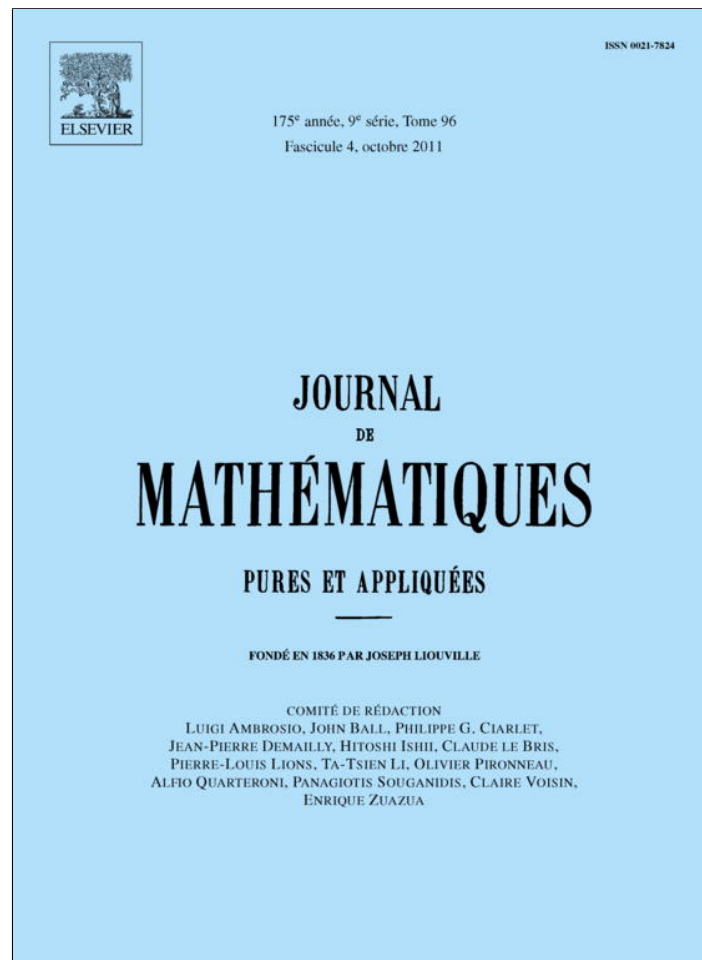


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# Strong trajectory attractors for dissipative Euler equations <sup>☆</sup>

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## Abstract

The 2D Euler equations with periodic boundary conditions and extra linear dissipative term  $Ru$ ,  $R > 0$  are considered and the existence of a strong trajectory attractor in the space  $L_{loc}^{\infty}(\mathbb{R}_+, \mathcal{H}^1)$  is established under the assumption that the external forces have bounded vorticity. This result is obtained by proving that any solution belonging to the proper weak trajectory attractor has a bounded vorticity which implies its uniqueness (due to the Yudovich theorem) and allows to verify the validity of the energy equality on the weak attractor. The convergence to the attractor in the strong topology is then proved via the energy method.

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## Résumé

On considère les équations d'Euler bi-dimensionnelles avec des conditions aux limites périodiques et un terme dissipatif  $Ru$ ,  $R > 0$ , supplémentaire et on montre l'existence de l'attracteur trajectoire fort dans l'espace  $L_{loc}^{\infty}(\mathbb{R}_+, \mathcal{H}^1)$  sous l'hypothèse que les forces extérieures ont un rotationnel borné. Ce résultat est obtenu en démontrant que toute solution appartenant à un attracteur trajectoire faible convenable a un rotationnel borné, ce qui entraîne son unicité (par le théorème de Yudovich) et permet de vérifier la validité de l'égalité d'énergie sur l'attracteur faible. La convergence vers l'attracteur dans la topologie forte est alors obtenue par la méthode d'énergie.

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## 1. Introduction

In the present paper, we study the attractors for the 2D Euler system,

$$\partial_t u + (u, \nabla_x)u + Ru + \nabla_x p = g, \quad \operatorname{div} u = 0, \quad (1.1)$$

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containing the additional dissipative term  $Ru$  and equipped by the periodic boundary conditions. These equations describe, for instance, a 2-dimensional fluid moving on a rough surface and are used in geophysical models for large-scale processes in atmosphere and ocean. The term  $Ru$  parameterizes the main dissipation occurring in the planetary boundary layer (see, e.g., [24]; see also [6] for the alternative source of damped Euler equations).

The mathematical features of these and related equations are studied in a number of papers (see, for instance, [3,5,7,16–18,26]) including the analytic properties (which are very similar to the classical Euler equations without dissipative term, see [4,19,20,30] and references therein), stability analysis, vanishing viscosity limit and various attractors.

Remind that, in contrast to the Navier–Stokes equations, the considered damped Euler system is *hyperbolic* (and is invertible in time), so one cannot expect any smoothing properties in a finite time. In addition, up to the moment, the questions related with smoothness of solutions of that equations are still badly understood. In a fact, to the best of our knowledge, only the modifications of the classical Yudovich result on the global existence of smooth solutions with possible double exponential growth in time are available in the literature and that is clearly insufficient for the attractors theory. Thus, it seems extremely difficult/impossible to obtain the asymptotic smoothing properties for that equations which are crucial for the classical theory of the attractors (see [1,28] and references therein). By this reason, only the existence of the attractor(s) in a *weak* topology has been verified before [16,5,7].

Another essential problem with the Euler equations is related with uniqueness. Indeed, the uniqueness result is known only for the solutions with bounded vorticity (due to Yudovich, see [32,34]) and is not known in the natural phase spaces  $\mathcal{H}$  or  $\mathcal{H}^1$ . So, for studying the long-time behavior in that spaces one has to deal either with the multi-valued semigroups [16] or with the so-called *trajectory* approach and trajectory attractors [5,7].

The main aim of the present paper is to verify the existence of the (trajectory) attractor for the damped equation in the phase space  $\mathcal{H}^1$  in a *strong* topology. The main difficulty here is, of course, to establish the asymptotic compactness. In order to gain it, we first construct the “usual” *weak* trajectory attractor and verify (using the maximum principle for the vorticity equation) that any solution, belonging to the attractor, has a bounded vorticity. To this end we use a slightly different (in comparison with [7] and [8–10]) construction of the trajectory attractor. Namely (following [35], see also [21]), a weak solution  $u(t)$  of the damped Euler equations is included in the trajectory phase space of the problem if and *only* if it can be obtained as a vanishing viscosity limit of the corresponding solutions of the Navier–Stokes equations. One of the advantages of this construction is that now every weak solution can be approximated by the regular ones and the justification of the maximum principle for such solutions (for the vorticity equations) becomes immediate.

As a result, following Yudovich (see [32]), we obtain the uniqueness on the (weak trajectory) attractor and this allows to establish the energy *equality* for the solutions belonging to the attractor. This equality is obtained from the corresponding energy *inequality* using the trick with reversing time (analogously to [15]).

Finally, we prove the desired asymptotic compactness using the so-called energy method (see [14,22] for the applications of this method to usual attractors; [2,23] for the multi-valued semigroups and [11,13] for the trajectory attractors). Trivial, but fruitful observation (in comparison with the previous works) which allows to obtain the result is that the energy equality is factually necessary on the attractor *only* (and it is sufficient to have the energy *inequality* outside of the attractor).

The paper is organized as follows. The existence of a weak vanishing viscosity solutions for the damped Euler equation as well as the construction of a weak trajectory attractor is given in Section 1. The boundedness of the vorticity on the attractor is verified in Section 2. Moreover, for the convenience of the reader, we reproduce the proof of the Yudovich uniqueness theorem here. Finally, the energy equality on the weak trajectory attractor and the existence of the attractor in a strong topology is proved in Section 3.

To conclude, we note that we consider the periodic boundary conditions only for simplicity. The difference with the case of a general bounded domain is only that one should equip the approximating Navier–Stokes problems by the proper boundary conditions in order to avoid the boundary layers. In contrast to that, the choice of the relatively strong phase space  $\mathcal{H}^1$  is crucial for us. Indeed, although one can construct the weak trajectory attractors in weaker (and, in a sense, more natural) phase space  $\mathcal{H}$  (e.g., using the concept of the so-called dissipative solution of the Euler equation for the vanishing viscosity limit of the proper Navier–Stokes solutions, see [19]), we do not know how to establish the strong convergence to that attractor as well as how to verify that it coincides with the attractor constructed in the phase space  $\mathcal{H}^1$ . By this reason, we do not develop this theory here.

## 2. Dissipativity and weak trajectory attractor

In this section, we construct the so-called *weak* trajectory attractor for the 2D damped Euler system:

$$\begin{cases} \partial_t u + (u, \nabla_x)u + Ru + \nabla_x p = g, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u_0, \end{cases} \quad (2.1)$$

in the domain  $\Omega := [-\pi, \pi]^2$  with periodic boundary conditions and with the external force  $g \in W^{1,\infty}(\Omega)$ . The attractors for that system in the weak topology of the appropriate phase space have been already studied in the literature (see [16,5,7]). However, in order to be able to verify the attraction property in the *strong* topology, we need to use (following to [35]) a slightly different construction of the trajectory attractor which allows to consider only the solutions of (2.1) which can be approximated by the smooth solutions of the Navier–Stokes system:

$$\begin{cases} \partial_t u + (u, \nabla_x)u + Ru + \nabla_x p - \nu \Delta_x u = g, \\ \operatorname{div} u = 0, \quad u|_{t=0} = u_0 \end{cases} \quad (2.2)$$

as  $\nu \rightarrow 0$ . Although we do not know whether or not the other solutions of the damped Euler system (2.1) exist, the vanishing viscosity approach not only looks more natural from the physical point of view, but also allows us to avoid a rather delicate problem of justification of the asymptotic  $L^\infty$ -estimates for the vorticity equation,

$$\partial_t \omega + (u, \nabla_x)\omega + R\omega = \operatorname{curl} g, \quad (2.3)$$

where  $\omega := \operatorname{curl} u = \partial_{x_2} u_1 - \partial_{x_1} u_2$ .

As usual, we denote by  $W^{l,p} = W^{l,p}(\Omega)$ ,  $1 \leq p \leq \infty$ , the Sobolev space of space-periodic distributions whose distributional derivatives up to order  $l$  belong to  $L^p$ . The Hilbert spaces  $W^{l,2}$  will be often denoted by  $H^l$ . Finally, we use the notations  $\mathcal{H}^l$  for the closed subspace of  $[H^l]^2$  generated by the divergence free vector fields. Since the norms in  $\mathcal{H}^l$  and  $[H^l]^2$  coincide, we will write  $\|u\|_{H^l}$  instead of  $\|u\|_{\mathcal{H}^l}$ .

We start with reminding the well-known uniform (with respect to  $\nu \rightarrow 0$ )  $\mathcal{H}^1$ -estimates for the solutions of the Navier–Stokes problem (2.2). Note that, in contrast to the classical case  $R = 0$ , for  $R > 0$ , we need not the assumption that the initial data  $u_0$  and external forces  $g$  have zero mean.

**Proposition 2.1.** *Let  $R, \nu > 0$  and  $u_0 \in \mathcal{H}^1$ . Then, problem (2.2) possesses a unique strong solution  $u \in C([0, T], \mathcal{H}^1) \cap L^2([0, T], \mathcal{H}^2)$  and the following dissipative estimate holds:*

$$\|u(T)\|_{H^1}^2 \leq \|u(S)\|_{H^1}^2 e^{-R(T-S)} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-R(T-S)}), \quad (2.4)$$

where  $T \geq S \geq 0$ .

Indeed, the dissipative estimate (2.4) follows immediately by multiplications of (2.2) by  $u$  and  $\Delta_x u$  (the inertial term vanishes since we are in the 2D case with periodic boundary conditions) and the uniqueness is also standard, see e.g., [28,30] for the details.

The obtained estimate (2.4) allows us to pass to the limit  $\nu \rightarrow 0$  and construct a solution of the limit Euler equation. To be more precise, we *define* the solution  $u$  of the Euler equations (2.1) via this procedure.

Let  $\Theta_+^{w,\text{loc}}$  be the local weak-star topology in the space  $L_{\text{loc}}^\infty(\mathbb{R}_+, \mathcal{H}^1)$ . By definition, a sequence  $v_n(t) \rightarrow v(t)$  ( $n \rightarrow \infty$ ) in the topology  $\Theta_+^{w,\text{loc}}$ , if, for every  $T > 0$ ,  $v_n(t) \rightarrow v(t)$  ( $n \rightarrow \infty$ )  $*$ -weakly in  $L_\infty(0, T; \mathcal{H}^1)$ .

**Definition 2.2.** A function  $u \in \mathcal{F}_+^b := L^\infty(\mathbb{R}_+, \mathcal{H}^1)$  is a solution of the damped Euler equation (2.1) with  $u_0 \in \mathcal{H}^1$  if it solves (2.1) in the sense of distributions and there exists a sequence  $v_n \rightarrow 0$  and a sequence  $u^{v_n}$  of strong solutions of the approximate Navier–Stokes system (2.2) with  $\nu = v_n$  such that

$$u = \Theta_+^{w,\text{loc}} - \lim_{n \rightarrow \infty} u^{v_n}. \quad (2.5)$$

Note that the convergence (2.5) implies in a standard way the weak-star convergence of  $\partial_t u^{v_n}$  in  $L_{\text{loc}}^\infty(\mathbb{R}_+, \mathcal{H}^{-1})$  (see [7,8]) and this gives the *strong* convergence in  $C_{\text{loc}}([0, \infty), \mathcal{H})$ . Thus, any solution  $u$  of the Euler equation is weakly continuous with values in  $\mathcal{H}^1$  ( $u \in C([0, T], \mathcal{H}_w^1)$ ) and, for any  $T \geq 0$ , we have the *weak* convergence,

$$u^{v_n}(T) \rightarrow u(T), \quad (2.6)$$

in the space  $\mathcal{H}^1$ . It is however important that we do not require the strong convergence  $u^{v_n}(T) \rightarrow u(T)$  in  $\mathcal{H}^1$  even for  $T = 0$ .

The following standard proposition gives the existence of a solution thus defined.

**Proposition 2.3.** *Under the above assumptions, for any  $u_0 \in \mathcal{H}^1$ , there exists at least one solution  $u$  of the damped Euler equation (2.1).*

**Proof.** Indeed, let  $u^v(t)$ ,  $v \rightarrow 0$ , be the solutions of the approximate Navier–Stokes problems (2.2) such that  $u^v(0) \rightarrow u_0$  as  $v \rightarrow 0$  (in particular, we may set  $u^v(0) = u_0$ ). Then, due to the dissipative estimate (2.4),  $u_v$  are uniformly bounded in  $\mathcal{F}_+^b$  and, since the unit ball of  $\mathcal{F}_+^b$  is compact in the local weak-star topology (of  $\Theta_+^{w,loc}$ , see [25]), there exists a sequence  $v_n \rightarrow 0$  and a function  $u \in \mathcal{F}_+^b$  such that (2.5) holds. In addition, using that  $\partial_t u^{v_n}$  are uniformly bounded in  $L^\infty(\mathbb{R}_+, \mathcal{H}^{-1})$ , we conclude that  $u^{v_n} \rightarrow u$  strongly in  $C_{loc}(\mathbb{R}_+, \mathcal{H})$ . This strong convergence, together with the weak-star convergence (2.5), is clearly enough in order to pass to the limit  $v_n \rightarrow 0$  (in the sense of distributions) in Eqs. (2.2) and verify that the obtained function  $u$  solves indeed the limit Euler problem (2.1).  $\square$

In order to write out the analogue of the dissipative estimate (2.4) for the limit Euler equations (2.1), it is convenient to introduce (following to [35]) the so-called  $M$ -functional on the solution  $u$ :

$$M_u(t) := \inf \left\{ \liminf_{n \rightarrow \infty} \|u^{v_n}(t)\|_{H^1}, u \in \Theta_+^{w,loc} - \lim_{n \rightarrow \infty} u^{v_n} \right\}, \tag{2.7}$$

where the external infimum is taken over all possible sequences of solutions of the approximate Navier–Stokes system which converge as  $v_n \rightarrow 0$  to the given solution  $u$  of the limit Euler equation. The next proposition collects some simple properties of the introduced functional.

**Proposition 2.4.** *Let  $u$  be a solution of the Euler problem and let  $M_u(t)$  be the associated  $M$ -functional. Then*

- 1)  $\|u(t)\|_{H^1} \leq M_u(t)$ , for all  $t \in \mathbb{R}_+$ ;
- 2) The following analogue of the dissipative estimate holds:

$$[M_u(T)]^2 \leq [M_u(S)]^2 e^{-R(T-S)} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-R(T-S)}), \tag{2.8}$$

where  $T \geq S \geq 0$ ;

- 3)  $M_{T(h)u}(t) \leq M_u(t+h)$ , for all  $h \geq 0$ , where  $(T(h)u)(t) := u(t+h)$ ;
- 4) The functional  $M_u(0)$  is weak lower semi-continuous with respect to  $u$ , i.e., if  $u_n$  be a sequence of solutions of Eq. (2.1) (in the sense of Definition 2.2) such that  $u = \Theta_+^{w,loc} - \lim_{n \rightarrow \infty} u_n$  and the sequence  $M_{u_n}(0)$  is bounded, then  $u$  is also a solution of (2.1), and

$$M_u(0) \leq \liminf_{n \rightarrow \infty} M_{u_n}(0). \tag{2.9}$$

**Proof.** For the convenience of the reader, we briefly remind the proof of that assertions (see [35] for more details).

Indeed, the first assertion is immediate (since the norm  $\|\cdot\|_{H^1}$  is weakly lower semi-continuous and the convergence  $u^{v_n}$  to  $u$  in  $\Theta_+^{w,loc}$  implies the weak convergence  $u^{v_n}(t) \rightarrow u(t)$  for every  $t$ ). To prove the second one, we note that, due to the energy equalities for the approximating Navier–Stokes problems, we have:

$$\|u^{v_n}(T)\|_{H^1}^2 \leq \|u^{v_n}(S)\|_{H^1}^2 e^{-R(T-S)} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-R(T-S)}), \tag{2.10}$$

for any approximating sequence  $u^{v_n}$ . By the definition of the  $M$ -functional, for any  $\varepsilon > 0$ , we may find the approximating sequence  $u^{v_n}$  such that

$$\liminf_{n \rightarrow \infty} \|u^{v_n}(S)\|_{H^1} \leq M_u(S) + \varepsilon.$$

Passing to the limit  $n \rightarrow \infty$  in (2.10), we have:

$$[M_u(T)]^2 \leq \liminf_{n \rightarrow \infty} \|u^{v_n}(T)\|_{H^1}^2 \leq [M_u(S) + \varepsilon]^2 e^{-R(T-S)} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-R(T-S)}),$$

and since  $\varepsilon$  is arbitrary, this gives the desired inequality (2.8).

The third assertion is also evident since the infimum in the definition of  $M_{T(h)u}(t)$  is taken over the *larger* set of admissible approximating sequences than the infimum in the definition of  $M_u(t+h)$ .

Let us prove the fourth assertion (which is, in a sense, the most important for the attractor theory). Indeed, since  $M_{u_n}(0)$  is bounded, from (2.8) we conclude that the sequence  $M_{u_n}(t)$  is bounded for every  $t \geq 0$ . Moreover, since every  $u_n$  is a solution of Eq. (2.1), there exists a sequence  $u_n^{v_k}$  of solutions of the Navier–Stokes problems (2.2) such that  $v_k = v_{n,k} \rightarrow 0$  as  $k \rightarrow \infty$ , and

$$u_n = \Theta_+^{w,loc} - \lim_{k \rightarrow \infty} u_n^{v_k}.$$

Without loss of generality, we may assume that, in addition,

$$|M_{u_n}(0) - \|u_n^{v_k}(0)\|_{H^1}| \leq 1/n, \quad v_k \leq 1/n, \tag{2.11}$$

for all  $k \in \mathbb{N}$  and, passing to the subsequence in  $n$  if necessary, we may suppose also that

$$M_0 := \liminf_{n \rightarrow \infty} M_{u_n}(0) = \lim_{n \rightarrow \infty} M_{u_n}(0).$$

Now, we only need to extract from the two-parametric sequence  $\{u_n^{v_k}\}_{k,n \in \mathbb{N}}$  the one-parametric sequence which will converge to the limit function  $u$ . To this end, we use the fact that the topology of  $\Theta_+^{w,loc}$  is *metrizable* on every bounded set of  $\mathcal{F}_+^b$ , see [25]. Indeed, since the sequence  $M_{u_n}(0)$  is bounded, according to (2.8),  $M_{u_n}(t)$  is uniformly bounded in  $n$  and  $t$ . Thus, due to the first property of Proposition 2.4, the sequence  $u_n$  is bounded in  $\mathcal{F}_+^b$ . Let  $\rho$  be such that all  $u_n$  belong to the closed  $\rho$ -ball  $B_\rho$  of  $\mathcal{F}_+^b$  and let  $d(\cdot, \cdot)$  be a metric on the ball  $B_{2\rho}$  which metrizes the topology of  $\Theta_+^{w,loc}$  on that ball. Then, obviously,  $u \in B_\rho \subset B_{2\rho}$  and, we may also assume that  $u_n^{v_k}$  belong to  $B_{2\rho}$  for all  $n$  and  $k$  (the sequence  $u_n^{v_k}(0)$  is uniformly bounded in  $n$  and  $k$  thanks to (2.11) and the fact that  $M_{u_n}(0)$  is bounded, this together with estimate (2.10) give the uniform boundedness of  $u_n^{v_k}(t)$  with respect to  $n, k$  and  $t$ ). Thus,

$$\lim_{n \rightarrow \infty} d(u, u_n) = 0, \quad \lim_{k \rightarrow \infty} d(u_n, u_n^{v_k}) = 0$$

for every  $n$ . So, for every  $n \in \mathbb{N}$ , we may find  $k = k(n)$  such that  $d(u_n, u_n^{v_{k(n)}}) \leq 1/n$ . Then, due to the triangle inequality, we have  $d(u, u_n^{v_{k(n)}}) \rightarrow 0$  as  $n \rightarrow \infty$  and, therefore,

$$u = \Theta_+^{w,loc} - \lim_{n \rightarrow \infty} u_n^{v_{k(n)}}.$$

Moreover, due to (2.11),  $v_{k(n)} \rightarrow 0$  as  $n \rightarrow \infty$ , so (analogously to Proposition 2.3)  $u$  is a solution of the Euler equation (2.1), and

$$M_u(0) \leq \lim_{n \rightarrow \infty} \|u_n^{v_{k(n)}}(0)\|_{H^1} = M_0.$$

Proposition 2.4 is proved.  $\square$

**Remark 2.5.** It would be nice to have the lower semi-continuity (2.9) not only for  $t = 0$ , but for any  $t \geq 0$ . However, the proof given above fails in that situation. Indeed, replacing  $t = 0$  by arbitrary  $t$  in (2.11), we will obtain the uniform boundedness of  $u_n^{v_k}(t)$  and, in order to verify that  $u_n^{v_k}$  are bounded in  $\mathcal{F}_+^b$ , we need the control of  $u_n^{v_k}(0)$ . In the situation of [35] (where the approximating equations were also hyperbolic) this boundedness was easy to verify just using the energy estimate with reversed time. But in our situation, the approximating equations are *parabolic* and the uniform boundedness at time moment  $t$  does not imply boundedness at  $t = 0$ . So, we do not know whether or not (2.9) holds for  $t \neq 0$ . Fortunately, the lower continuity at  $t = 0$  is enough for what follows.

We are now ready to construct the trajectory phase space, the trajectory semigroup and the kernel associated with the damped Euler equation (2.1).

**Definition 2.6.** Let  $\mathcal{K}_+ \subset \mathcal{F}_+^b$  be the set of all solutions of (2.1) (in the sense of Definition 2.2) which correspond to all  $u_0 \in \mathcal{H}^1$  and let  $T(h) : \mathcal{K}_+ \rightarrow \mathcal{K}_+$ ,  $h \geq 0$  be the translation semigroup ( $(T(h)u)(t) := u(t+h)$ ). Then, we will refer to  $\mathcal{K}_+$  and  $T(h) : \mathcal{K}_+ \rightarrow \mathcal{K}_+$  as a trajectory phase space and a trajectory dynamical system associated with the dissipative Euler equation respectively.

In addition, we endow the set  $\mathcal{K}_+$  by the topology induced by the embedding  $\mathcal{K}_+ \subset \Theta_+^{w,loc}$  and will say that a subset of trajectories  $B \subset \mathcal{K}_+$  is *bounded* ( $M$ -bounded) if

$$M_B(0) := \sup_{u \in B} M_u(0) < \infty. \tag{2.12}$$

Note that a set  $B \subset \mathcal{K}_+$  is  $M$ -bounded if there is a number  $\rho$  such that for every  $u \in B$  there exist a sequence  $u^{v_n}$  of solutions on the NS-system such that  $\|u^{v_n}(0)\|_{H^1} \leq \rho$  and  $u^{v_n} \rightarrow u$  ( $v_n \rightarrow 0+$ ) in  $\Theta_+^{w,loc}$ . In particular, any  $M$ -bounded set  $B \subset \mathcal{K}_+$  is bounded in the norm  $F_+^b$ . The converse statement, a priori, may not hold, i.e. boundedness (in  $F_+^b$ ) of a set  $B \subset \mathcal{K}_+$  may not imply the  $M$ -boundedness (to be more precise, it is not known).

**Definition 2.7.** A kernel  $\mathcal{K} \subset L^\infty(\mathbb{R}, \mathcal{H}^1)$  of the damped Euler equation (2.1), by definition, consists of all complete (defined for all  $t \in \mathbb{R}$ ) bounded solutions of (2.1) which can be obtained as a weak limit of the appropriate solutions of the Navier–Stokes problems (2.2) as  $\nu \rightarrow 0$ . Namely,  $u \in \mathcal{K}$  if and only if there exist a sequence  $\nu_n \rightarrow 0$ , a sequence of times  $t_n \rightarrow -\infty$ , and a bounded sequence of the initial data  $\xi_n \in \mathcal{H}^1$ ,  $\|\xi_n\|_{H^1} \leq C$  such that the corresponding solutions  $u^{v_n}$  of the Navier–Stokes problems (2.2) on the time intervals  $[t_n, \infty)$  and with the initial data  $u^{v_n}(t_n) = \xi_n$  converge weakly-star in  $L^\infty(\mathbb{R}, \mathcal{H}^1)$  to the complete solution  $u$  considered.

We now remind the definition of the associated trajectory attractor (see [8] for the detailed exposition).

**Definition 2.8.** A set  $\mathcal{A}^{tr} \subset \mathcal{K}_+$  is a (weak) trajectory attractor associated with the damped Euler equation (= global attractor for the trajectory dynamical system  $(T(h), \mathcal{K}_+)$ ) if the following conditions are satisfied:

- 1)  $\mathcal{A}^{tr}$  is compact in  $\mathcal{K}_+$  and is  $M$ -bounded;
- 2) It is strictly invariant:  $T(h)\mathcal{A}^{tr} = \mathcal{A}^{tr}$ ,  $h > 0$ ;
- 3) It attracts the images of bounded ( $M$ -bounded) sets as  $h \rightarrow \infty$ , i.e., for every  $B$  bounded in  $\mathcal{K}_+$  and every neighborhood  $\mathcal{O}(\mathcal{A}^{tr})$  of  $\mathcal{A}^{tr}$  (in the topology of  $\Theta_+^{w,loc}$ ) there exists  $H = H(B, \mathcal{O})$  such that

$$T(h)B \subset \mathcal{O}(\mathcal{A}^{tr}), \quad \forall h \geq H.$$

The next theorem which establishes the existence of the above defined attractor and gives some description of its structure can be considered as the main result of this section.

**Theorem 2.9.** *Let the above assumptions hold. Then, the damped Euler equation (2.1) possesses a trajectory attractor  $\mathcal{A}^{tr} \subset \mathcal{F}_+^b$  and the following description holds:*

$$\mathcal{A}^{tr} := \Pi_{t \geq 0} \mathcal{K}, \tag{2.13}$$

where  $\mathcal{K}$  is the kernel in the sense of Definition 2.7.

**Proof.** According to the general theory (see [8,21]), we only need to check that the trajectory dynamical system is continuous and that it possesses a compact and  $M$ -bounded absorbing set. The continuity is obvious since  $T(h)$  are simply shift operators and they are continuous on  $\Theta_+^{w,loc}$ . In addition, estimate (2.8), guarantees that the set,

$$\mathcal{B} := \{u \in \mathcal{K}_+, [M_u(0)]^2 \leq 2R^{-1} \|g\|_{H^1}^2\},$$

will be an absorbing set for the semigroup  $T(h)$  acting on  $\mathcal{K}_+$ . Moreover, this set is even semi-invariant. Indeed, due to Proposition 2.4,

$$\begin{aligned} [M_{T(h)u}(0)]^2 &\leq [M_u(h)]^2 \leq [M_u(0)]^2 e^{-Rh} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-Rh}) \\ &\leq 2R^{-1} \|g\|_{H^1}^2 e^{-Rh} + R^{-1} \|g\|_{H^1}^2 (1 - e^{-Rh}) = R^{-1} \|g\|_{H^1}^2 e^{-Rh} + R^{-1} \|g\|_{H^1}^2 \leq 2R^{-1} \|g\|_{H^1}^2, \end{aligned}$$

for all  $u \in \mathcal{B}$  and, therefore,  $T(h)\mathcal{B} \subset \mathcal{B}$ . Thus, we only need to check that this set is *compact*.

Since, due to inequality (2.8), the set  $\mathcal{B}$  is bounded in  $\mathcal{F}_+^b$  and, therefore, precompact in  $\Theta_+^{w,loc}$ , we need to prove that if  $u_n \in \mathcal{B}$  and  $u = \Theta_+^{w,loc} - \lim_{n \rightarrow \infty} u_n$ , then  $u \in \mathcal{B}$  as well. But this fact follows immediately from the forth assertion of Proposition 2.4.

Thus, all assumptions of the abstract attractor existence theorem are satisfied and, therefore, the weak trajectory attractor exists. The description (2.13) is a standard corollary of the fact that  $\mathcal{A}^{tr}$  is factually an  $\omega$ -limit set of  $\mathcal{B}$ .  $\square$

### 3. Additional regularity and uniqueness on the attractor

In this section, we first verify that the vorticity  $\omega(t)$  is uniformly bounded on the trajectory attractor  $\mathcal{A}^{tr}$  and then using that fact and following the Yudovich technique, we establish that the solution of the Euler equation is unique on the attractor. This fact will be used later in order to establish the energy equality on the attractor.

The following theorem gives the additional regularity of the solutions belonging to the attractor.

**Theorem 3.1.** *Let the assumptions of Theorem 2.9 hold. Then, for every  $u \in \mathcal{K}$ , the associated vorticity  $\omega = \partial_{x_2}u_1 - \partial_{x_1}u_2$  belongs to  $L^\infty(\mathbb{R} \times \Omega)$  and the following estimate holds:*

$$\|\omega(t)\|_{L^\infty(\Omega)} \leq R^{-1} \|g\|_{W^{1,\infty}} \tag{3.1}$$

for all  $t \in \mathbb{R}$ .

**Proof.** The proof is essentially based on the maximum principle for the vorticity equation and on the description of  $\mathcal{K}$  obtained in Theorem 2.9. Namely, let  $u \in \mathcal{K}$  be arbitrary and let  $v_n \rightarrow 0$ ,  $t_n \rightarrow -\infty$  and  $u^{v_n}(t)$  be the sequence of solutions of the Navier–Stokes system which approximates  $u$ . Then, the associated vorticities  $\omega_n := \text{curl } u^{v_n}$  solve the following equations:

$$\partial_t \omega_n - v_n \Delta_x \omega_n + (u^{v_n}, \nabla_x) \omega_n + R \omega_n = \text{curl } g, \quad \omega_n(t_n) = \omega_n^0, \tag{3.2}$$

and  $\|\omega_n^0\|_{L^2} \leq C$  (uniformly with respect to  $n$ ). Let us fix also arbitrary  $T \in \mathbb{R}$ . Then, from the convergence  $u_n \rightarrow u$ , we know also that

$$\omega_n(T) \rightharpoonup \omega(T) \tag{3.3}$$

weakly in  $L^2(\Omega)$ . Thus, we only need to split  $\omega_n(T)$  on the  $L^\infty$  and decaying parts. Indeed, let  $\omega_n(t) := U_n(t) + V_n(t)$ , where  $U_n(t)$  solves

$$\partial_t U_n - v_n \Delta_x U_n + (u^{v_n}, \nabla_x) U_n + R U_n = 0, \quad U_n(t_n) = \omega_n^0, \tag{3.4}$$

and the reminder  $V_n$  satisfies:

$$\partial_t V_n - v_n \Delta_x V_n + (u^{v_n}, \nabla_x) V_n + R V_n = \text{curl } g, \quad V_n(t_n) = 0. \tag{3.5}$$

Note that, in contrast to the limit case  $v_n = 0$ , Eqs. (3.4) and (3.5) are *parabolic* if  $v_n > 0$  and we have enough regularity to verify the uniqueness. In particular, clearly,  $\omega_n, U_n, V_n \in L^2(0, T, H^1)$  and the equations can be understood as equalities in  $L^2(0, T, H^{-1})$ . Thus, the key multiplication of (3.4) (or (3.5)) by  $U_n$  (resp.  $V_n$ ) is justified.

Applying now the comparison principle for second order parabolic equation (3.5), we see that

$$\|V_n(T)\|_{L^\infty} \leq R^{-1} \|\text{curl } g\|_{L^\infty}. \tag{3.6}$$

This estimate can be justified, e.g., as follows: multiplying Eq. (3.5) by,

$$w_+(t, x) := \max\{U_n(t, x) - R^{-1} \|\text{curl } g\|_{L^\infty}, 0\},$$

integrating by parts in a standard way (note that  $w_+ \in L^2(0, T, H^1)$ , so all integrals have sense) and using that

$$((u, \nabla_x)w, w_+) = ((u, \nabla_x)w_+, w_+) = 0$$

for all  $u \in \mathcal{H}^1$  and  $w_+ \in H^1$ , and we end up with

$$\frac{1}{2} \frac{d}{dt} \|w_+(t)\|_{L^2}^2 + v_n \|\nabla_x w_+(t)\|_{L^2}^2 + R \|w_+(t)\|_{L^2}^2 = (\text{curl } g - \|\text{curl } g\|_{L^\infty}, w_+) \leq 0.$$

This inequality together with the fact that  $w_+(t_n) = 0$  gives  $w_+(t) \equiv 0$  and  $U_n(t, x) \leq R^{-1} \|\text{curl } g\|_{L^\infty}$  almost everywhere. The lower bounds for  $U_n(t)$  can be justified analogously, see e.g., [31,27] for more details.



On the other hand, multiplying Eq. (3.4) by  $U_n$  integrating by  $x$  and using that  $\operatorname{div} u^{v_n} = 0$ , we establish that

$$\frac{d}{dt} \|U_n(t)\|_{L^2}^2 + R \|U_n(t)\|_{L^2}^2 \leq 0$$

and, therefore,

$$\|U_n(T)\|_{L^2}^2 \leq C e^{-R(T-t_n)}, \tag{3.7}$$

where  $C$  is independent of  $n$ . Thus,  $U_n(T) \rightarrow 0$  in  $L^2$  as  $n \rightarrow \infty$  and, without loss of generality, we may think that  $V_n(T) \rightharpoonup V_0$  (weakly-star) in  $L^\infty(\Omega)$  for some  $V_0 \in L^\infty(\Omega)$  and  $V_0$  satisfies (3.6) as well. Finally, we have established that  $\omega_n(T) \rightharpoonup V_0$  weakly in  $L^2(\Omega)$  which together with (3.3) gives that  $\omega(T) \in L^\infty(\Omega)$  and satisfies the analogue of (3.6). So, the theorem is proved.  $\square$

**Corollary 3.2.** *Under the assumptions of the previous theorem, any solution  $u(t)$  on the attractor  $\mathcal{A}^r$  belongs to  $W^{1,p}(\Omega)$  for all  $p < \infty$  and the following estimate holds:*

$$\|u(t)\|_{W^{1,p}(\Omega)} \leq Cp \|g\|_{W^{1,\infty}(\Omega)}, \quad p > 1, \tag{3.8}$$

where the constant  $C$  is independent of  $t$  and  $p$ .

Indeed, the solution  $u(t)$  can be expressed via the vorticity  $\omega$  using the Biot–Savart law:

$$u - \langle u \rangle = (\partial_{x_2}(-\Delta_x)^{-1}\omega, -\partial_{x_1}(-\Delta_x)^{-1}\omega),$$

where  $\langle u \rangle := \frac{1}{|\Omega|} \int_{\Omega} u \, dx$  and  $(-\Delta_x)^{-1}$  is the inverse Laplacian with periodic boundary conditions acting on the space of functions with zero mean. The desired estimate (3.8) is now an immediate corollary of (3.1) and the well-known maximal regularity for the Laplacian in  $L^p$ :

$$\|(-\Delta_x)^{-1}\|_{L^p \rightarrow W^{2,p}} \leq Cp, \quad p > 1,$$

see [33].

The next theorem gives the uniqueness of solutions of dissipative Euler equation in the class of solutions with bounded vorticity (note that, in this theorem, the uniqueness holds in the class of all such solutions and not only for solutions which can be obtained as a Navier–Stokes limit). Although that is just a variation of the famous Yudovich theorem (see [32,34]), for the convenience of the reader, we give below a simple proof of that result.

**Theorem 3.3.** *Let the above assumptions hold, let  $u_1, u_2 \in C([0, T], \mathcal{H})$  be two functions which solve the dissipative Euler equation (2.1) in the sense of distributions and let, in addition, the corresponding vorticities are bounded:  $\omega_1, \omega_2 \in L^\infty([0, T] \times \Omega)$ . Then,  $u_1(0) = u_2(0)$  implies  $u_1(t) = u_2(t)$  for all  $t \in [0, T]$ .*

**Proof.** Indeed, let  $u(t) = u_1(t) - u_2(t)$  and  $E(t) := \|u_1(t) - u_2(t)\|_{L^2}^2$ . Then, obviously,  $u \in C([0, T], \mathcal{H})$  and  $E \in C^1([0, T])$ . Moreover, analogously to Corollary 3.2, we know that  $u_i \in L^\infty([0, T], W^{1,p}(\Omega))$  and

$$\|\nabla_x u_i(t)\|_{L^p} \leq Cp, \quad t \in [0, T]. \tag{3.9}$$

Assume that  $E(t) > 0$  for  $t \in (0, \delta)$  for some  $\delta > 0$  (otherwise it is nothing to prove). The function  $u$  obviously solves the equation,

$$\partial_t u + (u_1, \nabla_x)u + (u, \nabla_x)u_2 + Ru + \nabla_x p = 0.$$

Multiplying this equation by  $u(t)$  and integrating over  $\Omega$ , we arrive at

$$\frac{d}{dt} E(t) + 2RE(t) \leq C \|\nabla_x u_2\|_{L^p} \|u(t)\|_{L^{2p^*}}^2$$

where  $p > 1$  is arbitrary and  $\frac{1}{p} + \frac{1}{p^*} = 1$ . Using now the interpolation inequality,

$$\|u\|_{L^{2p^*}} \leq \|u\|_{L^2}^{1-1/p} \|u\|_{L^\infty}^{1/p},$$

(see e.g., [29]) together with (3.9) and the fact that  $u$  is bounded in  $L^\infty$ , we obtain:

$$\frac{d}{dt} E(t) \leq Cp[E(t)]^{1-1/p}, \tag{3.10}$$

where the constant  $C$  is independent of  $p$ . The idea now is to fix  $p = p(E)$  in (3.10) in an optimal way. Namely, let

$$p = \log \frac{K}{E(t)},$$

where  $K$  is a sufficiently large number to be sure that  $p > 1$  for all  $t \in [0, T]$ . Using also the elementary fact that  $E^{-1/\log(K/E)} \leq C$ , we finally derive the following differential inequality,

$$\frac{d}{dt} E(t) \leq CE(t) \log \frac{K}{E(t)},$$

which is enough for the uniqueness. Indeed, integrating it we have:

$$E(t) \leq K [E(\varepsilon)/K] e^{-C(t-\varepsilon)},$$

for all small  $\varepsilon > 0$ . Passing to the limit  $\varepsilon \rightarrow 0$  ( $E(t)$  is continuous!), we get  $E(t) \equiv 0$  and finish the proof of the theorem.  $\square$

**Remark 3.4.** Note that the *backward* uniqueness theorem also holds for solutions of the damped Euler equation with bounded vorticity, namely, the equality  $u_1(T) = u_2(T)$  for some time  $T \geq 0$  two solutions of (2.1) (with bounded vorticities) implies that  $u_1(t) = u_2(t)$  for  $t \leq T$  as well. Indeed, reversing time in the (2.1) and repeating word by word the proof of the previous theorem, we end up with the following analogue of the differential inequality (3.10) for  $E(t) := \|u_1(T-t) - u_2(T-t)\|_H^2$ :

$$\frac{d}{dt} E(t) \leq Cp[E(t)]^{1-1/p} + 2RE(t).$$

Since the first term in the right-hand side dominates the second one (remind that  $E(t)$  is bounded on  $[0, T]$  and we need this estimate for  $E(t)$  *small* only). Thus, the second term in the right-hand side is not essential and, repeating word by word the rest of the proof, we obtain the backward uniqueness result. This simple observation is however crucial for our proof of the energy equality on the attractor (see next section).

#### 4. Energy equality and strong attraction

In that section, we verify that the so-called energy identity holds for every  $u \in \mathcal{A}^r$  and, based on this fact, establish that the attraction property and the compactness holds not only in a weak topology of  $\Theta_+^{w,loc}$  (where it is almost immediate), but also in a strong topology of the space  $\Theta_+^{s,loc} := L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^1)$ . We start with the following result:

**Theorem 4.1.** *Let  $u \in C([0, T], \mathcal{H}_w^1)$  solves the dissipative Euler equation (2.1) and be such that the corresponding vorticity  $\omega \in C([0, T], L_w^\infty(\Omega))$  is bounded. Then the function  $t \rightarrow \|\nabla_x u(t)\|_{L^2}$  is absolutely continuous and the following energy identity holds:*

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x u(t)\|_{L^2}^2 + R \|\nabla_x u(t)\|_{L^2}^2 + (\nabla_x g, \nabla_x u(t)) = 0, \tag{4.1}$$

for almost all  $t \in [0, T]$ .

**Proof.** It is more convenient for us to verify the absolute continuity not for  $\|\nabla_x u(t)\|_{L^2}^2$ , but for the equivalent function  $t \rightarrow e^{2Rt} \|\nabla_x u(t)\|_{L^2}^2$ . To this end, we introduce a new function  $v(t) := e^{Rt} u(t)$  which solves,

$$\partial_t v + \nabla_x \tilde{p} + e^{-Rt} (v, \nabla_x) v = \tilde{g}(t), \tag{4.2}$$

where

$$\tilde{p} := e^{Rt} p \quad \text{and} \quad \tilde{g}(t) := e^{Rt} g,$$

and will check the equivalent integral version of (4.1) for the function  $v$ , namely that

$$\frac{1}{2}(\|\nabla_x v(T)\|_{L^2}^2 - \|\nabla_x v(S)\|_{L^2}^2) = \int_S^T \tilde{g}(t) dt \tag{4.3}$$

holds for all  $T \geq S \geq 0$ . In turns, in order to verify the equality (4.3), we first verify the analogous *inequality*. To this end, we fix  $S$  being arbitrary and consider the approximation Navier–Stokes problems on the interval  $t \in [S, T]$ ,

$$\partial_t v_n + v_n \Delta_x v_n + e^{-Rt} (v_n, \nabla_x) v_n + \nabla_x \tilde{p}_n = \tilde{g}(t), \quad v_n(S) = v(S)$$

with  $v_n \rightarrow 0$ . Then, on the one hand, we have the energy equality for that Navier–Stokes problem which reads

$$\frac{1}{2}(\|v_n(T)\|_{L^2}^2 - \|v(S)\|_{L^2}^2) + v_n \int_S^T \|\Delta_x v_n(t)\|^2 dt = \int_S^T (\nabla_x \tilde{g}(t), \nabla_x v_n(t)) dt.$$

On the other hand, without loss of generality we may assume that  $v_n \rightarrow w$  (weakly in  $\Theta_+^{w,loc}$ ) where  $w$  solves the limit Euler problem. Moreover, since  $\text{curl } v_n(S)$  are uniformly bounded in  $L^\infty$ , we also conclude that  $\text{curl } w(t)$  is uniformly bounded in  $L^\infty$ . Thus, according to the uniqueness theorem,  $w = v$  and passing to the weak limit in the energy equality for  $v_n$ , we conclude that

$$\frac{1}{2}(\|\nabla_x v(T)\|_{L^2}^2 - \|\nabla_x v(S)\|_{L^2}^2) \leq \int_S^T \tilde{g}(t) dt. \tag{4.4}$$

Thus, the energy *inequality* is verified for all  $T \geq S \in [0, T]$ . In order to prove the inequality with the opposite sign, we use (following [15]) the reversibility of the Euler equation considered. Namely, note that the structure of Eq. (4.2) will not change if we replace  $t \rightarrow -t$ ,  $v \rightarrow -v$  and  $x \rightarrow -x$  (up to the non-essential change  $e^{-Rt} \rightarrow e^{Rt}$  in the vanishing inertial term). Thus, we are able to reverse the time and justify (exactly as before) the energy *inequality* (4.4) for the function  $\tilde{v}(t, x) = -v(-t, -x)$  with reversed time on the interval  $t \in [-T, -S]$ . Returning back to the function  $v$ , we see that this inequality coincides indeed with (4.4), but with the desired opposite sign. Thus, the inequality (4.4) is factually an identity and the theorem is proved.  $\square$

The following continuity property is the first standard corollary of the proved energy identity on the attractor.

**Corollary 4.2.** *Under the assumptions of Theorem 2.9 any function  $u \in \mathcal{K}$  belonging to the kernel of the dissipative Euler equation (2.1) is continuous as a  $\mathcal{H}^1$ -valued function:  $u \in C_{loc}(\mathbb{R}, \mathcal{H}^1)$ .*

Finally, we are able to state and prove the theorem about the strong attraction to the above constructed trajectory attractor  $\mathcal{A}^r$  which can be considered as the main result of the paper.

**Theorem 4.3.** *Let the assumptions of Theorem 2.9 hold. Then, the attractor  $\mathcal{A}_{tr}$  is compact in the strong topology of  $\Theta_+^{s,loc} := L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^1)$  and the attraction property holds in the strong topology of  $\Theta_+^{s,loc}$  as well.*

**Proof.** We first prove the analogous result for the projection  $\mathcal{A} := \mathcal{A}^r|_{t=0}$  to the usual phase space  $\mathcal{H}^1$ .

**Lemma 4.4.** *Let the above assumptions hold. Then the set  $\mathcal{A}$  is compact in  $\mathcal{H}^1$  and the following attraction property hold: for every  $M$ -bounded set  $B \subset \mathcal{K}_+$ ,*

$$\text{dist}_{\mathcal{H}^1}((T(h)B)|_{t=0}, \mathcal{A}) \rightarrow 0, \tag{4.5}$$

as  $h \rightarrow +\infty$  (here and below  $\text{dist}_V(A, B)$  denotes the Hausdorff semi-distance between sets  $A$  and  $B$  in a space  $V$ ).

**Proof.** Indeed, let  $h_n \rightarrow \infty$  and  $u_n \in T(h_n)B$  be arbitrary. Then, according to Theorem 2.9, without loss of generality, we may assume that

- 1)  $u_n \rightarrow u$  weakly in  $\Theta_+^{w,loc}$  and  $u \in \mathcal{A}^{tr}$ ;
- 2)  $u_n(0) \rightarrow u(0)$  weakly in  $\mathcal{H}^1$ ;
- 3) There are solutions  $\tilde{u}_n$  of Euler equation (2.1) defined on  $[-h_n, \infty)$  such that  $M_{\tilde{u}_n}(-h_n)$  are uniformly bounded and  $u_n = \tilde{u}_n$  and  $u_n(0) = \tilde{u}_n(0)$ ;
- 4)  $\tilde{u}_n \rightarrow \tilde{u}$  weakly in  $L_{loc}^\infty(\mathbb{R}, \mathcal{H}^1)$  and  $\tilde{u} \in \mathcal{K}$  be such that  $\tilde{u}(t) = u(t)$  for  $t \geq 0$ .

To prove the lemma, we only need to check that  $u_n(0) \rightarrow u(0)$  strongly in  $\mathcal{H}^1$ . To this end (since the weak convergence is already established), it is sufficient to establish that the norms  $\|\nabla_x u_n(0)\|_{L^2}^2 \rightarrow \|\nabla_x u(0)\|_{L^2}^2$  as  $n \rightarrow \infty$ . We will verify the last fact based on the energy equalities method.

Namely, we write out the energy inequality for the solution  $\tilde{u}_n$  of the dissipative Euler equation in the following form:

$$\|\nabla_x u_n(0)\|_{L^2}^2 \leq [M_{\tilde{u}_n}(-h_n)]^2 e^{-2Rh_n} + 2 \int_{-h_n}^0 e^{2Rs} (\nabla_x g, \nabla_x \tilde{u}_n(s)) ds. \tag{4.6}$$

Note that, using the technique of [12], one should be able to verify the  $\mathcal{H}^1$ -energy equality for any solution  $u \in \mathcal{K}_+$ . However, we really need in (4.6) the inequality only which clearly true and may be easily obtained by passing to the limit in the corresponding Navier–Stokes approximations.

Using now the fact that  $\tilde{u}_n$  are uniformly bounded in  $L^\infty(\mathbb{R}, \mathcal{H}^1)$  together with the weak convergence  $\tilde{u}_n \rightarrow \tilde{u}$  in  $L_{loc}^\infty(\mathbb{R}, \mathcal{H}^1)$ , one can easily obtain that

$$2 \int_{-h_n}^0 e^{2Rs} (\nabla_x g, \nabla_x u_n(s)) ds \rightarrow 2 \int_{-\infty}^0 e^{2Rs} (\nabla_x g, \nabla_x \tilde{u}(s)) ds \quad \text{as } n \rightarrow \infty. \tag{4.7}$$

Moreover, using that  $M_{\tilde{u}_n}(-h_n)$  are uniformly bounded, we see from (4.6) and (4.7) that

$$\limsup_{n \rightarrow \infty} \|\nabla_x u_n(0)\|_{L^2}^2 \leq 2 \int_{-\infty}^0 e^{2Rs} (\nabla_x g, \nabla_x \tilde{u}(s)) ds. \tag{4.8}$$

On the other hand, the energy equality holds for the function  $\tilde{u} \in \mathcal{K}$  (due to Theorem 4.1). So, multiplying the identity (4.1) by  $e^{2Rt}$  and integrating over  $t \in (-\infty, 0]$ , we arrive at

$$\|\nabla_x u(0)\|_{L^2}^2 = 2 \int_{-\infty}^0 e^{2Rs} (\nabla_x g, \nabla_x \tilde{u}(s)) ds. \tag{4.9}$$

Hence, inequality (4.8) implies:

$$\limsup_{n \rightarrow \infty} \|\nabla_x u_n(0)\|_{L^2}^2 \leq \|\nabla_x u(0)\|_{L^2}^2. \tag{4.10}$$

At the same time, since  $u_n(0) \rightarrow u(0)$  weakly in  $\mathcal{H}^1$ , as  $n \rightarrow \infty$ , we see that

$$\|\nabla_x u(0)\|_{L^2}^2 \leq \liminf_{n \rightarrow \infty} \|\nabla_x u_n(0)\|_{L^2}^2. \tag{4.11}$$

Equalities (4.10) and (4.11) yield

$$\|\nabla_x u_n(0)\|_{L^2}^2 \rightarrow \|\nabla_x u(0)\|_{L^2}^2 \quad \text{as } n \rightarrow \infty.$$

Thus, the strong convergence  $u_n(0) \rightarrow u(0)$  in  $\mathcal{H}^1$  is verified and the lemma is proved.  $\square$

Now, it is not difficult to finish the proof of the theorem. Indeed, to this end, it is sufficient to verify that if  $u_n \in T(h_n)B$  for some  $M$ -bounded set  $B \subset \mathcal{K}_+$ ,  $h_n \rightarrow \infty$  and  $u_n \rightarrow u \in \mathcal{A}^{tr}$  in  $\Theta_+^{w,loc}$ , then the strong convergence in  $L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^1)$  is actually holds.

By the definition of the local topology in  $\Theta_+^{s,loc}$ , it is sufficient to verify that

$$\|u_n - u\|_{L^\infty([0,L], \mathcal{H}^1)} \rightarrow 0 \tag{4.12}$$

as  $n \rightarrow \infty$  for every fixed  $L$ . In order to establish (4.12), we note that Lemma 4.4 together with the obvious facts that the set  $\{T(h)u, h \in [0, L]\}$  is  $M$ -bounded implies that

$$\sup_{t \in [0, L]} \text{dist}_{\mathcal{H}^1}(u_n(t), \mathcal{A}) \rightarrow 0 \tag{4.13}$$

as  $n \rightarrow \infty$ .

Let now  $P_N$  be the orthoprojector on the first  $N$  eigenvectors of the Stokes operator in  $\Omega$  and  $Q_N := 1 - P_N$ . Then, the compactness of the attractor  $\mathcal{A}$  in  $\mathcal{H}^1$  (proved in Lemma 4.4) together with the convergence (4.13) imply that, for every  $\delta > 0$  there exists  $N = N(\delta)$  such that

$$\limsup_{n \rightarrow \infty} \|Q_N u_n\|_{L^\infty([0, L], \mathcal{H}^1)} + \|Q_N u\|_{L^\infty([0, L], \mathcal{H}^1)} \leq \delta. \tag{4.14}$$

On the other hand, due to the control of the  $\partial_t u_n$ -norm through the Euler equation (2.1), we have the uniform convergence  $u_n \rightarrow u$  in  $C([0, L], \mathcal{H})$  and, therefore, since  $P_N$  is finite-dimensional, for every fixed  $N$ , we have the convergence,

$$\|P_N(u_n - u)\|_{L^\infty([0, L], \mathcal{H}^1)} \rightarrow 0 \tag{4.15}$$

as  $n \rightarrow \infty$ . The convergences (4.14) and (4.15) give the strong convergence  $u_n \rightarrow u$  in  $L^\infty([0, L], \mathcal{H}^1)$  which finishes the proof of the theorem.  $\square$

**Corollary 4.5.** *Under the assumptions of the previous theorem, the attractor  $\mathcal{A}^{tr}$  is compact in  $C_{loc}(\mathbb{R}_+, W^{1,p}(\Omega))$  for all finite  $p > 1$ .*

Indeed, the result is an immediate corollary of the compactness in  $\mathcal{H}^1$ , interpolation inequality,

$$\|u_1 - u_2\|_{W^{1,p}} \leq C_p \|u_1 - u_2\|_{\mathcal{H}^1}^{\theta_p} \|u_1 - u_2\|_{W^{1,2p}}^{1-\theta_p},$$

and the boundedness of the attractor in  $L^\infty(\mathbb{R}_+, W^{1,2p})$  established in Corollary 3.2.

We conclude the paper by the following corollary which establishes the strong convergence of the time derivatives of solutions to the “time derivative” of the trajectory attractor.

**Corollary 4.6.** *For an arbitrary  $M$ -bounded set  $B \subset \mathcal{K}_+$  the corresponding set,*

$$\partial_t B := \{\partial_t u, u \in B\},$$

*converges to  $\partial_t \mathcal{A}^{tr}$  in the strong topology of  $L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^{-1})$ . Moreover, the set  $\partial_t \mathcal{A}^{tr}$  is compact in  $C_{loc}(\mathbb{R}_+, L^p(\Omega))$  for all  $p < \infty$ .*

Indeed, let

$$T(u) := \Pi(g - Ru - (u, \nabla_x)u),$$

where  $\Pi$  is a Leray projector to the divergence free vector fields. Then, obviously,  $\partial_t u = T(u)$  for all  $u \in \mathcal{K}_+$  and rewriting the inertial term in the standard form,

$$(u, \nabla_x)u = \text{div}(u \otimes u) := \partial_{x_1}(u^1 u) + \partial_{x_2}(u^2 u),$$

we see that  $T$  is continuous from  $L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^1) \subset L_{loc}^\infty(\mathbb{R}_+, L^4(\Omega))$  to  $L_{loc}^\infty(\mathbb{R}_+, \mathcal{H}^{-1})$ . This fact, together with the established convergence in  $\Theta_+^{s,loc}$  gives the first statement of the corollary. The second statement follows analogously using the continuity of the map  $T$  from  $C_{loc}(\mathbb{R}_+, W^{1,p}(\Omega))$  to  $C_{loc}(\mathbb{R}_+, L^p(\Omega))$  for  $p > 2$  and the compactness of  $\mathcal{A}^{tr}$  in  $C_{loc}(\mathbb{R}_+, W^{1,p}(\Omega))$  verified the previous corollary.

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