

ZHU'S ALGEBRAS, C_2 -ALGEBRAS AND ABELIAN RADICALS

BORIS FEIGIN, EVGENY FEIGIN AND PETER LITTELMANN

ABSTRACT. This paper consists of three parts. In the first part we prove that Zhu's and C_2 -algebras in type A have the same dimensions. In the second part we compute the graded decomposition of the C_2 -algebras in type A , thus proving the Gaberdiel-Gannon's conjecture. Our main tool is the theory of abelian radicals, which we develop in the third part.

INTRODUCTION

The theory of vertex operator algebras plays a crucial role in the mathematical description of the structures arising from the conformal field theories. In particular, the representation theory of VOAs allows one to study correlation functions, partition functions and fusion coefficients of various theories (see [BF], [LL], [K2]). The Zhu's and C_2 -algebras are powerful tools in the representation theory of vertex operator algebras.

Let \mathcal{V} be a vertex operator algebra. In [Z] Zhu introduced certain associative algebra $A(\mathcal{V})$ attached to \mathcal{V} , which turned out to be very important for the representation theory of \mathcal{V} . In particular, the rationality of \mathcal{V} is conjectured to be equivalent to finite-dimensionality and semi-simplicity of $A(\mathcal{V})$. Moreover, for rational VOAs the representations of the vertex algebra and of the Zhu's algebra are in one-to-one correspondence.

There is another algebra (also introduced in [Z]) one can attach to \mathcal{V} . This commutative (Poisson) algebra is called the C_2 -algebra and is denoted by $A_{[2]}(\mathcal{V})$ (we follow the notations from [GG], [GabGod]). The C_2 -algebra as a vector space is the quotient of \mathcal{V} by its C_2 -subspace. The finite-dimensionality condition of $A_{[2]}(\mathcal{V})$ (which is called the C_2 -cofiniteness condition) is a very important characterization of \mathcal{V} ([ABD], [CF], [DM], [M1],[M2]). In particular, it implies that the number of isomorphism classes of irreducible \mathcal{V} -modules is finite and their characters have certain modular properties. We note however that finite-dimensionality of the C_2 -algebra does not mean that \mathcal{V} is rational (see [GK]).

The algebras $A(\mathcal{V})$ and $A_{[2]}(\mathcal{V})$ are very closely related. It has been shown in many examples that the Zhu's and C_2 -algebras have the same dimensions and the former can be regarded as a non-commutative deformation of the latter. But there are also examples where the equality of dimensions does not hold. Note that these two algebras can be included into the larger family of spaces and thus be defined in a uniform way (see [GabGod], [GG]). One

of the special features of the $A_{[2]}(\mathcal{V})$ (compared with Zhu's algebra) is that the C_2 -algebra comes equipped with the special grading, induced by the conformal weights (energy) grading on \mathcal{V} . Thus $A_{[2]}(\mathcal{V}) = \bigoplus_{m \geq 0} A_{[2]}^m(\mathcal{V})$. The spaces $A_{[2]}^m(\mathcal{V})$ play the central role in this paper.

In this paper we are only interested in the case of vertex-operator algebras, associated with affine Kac-Moody algebras on the integer level (see [K2], [BF]). So let \mathfrak{g} be a simple Lie algebra, $\widehat{\mathfrak{g}}$ be the corresponding affine Kac-Moody Lie algebra (see [K1]). For any non-negative integer k (which is called the level) we denote by $\mathcal{V}(\mathfrak{g}; k)$ the vertex operator algebra associated with $\widehat{\mathfrak{g}}$ on the level k . In particular, as a vector space, $\mathcal{V}(\mathfrak{g}; k)$ is isomorphic to the level k basic (vacuum) representation L_k of $\widehat{\mathfrak{g}}$. It is proved in [FZ] that the corresponding Zhu's algebra is isomorphic to the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the two-sided ideal generated by the power of the highest root e_θ :

$$A(\mathcal{V}(\mathfrak{g}; k)) = U(\mathfrak{g}) / \langle e_\theta^{k+1} \rangle.$$

The C_2 -algebra $A_{[2]}(\mathcal{V}(\mathfrak{g}; k))$ is the quotient of L_k by the subspace, spanned by the vectors $(a \otimes t^{-n})v$, where $a \in \mathfrak{g}$, $n \geq 2$ and $v \in L_k$. In particular, each space $A_{[2]}^m(\mathcal{V}(\mathfrak{g}; k))$ is equipped with the structure of a \mathfrak{g} -module. One can see from the definition that $A_{[2]}(\mathcal{V}(\mathfrak{g}; k))$ is the quotient of the symmetric algebra $S(\mathfrak{g}) \simeq S(\mathfrak{g} \otimes t^{-1})$ by some ideal. Therefore, it is a very natural question to ask whether the dimensions of Zhu's and C_2 -algebras do coincide. The answer is believed to be positive in many cases (though to the best of our knowledge is not proved outside the level 1 case). Note however that for \mathfrak{g} of the type E_8 the answer is negative (see [GG]).

We show that if \mathfrak{g} is of type A , then the C_2 -algebra is a degeneration of the Zhu's algebra exactly like $S(\mathfrak{g})$ is the degeneration of $U(\mathfrak{g})$. We also prove the conjecture from [GG], computing the structure of each $A_{[2]}^m(\mathcal{V}(\mathfrak{g}; k))$ as \mathfrak{g} -module (see also [F] for the affine version in type A_1).

Our approach is based on the following observation: let ω_n be the "middle" (in the standard Bourbaki numeration [B]) fundamental weight of the Lie algebra \mathfrak{gl}_{2n} . The representation $V(k\omega_n)$ is equipped with the structure of a \mathfrak{sl}_n -module via the embeddings

$$\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_n \oplus \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{2n}$$

(the first embedding is trivial, the second is the diagonal one and the last embedding comes from the identification of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ with the Levi subalgebra of the parabolic subalgebra $\mathfrak{p}_n \subset \mathfrak{gl}_{2n}$ corresponding to ω_n). We construct an isomorphism of \mathfrak{sl}_n -modules (not of algebras)

$$V(k\omega_n) \simeq \bigoplus_{i=0}^k A_{[2]}(\mathcal{V}(\mathfrak{sl}_n; i),$$

We derive the graded decomposition of the C_2 -algebras by describing $V(k\omega_i)$ as a representation of enveloping algebra of nilpotent radical of \mathfrak{p} , which is isomorphic to the symmetric algebra $S(\mathfrak{gl}_n)$.

The paper is organized as follows:

In Section 1 we fix the main definitions and prove that the dimensions of Zhu's and the C_2 -algebras coincide in type A . The main tool is a deformation argument.

Knowing the dimension formula, in Section 2 we compute the graded decomposition of $A_{[2]}(\mathcal{V}(\mathfrak{sl}_n; k))$ as \mathfrak{sl}_n -module.

In Section 3 we discuss some properties of abelian radicals and spherical representations used in the sections before. This discussion is independent of the type of the algebra and should be helpful to generalize the arguments to other groups of classical types. We work out explicitly the type A case.

1. ZHU'S AND C_2 -ALGEBRAS IN TYPE A

1.1. Definitions. Let \mathfrak{g} be a simple Lie algebra. Let θ be the highest root of \mathfrak{g} and let $e_\theta \in \mathfrak{g}$ be a highest weight vector in the adjoint representation. Fix a non-negative integer k . Let $P_k^+(\mathfrak{g})$ be the set of level k integrable \mathfrak{g} weights, i.e the set of dominant integral \mathfrak{g} weights β satisfying $(\beta, \theta) \leq k$. We denote by $V(\beta)$ the irreducible \mathfrak{g} -module with highest weight β . The following Theorem is proved in [FZ]:

Theorem 1.1. *The level k Zhu's algebra $A(\mathfrak{g}; k)$ is the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the two-sided ideal generated by e_θ^{k+1} :*

$$A(\mathfrak{g}; k) = U(\mathfrak{g}) / \langle e_\theta^{k+1} \rangle.$$

In addition, one has the isomorphism of \mathfrak{g} -modules:

$$A(\mathfrak{g}; k) \simeq \bigoplus_{\beta \in P_k^+(\mathfrak{g})} V(\beta) \otimes V(\beta)^*.$$

The form of the description of $A(\mathfrak{g}; k)$ arises ultimately because of the Peter-Weyl Theorem.

Notation 1.2. Let $S(\mathfrak{g}) = \bigoplus_{m=0}^{\infty} S^m(\mathfrak{g})$ be the symmetric algebra of \mathfrak{g} . For $v \in S^m(\mathfrak{g})$ and $a \in \mathfrak{g}$ let $av \in S^{m+1}(\mathfrak{g})$ be the product in the symmetric algebra. The adjoint action on \mathfrak{g} makes each homogeneous summand $S^m(\mathfrak{g})$ into a \mathfrak{g} -module. For $v \in S^m(\mathfrak{g})$ and $a \in \mathfrak{g}$ we denote by $a \circ v \in S^m(\mathfrak{g})$ the adjoint action of a .

The C_2 -algebra associated with $\mathcal{V}(\mathfrak{g}; k)$ can be described as follows. The level k C_2 -algebra $A_{[2]}(\mathfrak{g}; k)$ is the quotient of the symmetric algebra $S(\mathfrak{g})$ by the ideal generated by the subspace $V_{k+1} = U(\mathfrak{g}) \circ e_\theta^{k+1} \hookrightarrow S^{k+1}(\mathfrak{g})$:

$$A_{[2]}(\mathfrak{g}; k) = S(\mathfrak{g}) / \langle V_{k+1} \rangle.$$

Remark 1.3. The subspace $V_{k+1} \hookrightarrow S^{k+1}(\mathfrak{g})$ is isomorphic to the irreducible \mathfrak{g} -module $V((k+1)\theta)$ of highest weight $(k+1)\theta$. The algebra $A_{[2]}(\mathfrak{g}; k)$ is naturally a \mathfrak{g} -module, the module structure being induced by the adjoint action. Note that $A_{[2]}(\mathfrak{g}; k)$ is **not** a $\mathfrak{g} \oplus \mathfrak{g}$ -module, differently from $A(\mathfrak{g}; k)$.

Consider the standard filtration F_\bullet on the universal enveloping algebra $U(\mathfrak{g})$, such that $\text{gr}_\bullet F \simeq S(\mathfrak{g})$. Let $F_\bullet(k)$ be the induced filtration on the quotient algebra $A(\mathfrak{g}; k)$. We have an obvious surjection

$$(1.1) \quad A_{[2]}(\mathfrak{g}; k) \rightarrow \text{gr}_\bullet F(k).$$

Therefore, we have a surjective homomorphism of \mathfrak{g} -modules

$$(1.2) \quad A_{[2]}(\mathfrak{g}; k) \rightarrow A(\mathfrak{g}; k)$$

and thus $\dim A_{[2]}(\mathfrak{g}; k) \geq \sum_{\beta \in P_k^+(\mathfrak{g})} (\dim V(\beta))^2$. A natural question is: when does this inequality turn into an equality? In this paper we are also interested in the degree grading on $A_{[2]}(\mathfrak{g}; k)$ and in the corresponding graded decomposition into the direct sum of \mathfrak{g} -modules. Let

$$S(\mathfrak{g}) = \bigoplus_{m \geq 0} S^m(\mathfrak{g})$$

be the degree decomposition of the symmetric algebra. This decomposition induces the decomposition of the C_2 -algebra:

$$(1.3) \quad A_{[2]}(\mathfrak{g}; k) = \bigoplus_{m \geq 0} A_{[2]}^m(\mathfrak{g}; k).$$

Each space $A_{[2]}^m(\mathfrak{g}; k)$ is naturally a representation of \mathfrak{g} . Our main question is as follows: *Find the decomposition of $A_{[2]}^m(\mathfrak{g}; k)$ into the direct sum of irreducible \mathfrak{g} -modules.* The conjectural answer in type A is given in [GG]. We prove the conjecture in the next sections.

1.2. Comparing dimensions in type A . In what follows we restrict ourselves to the case $\mathfrak{g} = \mathfrak{sl}_n$. So we omit \mathfrak{g} in the notation of Zhu's and C_2 -algebras:

$$A(k) = A(\mathfrak{sl}_n; k), \quad A_{[2]}(k) = A_{[2]}(\mathfrak{sl}_n; k), \quad A_{[2]}^m(k) = A_{[2]}^m(\mathfrak{sl}_n; k).$$

It turns out that it is very convenient to consider the algebras \mathfrak{sl}_n and \mathfrak{gl}_n and their representations simultaneously. We fix some notation: let $\omega_1, \dots, \omega_{n-1}$ be the fundamental weights for \mathfrak{sl}_n and let ω_n be the additional fundamental weight for \mathfrak{gl}_n . For a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ with $\lambda_i \in \mathbb{Z}$, $\lambda_n \geq 0$ we denote by $V(\lambda)$ the corresponding \mathfrak{gl}_n -module of highest weight:

$$(\lambda_1 - \lambda_2)\omega_1 + \dots + (\lambda_{n-1} - \lambda_n)\omega_{n-1} + \lambda_n\omega_n.$$

(We note that $V(\lambda)$ can be obtained by applying the Schur functor S_λ to the standard \mathfrak{gl}_n -module V , see for example [FH], §6). For a partition $\beta = (\beta_1 \geq \dots \geq \beta_{n-1})$, $\beta_i \in \mathbb{Z}$, $\beta_{n-1} \geq 0$ let $V(\beta)$ be the corresponding \mathfrak{sl}_n -module of highest weight.

$$(\beta_1 - \beta_2)\omega_1 + \dots + (\beta_{n-2} - \beta_{n-1})\omega_{n-2} + \beta_{n-1}\omega_{n-1}.$$

We have the following restriction isomorphism

$$(1.4) \quad V(\lambda_1, \dots, \lambda_n)|_{\mathfrak{sl}_n} \simeq V(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n).$$

Note also that

$$P_k^+ = P_k^+(\mathfrak{sl}_n) = \{(\beta_1, \dots, \beta_{n-1}) : \beta_1 \leq k\}.$$

Our goal is to prove the following theorem:

Theorem 1.4. $A_{[2]}(k)$ and $A(k)$ are isomorphic as \mathfrak{sl}_n -modules.

Because of (1.2), it is enough to prove that

$$(1.5) \quad \dim A_{[2]}(k) \leq \sum_{\beta \in P_k^+} (\dim V(\beta))^2.$$

In what follows we denote by V the n -dimensional vector representation of \mathfrak{sl}_n . Let $\mathfrak{gl}_{n-1} \hookrightarrow \mathfrak{sl}_n$ be the standard embedding, i.e. $A \in \mathfrak{gl}_{n-1}$ is mapped onto the matrix

$$\begin{pmatrix} A & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{sl}_n$$

where $a = \text{Tr}(A)$ is the trace of A . We denote by U the standard $(n-1)$ -dimensional vector representation of \mathfrak{gl}_{n-1} . Then

$$(1.6) \quad \mathfrak{sl}_n \simeq U \oplus \mathfrak{gl}_{n-1} \oplus U^*.$$

To be precise, the actions on U and U^* are twisted by one-dimensional representations given by the characters Tr and $-\text{Tr}$. But since these actions are not important for the subsequent dimension counting, we omit to mention them explicitly. Note also that

$$\mathfrak{gl}_{n-1} \simeq \mathfrak{gl}(U) \simeq U \otimes U^*, \quad V \simeq U \oplus \mathbb{C}.$$

We introduce an ‘‘intermediate’’ algebra $B(k)$. In the following we often identify $S(\mathfrak{sl}_n)$ with the symmetric algebra $S(U \oplus \mathfrak{gl}_{n-1} \oplus U^*)$. Since

$$U \otimes V^* \simeq U \oplus U \otimes U^* \simeq U \oplus \mathfrak{gl}(U),$$

we have embeddings

$$(1.7) \quad \iota : S^{k+1}(U) \otimes S^{k+1}(V^*) \hookrightarrow S^{k+1}(U \oplus \mathfrak{gl}(U)) \hookrightarrow S^{k+1}(U \oplus \mathfrak{gl}(U) \oplus U^*).$$

Similarly, we have embeddings

$$(1.8) \quad \iota^* : S^{k+1}(U^*) \otimes S^{k+1}(V) \hookrightarrow S^{k+1}(\mathfrak{gl}(U) \oplus U^*) \hookrightarrow S^{k+1}(U \oplus \mathfrak{gl}(U) \oplus U^*).$$

Definition 1.5. Define the algebra $B(k)$ as the following quotient:

$$B(k) = S(U \oplus \mathfrak{gl}(U) \oplus U^*) / \langle \iota(S^{k+1}(U) \otimes S^{k+1}(V^*)), \iota^*(S^{k+1}(V) \otimes S^{k+1}(U^*)) \rangle.$$

Remark 1.6. Let $J = \langle V_{k+1} \rangle \subset S(\mathfrak{g})$ be the ideal defining $A_{[2]}(k)$. We are going to define a new grading on $S(\mathfrak{sl}_n)$. Combining the results below, one can view the algebra $B(k)$ as the quotient of the symmetric algebra $S(\mathfrak{sl}_n)$

by the ideal \tilde{J} obtained from J by the method of the associated cone with respect to this new grading. The new ideal is stable under the Levi subgroup

$$(1.9) \quad L = GL_{n-1} \subset SL_n, \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & (\det g)^{-1} \end{pmatrix}$$

so $B(k)$ admits an L -action. The advantage of the new ideal is the following:

Lemma 1.7. *$B(k)$ admits a $GL_{n-1} \times GL_{n-1}$ -action which, restricted to the diagonally embedded L , is isomorphic to the action of L on $B(k)$ induced by the adjoint action of L on \mathfrak{sl}_n .*

Proof. The canonical linear map $\mathfrak{sl}_n \rightarrow \mathfrak{k} = \{A = (a_{i,j}) \in M_n \mid a_{n,n} = 0\}$ given by

$$\begin{pmatrix} A & B \\ C & -\text{Tr}(A) \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is an isomorphism of L -representations. Now \mathfrak{k} admits an obvious $L \times L$ -action: $(\ell_1, \ell_2) \circ A := \ell_1 A \ell_2^{-1}$, which, restricted to the diagonal L gives back the adjoint action. The $L \times L$ -action also respects the decompositions $\mathfrak{k} = U \oplus \mathfrak{gl}(U) \oplus U^* = U \oplus V \otimes U^* = U \otimes V^* \oplus U^*$ and hence

$$B(k) \simeq S(\mathfrak{k}) / \langle S^{k+1}(U) \otimes S^{k+1}(V^*), S^{k+1}(V) \otimes S^{k+1}(U^*) \rangle$$

admits an $L \times L$ action with the desired properties. \square

Lemma 1.8. $\dim A_{[2]}(k) \leq \dim B(k)$.

Proof. We prove our lemma by introducing a certain grading on $A_{[2]}(k)$ and keeping only the highest degree terms of the relations. We then show that the resulting space of relations contains the relations of $B(k)$.

We have on $S(\mathfrak{sl}_n) = \bigoplus_{m=0}^{\infty} S^m(\mathfrak{sl}_n)$ the standard degree grading. We introduce a new grading called the UU^* -grading, by setting

$$(1.10) \quad \mathfrak{sl}_n = (\mathfrak{sl}_n)_0 \oplus (\mathfrak{sl}_n)_1, \quad (\mathfrak{sl}_n)_0 = U \oplus U^*, \quad (\mathfrak{sl}_n)_1 = \mathfrak{gl}_{n-1}.$$

This decomposition into degree zero and degree one elements induces the UU^* -grading on the symmetric algebra

$$S(\mathfrak{sl}_n) = \bigoplus_{l \geq 0} S_l(\mathfrak{sl}_n).$$

For an element $f \in S(\mathfrak{sl}_n)$ let $f = \sum_{j=0}^p f_j$, $f_p \neq 0$ be a decomposition of f into its homogeneous parts (with respect to the UU^* -grading), and denote by $gr_{UU^*} f := f_p$ its homogeneous part of highest degree.

For a subspace $I \hookrightarrow S(\mathfrak{sl}_n)$ let \tilde{I} be the subspace spanned by the highest degree parts of the elements in I :

$$\tilde{I} = \langle gr_{UU^*} f \mid f \in I \rangle.$$

If I is an ideal, then so is \tilde{I} . Moreover, if I is homogeneous with respect to the standard degree grading, then so is \tilde{I} , and for any $m \geq 0$ we have:

$$(1.11) \quad \dim(S(\mathfrak{sl}_n)/I)^m = \dim(S(U \oplus \mathfrak{gl}(U) \oplus U^*)/\tilde{I})^m.$$

The upper index m is used to denote the (standard) degree grading of the quotient space. We are going to apply the procedure above to the ideal of relations of the algebra $A_{[2]}(k)$.

Recall that $A_{[2]}(k) = S(\mathfrak{sl}_n)/\langle V_{k+1} \rangle$. Because of (1.11), Lemma 1.8 follows from the statement that the generators

$$\iota(S^{k+1}(U) \otimes S^{k+1}(V^*)), \quad \iota^*(S^{k+1}(V) \otimes S^{k+1}(U^*))$$

of the ideal of relations of $B(k)$ are contained in \tilde{V}_{k+1} . Let us show that

$$(1.12) \quad \iota(S^{k+1}(U) \otimes S^{k+1}(V^*)) \hookrightarrow \tilde{V}_{k+1}$$

(the arguments for the second subspace of relations of $B(k)$ are similar, see Remark 1.9 for more details). Let

$$\phi_j : S^{k+1}(U) \otimes S^j(U^*) \hookrightarrow S^{k+1}(U) \otimes S^{k+1}(V^*), \quad j = 0, \dots, k+1$$

be the embedding coming from the decomposition

$$S^{k+1}(V^*) \simeq S^{k+1}(U^* \oplus \mathbb{C}) = \bigoplus_{j=0}^{k+1} S^j(U^*).$$

Let us show that

$$(1.13) \quad \iota\phi_j(S^{k+1}(U) \otimes S^j(U^*)) \hookrightarrow (\tilde{V}_{k+1})_j$$

(the j -th UU^* -degree part of \tilde{V}_{k+1}). We start with $j = 0$. Note that since $e_\theta \in U$, the power e_θ^{k+1} belongs to $(\tilde{V}_{k+1})_0$. Since the UU^* -grading on $S(\mathfrak{sl}_n)$ is \mathfrak{gl}_{n-1} invariant, we obtain

$$\iota\phi_0(S^{k+1}(U)) \hookrightarrow (\tilde{V}_{k+1})_0.$$

Note: by definition, $e_\theta^{k+1} \in V_{k+1}$ and hence $\iota\phi_0(S^{k+1}(U)) \hookrightarrow V_{k+1}$. So in the following we just write $S^{k+1}(U) \hookrightarrow V_{k+1}$. Now let us prove (1.13) for $j = 1, \dots, k+1$. Let u_1^*, \dots, u_j^* be some elements of $U^* \hookrightarrow \mathfrak{sl}_n$. Then for any $u_1 \dots u_{k+1} \in S^{k+1}(U) \subseteq V_{k+1}$ we have

$$(1.14) \quad [u_j^*, [u_{j-1}^*, \dots, [u_1^*, u_1 \dots u_{k+1}] \dots]] \in V_{k+1}.$$

The highest UU^* -degree term of (1.14) is equal to

$$(1.15) \quad \sum_{1 \leq i_1 < \dots < i_j \leq k+1} u_1 \dots u_{i_1-1} [u_1^*, u_{i_1}] \dots [u_j^*, u_{i_j}] u_{i_j+1} \dots u_{k+1},$$

because $[U^*, U] \subseteq \mathfrak{gl}_{n-1} = (\mathfrak{sl}_n)_1$ and $[U^*, \mathfrak{gl}_{n-1}] = U^* \hookrightarrow (\mathfrak{sl}_n)_0$. It is easy to see that the linear span of the elements (1.15) (for all $u_i \in U$ and $u_i^* \in U^*$) coincides with $\iota\phi_j(S^{k+1}(U) \otimes S^j(U^*))$. Therefore

$$\iota\phi_j(S^{k+1}(U) \otimes S^j(U^*)) \hookrightarrow (\tilde{V}_{k+1})_j$$

which finishes the proof of the lemma. \square

Remark 1.9. In order to prove the inclusion

$$(1.16) \quad \iota^*(S^{k+1}(U^*) \otimes S^{k+1}(V)) \hookrightarrow \tilde{V}_{k+1}$$

one starts with the element f_θ^{k+1} ($f_\theta \in \mathfrak{sl}_n$ is the lowest weight vector) and introduce the embeddings

$$\phi_j^* : S^{k+1}(U^*) \otimes S^j(U) \hookrightarrow S^{k+1}(U^*) \otimes S^{k+1}(V), \quad j = 0, \dots, k+1.$$

Then $\tilde{V}_{k+1} = \bigoplus_{j=0}^{k+1} (\tilde{V}_{k+1})_j$, where

$$(\tilde{V}_{k+1})_j = \iota\phi_j(S^{k+1}(U) \otimes S^j(U^*)) \oplus \iota^*\phi_j^*(S^{k+1}(U^*) \otimes S^j(U)), \quad j = 0, \dots, k$$

and

$$(\tilde{V}_{k+1})_{k+1} = \iota\phi_{k+1}(S^{k+1}(U) \otimes S^{k+1}(U^*)) = \iota^*\phi_{k+1}^*(S^{k+1}(U^*) \otimes S^{k+1}(U)).$$

In what follows we often consider quotient algebras of the type

$$S(W_1 \otimes W_2) / \langle S^{k+1}(W_1) \otimes S^{k+1}(W_2) \rangle.$$

In Section 3 (see subsection 3.3) we describe the $\mathfrak{gl}(W_1) \oplus \mathfrak{gl}(W_2)$ -module structure of this algebra explicitly, see also [DEP]. Below we give the description in three special cases: $W_1 = W_2 = U$; $W_1 = U$, $W_2 = V^*$ and $W_1 = V$, $W_2 = U^*$.

For a partition λ let $U(\lambda)$ (resp. $V(\lambda)$) be the irreducible \mathfrak{gl}_{n-1} - (resp. \mathfrak{gl}_n -) module with highest weight λ . Then

$$(1.17) \quad S(U \otimes U^*) / \langle S^{k+1}(U) \otimes S^{k+1}(U^*) \rangle \simeq \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} U(\lambda) \otimes U(\lambda)^*,$$

$$(1.18) \quad S(U \otimes V^*) / \langle S^{k+1}(U) \otimes S^{k+1}(V^*) \rangle \simeq \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} U(\lambda) \otimes V(\lambda)^*,$$

$$(1.19) \quad S(V \otimes U^*) / \langle S^{k+1}(V) \otimes S^{k+1}(U^*) \rangle \simeq \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} V(\lambda) \otimes U(\lambda)^*.$$

Proposition 1.10. $\dim B(k) \leq \sum_{\beta \in P_k^+} (\dim V(\beta))^2$.

Proof. We regard $B(k)$ as a $L \times L$ -module, recall that \mathfrak{sl}_n decomposes as

$$\mathfrak{sl}_n = U \otimes U^* \oplus U \otimes \mathbb{C}_{\det} \oplus \mathbb{C}_{\det^{-1}} \otimes U^*.$$

Here \mathbb{C}_{\det} and $\mathbb{C}_{\det^{-1}}$ denote the one-dimensional representations given by the determinant respectively its inverse. In the following we omit to indicate the twists by these characters since they are not relevant for the dimension counting. Let $B_0(k) \hookrightarrow B(k)$ be the subalgebra generated by $\mathfrak{gl}_{n-1} \simeq U \otimes U^*$. Since the subspace

$$S^{k+1}(U) \otimes S^{k+1}(U^*) \hookrightarrow S^{k+1}(\mathfrak{gl}_{n-1})$$

is sitting inside the space of relations of $B(k)$, there exists an embedding of $L \times L$ -modules

$$(1.20) \quad B_0(k) \hookrightarrow \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} U(\lambda) \otimes U(\lambda)^*.$$

We have a surjective homomorphism of $L \times L$ -modules

$$(1.21) \quad S(U) \otimes B_0(k) \otimes S(U^*) \rightarrow B(k).$$

By (1.20), we obtain the $L \times L$ -equivariant surjection

$$(1.22) \quad \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} S(U) \otimes (U(\lambda) \otimes U(\lambda)^*) \otimes S(U^*) \rightarrow B(k).$$

In the following we write $\mu \succeq \lambda$ for μ is larger and in between λ , i.e.:

$$\mu \succeq \lambda \Leftrightarrow \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_{n-1},$$

and we write $|\lambda|$ for the sum $\sum_i \lambda_i$. By the Pieri formula (see for example [FH], Appendix A) the left hand side of (1.22) is, as $L \times L$ -module, isomorphic to the direct sum

$$(1.23) \quad \bigoplus_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} \bigoplus_{\mu, \nu \succeq \lambda} U(\mu) \otimes U(\nu)^*.$$

Because of the relations in $B(k)$, not all the tensor products as in (1.23) do really appear in the decomposition of $B(k)$ as $L \times L$ -module. In fact, (1.18) and (1.19) imply that if $U(\mu) \otimes U(\nu)^*$ appears in $B(k)$ (up to twists by a character), then μ and ν are restricted by $\mu_1 \leq k$ and $\nu_1 \leq k$. Thus our proposition follows from the equality

$$(1.24) \quad \sum_{\beta \in P_k^+} (\dim V(\beta))^2 = \sum_{\lambda: k \geq \lambda_1} \sum_{\substack{k \geq \mu_1, \nu_1 \\ \mu, \nu \succeq \lambda}} \dim U(\mu) \dim U(\nu)^*,$$

which is going to be proved in the following lemma. \square

Lemma 1.11. *The equality in (1.24) holds.*

Proof. We decompose each $V(\beta)$ and $V(\beta)^*$ into the direct sum of \mathfrak{gl}_{n-1} -modules following the Gelfand-Tsetlin rule (see [GT]) and keep in mind the embedding of $L = GL_{n-1}$ in SL_n (see (1.9)). Thus our lemma is equivalent to the following equation (again we omit twists by characters):

$$(1.25) \quad \sum_{\substack{\beta=(\beta_1 \geq \dots \geq \beta_{n-1}) \\ k \geq \beta_1, \beta_{n-1} \geq 0}} \sum_{\mu, \nu \preceq \beta} \dim U(\mu) \dim U(\nu)^* \\ = \sum_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_{n-1}) \\ k \geq \lambda_1, \lambda_{n-1} \geq 0}} \sum_{\substack{\bar{\mu}, \bar{\nu} \succeq \lambda \\ \bar{\mu}_1, \bar{\nu}_1 \leq k}} \dim U(\bar{\mu}) \dim U(\bar{\nu})^*.$$

To prove the latter we use the bijection between the parameter sets

$$\begin{aligned} (\beta, \mu, \nu) &\mapsto (\lambda, \bar{\mu}, \bar{\nu}), \\ \lambda_i &= k - \beta_{n-i-1}, \quad \bar{\mu}_i = k - \mu_{n-1-i}, \quad \bar{\nu}_i = k - \nu_{n-1-i}. \end{aligned}$$

□

Corollary 1.12. *$A(k)$ and $A_{[2]}(k)$ are isomorphic as \mathfrak{sl}_n -modules.*

Proof. We have

$$\begin{aligned} \sum_{\beta \in P_k^+} (\dim V(\beta))^2 &= \dim A(k) \leq \dim A_{[2]}(k) \\ &\leq \dim B(k) \leq \sum_{\beta \in P_k^+} (\dim V(\beta))^2. \end{aligned}$$

Hence $\dim A(k) = \dim A_{[2]}(k)$, which by (1.2) implies the corollary. □

Proof. (of Theorem 1.4) The theorem is a consequence of Corollary 1.12. □

As another consequence we get: the inequality $\dim S(\mathfrak{sl}_n)/\langle \tilde{V}_{k+1} \rangle \leq \dim B(k)$ in the proof of Lemma 1.8 is an equality and hence, as claimed in Remark 1.6:

Corollary 1.13. $B(k) = S(\mathfrak{sl}_n)/\langle \tilde{V}_{k+1} \rangle$.

Corollary 1.14. $\dim A_{[2]}^m(k) = \dim B^m(k)$ for all $m \geq 0$.

2. THE GRADED DECOMPOSITION OF $A_{[2]}(k)$

In this section we compute the graded decomposition of C_2 -algebras, thus proving the conjecture of [GG].

Definition 2.1. We define the algebra $C(k)$ as follows:

$$C(k) = S(V \otimes V^*) / \langle S^{k+1}(V) \otimes S^{k+1}(V^*) \rangle$$

Remark 2.2. Let $E_{a,b} \in \mathfrak{gl}_n$ be the standard matrix unit. Identifying $V \otimes V^*$ with $\mathfrak{gl}(V)$ we can describe the subspace $S^{k+1}(V) \otimes S^{k+1}(V^*)$ as the linear span of the monomials of the form

$$\begin{aligned} \sum_{\sigma \in S_{k+1}} E_{i_1, j_{\sigma_1}} E_{i_2, j_{\sigma_2}} \cdots E_{i_{k+1}, j_{\sigma_{k+1}}}, \\ 1 \leq i_1 \leq \cdots \leq i_{k+1} \leq n, 1 \leq j_1 \leq \cdots \leq j_{k+1} \leq n, \end{aligned}$$

where the sum is taken over the permutation group S_{k+1} .

Remark 2.3. Using the identification in Lemma 1.7, we see that with respect to the embeddings $L = GL_{n-1} \subset SL_n \subset GL_n$ we have: $B(k) \simeq C(k)/\langle E_{n,n} \rangle$, also as a $L \times L$ -module.

The standard degree grading S^m on the symmetric algebra induces a grading on $C(k)$:

$$C(k) = \bigoplus_{m \geq 0} C^m(k).$$

Recall that we have the decomposition with respect to the action of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$:

$$(2.1) \quad C(k) \simeq \bigoplus_{\substack{\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \\ k \geq \lambda_1, \lambda_n \geq 0}} V(\lambda) \otimes V(\lambda)^*.$$

Moreover, using the Cauchy formula (see for example [P], §9) for $S^m(V \otimes V^*)$ we see:

$$(2.2) \quad C^m(k) \simeq \bigoplus_{\substack{\lambda: k \geq \lambda_1, \lambda_n \geq 0 \\ \lambda_1 + \dots + \lambda_n = m}} V(\lambda) \otimes V(\lambda)^*.$$

We will extract the information about $A_{[2]}^m(k)$ from (2.2).

Let $\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n$ be again the standard embedding. As \mathfrak{sl}_n -module (adjoint action) the Lie algebra $\mathfrak{gl}_n \simeq V \otimes V^*$ decomposes as

$$(2.3) \quad \mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}c,$$

where $\mathbb{C}c$ is trivial one-dimensional module with fixed non-trivial vector c (say, $c \in V \otimes V^*$ is the identity operator). The algebra $C(k)$ is by construction a $S(\mathfrak{gl}_n)$ - as well as a $S(\mathfrak{sl}_n)$ -module. For $i \geq 0$ let $D_i \subset C(k)$ be the $S(\mathfrak{sl}_n)$ -submodule of $C(k)$ generated c^i :

$$D_i = S(\mathfrak{sl}_n) \cdot c^i \subset C(k).$$

Lemma 2.4. *The canonical surjective map $S(\mathfrak{sl}_n) \rightarrow D_0 \subset C(k)$, $f \mapsto f \cdot 1$, induces a surjective homomorphism $A_{[2]}(k) \rightarrow D_0$ of \mathfrak{sl}_n -modules.*

Proof. By definition, the image of $S(\mathfrak{sl}_n)$ in $C(k)$ is D_0 . Now $e_\theta = E_{1,n}$ and by Remark 2.2 $e_\theta^{k+1} \in S^{k+1}(V) \otimes S^{k+1}(V^*)$, which proves the lemma. \square

In general the following proposition holds.

Proposition 2.5. *For all $i = 0, 1, \dots, k$ we have a surjective homomorphism of \mathfrak{sl}_n -modules*

$$A_{[2]}(k-i) \rightarrow \sum_{j=0}^i D_j \Big/ \sum_{j=0}^{i-1} D_j$$

and $D_{k+1} \hookrightarrow \sum_{j=0}^k D_j$.

Proof. It suffices to show that for any $0 \leq i \leq k+1$ the following is true in the symmetric algebra $S(V \otimes V^*)$:

$$(2.4) \quad e_\theta^{k+1-i} c^i \in S^{k+1}(V) \otimes S^{k+1}(V^*) + \sum_{l=0}^{i-1} D_l.$$

Let $E_{a,b} \in \mathfrak{gl}_n$ be the matrix with entries $(\delta_{i,a}\delta_{b,j})_{1 \leq i,j \leq n}$. Note that

$$e_\theta = E_{1,n}, \quad c = \sum_{a=1}^n E_{a,a}.$$

We first write

$$(2.5) \quad e_\theta^{k+1-i} c^i = \left[(c - nE_{n,n})^i + \sum_{j=0}^{i-1} n^{i-j} \binom{i}{j} (c - nE_{n,n})^j E_{n,n}^{i-j} \right] E_{1,n}^{k+1-i}.$$

Let us show that for any $j = 0, \dots, i-1$

$$(2.6) \quad E_{n,n}^{i-j} E_{1,n}^{k+1-i} \in S^{k+1}(V) \otimes S^{k+1}(V^*) + \sum_{l=0}^{i-1} D_l.$$

For the $j = 0$ term (because of Remark 2.2) we have

$$E_{n,n}^i E_{1,n}^{k+1-i} \in S^{k+1}(V) \otimes S^{k+1}(V^*).$$

Note that since $E_{1,n} \in \mathfrak{sl}_n$ the general j case of (2.6) follows from the statement that for any $p \geq 0$

$$(2.7) \quad E_{n,n}^p \in S(\mathfrak{sl}_n) \text{span}\{1, c, \dots, c^p\}$$

We prove (2.7) by induction on p . For any $\alpha = 1, \dots, n$ we have

$$E_{n,n}^p - E_{n,n}^{p-1} E_{\alpha,\alpha} = E_{n,n}^{p-1} (E_{n,n} - E_{\alpha,\alpha}).$$

By induction assumption $E_{n,n}^{p-1} \in S(\mathfrak{sl}_n) \text{span}\{1, c, \dots, c^{p-1}\}$ and thus summing up over all α

$$nE_{n,n}^p - cE_{n,n}^{p-1} \in S(\mathfrak{sl}_n) \text{span}\{1, c, \dots, c^{p-1}\}.$$

Using the induction assumption again we arrive at

$$E_{n,n}^p \in S(\mathfrak{sl}_n) \text{span}\{1, c, \dots, c^p\}.$$

This finishes the proof of the proposition. \square

Recall (see (3.1)) that $C(k)$ is isomorphic to the \mathfrak{gl}_{2n} -module

$$V(\underbrace{k, \dots, k}_n, \underbrace{0, \dots, 0}_n).$$

We endow $C(k)$ with a structure of \mathfrak{sl}_n -module via the chain of embeddings

$$\mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_n \oplus \mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{2n}.$$

Here the first embedding is the standard one, the second is the diagonal embedding and the last one comes from the isomorphism of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ and the Levi subalgebra of \mathfrak{gl}_{2n} , corresponding to $\omega_n (E_{i,j} \oplus E_{k,l} \mapsto E_{i,j} + E_{n+k,n+l})$. So in what follows we consider $C(k)$ equipped with this structure of \mathfrak{sl}_n -module.

Theorem 2.6. *We have an isomorphism of \mathfrak{sl}_n -modules*

$$C(k) \simeq \bigoplus_{i=0}^k A_{[2]}(k-i).$$

Moreover,

$$C^m(k) \simeq \bigoplus_{i=0}^{\min(m,k)} A_{[2]}^{m-i}(k-i).$$

Proof. Because of Lemma 2.4 and Proposition 2.5 it suffices to prove that

$$\sum_{j=0}^k \dim A_{[2]}(j) = \dim C(k).$$

Because of Corollary 1.12 this is equivalent to

$$\sum_{j=0}^k \dim A(j) = \dim C(k).$$

Recall that

$$\begin{aligned} (2.8) \quad \dim C(k) &= \sum_{\substack{\lambda=(\lambda_1 \geq \dots \geq \lambda_n) \\ k \geq \lambda_1, \lambda_n \geq 0}} (\dim V(\lambda))^2 \\ &= \sum_{\lambda_n=0}^k \sum_{\beta=(\lambda_1-\lambda_n, \dots, \lambda_{n-1}-\lambda_n)} (\dim V(\beta))^2, \end{aligned}$$

where $V(\beta)$ in the last line are irreducible \mathfrak{sl}_n -modules. We note that if $\beta_i = \lambda_i - \lambda_n$ as above, then the \mathfrak{sl}_n -module $V(\beta)$ and \mathfrak{gl}_n -module $V(\lambda)$ are isomorphic as vector spaces. Note also that if λ_n is fixed, the restriction $k \geq \lambda_1$ turns into $\beta_1 \leq k - \lambda_n$. Therefore, (2.8) can be rewritten as

$$\dim C(k) = \sum_{\lambda_n=0}^k \dim A(k - \lambda_n),$$

which proves the theorem. \square

Corollary 2.7. *We have an isomorphism of \mathfrak{sl}_n -modules*

$$A_{[2]}^m(k) \simeq C^m(k)/C^{m-1}(k-1).$$

Corollary 2.8. *We have an isomorphism of \mathfrak{sl}_n -modules*

$$A_{[2]}^m(k) = \frac{\bigoplus_{\substack{\lambda: k \geq \lambda_1, \lambda_n \geq 0 \\ \lambda_1 + \dots + \lambda_n = m}} V(\lambda) \otimes V(\lambda)^*}{\bigoplus_{\substack{\lambda: k-1 \geq \lambda_1, \lambda_n \geq 0 \\ \lambda_1 + \dots + \lambda_n = m-1}} V(\lambda) \otimes V(\lambda)^*},$$

where the \mathfrak{gl}_n -module $V(\lambda)$ is regarded as the \mathfrak{sl}_n -module with the highest weight $(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n)$.

Corollary 2.8 is a restatement of the Conjecture of Gaberdiel and Gannon (see [GG], formulas (4.4) and (4.5)). Recall that, despite the suggesting formula, $A_{[2]}^m(k)$ is **not** a $\mathfrak{sl}_n \oplus \mathfrak{sl}_n$ -module.

3. ABELIAN RADICALS AND SPHERICAL MODULES

Let G be a simple simply connected algebraic group with Lie algebra \mathfrak{g} . We fix a Borel subgroup B with Lie algebra \mathfrak{b} and a maximal torus $T \subset B$ with Lie algebra \mathfrak{t} . Let Φ be the root system and let Φ^+ be the set of positive roots and Δ the set of simple roots corresponding to the choice of \mathfrak{b} .

Let \mathfrak{p} be a maximal standard parabolic subalgebra of the simple Lie algebra \mathfrak{g} , i.e. \mathfrak{p} contains the fixed Borel subalgebra \mathfrak{b} . Then \mathfrak{p} is completely determined by the only simple root α such that the root space $\mathfrak{g}_{-\alpha}$ is not contained in \mathfrak{p} , we call α the simple root associated to \mathfrak{p} .

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}$ be a Levi decomposition, i.e. \mathfrak{l} is a reductive Lie subalgebra containing \mathfrak{t} and \mathfrak{m} is the nilpotent radical of \mathfrak{p} .

Let P be the corresponding maximal parabolic subgroup of G with Levi decomposition $P = LP^u$, $L \supset T$. Then L and \mathfrak{l} act on \mathfrak{m} via the adjoint action.

The opposite parabolic subalgebra is denoted by \mathfrak{p}^- . Let $\mathfrak{p}^- = \mathfrak{l} \oplus \mathfrak{m}^-$ be its Levi decomposition, then $\mathfrak{m}^- \simeq \mathfrak{m}^*$ as L -module.

3.1. Certain annihilators. Corresponding to the choice of a maximal parabolic and its associated root α let ω be the associated fundamental weight. Let \mathfrak{n} be the nilpotent radical of \mathfrak{b} and let \mathfrak{n}^- be the nilpotent radical of the opposite Borel subalgebra \mathfrak{b}^- .

We define a \mathbb{Z} -grading on the root system Φ by setting for the simple roots $\deg \alpha = 1$ and $\deg \gamma = 0$ for $\gamma \neq \alpha$. Let $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be the corresponding \mathbb{Z} -grading, then

$$\mathfrak{l} = \mathfrak{g}_0, \quad \mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j \quad \mathfrak{m} = \bigoplus_{j \geq 1} \mathfrak{g}_j \quad \text{and} \quad \mathfrak{m}^- = \bigoplus_{j \leq -1} \mathfrak{g}_j$$

We choose a basis $X_{-\beta} \in \mathfrak{g}_{-\beta}$, $\beta \in \Phi^+$, of root vectors for \mathfrak{n}^- . Fix a highest weight vector $v_{k\omega}$ in $V(k\omega)$. For all $k \in \mathbb{N}$ we have a surjective map

$$\Psi : U(\mathfrak{n}^-) \rightarrow V(k\omega), \quad n \mapsto nv_{k\omega},$$

the kernel being the left ideal in $U(\mathfrak{n}^-)$ generated by

$$\{X_{-\alpha}^{k+1}, X_{-\gamma} \mid \gamma \in \Delta, \deg \gamma = 0\}.$$

The restriction of this map to $U(\mathfrak{m}^-)$ is already surjective:

$$\pi_k : U(\mathfrak{m}^-) \rightarrow V(k\omega), \quad m \mapsto mv_{k\omega}.$$

To describe a generating system for the kernel of π_k , recall that \mathfrak{m}^- is a L -module via the adjoint action of \mathfrak{l} and L on \mathfrak{g} . We write $\ell \circ m$ for the adjoint action. Set $\mathfrak{l}' := [\mathfrak{l}, \mathfrak{l}]$.

Lemma 3.1. *The kernel of π_k is the left $U(\mathfrak{m}^-)$ -ideal generated by the irreducible L -module $\langle L \circ X_{-\alpha}^{k+1} \rangle = U(\mathfrak{l}') \circ X_{-\alpha}^{k+1}$, and π_k induces an isomorphism of L -modules $U(\mathfrak{m}^-)/\ker \pi_k \simeq V(k\omega) \otimes \mathbb{C}_{-k\omega}$, where $\mathbb{C}_{-k\omega}$ denotes the one-dimensional representation associated to the L -character $-k\omega$.*

Proof. If $p_1, \dots, p_r \in \mathfrak{m}^-$ and $X \in \mathfrak{l}'$, then

$$\begin{aligned} X.p_1 \cdots p_r &= \sum_{j=1}^r p_1 \cdots [X, p_j] \cdots p_r + p_1 p_2 \cdots p_r X \\ &= X \circ (p_1 \cdots p_r) + p_1 p_2 \cdots p_r X. \end{aligned}$$

So if m_1, \dots, m_q are monomials in $U(\mathfrak{m}^-)$ and $\sum_{j=1}^q a_j m_j \in \ker \pi_k$, then

$$X \circ \left(\sum_{j=1}^q a_j m_j \right) = X \cdot \left(\sum_{j=1}^q a_j m_j \right) - \left(\sum_{j=1}^q a_j m_j \right) \cdot X$$

annihilates $v_{k\omega}$ too since X annihilates $v_{k\omega}$. Hence the left ideal generated by $U(\mathfrak{l}') \circ X_{-\alpha}^{k+1}$ is contained in $\ker \pi_k$. Set $\mathfrak{n}_0^- = \mathfrak{n}^- \cap \mathfrak{l}$, then

$$\begin{aligned} \ker \Psi &= U(\mathfrak{n}^-)X_{-\alpha}^{k+1} + \sum_{\gamma \in \Delta, \deg \gamma=0} U(\mathfrak{n}^-)X_{-\gamma} \\ &= U(\mathfrak{m}^-)U(\mathfrak{n}_0^-)X_{-\alpha}^{k+1} + \sum_{\gamma \in \Delta, \deg \gamma=0} U(\mathfrak{n}^-)X_{-\gamma} \\ &= U(\mathfrak{m}^-)(U(\mathfrak{n}_0^-) \circ X_{-\alpha}^{k+1}) + \sum_{\gamma \in \Phi^+, \deg \gamma=0} U(\mathfrak{n}^-)X_{-\gamma}. \end{aligned}$$

Now $X_{-\alpha}$ (and hence $X_{-\alpha}^{k+1}$ too) is a highest weight vector for the action of $U(\mathfrak{l}')$ and hence $U(\mathfrak{l}') \circ X_{-\alpha}^{k+1} = U(\mathfrak{n}_0^-) \circ X_{-\alpha}^{k+1}$. So the image of $\ker \Psi$ in $U(\mathfrak{n}^-)/U(\mathfrak{n}^-).\mathfrak{n}_0^- \simeq U(\mathfrak{m}^-)$ coincides with the left ideal described in the lemma, which shows the equality.

To make the induced isomorphism $U(\mathfrak{m}^-)/\ker \pi_k \simeq V(k\omega)$ L -equivariant, we need to twist the representation by the appropriate character. \square

3.2. Abelian and spherical radicals. The action of a reductive group H on a (possibly infinite dimensional) locally finite representation space V is called *multiplicity free* if the decomposition of the H -module V into the direct sum of irreducible finite dimensional H -modules contains any irreducible L -module with multiplicity at most one. The action of H on an affine variety M is called *multiplicity free* if $\mathbb{C}[M]$ is multiplicity free.

Note that the action of L on $\mathbb{C}[\mathfrak{m}^-]$ is multiplicity free if and only the action of L on $U(\mathfrak{m}^-)$ is multiplicity free. In particular, in this case the irreducible L -module in $U(\mathfrak{m}^-)$ spanned by the generating set for the ideal $\ker \pi_k$ (see Lemma 3.1) is uniquely determined by the highest weight $-k\alpha$.

Let θ denote the highest root of \mathfrak{g} . A simple root α is called *co-minuscule* if α occurs with coefficient 1 in the expression of θ as a sum of simple roots.

Proposition 3.2. *i) The nilpotent radical \mathfrak{m}^- of \mathfrak{p}^- is abelian if and only if the simple root α associated to \mathfrak{p} is co-minuscule. In this case the action of L on \mathfrak{m}^- is irreducible. (For a list of co-minuscule weights see Remark 3.3.)*

ii) The action of L on $\mathbb{C}[\mathfrak{m}^-]$ is multiplicity free if and only if either \mathfrak{m}^- is abelian, or $\alpha = \alpha_n$ for \mathfrak{g} of type B_n , or $\alpha = \alpha_1$ for \mathfrak{g} of type C_n .

Remark 3.3. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the simple roots, indexed as in [B], Planches I-IX. According to the tables in [B] we find the following list of co-minuscule roots α_i , or, equivalently, the list of roots such that the corresponding maximal parabolic subalgebra has an abelian nilpotent radical: in type A_n all roots are co-minuscule, in type B_n only the root α_1 and in type C_n only the root α_n are co-minuscule. In type D_n the roots $\alpha_1, \alpha_{n-1}, \alpha_n$ and in type E_6 the roots α_1, α_6 are co-minuscule. In type E_7 only the root α_7 is co-minuscule. For the types E_8 , F_4 and G_2 there are no maximal parabolic subalgebras with abelian nilpotent radical.

Proof. The first part *i)* is well known: Let $\Phi_{\mathfrak{m}}^+$ be the positive roots occurring in \mathfrak{m} . If the last condition holds, then for two roots $\beta, \gamma \in \Phi_{\mathfrak{m}}^+$ the sum is never a root and hence the commutator of the two root subspaces is zero. If the condition fails, then we can find two roots $\beta, \gamma \in \Phi_{\mathfrak{m}}^+$ such that the sum is a root, and hence the corresponding root subspaces fail to commute. It follows now from elementary properties of root systems that the \mathfrak{l} -action on \mathfrak{m} is irreducible, having the root α as lowest weight vector.

A comparison of the list of co-minuscule roots with the tables in [K3] shows that in these cases the representations of L on \mathfrak{m} are multiplicity free.

Another approach, which gives the complete list at once is the following: the action of a reductive group H on a variety Y is called *spherical* if a Borel subgroup of H has a dense orbit in Y . It is known for Y affine that the conditions spherical and multiplicity free are equivalent. The action of L on \mathfrak{m} is spherical if and only if it is so for the action on \mathfrak{m}^- . Via the exponential we see that the action of L on \mathfrak{m}^- is spherical if and only if the action of L on the unipotent radical $P^{-,u}$ of the opposite parabolic subgroup is so. Now $P^{-,u}$ can be identified in a L -equivariant way with the open cell $P^{-,u}.id \subset G/P$, so we see that the action of L on \mathfrak{m} is multiplicity free if and only if the action of L on G/P is spherical. These maximal parabolic subgroups have been classified in [L1]. \square

3.3. Type A case. In this subsection we work out explicitly the type A case of the general constructions explained in Section 3 above.

So let $\mathfrak{g} = \mathfrak{gl}_N$ and let

$$\omega_i = \underbrace{(1, \dots, 1)}_i, \quad i = 1, \dots, N$$

be some fundamental weight. Then $\mathfrak{l} = \mathfrak{gl}_i \oplus \mathfrak{gl}_{N-i}$ and the abelian radical \mathfrak{m}^- is isomorphic to $(\mathbb{C}^i)^* \otimes \mathbb{C}^{N-i}$ as \mathfrak{l} -module (\mathbb{C}^i and \mathbb{C}^{N-i} are vector representations of \mathfrak{gl}_i and \mathfrak{gl}_{N-i}). In what follows we deal with \mathfrak{gl}_N , \mathfrak{gl}_i and \mathfrak{gl}_{N-i} -modules simultaneously, so we use the notation $V_j(\lambda)$ to denote the irreducible \mathfrak{gl}_j -module with highest weight λ .

The \mathfrak{l} -module $U(\mathfrak{l}) \circ X_{-\alpha_i}^{k+1}$ is isomorphic to

$$(3.1) \quad S^{k+1}(\mathbb{C}^{i*}) \otimes S^{k+1}(\mathbb{C}^{N-i}) \hookrightarrow S^{k+1}(\mathfrak{m}^-).$$

We write (k^i) for the partition $\underbrace{(k, \dots, k)}_i$. Then Lemma 3.1 in type A reads as:

$$(3.2) \quad V_N((k^i)) \otimes \mathbb{C}_{-k\omega_i} \simeq S(\mathbb{C}^{i*} \otimes \mathbb{C}^{N-i}) / \langle S^{k+1}(\mathbb{C}^{i*}) \otimes S^{k+1}(\mathbb{C}^{N-i}) \rangle.$$

In the following lemma we describe $V_N((k^i))$ as $\mathfrak{l} = \mathfrak{gl}_i \oplus \mathfrak{gl}_{N-i}$ -module (the \mathfrak{l} action on $V_N(k\omega_i)$ coming from the adjoint action of \mathfrak{l} on \mathfrak{m}^-).

Lemma 3.4. *The module $S(\mathbb{C}^{i*} \otimes \mathbb{C}^{N-i}) / \langle S^{k+1}(\mathbb{C}^{i*}) \otimes S^{k+1}(\mathbb{C}^{N-i}) \rangle$ (respectively $V_N((k^i)) \otimes \mathbb{C}_{-k\omega_i}$) is isomorphic (as $\mathfrak{gl}_i \oplus \mathfrak{gl}_{N-i}$ -module) to the direct sum*

$$(3.3) \quad \bigoplus_{\lambda} V_i(\lambda)^* \otimes V_{N-i}(\lambda),$$

where the sum is running over all partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_{\min(i, N-i)})$ such that $k \geq \lambda_1$.

Proof. Using (3.2), the lemma can be deduced from [DEP]. An alternative approach is to use branching rules for Levi subgroups of GL_N . The rule is due to Littlewood, the formulation used here can be found in [L2].

We briefly explain how the formula (3.3) shows up in type A . The partition (k^i) is represented by the rectangular Young diagram having i -rows and k -columns. The restriction formula for Levi subalgebras can in this case be read as follows: the irreducible components of $V_N((k^i))$ as $\mathfrak{gl}_i \oplus \mathfrak{gl}_{N-i}$ -module are of the form $V_i((k^i - \lambda)) \otimes V_{N-i}(\lambda)$ where the partition λ is obtained by cutting the rectangle (k^i) into two partitions: $\lambda = (\lambda_1, \dots, \lambda_i)$ and $(k^i - \lambda) = (k - \lambda_i, \dots, k - \lambda_1)$. Here we assume $\lambda_{N-i+1} = \dots = \lambda_i = 0$ if $i > N - i$. But note that $V_i((k^i - \lambda)) \simeq V_i(\lambda)^* \otimes \mathbb{C}_{k\omega_i}$, which proves the lemma. \square

ACKNOWLEDGMENTS

The work of EF was partially supported by the Russian President Grant MK-281.2009.1, by the RFBR Grants 09-01-00058, 07-02-00799 and NSh-3472.2008.2, by Pierre Deligne fund based on his 2004 Balzan prize in mathematics and by Alexander von Humboldt Fellowship. The work of BF was partially supported by RFBR initiative interdisciplinary project grant 09-02-12446-ofi-m and by RFBR-CNRS grant 09-02-93106. The work of PL was partially supported by the priority program SPP 1377 of the German Science Foundation.

REFERENCES

- [ABD] T. Abe, G. Buhl, and C. Dong, *Rationality, regularity, and C_2 -cofiniteness*, Trans. Amer. Math. Soc. 356 (2004) 3391-3402.

- [BF] D. Ben-Zvi and E. Frenkel, Vertex algebras and algebraic curves, *Mathematical Surveys and Monographs*, **88**, AMS, 2001.
- [B] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV, V, VI*, Actuelles Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.
- [CF] N. Carqueville, M. Flohr, *Nonmeromorphic operator product expansion and C_2 -cofiniteness for a family of W -algebras*, arXiv:math-ph/0508015.
- [DEP] C. DeConcini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, *Inv. math.* 56 (1980), 129 - 165.
- [DLM] C. Dong, H. Li, G. Mason, *Twisted representations of vertex operator algebras*, *Math. Annalen.* 310 (1998), 571-600.
- [DM] C. Dong, G. Mason, *Integrability of C_2 -cofinite vertex operator algebras*, arXiv:math/0601569.
- [F] E. Feigin, *The PBW filtration*, *Represent. Theory* 13 (2009), 165-181.
- [FH] W. Fulton, J. Harris, *Representation Theory*, Graduate Texts in Mathematics, Springer Verlag, New York 1991.
- [FZ] I. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, *Duke Math. J.* 66 (1992), 123-168.
- [GG] M. R. Gaberdiel, T. Gannon, *Zhu's algebra, the C_2 algebra, and twisted modules*, arXiv:0811.3892
- [GabGod] M. R. Gaberdiel, P. Goddard, *Axiomatic conformal field theory*, *Commun. Math. Phys.* 209 (2000), 549-594.
- [GK] M. R. Gaberdiel, H. G. Kausch, *A rational logarithmic conformal field theory*, *Phys. Lett.* B386 (1996), 131-137.
- [GN] M. R. Gaberdiel, A. Neitzke, *Rationality, quasirationality and finite W -algebras*, *Commun. Math. Phys.* 238 (2003), 305-331.
- [GT] I. Gelfand, M. Tseitlin, *Finite-dimensional representations of the group of unimodular matrices*, pp. 653-656 in: I. Gelfand, *Collected papers*, volume II, Springer, 1988. Originally appeared in *Doklady akademii nauk SSSR*, nowaja serija 71 (1950), 825-828.
- [HK] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, *Grad. Stud. Math.* vol. 42, Amer. Math. Soc., Providence, RI (2002).
- [K1] V. Kac, *Infinite dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.
- [K2] V. Kac, *Vertex algebras for beginners*, University Lecture Series, **10**, 1997.
- [K3] V. Kac, *Some remarks on nilpotent orbits*, *J. Algebra* **64** (1980), pp. 190-213.
- [Li] H. Li, *Some finiteness properties of regular vertex operator algebras*, *J. Algebra* 212 (1999), 495-514.
- [L1] P. Littelmann, *On spherical double cones*, *J. Algebra* **166** (1994), pp. 142-157.
- [L2] P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, *Inventiones mathematicae* 116 (1994), 329-346.
- [LL] J. Lepowsky and H.-S. Li, *Introduction to Vertex Operator Algebras and Their Representations*, *Progress in Math.* 227, Birkhäuser, Boston, 2003.
- [M1] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C_2 -cofiniteness*, *Duke Math. J.* 122 (2004), 51-91.
- [M2] M. Miyamoto, *A theory of tensor products for vertex operator algebra satisfying C_2 -cofiniteness*, math.QA/0309350.
- [N] A. Neitzke, *Zhu's theorem and an algebraic characterization of chiral blocks*, arXiv:hep-th/0005144.
- [P] C. Procesi, *Lie groups, An approach to Lie Theory through Invariants and Representations*, Universitext, Springer Verlag, New York, (2007).
- [Z] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, *J. Amer. Math. Soc.* 9 (1996), 237-302.

BORIS FEIGIN:

LANDAU INSTITUTE FOR THEORETICAL PHYSICS,
PROSP. AKADEMIKA SEMENOVA, 1A, 142432, CHERNOGOLOVKA, RUSSIA,

and

HIGHER SCHOOL OF ECONOMICS,
MYASNITSKAYA UL., 20, 101000, MOSCOW, RUSSIA,

and

INDEPENDENT UNIVERSITY OF MOSCOW,
BOL'SHOI VLAS'EVSKI PER., 11, 119002, MOSCOW, RUSSIA

E-mail address: `bfeigin@gmail.com`

EVGENY FEIGIN:

TAMM THEORY DIVISION, LEBEDEV PHYSICS INSTITUTE,
LENINISKY PROSPECT, 53, 119991, MOSCOW, RUSSIA,

and

FRENCH-RUSSIAN PONCELET LABORATORY, INDEPENDENT UNIVERSITY OF MOSCOW

E-mail address: `evgfeig@gmail.com`

PETER LITTELMANN:

MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN,
WEYERTAL 86-90, D-50931 KÖLN, GERMANY

E-mail address: `littelma@math.uni-koeln.de`