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SHORT COMMUNICATIONS

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On the Solenoidal Representation of the Hyperbolic Attractor of a Diffeomorphism of the Sphere

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The study of one-dimensional hyperbolic attractors of diffeomorphisms of a surface stems from the example due to Smale [1] of a DA (derived from Anosov) diffeomorphism of the torus. Williams [2], [3] gave an abstract description of expanding hyperbolic attractors by using inverse spectra of mappings of smooth ramified manifolds. These objects were dubbed Williams solenoids (generalized solenoids).

An example of a one-dimensional hyperbolic attractor on the sphere was constructed by Plykin [4]. Plykin's approach was further developed in his paper [5] and in Zhirov's papers (see [6] and references therein).

The main obstructions to one-dimensional generalized solenoids arise when they are realized in two-dimensional dynamical systems. The papers [7] and [8] establish some necessary conditions as well as sufficient conditions for the existence of such a realization. The aim of the present paper is to carry out further study of solenoidal representations of hyperbolic attractor on the sphere.

Let $K = V_{i=1}^{n} S_i^1$ be a bouquet of circles, and let $\varphi\colon K \to K$ be a mapping such that

(1) The circles have identical lengths in the standard metric generating the metric of the bouquet K.

(2) $\varphi(\omega) = \omega$, where ω is the branching point of the bouquet.

(3) The mapping φ is uniformly expanding in this metric.

(4) For two arbitrary circles S_i^1 and S_j^1, there exists a power p of φ such that $\varphi^p(S_i^1) \supset S_j^1$.

The system $K \xleftarrow{\varphi} K \xleftarrow{\varphi} \cdots$ forms an inverse spectrum. Let $K_\infty = \varprojlim K$ be its projective limit, and let $\varphi_\infty\colon K_\infty \to K_\infty$ be the shift homomorphism

$$\varphi_\infty(x_1, x_2, \dots) = (\varphi(x_1), x_1, x_2, \dots).$$

The pair $(K_\infty, \varphi_\infty)$ is called a *generalized solenoid* (a *Williams solenoid*). By orienting the circles of the bouquet K, we obtain the standard system of generators $a_1, \dots, a_n$ of the fundamental group $\pi_1(K, \omega) = F_n = F_n(a)$; $\varphi_{*a}\colon \pi_1(K, \omega) \to \pi_1(K, \omega)$ is the endomorphism generated by the mapping φ. The notation $F_n(a)$ and φ_{*a} emphasizes the system of generators to which these objects are related.

Let $A = a_1, \dots, a_n$ be an n-element set, $n \geq 3$. Let $\leq_1$ be a partial order on A. If two elements a_i and a_j are related by $a_i <_1 a_j$ (i.e., $a_i \leq_1 a_j$ and $a_i \neq a_j$) and there are no elements a_k such that $a_i <_1 a_k <_1 a_j$, then a_j is said to *cover* a_i and a_i is said to be *covered* by a_j.

Assume that every element a_i is covered by at most one element a_j. Then A is a union of trees T_α, that is, a forest.

In connection with the above-considered partial order, let us introduce a partial order $\leq_2$ such that, for two arbitrary distinct elements a_i and a_j covered by one and the same element one has either $a_i <_2 a_j$, or $a_j <_2 a_i$. Thus, a linear order is introduced on every subset formed by elements covered by one and the same element.

Let $h\colon A \to \mathbb{R}^1$ be an injection preserving both orders and satisfying the following conditions:

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(1) If $a_i <_1 a_j$ and $h(a_i) < h(a_k) < h(a_j)$, then $a_i <_1 a_k <_1 a_j$; thus, a linear order is defined on the set of trees T_α ($T_\alpha < T_\beta$ if $a_\alpha <_1 a_\beta$, where $a_\alpha \in T_\alpha$ and $a_\beta \in T_\beta$).

(2) h takes both positive and negative values, and the values of h on all elements of any given tree T_α have the same sign.

Finally ,we assume that the elements of A are always measured in such a way that

$$h(a_1) > h(a_2) > \cdots > h(a_n).$$

The quadruple $(A; \leq_1; \leq_2; h)$ will be called an *ordering of* A.

We treat the elements of A as generators of the free group $F_n(a)$ and construct the new system of generators

$$b_i = a_i(a_k \cdot a_{k+1} \cdots a_{k+p})^{-1}, \qquad i = 1, \dots, n,$$

where $a_k, \dots, a_{k+p}$ are all elements covered by a_i and multiplied in this order. In particular, for the elements a_i minimal with respect to the partial order $\leq_1$, one has $b_i = a_i$.

Consider the automorphism

$$\psi \colon F_n(a) \to F_n(a), \qquad \psi(a_i) = b_i, \quad i = 1, \dots, n.$$

This permits writing out the action of an arbitrary automorphism $\varphi_a \in \operatorname{Aut}(F_n(a))$ in the new system of generators $\varphi_b = \psi \cdot \varphi_a \cdot \psi^{-1}$.

The *braid group* $B(n)$, $n \geq 1$, is the group with $n-1$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and the braid relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| \geq 2, \quad i, j = 1, 2, \dots, (n-1),$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad i = 1, 2, \dots, (n-2)$$

(see [9]).

If $F_n = F_n(b)$ is the free group with generators $b_1, b_2, \dots, b_n$, then to each element $\sigma_i \in B(n)$ one assigns the automorphism $\overline{\sigma}_i \in \operatorname{Aut}(F_n)$ such that

$$\overline{\sigma}_i(b_i) = b_{i+1}, \quad \overline{\sigma}_i(b_{i+1}) = b_{i+1}^{-1} \cdot b_i \cdot b_{i+1}, \quad \overline{\sigma}_i(b_k) = b_k, \qquad i = 1, \dots, (n-1), \quad k \neq i, i+1.$$

This correspondence generates a faithful representation $B(n) \to \operatorname{Aut}(F_n)$ whose range consists of automorphisms β such that $\beta(b_i) = T_i^{-1} b_{\pi(i)} T_i$ and

$$\beta(b_1 \cdot b_2 \cdot \cdots \cdot b_n) = b_1 \cdot b_2 \cdots b_n.$$

Here $T_i \in F_n$, π is a permutation of the numbers $1, 2, \dots, n$, and $i = 1, \dots, n$. Such automorphisms β form the subgroup $\overline{B}_n$ of braid automorphisms in $\operatorname{Aut}(F_n)$ (see [9]).

Now let $A = \{a_1, \dots, a_n\}$ be the above-considered system of generators of the fundamental group $\pi_1(K, \omega)$, and let $\varphi_{*a} \colon \pi_1(K, \omega) \to \pi_1(K, \omega)$ be the endomorphism generated by φ of the fundamental group.

{t1:x155}

Theorem 1. *The following conditions are sufficient for the generalized solenoid* $(K_\infty, \varphi_\infty)$ *to be realizable as a hyperbolic attractor of the sphere* S^2:

(1). *There exists an ordering* $(A; \leq_1; \leq_2; h)$ *such that* $\varphi_{*b} = \psi \varphi_{*a} \psi^{-1} \in \overline{B}(n)$; *hence* φ_{*a} *is an automorphism.*

(2). *The irreducible words* $\varphi_{*a}(a_i)$ *in the alphabet* A, $i = i, \dots, n$, *have the following structure*:

$$\varphi_{*a}(a_i) = \begin{cases} a_1 c_i a_1^{-1} & \text{if } h(a_i) > 0, \\ a_n^{-1} d_i a_n & \text{if } h(a_i) < 0, \end{cases}$$

$c_i, d_i \in \pi_1(K, \omega)$, *and in each of the words* $\varphi_{*a}(a_i)$, $i = 1, \dots, n$, *the function* h *takes values of opposite signs on neighboring elements.* (*Here it is assumed that* $h(a_i) = h(a_i^{-1})$, $i = 1, \dots, n$.)

Let Λ be a connected one-dimensional hyperbolic attractor of a diffeomorphism

$$f\colon S^2 \to S^2, \qquad n = \operatorname{rank} H_1(\Lambda, \mathbb{Z}) = \frac{1}{2}(m+2) = N - 1,$$

where $H_1(\Lambda, \mathbb{Z})$ is the first Aleksandrov–Čech homology group of Λ, m is the number of its boundary points, and N is the number of connected components of the complement $S^2 \setminus \Lambda$.

{t2:x155}

Theorem 2. *As a topological space, the attractor Λ is homeomorphic to a generalized solenoid consisting of a bouquet of circles $K = \bigvee_{i=1}^{n} S_i^1$ and an expanding mapping $\varphi\colon K \to K$ satisfying the assumptions of Theorem* 1. *Further, some power of the restriction $f|_\Lambda$ is conjugate to the shift $\varphi_\infty\colon K_\infty \to K_\infty$ of this solenoid.*

The proofs of the theorems are based an an analysis of properties of the linear self-intersection form and the bilinear intersection form on the group ring $\mathbb{Z}[\pi_1(\Omega, x_0)]$ of the fundamental group of the multiply connected planar domain Ω (see cite [8]) and on the classical mapping class theory [9, Theorem 1.33].

Theorem 2 refines Theorem 3.1 in [5] owing to the presence of additional conditions and establishes that the solenoidal representation described in Theorem 1 is typical.

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