A Direct Lyapunov Method for Delay Differential Equations

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This presentation is devoted to well-known tools of stability theory: Lyapunov functions and functionals. The development of delay equation theory through time based on some results of Russian mathematicians is discussed. In particular, there are two different approaches: one could measure the distance between points in a finite dimension space or between short curve segments (points in a functional space). Some definitions, theorems and examples are presented

1 Stability

Let's assume that the basic concepts and results of the second Lyapunov method for systems of ordinary differential equations are well known. Delay differential equations are not so famous. Any way there are a numerous good books on this topic, for example Bellman R. and K.L. Cooke. - Differential-Difference equations. New York, Academic Press, 1963; Hale, J.K. Theory of Functional Differential Equations. Springer-Verlag, New York, 1977; Kuang, Y. Delay Differential Equations with Applications in Population Dynamics. Academic Press, 1993, Kharitonov V.L. Time-delay systems. Lyapunov functionals and matrices. Birkhauser, Basel, 2013. The use of Lyapunovs functions for studying the stability of solutions of systems with a lag of the general form

$$\dot{x}_i = F_i(t, X(\cdot)); i = \overline{1, n} \tag{1}$$

did not start with very successful results: L.E. Elzgoltz [Elzgoltz L.E. Stability of solutions of Difference-Differential Equations Moscow, Uspekhi Matematicheskih Nauk, 1954, v.9, issue 4, pp 95-112] stated in 1954 the Lyapunov theorems for the system (1) without any changes. The method of Lyapunov functions was considered to be unpromising because no inverses of the theorems had been proven at this time. One of Elzgoltzs theorems is:

Theorem 1. *If there exists a definitely positive function, whose trajectory derivative with respect to the system (1) is not positive, then the zero solution of the system (1) is stable according to Lyapunov.*

Example 1. For a given delay equation

$$\frac{dx(t)}{dt} = -ax(t)x^2(t-h), \ a, h > 0$$

we could use the Lyapunov function $V(x) = x^2$, and the corresponding trajectory derivative is always negative: $\frac{dV}{dt} = -ax^2(t)x^2(t-h) < 0$. So, the zero solution is stable in accordance with the theorem.

To proceed with the discussion of this method, we will assume that the following conditions must be met: (i) the zero solution exists, (ii) the solution of the Cauchy problem of the system (1) must exist and be unique, (iii) this solution can be extended to positive infinity from some neighbourhood of zero. Theorem 1 is not practically important, as the question of the existence of such Lyapunov functions $V(t, X)$ is still open. Furthermore, the trajectory derivative of function $V(t, X)$ with respect to the system (1) will be a functional which is determined on the segments of the trajectory, thus complicating the check of the property that the derivative has a fixed sign.

In 1956 N. N. Krasovskii offered a method of studying the stability of solutions of the system (1) by using functionals. Instead of controlling the deviation of the vector $X(t)$ from zero, he suggested separating the segment of the trajectory $X_t(\theta), \theta \in [0, h]$ and considering some functionals $V(t, X(\cdot))$ as a generalised distance, i.e., to go from the space E^n to the space of piecewise-continuous vector-functions. All of the definitions of the second method of Lyapunov in this space can be transferred without any special changes. N. N. Krasovskii proved a number of theorems about stability and asymptotic stability of the zero solution of the system (1), and these theorems can be inversed [Krasovskii N. N. Some problems of stability motion theory Moscow, PhysMatGiz, 1959. See Krasovskii N. N. Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay.- Stanford (Calif.): Stanford Univ. Press, 1963.- 188 p]. We will state one of these theorems without proof.

Theorem 2. *For the trivial solution of the system (1) to be uniformly and asymptotically stable with respect to* t_0 *and* $\varphi_{t_0}(\cdot)$ *, it is necessary and sufficient for the functional* $V(\psi(\cdot), t)$*, to exist and have the following properties:*
1. $V(\psi(\cdot), t)$ *is determined in the domain*

$$||\psi(\cdot)||^{(h)} < H, \, t > t_0 \tag{2}$$

2. $V(\psi(\cdot), t)$*, is continuous with respect to* t *and satisfies a Lipschitz condition with respect to* $\psi(\cdot)$ *in the domain (2);*

3. $V(\psi(\cdot),t)$, is definitely positive and uniformly continuous at zero in the domain (2); 4. the upper trajectory derivative from the right with respect to system (1)

$$\lim_{\Delta t \to +0} sup \frac{\Delta V}{\Delta t}\bigg|_{(1)}$$

is a definitely negative functional in the domain (2).

B. S. Razumikhin in 1956 [Razumikhin B. S. On stability of delay systems - Applied Math. and Mech., Moscow, 1956, v. 20, issue 4, pp 500-512.] continued the study of the use of Lyapunov functions to determine the stability of the system (1) because the problem of the construction of such Krasovskii functionals was still, apparently, far from a constructive solution of many technical problems. However, the Lyapunov functions had been constructed in specific areas of research. We will talk about Razumikhins results in detail. Lets impose some restrictions on the functionals $F_i(t, X(\cdot))$ that will guarantee the existence, uniqueness and continuability of solutions of the Cauchy problem for the system (1) in the domain (2), namely $F_i(t, \varphi(\cdot)) \equiv 0$ at $\varphi(\cdot) \equiv 0$, and

$$||F_i(t, \varphi(\cdot)) - F_i(t, \psi(\cdot))|| \leq L||\varphi(\cdot) - \psi(\cdot)||^{(h)},\ L > 0$$

First we will consider the question of stability of the zero solution for the system (1).

Theorem 3. *The zero solution of the system (1) will be stable in the Lyapunov sense, if there exists such a function $V(t, X)$, that*
1. $V(t, X)$ is determined and continuously differentiable in the domain

$$||X|| < H,\ t \geq t_0; \tag{3}$$

2. $V(t, X)$ is positively determined in (3);

3. the trajectory derivative of $V(t, X)$ with respect to the system (1)

$$\frac{dV}{dt} = U(t, X_t(\cdot)) = \frac{\partial V(t, X(t))}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, X(t))}{\partial x_i} F_i(t, X(\cdot))$$

is a functional, which is determined, generally speaking, on any continuous curves $Y_t(\cdot)$ of (2), and which has the property

$$U(t, Y_t(\cdot)) \leq 0 \tag{4}$$

along any continuous curves that satisfy the condition

$$Y(t) = X(t); V(s, Y(s)) \leq V(t, X(t))\, in\, s \in [t-h, t] \tag{5}$$

Theorem 4. *If the conditions of Theorem 3 are satisfied and $V(t, X)$ is uniformly continuous, then the zero solution of the system (1) will be uniformly stable with respect t_0 according to Lyapunov.*

Let's state now a theorem about asymptotic stability.

Theorem 5. *The zero solution of the system (1) will be asymptotically stable, if there exists a function $V(t, X)$ in (3) and having the properties: 1. $V(t, X)$ has bounded and continuous partial derivatives in (3); 2. $V(t, X)$ is definitely positive and uniformly continuous at zero; 3. the trajectory derivative of $V(t, X)$ with respect to the system (1)*

$$\frac{dV}{dt} = U(t, X_t(\cdot))$$

is a functional which is determined, generally speaking, on any continuous curves $y_t(\cdot)$ of (2) and which has a property

$$U(t, y_t(\cdot) \leq -W(||y(t)||)$$

along any continuous curves satisfying the condition (5). Here $W(r) > 0$ at $r \neq 0, W(0) = 0$.

B. S. Razumihkin showed that the set of continuous curves $y(t)$ satisfying the inequality (5), on which the derivative $V(t, x)$ is negative, can be reduced. Namely, we will designate $S_i(t, x)$ as the largest value of $|F_i(t, y(t + \cdot))|$, for all continuous curves satisfying (5). Then it is possible to assert that Theorems 3, 4 and 5 will remain valid, if instead of set of the curves $y_t(\cdot)$ we will consider a set of continuously differentiatable curves $z_t(\cdot)$ satisfying the inequalities:

$$V(s, z(s)) \leq V(t, z(t)),\ s \in [t - h, t],\ z(t) = X(t)$$

and

$$\left| \frac{dz_i}{ds} \right| \leq S_i(t, X). \tag{6}$$

This process of reduction of the set of the curves can be continued in a similar way. Let $K_i(t, X) = sup|F_i(t, z_t(\cdot))|$ at every $z_t(\cdot)$ satisfying (5) and (6). It is obvious that $K_i(t, X) \leq S_i(t, X)$. The negativeness of $U(t, X_t(\cdot))$ will be checked now only on such continuously differentiatable curves $\xi_t(\cdot)$ such that

$$\xi(t) = X(t),\ V(s, \xi(s)) \leq V(t, X(t)),\ s \in [t - h.t],\ \left| \frac{d\xi_i(t)}{dt} \right| \leq K_i(t, X).$$

Simple examples show that this process makes the estimation of the area of stability in the space of parameters of the system more precise, but whether it gives the exact solution of the problem of stability hasnt been determined as yet.

Example 2. Let it be required to find an area in the space of parameters a and b, for which the zero solution of the equation

$$\dot{x} = ax + bx(t-h) \tag{7}$$

will be asymptotically stable. Applying Theorem 5 with the function $V(t,X) = \frac{x^2}{2}$, we have $\dot{V} = ax^2 + bxx_h$, where $x_h = x(t-h)$. From the inequality (5) we will have for $t > t_0$

$$|x_h| \leq |x|, \tag{8}$$

and then $\dot{V} \leq (a + |b|)x^2$, i.e., for $a < -|b|$ the zero solution is asymptotically stable (Fig. 1).

Figure 2 shows the exact boundary of the area of stability in the plane (a, b). Figure 3 shows the area specified by the method described above. From Equation (7) we can obtain an estimate with the help of (8):

$$|x(t) - x(t-h)| = |\dot{x}(\xi)|h \leq h \sup_{\xi \in [t-h,t]} |\dot{x}(\xi)|$$

or

$$|x - x_h| \leq (|a| + |b|)|x|,$$

that is

$$-2xx_h \leq h^2(|a| + |b|)^2x^2 - x^2 - x_h^2.$$

We use this inequality to estimate $\dot{V}$ at negative values of b:

$$\dot{V} \leq ax^2 + \left(-\frac{b}{2}\right)[x^2\left(h^2(|a| - b)^2 - 1\right) - x_h^2].$$

It is obvious that the right part of the inequality will be negative for

$$2a - 2h^2(a^2 - 2|a|b + b^2) + b < 0.$$

Getting rid of the parameter h by introducing new variables $\xi = bh$ and $\eta = ah$, we obtain the area represented in Figure 3.

Together with Theorem 5, the following theorem of N. N. Krasovskii appeared [Krasovskii N. N. On asymptotic stability of systems with after-effect // Applied Math. and Mech., Moscow, 1956, v. 20, issue 4, pp 513-518]. In this theorem the role of the condition (5) was given to the inequality

$$V(s, y(s)) \leq f(V(t, y(t))), \tag{9}$$

where $f(r) > r$ and f is strongly monotonic. The theorems with the conditions (5) and (9) were applied and generalised by various Russian and foreign mathematicians but preference was given to the condition (9).

2 Instability

Let's consider the criteria of instability of solutions. Let's say that the instability of the zero solution of the system (1) takes place, if negation of the definition of stability according to Lyapunov is satisfied. If the negation of the definition of the uniform stability is satisfied, we will say, that the instability takes place in a broad sense.

In 1960 S. N. Shimanov applied the method of functionals of Lyapunov-Krasovski to studying the zero solution of the system (1) [Shimanov S. N. On motion instability with time delay - Applied Math. and Mech., Moscow, 1960, v. 24, issue 1, pp 55-63]. He proved analogues of the theorems of Lyapunov and Chetaev in the following formulations.

Theorem 7. *If the functional* $V(t, X_t(\cdot))$ *exists and is determined and bounded in the domain*

$$||X(t+\cdot)||^{(h)} < H,\ V(t, X(t+\cdot)) > 0,\ t \geq t_0,$$

being uniformly continuous at zero and such that its trajectory derivative with respect to the system (1)

$$\lim_{\Delta t \to 0} sup \left(\frac{\Delta V}{\Delta t} \right)$$

is a positively determined functional, then the zero solution is unstable.

Theorem 8. *If there exists a functional* $V(t, X(t, X(t+\cdot))$ *bounded in (2) satisfying the criteria:*

1. for any $t \geq t_0$ *there exists a curve* $X(\cdot)$ *with an arbitrarily small norm and* $V(t, X(\cdot)) > 0$*;*

2. the trajectory derivative of $V(t, X(t+\cdot))$ *with respect to the system (1) can be represented as* $\lambda V(t, X(t+\cdot)) + W(t, X(t+\cdot))$*, where* $\lambda = const > 0$, $W \geq 0$*, then the zero solution is unstable.*

The analogue of the first theorem of Lyapunov about instability is similarly formulated. For studying the instability in a broad sense we will present a theorem **Prasolov** A. V. Reversibility of Shimanovs theorem on instability of delay systems.- Vestnik LGU, 1981, 1, p. 116-117], which has an inverse.

Theorem 9. *If the zero solution of the system (1) is unstable in a broad sense then there is a functional* $V(t, X(t+\cdot))$ *with the properties:*

1. it is bounded in the domain (2);

2. for any $\varepsilon > 0$ *there exists a time* t^* *and a curve* $X^*(\cdot)$*, such that* $t^* > 0$, $||X^*(\cdot)||^{(h)} < \varepsilon\, V(t^*, X^*(\cdot)) > 0$*;*

3. the trajectory derivative of $V(t, X(t+\cdot))$ *with respect to the system (1) can be represented as:*

$$\dot{V} = \lambda V(t, X_t(\cdot)) + W(t, X_t(\cdot)), \tag{10}$$

where $\lambda = const > 0$, $W \geq 0$.

The following theorems give the reasons for using Lyapunov functions for studying the instability [Prasolov A. V. On Lyapunov function application for checking of delay system instability.- Vestnik LGU, 1981, 19, p. 116-118].

Theorem 10. *The zero solution of the system (1) will be unstable in a broad sense, if there exists a uniformly continuous function $V(t, X)$ in the domain (3) which has the properties:*
1. in an arbitrary small neighbourhood of zero there will be a such point X^* and a time $t^* > h$, such that $V(t^*, X^*) > 0$;
2. the right lower trajectory derived number $V(t, X)$ with respect to the system (1) is a functional $U(t, X(t+\cdot))$, which is assigned, generally speaking, on any piecewise-continuous curves $y(t+\cdot)$ of the domain (2) and, furthermore, it is positively determined on the curves satisfying the condition (5).

By the positive definiteness of the functional we mean the existence of a positively defined function ω, such that

$$U(t, X(t+\cdot)) \geq \omega\left(||X(t)||\right). \tag{11}$$

The following theorem for the system (1) is proved in the same way as Theorem 10 with those natural modifications that distinguish the first theorem of Lyapunov about instability from the second one.

Theorem 11. *The zero solution of the system (1) will be unstable in a broad sense, if there exists a function $V(t, X)$ which is determined and uniformly continuous in the domain (3) and has the properties:*
1. for any $\varepsilon > 0$ there exists a point X^* and a time $t^* > t_0$, such that and $||X^*|| < \varepsilon < H$ and $V(t^*, X^*) > 0$.
2. the right lower trajectory derived number V with respect to the system (1) can be represented as

$$D_+V = \lambda V + W(t, X(t+\cdot)),$$

where $\lambda > 0$, and the functional W, determined on any piecewise-continuous curves $y(t+\cdot)$ from the domain (2), satisfy the inequality $W(t, y(t+\cdot)) \geq 0$ for all curves $y(t+\cdot)$ subject to the condition (5).

Example 3. We can obtain the following solution for the equation $\dot{x} = ax + bx_h$ by applying Theorem 10. Let $V = \frac{x^2}{2}$. Then $\dot{V} = ax^2 + bxx_h$. The condition (5) will give inequality $|x(t-h)| \leq |x(t)|$. Let's find a lower estimate $\dot{V}$ by using the last inequality: $\dot{V} \geq (a - |b|)x^2$; it means, that $\dot{V}$ will be positively determined, if $a > |b|$.

The determination of the property of instability of the zero solution of the system (1) by using functions is more convenient than by using functionals. The following theorem supports the existence of such functions [Prasolov A. V. Instability properties of delay systems.- Vestnik LGU, 1984, 1, p. 37-42].

Theorem 12. *For the zero solution of the system (1) to be unstable in a broad sense, it is necessary and sufficient that there should be a function $V(t, X)$ with*

the following properties:
1. $V(t,X)$ *is determined in the semi-cylinder (3);*
2. it is bounded in (3) and at any fixed $t>0$

$$\lim_{||X||\to 0}|V(t,X)|=0;$$

3. for any $\varepsilon>0$ *there exists a pair* $\{t^*,X^*\}$*, such that*

$$t^*\geq 0,\ ||X^*||<\varepsilon,\ V(t^*,X^*)>0;$$

4. the trajectory derivative with respect to the system (1) has the form:

$$\frac{d}{dt}V(t,X)=\lambda(t,X(t+\cdot))V(t,X(t))+W(t,X(t+\cdot)),$$

where the functionals λ *and* W *are determined along different piecewise-continuous curves* $y(t+\cdot)$ *of the domain (2) and* λ *is definitely limited from below by a positive number* $\alpha, W\geq 0$*, when the curves* $y(t+\cdot)$ *satisfy the inequality:*

$$e^{\alpha(t-s)}V(s,y(s))\leq V(t,y(t))\,for\,s\in[t-h,t]. \qquad (12)$$

3 Perturbed Systems

In the qualitative theory of differential and differential-difference equations of great importance are theorems of stability by using a first approximation. Below we will give, practically without any changes, the corresponding material of N. N. Krasovskii [Krasovskii N. N. Some problems of stability motion theory Moscow, PhysMatGiz, 1959]. In doing so we will repeat the introductory assumptions to keep the original designations of N. N. Krasovskii.

Let's consider the system of equations with a lag.

$$\frac{dx_i}{dt}=F_i(x_1(t+\vartheta),...,x_n(t+\vartheta),t), \qquad (13)$$

where the functionals F_i satisfy the following conditions:
1. the functionals F_i are determined and piecewise-continuous in the domain $||x||^{(h)}<H, t\geq 0$ (where H is a fixed constant) in the following sense: there exists a sequence of numbers $t_k, k=1,2,3$ such that in each domain $||x(\vartheta)||^{(h)}<H,\ t_k\leq t<t_{k+1}$ the functionals F_i are continuous with respect to t can be extended (with preservation of continuity) to the whole domain $||x||^{(h)}<H, t_k\leq t<t_{k+1}$ in such way, that for every $t^*\in[t_k,t_{k+1}]$ and for each

continuous function $x_i^*(\vartheta)$, $(i = \overline{1,n})$, for any given number $\varepsilon > 0$ it would be possible to indicate a $\delta > 0$, such that

$$|F_i(x_1^*(\vartheta), ..., x_n^*(\vartheta), t^*) - F_i(x_1^*(\vartheta), ..., x_n^*(\vartheta), t)| < \varepsilon, \ i = \overline{1,n},$$

if only $|t^* - t| < \delta$ and $t \in [t_k, t_{k+1}]$; as the functionals F_i can have jumps at some moments of time t_k, and where for a number of other reasons, we will consider everywhere in the equations only the right derivative of x with respect time t as $\frac{dx_i}{dt}$.

2. The functionals F_i satisfy Lipschitz conditions in x, i.e.,

$$|F_i(x_1^{''}(\cdot), ..., x_n^{''}(\cdot), t) - F_i(x_1^{'}(\cdot), ..., x_n^{'}(\cdot), t)| <$$

$$< L||x^{''}(\cdot) - x^{'}(\cdot)||^{(h)}, \ (L = const, \ i = \overline{1,n}).$$

We introduce the norm as follows:

$$||x||^{(h)} = sup(|x_i(\vartheta)| \, for \ -h \leq \vartheta \leq 0, \ i = \overline{1,n})$$

As usual we assume, that $F_i(x(\vartheta), t) = 0$ at $x_j(\vartheta) \equiv 0$, $i = \overline{1,n}$, $j = \overline{1,n}$, $-h \leq \vartheta \leq 0$.

Lemma 1. *If the solutions $x(x_0(\vartheta_0), t_0, t)$ of the system (13) satisfy the condition*

$$||x(x_0(\vartheta_0), t_0, t)||^{(h)} \leq B||x_0(\vartheta_0)||^{(h)} e^{-\alpha(t-t_0)} \, at \, t \geq t_0, \tag{14}$$

$$||x_0(\vartheta_0)||^{(h)} < H_0 = \frac{H}{B}, \tag{15}$$

then, in the domain (15) a functional $V(x(\vartheta), t)$ satisfying the following conditions can be constructed:

$$c_1||(x(\vartheta))||^{(h)} \leq V(x(\vartheta), t) \leq c_2||(x(\vartheta))||^{(h)}, \tag{16}$$

$$\lim_{\Delta t \to +0} sup \left(\frac{\Delta V}{\Delta t} \right)_{(48)} \leq -c_3||(x(\vartheta))||^{(h)}, \tag{17}$$

$$|V(x^{''}(\vartheta), t) - V(x'(\vartheta), t)| \leq c_4||x^{''}(\vartheta) - x^{'}(\vartheta)||^{(h)} \tag{18}$$

($c_1, ..., c_4$ - are positive constants).

The consequence of the lemma 1 is the following statement

Theorem 13. *Let $F_i(x_1(\vartheta), ..., x_n(\vartheta))$ be linear functionals. If the roots of the "characteristic" equation*

$$\begin{vmatrix} F_{11} - \lambda & ... & F_{1n} \\ ... & ... & ... \\ F_{n1} & ... & F_{nn} - \lambda \end{vmatrix} = 0 \tag{19}$$

$\big(F_{ij} = F_i(0, ..., e^{\lambda\vartheta}, ...0)\big)$ *satisfy the inequality.*

$$Re\lambda < -\gamma\,(\gamma > 0 - const), \tag{20}$$

then there exists a functional $V(x(\vartheta), t)$ satisfying the conditions (16), (17), (18).

Theorem 13 allows the proof of theorems of stability by the first approximation for equations with a lag. Here we will give one such theorems.

Let's consider the system of equations:

$$\frac{dx_i}{dt} = F_i(x_1(\vartheta), ..., x_n(\vartheta)) + R_i(x_1(\vartheta), ...x_n(\vartheta), t), \tag{21}$$

where the F_i are linear functionals and the R_i are some continuous functionals. Let's assume, that the functionals R_i satisfy the inequality

$$|R_i(x_1(\vartheta)), ..., x_n(\vartheta), t)| \leq \beta||x(\vartheta)||^{(h)}. \tag{22}$$

Theorem 14. *If the roots λ of the equation (19) satisfy the inequality (20), it is possible to specify a constant $\beta > 0$, such that the solution $x = 0$ of the system (13) will be asymptotically stable for any choice of the continuous functionals R_i satisfying the conditions (22).*

In the specific case of equations with lags

$$\begin{gathered}\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j(t) + b_{ij}x_j(t - h_{ij}) + \\ +R_i(x_1, ..., x_n, x_1(t - h^*_{ij}(t)), ...x_n(t - h^*_{ij}(t)), t) \\ (h_{ij} - const, \quad 0 \leq h_{ij} \leq h, \quad 0 \leq h^*_{ij}(t) \leq h)\end{gathered} \tag{23}$$

Theorem 14 can be stated as follows.
If the roots of the "characteristic" equation

$$\begin{vmatrix} a_{11}e^{-\lambda h_{11}} - \lambda & ... & a_{1n}e^{-\lambda h_{1n}} \\ ... & ... & ... \\ a_{n1}e^{-\lambda h_{n1}} & ... & a_{nn}e^{-\lambda h_{nn}} - \lambda \end{vmatrix} = 0$$

satisfy the inequality

$$Re\lambda < -\gamma \quad (\gamma > 0 - const),$$

it is possible to specify a constant $\beta > 0$, such that the solution $x = 0$ of the system (23) will be asymptotically stable for any continuous functions $R_i(x_1, ..., x_n, y_1, ..., y_n, t)$ and delays $h^*_{ij}(t)$ satisfying the inequality $R_i(x_1, ..., x_n, y_1, ..., y_n, t) \leq \beta(||x|| + ||y||)$.

Example 4. Let us consider the problem of single-axle stabilization of a solid body rotation with delayed control feedback moment. As it is known corresponding equations have a form

$$\Theta\dot{\omega} + \omega \times \Theta\omega = M, \tag{24}$$

where Θ – an inertia tensor of the body, $\omega = \{\omega_1, \omega_2, \omega_3\}$ – angle velocity vector, M – external forces moment. The moment of forces

$$M = -a\omega + br \times s \tag{25}$$

was introduced by V.I. Zubov (1974). Here $a, b > 0, r = \{r_1, r_2, r_3\}$ –unit vector being constant regarding a coordinate system fixed in the body, $s = \{s_1, s_2, s_3\}$ – an unit vector being constant in inertia space. To make the mathematical model (24), (25) complete one it needs to add a kinematical vector equation

$$\dot{s} = s \times \omega, \tag{26}$$

The moment of forces (25) brings to the body motions satisfying the limit relations: $\omega(t) \to 0, s(t) \to r$, when $t \to \infty$, if the initial data belong to some neighborhood of the relative equilibrium point

$$\omega = 0, \quad s = r. \tag{27}$$

Note the joint system (24), (25), (26) has the first integral $||s|| = const$, and the problem sense gives us $||s|| \equiv 1$. So further we shell talk about conditional stability only.

Suppose the control moment of forces M is formed with time lag $h > 0$. This is more adequate description from engineering point of view. Thus, we have the system

$$\Theta\dot{\omega} + \omega \times \Theta\omega = -a\omega_h + b\xi_h, \quad \dot{s} = s \times \omega, \tag{28}$$

where $\omega_h = \omega(t-h)$, $\xi_h = r \times s(t-h)$.Every integral curve of system (28) is unique with continues initial functions $\{\omega_0(\tau, s_0(\tau)\}$ at $\tau \in [-h, 0]$. Let us research the relative equilibrium state $\omega \equiv 0, s(t) \equiv r$ with respect to stability by Lypunov function and Razumikhin condition. Choose the Lypunov function in form

$$V(\omega, s, s_h) = \frac{1}{2}\omega^T\Theta\omega + \alpha(s-r)^2 - \beta\omega^T\Theta\xi + \gamma(s_h - r)^2,$$

where upper index "T" means transposition and α, β, γ are constants. It is possible to show that if the following inequality is hold: $2\alpha\Theta_1 - \beta^2\Theta_2^2 > 0$, where Θ_1 and Θ_2 arethe least and the most Eigen values of the matrix Θ, then is true inequality

$$\mu_1(\omega^2 + \xi^2 + \xi_h^2) \le V(\omega, s, s_h) \le \mu_2(\omega^2 + \xi^2 + \xi_h^2),$$

where $0 < \mu_1 \leq \mu_2$. It means the function $V(\omega, s, s_h)$ is a definitely positive in some neighborhood of the relative equilibrium point $\omega \equiv 0, s(t) \equiv r$. Let us consider the derivative of the function V by virtue of the system (28):

$$\dot{V}\Big|_{(28)} = \omega^T(-a\omega_h + b\xi_h) - 2\alpha\omega^T\xi - 2\gamma\omega^T\xi_h$$

$$-\beta\{\omega^T\Theta[r \times (s \times \omega)] - \xi^T(\omega \times \Theta\omega + a\omega_h - b\xi_h)\}.$$

Estimating the terms of the third order with quadratic components we come to

$$\dot{V}\Big|_{(28)} \leq \beta\theta_2\omega^2 - a\omega^T\omega_h + b\omega T\xi_h - 2\alpha\omega^T\xi - 2\gamma\omega_h^T\xi_h + \beta a\xi^T\omega_h - \beta b\xi^T\xi_h.$$

Using the Razumikhin conditions (inequalities) we have got the estimation

$$\dot{V}\Big|_{(28)} \leq -\frac{1}{2}\{\omega^2[a(1-h^2L_1^2) - 2\beta\theta_1 - \beta bh^2L_2^2 - 2bL_2^2h - 2\beta aL_2^2h] +$$

$$+\xi^2[-ah^2L_1^2 + \beta b(1-h^2L_2^2) - 2L_2^2(b+\beta a)h]a\omega_h^2$$

$$+\xi^2[-ahL_1^2 + \beta b(1-h^2L_2^2) - 2L_2^2(b+\beta a)h]\}.$$

This estimation is definitely negative if the following relations are true:

$$h^2p_1 + 2p_2h + 2\beta\theta_2 - a < 0, \quad h^2p_1 + 2p_2h - b\beta < 0,$$

where $p_1 = aL_1^2 + \beta bL_2^2$, $\quad p_2 = (b + \alpha\beta)L_2^2$, $\quad L1 = \frac{2}{\theta_1}\sqrt{\frac{2\mu_2}{\mu_1(4-H)}}(H\theta + a^2 + b^2)$, $L_2 = 2\sqrt{\frac{2\mu_2}{\mu_1(4-H)}}, \theta = \sqrt{\theta_2^2 - \theta_1^2}$. The constant H defines the neighborhood of the relative equilibrium point $\equiv 0, s(t) \equiv r$:

$$\omega_0^2(\tau) + [s_0(\tau) - r]^2 \leq H.$$

This example shows how effective the direct Lypunov method is: we considered a non-linear system of six order (in more details one could see A.V. Prasolov (2010)).

4 Conclusion

We talked briefly about one small part of the delay differential equations theory, namely, the Lyapunov second method (or direct Lyapunovs method) of stability and instability study. This description certainly is not full either from theoretical or from practical points of view. A lot of names and interesting ideas are not mentioned here. More than this, the theory consists of numerical chapters, existence and uniqueness theorems, the problem of periodic solutions and many other staying outside of our attention.

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