finally, we conclude that $H_n \lesssim V_n^{c+1}$, if $r = 1,2,...,$ and $H_n \lesssim V_n^{c+3/2}$ if $r = 0$.

Appendix C. Mixing properties of Levy based MA processes

Theorem 3. *Let (L_t) be a Lévy subordinator with Lévy measure ν on $\mathbb{R}_+$. Consider a Lévy-based moving average process of the form*

$$Z_t = \int K(s-t)\, dL_t$$

with a symmetric kernel K. Fix some $\Delta > 0$ and denote

$$Z_S \doteq (Z_{j\Delta})_{j \in S}$$

for any subset S of $\{1, \ldots, n\}$. Fix two natural numbers m and p such that $m + p \leq n$. For any subsets $S \subseteq \{1, \ldots, m\}$ and $S' \subseteq \{p+m, \ldots, n\}$, let g and g' be two real valued functions on $\mathbb{R}^{|S|}$ and $\mathbb{R}^{|S'|}$ satisfying

$$\max\left\{ \left\| e^{-R_S^\top \cdot} g \right\|_{L^1}, \left\| e^{-R_{S'}^\top \cdot} g' \right\|_{L^1} \right\} < \infty$$

for some $R_S \in \mathbb{R}_+^{|S|}$ and $R_{S'} \in \mathbb{R}_+^{|S'|}$, and denote $C_\circ := \left\| e^{-R_S^\top \cdot} g \right\|_{L^1} \cdot \left\| e^{-R_{S'}^\top \cdot} g' \right\|_{L^1}$. Suppose that the Fourier transform $\widehat{K}$ of K fulfils

$$K^* \doteq \sum_{j=-\infty}^{\infty} \left| \widehat{K}\left(2\pi \frac{j}{\Delta} \right) \right| < \infty$$

and

$$\int_{|x|>1} e^{R^* x} \nu(dx) < A_R, \quad \int_{|x| \leq \varepsilon} x^2 e^{R^* x} \nu(dx) \geq B_R \varepsilon^{2-\beta}$$

for $R^* = \frac{\|R_{S \cup S'}\|_\infty K^*}{\Delta}$ and any $\varepsilon \in (0,1)$. Then

$$\begin{aligned} |\mathrm{Cov}\,(g(Z_S), g'(Z_{S'}))| \quad &\leq \quad C_R C_\circ \max_{|l|>p} (K \star K)(l\Delta), \\ &\int \|u_{S \cup S'} - i R_{S \cup S'}\|^2 \exp\left(-\varkappa B_R \|u_{S \cup S'}\|^\beta \right) du_{S \cup S'}, \end{aligned}$$

where $\varkappa \doteq \left(\int \lambda_{S \cup S'}^{\beta/2}(s)\, ds \right)$ and $\lambda_S(t)$ stands for minimal eigenvalue of the positive definite matrix

$$\mathcal{K}_S(t) \doteq (K(t - j\Delta) K(t - l\Delta))_{j,l \in S}.$$

Proof. We have for any $S \subseteq \{1, \ldots, n\}$

$$\begin{aligned}
\Phi_S(u_S - iR_S) &\doteq \mathrm{E}\left[\exp\left(i\sum_{j\in S} u_j Z_{j\Delta} + \sum_{j\in S} R_j Z_{j\Delta}\right)\right] \\
&= \exp\left(\int \psi\left(\sum_{j\in S}(u_j - iR_j)\, K(t - j\Delta)\right) dt\right),
\end{aligned}$$

where $u_S \doteq (u_j \in \mathbb{R},\, j \in S)$ and $R_S \doteq (R_j \in \mathbb{R}_+,\, j \in S)$, provided

$$\mathrm{E}\left[\exp\left(\sum_{j\in S} R_j Z_{j\Delta}\right)\right] < \infty.$$

Denote for any subsets $S \subseteq \{1, \ldots, m\}$ and $S' \subseteq \{p+m, \ldots, n\}$,

$$D(u_S - iR_S, u_{S'} - iR_{S'}) \doteq \Phi_{S,S'}(u_S - iR_S, u_{S'} - iR_{S'}) - \Phi_S(u_S - iR_S)\Phi_{S'}(u_{S'} - iR_{S'}),$$

where it is assumed that

$$\mathrm{E}\left[\exp\left(\sum_{j\in S\cup S'} R_j Z_{j\Delta}\right)\right] < \infty$$

Then using the inequality $|e^z - e^y| \le (|e^z| \vee |e^y|)\, |y - z|\,,\ y, z \in \mathbb{C}$, we derive

$$\begin{aligned}
|D(u_S - iR_S, u_{S'} - iR_{S'})| \le\ & \{|\Phi_{S,S'}(u_S - iR_S, u_{S'} - iR_{S'})| \vee |\Phi_S(u_S - iR_S)\Phi_{S'}(u_{S'} - iR_{S'})|\} \times \\
& \left|\int \left\{\psi\left(\sum_{j\in S\cup S'}(u_j - iR_j)\, K(x - j\Delta)\right) - \psi\left(\sum_{j\in S}(u_j - iR_j)\, K(x - \right.\right.\right. \\
& \left.\left.\left. -\psi\left(\sum_{j\in S'}(u_j - iR_j)\, K(x - j\Delta)\right)\right\} dx\right|.
\end{aligned}$$

Due to Lemma 1 and the Poisson summation formula, we derive

$$\begin{aligned}
|D(u_S - iR_S, u_{S'} - iR_{S'})| \le\ & \{|\Phi_{S,S'}(u_S - iR_S, u_{S'} - iR_{S'})| \vee |\Phi_S(u_S - iR_S)\Phi_{S'}(u_{S'} - iR_{S'})|\} \times \\
& \left[\sum_{j\in S}\sum_{l\in S'} |(u_l - iR_l)\,(u_j - iR_j)|\, (K \star K)\,((j-l)\Delta)\right] \\
& \times \int y^2 e^{\frac{y\|R\|_\infty K^*}{\Delta}}\, \nu(dy).
\end{aligned}$$

□

We have

$$\operatorname{Cov}\left(g(Z_S), g'(Z_{S'})\right) = \int_{\mathbb{R}_+^{|S|}} \int_{\mathbb{R}_+^{|S'|}} g(x_S) g'(x_{S'}) \left(p_{S,S'}(x_S, x_{S'}) - p_S(x_S) p_{S'}(x_{S'})\right) \, dx_S \, dx_{S'}.$$

and the Parseval's identity implies

$$\begin{aligned} \operatorname{Cov}\left(g(Z_S), g(Z_{S'})\right) &= \frac{1}{(2\pi)^{|S|+|S'|}} \int_{\mathbb{R}^{|S|}} \int_{\mathbb{R}^{|S'|}} \widehat{g}(iR_S - u_S) \widehat{g'}(iR_{S'} - u_{S'}). \\ & \times D(u_S - iR_S, u_{S'} - iR_{S'}) \, du_S \, du_{S'}. \end{aligned}$$

Hence

$$\left|\operatorname{Cov}\left(g(Z_S), g'(Z_{S'})\right)\right| \leq C_\circ \int_{\mathbb{R}^{|S|}} \int_{\mathbb{R}^{|S'|}} \left|D(u_S - iR_S, u_{S'} - iR_{S'})\right| \, du_S \, du_{S'}.$$

Furthermore, for any set $S \in \{1, \ldots, n\}$, we have due to Lemma~2

$$\begin{aligned} \psi\left(\sum_{j\in S} (u_j - iR_j) \, K(s - j\Delta)\right) &\leq A_R - B_R \left|\sum_{j\in S} u_j K(s - j\Delta)\right|^\beta \\ &= A_R - B_R \left(\sum_{j,l\in S} u_j u_l \mathcal{K}_{jl}(s)\right)^{\beta/2} \\ &\leq A_R - B_R \lambda_S^{\beta/2}(s) \left(\sum_{j\in S} u_j^2\right)^{\beta/2}. \end{aligned}$$

As a result

$$|\Phi_S(u_S - iR_S)| \leq C_R \exp\left(-B_R \left(\int \lambda_S^{\beta/2}(s) \, ds\right) \|u_S\|^{\beta/2}\right)$$

and

$$\begin{aligned} \left|D(u_S - iR_S, u_{S'} - iR_{S'})\right| &\leq \max_{|l|>p} (K \star K)(l\Delta) \sum_{j\in S} \sum_{l\in S'} \left|(u_l - iR_l)(u_j - iR_j)\right| \\ & \exp\left(-C_R \left(\int \lambda_{S\cup S'}^{\beta/2}(s) \, ds\right) \left(\sum_{j\in S\cup S'} u_j^2\right)^{\beta/2}\right). \end{aligned}$$

Lemma 1. *Set*

$$\psi(z)=\int_0^\infty(\exp(zx)-1)\nu(dx)$$

for any $z\in\mathbb{C}$*, such that the integral* $\int_{|x|>1}\exp(Re(z)x)\nu(dx)$ *is finite. Then*

$$|\psi(z_1+z_2)-\psi(z_1)-\psi(z_2)|\le 2\,|z_1|\,|z_2|\int x^2e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx),$$

provided the integral $\int x^2e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx)$ *is finite.*

Proof. We have

$$\begin{aligned}\psi(z_1+z_2)&-\psi(z_1)-\psi(z_2)\\&=\int_0^\infty(\exp(z_1x)-1)(\exp(z_2x)-1)\nu(dx)\\&=\int_0^\infty(\exp((z_1+z_2)x)-\exp(z_1x)-\exp(z_2x)+1)\nu(dx)\end{aligned}$$

Since

$$\begin{aligned}|\exp(z)-1|&=\left|e^{\mathrm{Re}(z)}e^{i\mathrm{Im}(z)}-1\right|\\&=\left|e^{\mathrm{Re}(z)}\left(e^{i\mathrm{Im}(z)}-1\right)+e^{\mathrm{Re}(z)}-1\right|\\&\le|\mathrm{Im}(z)|\,e^{\mathrm{Re}(z)}+\left|e^{\mathrm{Re}(z)}-1\right|\\&\le(|\mathrm{Re}(z)|+|\mathrm{Im}(z)|)\,e^{\mathrm{Re}(z)}\\&\le\sqrt{2}\,|z|\,e^{\mathrm{Re}(z)},\end{aligned}$$

we get

$$\begin{aligned}|\psi(z_1+z_2)-\psi(z_1)-\psi(z_2)|&\le\int_0^\infty|\exp(z_1x)-1|\,|\exp(z_2x)-1|\,\nu(dx)\\&\le 2\,|z_1|\,|z_2|\int x^2e^{x(\mathrm{Re}(z_1)+\mathrm{Re}(z_2))}\nu(dx).\end{aligned}$$

□

Lemma 2. *Let* (L_t) *be a Levy process with Lévy triplet* $(b,0,\nu)$*. Assume that the Lévy measure* ν *satisfyes*

$$\int_{|x|>1}e^{Rx}\nu(dx)<A_R,\qquad\int_{|x|\le\varepsilon}x^2e^{Rx}\nu(dx)\ge B_R\varepsilon^{2-\beta}\tag{C.1}$$

for some $\beta \in (0,1)$, $R > 0$ *and any* $\varepsilon \in (0,1)$. *Then it holds for* $\psi(u) = \log\left(t^{-1}\mathrm{E}\left[\exp\left(iuL_t\right)\right]\right)$,

$$\mathrm{Re}\left[\psi(u - iR)\right] \leq C_1 - C_2|u|^{\beta}, \quad |u| > 1$$

where $C_1 := bR + A_R + \left(e^R + 1/2\right) R^2 \int_{|x|<1} x^2 \nu(dx)$, *and* $C_2 = B_R/2$.

Proof. Note that

$$\begin{aligned}
\mathrm{Re}\left[\psi(u - \mathrm{i}R)\right] &= bR + \int_R \left(e^{Rx}\cos(ux) - 1 - Rx\mathbb{I}\left\{|x| < 1\right\}\right)\nu(dx) \\
&= bR + \int_{|x|>1} \left(e^{Rx}\cos(ux) - 1\right)\nu(dx) \\
&\qquad + \int_{|x|<1} \left(e^{Rx}\cos(ux) - 1 - Rx\right)\nu(dx) \\
&= bR + I_1 + I_2.
\end{aligned}$$

Next, we separately consider the integrals I_1 and I_2.

$$\begin{aligned}
I_1 := \int_{|x|>1} \left(e^{Rx}\cos(ux) - 1\right)\nu(dx) \leq \int_{|x|>1} \left(e^{Rx} - 1\right)\nu(dx) \\
\leq \int_{|x|>1} e^{Rx}\nu(dx) < A_R
\end{aligned}$$

due to the first assumption in (C.1). As for the second integral, we get

$$\begin{aligned}
I_2 &:= \int_{|x|<1} \left(e^{Rx}\cos(ux) - 1 - Rx\right)\nu(dx) \\
&= -\int_{|x|<1} (1 - \cos(ux))\,\overline{\nu}(dx) + \int_{|x|<1} \left(e^{Rx} - 1 - Rx\right)\nu(dx) \\
&\leq -\int_{|x|<1/u} (1 - \cos(ux))\,\overline{\nu}(dx) + \int_{|x|<1} \left(e^{Rx} - 1 - Rx\right)\nu(dx),
\end{aligned}$$

where $u \geq 1$ and $\overline{\nu}(dx) = e^{Rx}\nu(dx)$. Since $1 - \cos(w) \geq |w|^2/2$ for any $w \in (0,1)$, we get

$$\int_{|x|<1/u} (1 - \cos(ux))\,\overline{\nu}(dx) \geq \frac{u^2}{2}\int_{|x<1/u} x^2\overline{\nu}(dx) \geq \frac{B_R}{2}u^{\beta}$$

due to the second assumption in (C.1). Moreover,

$$\begin{aligned}\int_{|x|<1}\left(e^{Rx}-1-Rx\right)\nu(dx) &= \left(\int_{-1}^{0}+\int_{0}^{1}\right)\left(e^{Rx}-1-Rx\right)\nu(dx)\\ &\le \frac{R^2}{2}\int_{-1}^{0}x^2\nu(dx)+R^2\int_0^1 e^{Rx}x^2\nu(dx)\\ &< \left(1/2+e^R\right)R^2\int_{|x|<1}x^2\nu(dx),\end{aligned}$$

where we use the inequalities $e^x < 1+x+x^2/2,\ x<0$, and $e^x < 1+x+x^2e^x$, $x\in(0,1)$. Combining all of the derived inequalities, we arrive at the desired result. □

Lemma 3. *Let* $K(x)=|x|^r e^{-|x|}$ *with some* $r\in\mathbb{N}\cup\{0\}$ *and* $K_t(s)=K(s-t),\ \forall s,t\in\mathbb{R}_+$ *. Then*

$$\int_{\mathbb{R}} K_{\Delta j}(v)K_{\Delta k}(v)dv \le \kappa_0\,(j-k)^{\kappa_1}e^{-\kappa_2(j-k)},\quad \forall\, j>k \tag{C.2}$$

with $\kappa_2=\Delta,\ \kappa_1=2r+1,$ *and*

$$\kappa_0 = (2r+3)\max\left\{\frac{\Delta^{2r+1}}{2^{2r}},\max_{m=0..r}\left\{C_r^m(r+m)!\frac{\Delta^{r-m}}{2^{r+m+1}}\right\}\right\}.$$

Proof. Integral in the l.h.s. of (C.2) can be decomposed into the sum of three terms:

$$\int_{\mathbb{R}} K_{\Delta j}(v)K_{\Delta k}(v)dv = \left(\int_{-\infty}^{\Delta k}+\int_{\Delta k}^{\Delta j}+\int_{\Delta j}^{\infty}\right)K_{\Delta j}(v)K_{\Delta k}(v)dv$$
$$=: I_1+I_2+I_3.$$

In the sequel we separately consider integrals I_1, I_2, I_3.

$$\begin{aligned}I_1 &= \int_{\Delta j}^{\infty}(v-\Delta j)^r(v-\Delta k)^r e^{-2v+\Delta(j+k)}dv = \left[v-\Delta j=u\right]\\ &= \int_{\mathbb{R}_+}u^r\left(u+\Delta(j-k)\right)^r e^{-2u-\Delta(j-k)}du\\ &= e^{-\Delta(j-k)}\int_{\mathbb{R}_+}u^r\left(\sum_{m=0}^{r}C_r^m u^m\left(\Delta(j-k)\right)^{r-m}\right)e^{-2u}du\\ &= \left[\sum_{m=0}^{r}C_r^m(r+m)!\frac{\Delta^{r-m}}{2^{r+m+1}}(j-k)^{r-m}\right]e^{-\Delta(j-k)},\end{aligned}$$

because $\int_{\mathbb{R}_+} u^{r+m} e^{-2u} du = 2^{-(r+m+1)}\Gamma(r+m+1) = 2^{-(r+m+1)}(r+m)!$.

$$I_2 = \int_{\Delta k}^{\Delta j} \left[-\left(v-\Delta j\right)\left(v-\Delta k\right)\right]^r e^{-\Delta(j-k)} dv$$
$$\leq \frac{\Delta^{2r+1}}{2^{2r}} \left(j-k\right)^{2r+1} e^{-\Delta(j-k)},$$

because maximum of the quadratic function $f(v) := -\left(v-\Delta j\right)\left(v-\Delta k\right)$ is attained at the point $v = \Delta\left(k+j\right)/2$ and is equal to $\left(\Delta^2/4\right)\left(j-k\right)^2$.

$$\begin{aligned} I_3 &= \int_{-\infty}^{\Delta k} \left(\Delta j - v\right)^r \left(\Delta k - v\right)^r e^{2v-\Delta(j+k)} dv = \Big[\Delta k - v = u\Big] \\ &= \int_{\mathbb{R}_+} \left(u + \Delta(j-k)\right)^r u^r e^{-2u-\Delta(j-k)} du = I_1. \end{aligned}$$

□

Appendix D. Proof of Theorem 2

The proof basically follows the same lines as the proof of Proposition 3.3 from [14]. First note that due to (A.4),

$$\max_{|u|\leq U_n} \frac{|\Phi_n(u) - \Phi(u)|}{|\Phi(u)|}$$
$$\leq \exp\left\{c_1 U_n^{\beta} \int_{\mathbb{R}} (\mathcal{K}(x))^{\beta} dx\right\} \cdot \max_{|u|\leq U_n} \Big|\Phi_n(u) - \Phi(u)\Big|.$$

Next, we separately consider the real and imaginary parts of the difference between $\Phi_n(u)$ and $\Phi(u)$. Denote

$$S_n(u) := n \operatorname{Re}\left(\Phi_n(u) - \Phi(u)\right) = \sum_{k=1}^{n} \left[\cos\left(u Z_{k\Delta}\right) - \mathbb{E}\left[\cos\left(u Z_{k\Delta}\right)\right]\right]$$

Since $S_n(u)$ is a sum of i.i.d. centered real-valued random variables, bounded by 2, there exist a positive constant c_1 such that

$$\mathbb{P}\left\{|S_n(u)| \geq x\right\} \leq \exp\left\{\frac{-c_1 x^2}{2n + x\log(n)\log\log(n)}\right\}, \quad \forall\, x \geq 0, \qquad \text{(D.1)}$$

see Theorem 1 from [15]. In order to apply classic chaining argument, we divide the interval $[-U_n, U_n]$ by $2J$ equidistant points $(u_j) =: \mathcal{G}$, where $u_j = U_n(-J+j)/J, \quad j = 1..(2J)$. Applying (D.1), we get for any $x \geq 0$,

$$\mathbb{P}\left\{\max_{u_j \in \mathcal{G}} |S_n(u_j)| \geq x/2\right\} \leq 2J \exp\left\{\frac{-c_2 x^2}{4n + x\log(n)\log\log(n)}\right\}, \qquad \text{(D.2)}$$

where $c_2 = c_1/2$. Note that for any $u \in [-U_n, U_n]$ there exists a point $u^\star \in \mathcal{G}$ such that $|u - u^\star| \leq U_n/J$ and therefore for all $k \in 1..n$

$$|\cos(uZ_{k\Delta}) - \cos(u^\star Z_{k\Delta})| \leq |Z_{k\Delta}| \cdot |u - u^\star| \leq |Z_{k\Delta}| \cdot U_n/J.$$

Next, we get

$$\begin{aligned}\mathbb{P}&\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\} \\ &\leq \mathbb{P}\left\{\max_{u_j \in \mathcal{G}} |S_n(u_j)| \geq x/2\right\} + \mathbb{P}\left\{\sum_{k=1}^n \left(|Z_{k\Delta}| + \mathbb{E}\left[|Z_{k\Delta}|\right]\right) U_n/J \geq x/2\right\}.\end{aligned}$$

Applying (D.2) and the Markov inequality, we arrive at

$$\begin{aligned}\mathbb{P}&\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\} \\ &\qquad \leq 2J \exp\left\{\frac{-c_2 x^2}{4n + x\log(n)\log\log(n)}\right\} + \frac{4U_n}{xJ} n\mathbb{E}\left|Z_\Delta\right|,\end{aligned}$$

where $\mathbb{E}\left[Z_\Delta\right] = i^{-1}\Phi'(0) = 2i^{-1}\psi'(0) \cdot \int_{\mathbb{R}} K(s)ds$ due to (6), and $Z_\Delta \geq 0$ a.s. since L is a subordinator. The choice

$$J \quad = \quad \sqrt{\frac{U_n n}{x} \cdot \exp\left\{\frac{c_2 x^2}{4n + x\log(n)\log\log(n)}\right\}}$$

leads to the order

$$\begin{aligned}\mathbb{P}\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\} &\leq c_3\sqrt{\frac{U_n n}{x}} \exp\left\{\frac{-c_2 x^2}{8n + 2x\log(n)\log\log(n)}\right\} \\ &\leq c_3\sqrt{\frac{U_n n}{x}} \exp\left\{\frac{-c_4 x^2}{n}\right\},\end{aligned}$$

which holds for n large enough with $c_3 = 2\left(1 + \mathbb{E}\left[Z_\Delta\right]\right), c_4 = c_2/9$, provided $x \lesssim n^{1-\varepsilon}$ with some $\varepsilon > 0$. Finally,

$$\mathbb{P}\left\{\max_{|u|\leq U_n} |S_n(u)| \geq x\right\}$$
$$\geq \mathbb{P}\left\{\max_{|u|\leq U_n} \left|\mathrm{Re}\left(\frac{\Phi_n(u) - \Phi(u)}{\Phi(u)}\right)\right| \geq \frac{x}{n}\exp\left\{c_1 U_n^\beta \int_{\mathbb{R}} (\mathcal{K}(x))^\beta dx\right\}\right\}.$$

Therefore, the choice

$$x = Kn\exp\left\{-c_1 U_n^\beta \int_{\mathbb{R}} (\mathcal{K}(x))^\beta dx\right\}\varepsilon_n/2 = K\sqrt{n\log(n)}/2$$

with any positive K leads to

$$\mathbb{P}\left\{\max_{|u|\leq U_n} \left|\mathrm{Re}\left(\frac{\Phi_n(u) - \Phi(u)}{\Phi(u)}\right)\right| \geq \frac{K\varepsilon_n}{2}\right\} \leq \frac{c_5}{\sqrt{K}} \frac{\sqrt{U_n} n^{(1/4)-c_4(K/2)}}{\log(n)}$$

with $c_5 = \sqrt{2}c_4$. Since the same statement holds for the imaginary bound of $\left(\Phi_n(u) - \Phi(u)\right)/\Phi_u$, we arrive at the desired result.

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