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On the Infinity of the Set of Boundary Classes for the Edge 3-Colorability Problem

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Abstract—The set of boundary classes for the edge 3-colorability problem is proved to be infinite.

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INTRODUCTION

A series of articles [1–4] studies the boundary between the "simple" and "complex" classes of graphs as regards various problems concerning graphs in the family of *hereditary classes of graphs*, which are the classes closed under isomorphisms and vertex deletion. Every hereditary class $\mathbf{X}$ of graphs can be defined by a set of its forbidden induced subgraphs S, and the usual notation is $\mathbf{X} = \mathrm{Free}(S)$. If S is finite then the class $\mathbf{X}$ is called *finitely defined*.

In all articles of the series, the study of the boundary rests on the concepts of simple, complex, and boundary classes of graphs for the problem under consideration. A hereditary class of graphs is called Π-*simple* whenever the problem Π in this class is polynomially solvable, and otherwise it is called Π-*hard*. A hereditary class $\mathbf{X}$ of graphs is called Π-*limit* whenever there exists an infinite sequence $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$ of Π-hard classes of graphs with

$$\mathbf{X} = \bigcap_{i=1}^{\infty} \mathbf{X}_i.$$

An inclusion minimal Π-limit class is called a Π-*boundary* class. The following is proved in [4]:

Theorem 1. *If* P≠NP *then a finitely defined class* $\mathbf{X}$ *of graphs is* Π*-hard if and only if* $\mathbf{X}$ *includes some* Π*-boundary class.*

The claims of this article and some cited results of other articles hold on assuming that P≠NP. In this article, we always assume the validity of this conjecture, but henceforth omit the inequality P≠NP from the statements.

Theorem 1 reveals the meaning of the concept of a boundary class of graphs and shows that, when the set of Π-boundary classes is known, we can completely classify the classes of graphs in the family of finitely defined classes into Π-simple and Π-hard classes. Therefore, it is definitely of interest to find the boundary classes of various problems.

The boundary classes of the independent set problem were considered in [2] which showed that a certain class of graphs is a boundary class for the independent set problem. The concept of a boundary class was applied in [3] to the dominating set problem and three boundary classes were found. Two well-known classes were shown in [1, 4] to be the boundary classes for a series of problems on graphs.

At the same time there are some classical problems for which no boundary classes have been described. These include the Hamiltonian cycle problem, the vertex 3-colorability and edge 3-colorability problems, and the maximum clique problem. However, for some of these problems, it is possible to

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estimate the cardinality of the set of boundary classes. For instance, it was shown [4] that there exist at least five boundary classes for the Hamiltonian cycle problem. A similar estimate holds [4] for the vertex 3-colorability problem. This circumstance served as a basis for stating the hypothesis [4] that there exists a problem concerning graphs with an infinite set of boundary classes.

In this article, we point out a concrete problem on graphs with infinitely many boundary classes. Namely, we prove nonconstructively that the set of boundary classes for the edge 3-colorability problem (or the 3-EC problem for short) is infinite.

We use the following notation: $[\mathbf{K}]$ stands for the *hereditary closure* of a class $\mathbf{K}$, i.e., the class of graphs isomorphic to the induced subgraphs of the graphs in $\mathbf{K}$; $\mathbf{K}^+$ is the set of graphs in which every connected component belongs to $\mathbf{K}$; $\mathbf{Deg}(3)$ denotes the class of graphs in which the degrees of vertices are at most 3; nG is the disjoint union of n copies of a graph G; the graph $K_4 - e$ results from the graph K_4 by deleting an arbitrary edge e; the graph B results from the graph $2K_3$ by adding the two edges that join the pairs of degree 2 vertices in the different triangles.

1. ON SOME 3-EC-LIMIT CLASSES

Introduce the concept of a replacement of an edge by the graph G containing precisely two degree 2 vertices. The replacement of an edge $e = (a, b)$ in some graph by G amounts to deleting this edge followed by identifying a with one degree 2 vertex of G, and b with another degree 2 vertex of G. Assume that G has an automorphism taking the degree 2 vertices into each other; thus, the resulting graph is independent of the choice of a degree 2 vertex of G identified with a.

Refer as the *nonsymmetric* (i, j)-*bunch* to the graph obtained from the simple path $P_{2i+2j+1}$ by replacing the first i edges with even indices by the graph $K_4 - e$ and by replacing the next j edges with even indices by the graph B. Refer as the *symmetric* (i, j)-*bunch* to the graph obtained from the simple path $P_{4i+4j+1}$ by replacing the first i edges with even indices by $K_4 - e$, the next $2j$ edges with even indices by B, and the last i edges with even indices by $K_4 - e$. It is obvious that the symmetric (i, j)-bunch results by identifying the last edges of two copies of the nonsymmetric (i, j)-bunch.

Refer as the (i, j)-*transformation* of a graph to the replacement of each edge by the symmetric (i, j)-bunch. Denote the graph obtained by applying this operation to G by $G(i, j)$. Let $\mathbf{S}(i, j)$ be the set of graphs obtained by applying the (i, j)-transformation to the graphs in $\mathbf{Deg}(3)$.

The following is easy:

Lemma 1. *For all nonnegative integers i and j, the graph $G(i, j)$ is edge 3-colorable if and only if G is edge 3-colorable.*

Let $\widehat{T}(i, j)$ denote the graph obtained from the three copies of the nonsymmetric (i, j)-bunch by identifying three degree 1 vertices whose adjacent vertices belong to the graphs $K_4 - e$. Let $\widetilde{T}(i, j)$ be the graph obtained in the same fashion from the two copies of the nonsymmetric (i, j)-bunch. Put

$$\widehat{\mathbf{T}}_i = \left[\bigcup_{j=1}^{\infty} \{\widehat{T}(i, j)\} \right]^+.$$

Lemma 2. *For every positive integer i, the class $\widehat{\mathbf{T}}_i$ is a 3-EC-limit class.*

Proof. It is known that $\mathbf{Deg}(3)$ is a 3-EC-complex class [5]. By Lemma 1, this implies that, for all i and j, the class $\mathbf{S}(i, j)$ is 3-EC-complex. Thus, the class

$$\mathbf{S}^*(i, s) = \left[\bigcup_{j=s}^{\infty} \mathbf{S}(i, j) \right]$$

is 3-EC-complex for all i and s. Verify that

$$\widehat{\mathbf{T}}_i = \bigcap_{j=1}^{\infty} \mathbf{S}^*(i, j)$$

for each i.

Given $G \in \widehat{\mathbf{T}}_i$, there exist positive integers n and k such that G is an induced subgraph of $n\widehat{T}(i,j)$ for every $j \geq k$. It is obvious that, for all n, i, j, and s, the graph $n\widehat{T}(i,j)$ belongs to the class $\mathbf{S}^*(i,s)$ since $\mathbf{S}^*(i,s)$ is a hereditary class for all i and s. Therefore, every graph $G \in \widehat{\mathbf{T}}_i$ belongs to the class $\bigcap_{j=1}^{\infty} \mathbf{S}^*(i,j)$. Thus, for all i, we have

$$\widehat{\mathbf{T}}_i \subseteq \bigcap_{j=1}^{\infty} \mathbf{S}^*(i,j).$$

Consider an arbitrary graph

$$G \in \bigcap_{j=1}^{\infty} \mathbf{S}^*(i,j).$$

It is clear that $G \in \mathbf{S}^*(i,1)$, $G \in \mathbf{S}^*(i,2)$, $\ldots$. Thus, there exists an infinite monotonely increasing sequence $\{j_d\}$ such that G belongs to $\mathbf{S}^*(i,j_d)$ for every positive integer d. Put $d^{'} = |V(G)| + 1$. Then $G \in \mathbf{S}^*(i,j_{d'})$. This implies that G is an induced subgraph of $n\widehat{T}(i,j)$ for some n and j. Therefore, G belongs to $\widehat{\mathbf{T}}_i$. Thus, for each i, we have

$$\widehat{\mathbf{T}}_i \supseteq \bigcap_{j=1}^{\infty} \mathbf{S}^*(i,j).$$

These inclusions yield

$$\widehat{\mathbf{T}}_i = \bigcap_{j=1}^{\infty} \mathbf{S}^*(i,j)$$

for all i. Therefore, $\widehat{\mathbf{T}}_i$ is a 3-EC-limit class for every i. The proof of Lemma 2 is complete. □

2. THE NUMBER OF BOUNDARY CLASSES FOR THE EDGE 3-COLORING PROBLEM

A vertex x of a graph G is called *annihilated* whenever one of the following holds:

- $\deg(x) \leq 1$;
- $\deg(x) = 2$ and there exists a vertex y of G such that $\deg(y) \leq 2$ and $(x,y) \in E(G)$;
- $\deg(x) = 2$ and x belongs to an induced subgraph $K_4 - e$ of G;
- $\deg(x) = 2$ and x belongs to an induced subgraph B of G.

Given a hereditary class $\mathbf{X}$ of graphs, let $(\mathbf{X})^a$ denote the set of graphs in $\mathbf{X} \cap \mathbf{Deg}(3)$ without annihilated vertices.

Lemma 3. *For every hereditary class $\mathbf{X}$ of graphs, the* 3-EC *problem is polynomially reducible to the same problem for the graphs of the class* $(\mathbf{X})^a$.

Proof. Take $G \in \mathbf{X}$. We may assume that $G \in \mathbf{Deg}(3)$. Suppose that G contains an annihilated vertex x. It is obvious that if $\deg(x) \leq 1$ then G is an edge 3-colorable if and only if the graph $G^{'} = G \setminus \{x\}$ enjoys this property. The same holds when G contains an edge (x,y) incident to the vertices of degree at most 2.

Suppose that x has degree 2 and lies in an induced subgraph $K_4 - e$ of G. It is clear that, in every edge 3-coloring of G, the edges of this subgraph which are not incident to x are assigned to distinct colors. Thus, G is edge 3-colorable if and only if $G^{'}$ is edge 3-colorable.

Suppose that $\deg(x) = 2$ and x lies in an induced subgraph B of G. Let y and z be the two vertices of G adjacent to x. Take two edges e_1 and e_2 incident to y and z respectively and lying outside the set $\{(x,y),(x,z),(y,z)\}$. It is easy to verify that, in every edge 3-coloring of G, the edges e_1 and e_2 are assigned to different colors. Therefore, a proper edge coloring of G with 3 colors exists if and only if a coloring of this type exists for $G^{'}$. The proof is complete. □

Lemma 4. *Take some* 3-EC-*boundary class* $\mathbf{B}$ *and suppose that* $G_1 \in \mathbf{B}$ *contains an annihilated vertex* x. *Then there exists* $G_2 \in \mathbf{B}$ *such that* G_1 *is an induced subgraph of* G_2 *and* x *is not annihilated in* G_2.

Proof. Since $\mathbf{B}$ is a 3-EC-boundary class, it follows that there exist hereditary 3-EC-complex classes

$$\mathbf{B}_1 \supseteq \mathbf{B}_2 \supseteq \dots, \qquad \bigcap_{i=1}^{\infty} \mathbf{B}_i = \mathbf{B}.$$

Put $\mathbf{B}_i' = [(\mathbf{B}_i)^a]$. It is clear that $\mathbf{B}_i' \supseteq \mathbf{B}_{i+1}'$ for every i. The previous lemma implies that $\mathbf{B}_i' \subseteq \mathbf{Deg}(3)$ is a 3-EC-complex class for each i. Thus, if

$$\mathbf{B}' = \bigcap_{i=1}^{\infty} \mathbf{B}_i'$$

then $\mathbf{B}'$ is a limit class for the edge 3-colorability problem. Since $\mathbf{B}_i' \subseteq \mathbf{B}_i$ for every i, it follows that $\mathbf{B}' \subseteq \mathbf{B}$. However, $\mathbf{B}$ is a minimal 3-EC-limit class; thus, $\mathbf{B}' = \mathbf{B}$.

Take a graph G_1 in $\mathbf{B}$ containing an annihilated vertex x. Then $G_1 \in \mathbf{B}_1'$, $G_1 \in \mathbf{B}_2'$, ... By the construction of $\mathbf{B}_i'$, for each i, there exists $G_i' \in \mathbf{B}_i'$ such that G_1 is induced by G_i' and x is not annihilated in G_i'. Inspect the possible cases:

1. The degree of x in G_i' is equal to 3. Consider the graph G_i'' resulting from G_i' by the deletion of all vertices outside G_1 lying at the distance at least 2 from x. It is clear that x is not annihilated in G_i'', and $|V(G_i'')| - |V(G_1)| \leq 3$.

2. The degree of x in G_i' is equal to 2. Since x is not annihilated in G_i', it follows that x lies outside all induced subgraphs $K_4 - e$ of G_i'. Consider the graph G_i'' resulting from G_i' by the deletion of all vertices outside G_1 at the distance at least 3 from x. It is easy that x is not annihilated in G_i'', and

$$|V(G_i'')| - |V(G_1)| < 7.$$

Therefore, for each i, there exists a graph $G_2^i \in \mathbf{B}_i'$ such that G_1 is induced by G_2^i, while $|V(G_2^i)| - |V(G_1)| < 7$ and x is not annihilated in G_2^i. Put $M = \{G_2^1, G_2^2, \dots\}$. It is obvious that M contains only finitely many distinct graphs. Thus, there is a graph G_2 belonging to $\mathbf{B}_s'$ for infinitely many s. By the inclusion $\mathbf{B}_1' \supseteq \mathbf{B}_2' \supseteq \dots$, this implies $G_2 \in \mathbf{B}_i'$ for each i; and so $G_2 \in \mathbf{B}$. The proof of Lemma 4 is complete. □

The main result of this articles is

Theorem 2. *The set of boundary classes for the edge* 3-*colorability problem is infinite.*

Proof. Assume on the contrary that the set of boundary classes for the edge 3-colorability problem is finite. Then, for some infinite monotonely increasing sequence $\{j_s\}$, some 3-EC-boundary class $\mathbf{B}$ lies in every class $\widehat{\mathbf{T}}_{j_s}$. Let G' and G'' be the graphs resulting respectively from $\widehat{T}(j_1+1, 0)$ and $\widetilde{T}(j_1+1, 0)$ by the deletion of all dangling vertices and those adjacent to them. Take a graph G in $\mathbf{B}$ which is an inclusion maximal subgraph of G'. The graph G' contains precisely six annihilated degree 2 vertices which we denote by a_1, a_2, b_1, b_2, c_1, and c_2. Assume that

$$(a_1, a_2) \in E(G'), \qquad (b_1, b_2) \in E(G'), \qquad (c_1, c_2) \in E(G'),$$

and $\{a_1, a_2, b_1, b_2\} \subseteq V(G'')$. Consider the three cases:

If G is a proper induced subgraph of G' and $G \neq G''$ then G contains an annihilated vertex outside the set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$. By Lemma 4 and the inclusion

$$\mathbf{B} \subseteq \bigcap_{s=1}^{\infty} \widehat{\mathbf{T}}_{j_s},$$

there exists an induced subgraph $G^* \in \mathbf{B}$ of G' such that G is a proper induced subgraph of G^*. We arrive at a contradiction with the maximality of G.

If $G = G''$ then $G'' \in \mathbf{B}$. Thus, given a graph $G_1 = G''$ and its vertex a_1, there must exist a graph $G_2 \in \mathbf{B}$ such that G_1 is an induced subgraph of G_2 and a_1 is not annihilated in G_2. Therefore, G_2 contains a vertex x such that $(x, a_1) \in E(G_2)$ and $x \notin V(G_1)$. Since $\mathbf{B} \subseteq \widehat{\mathbf{T}}_{j_1}$, this implies that the vertex x is not adjacent to a_2 in G_2. However, then G_2 is outside the class $\widehat{\mathbf{T}}_{j_2}$; thus, $G_2 \notin \mathbf{B}$. We arrive at a contradiction.

The case $G = G'$ is considered in the same fashion.

Therefore, the original assumption is false, and the set of boundary classes for the edge 3-colorability problem is infinite. The proof of Theorem 2 is complete. □

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