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Collected papers
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The collection contains papers accepted for the Sixth International Conference Game Theory and Management (June 27-29, 2012, St. Petersburg University, St. Petersburg, Russia). The presented papers belong to the field of game theory and its applications to management.

The volume may be recommended for researches and post-graduate students of management, economic and applied mathematics departments.

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Сборник статей содержит работы участников шестой международной конференции "Теория игр и менеджмент" 27-29 июня 2012 года, Высшая школа менеджмента, Санкт-Петербургский государственный университет, Санкт-Петербург, Россия). Представленные статьи относятся к теории игр и ее приложениям в менеджменте.

Издание представляет интерес для научных работников, аспирантов и студентов старших курсов университетов, специализирующихся по менеджменту, экономике и прикладной математике.

Электронные версии серии "Теория игр и менеджмент" размещены и рецензируются в Высшей школе менеджмента СПбГУ; Math-Net.Ru; Mathematical Reviews; Zentralblatt MATH и ВИНИТИ. Аннотации и ссылки на статьи цитируются в следующих базах данных: Mathematical Reviews; Zentralblatt MATH, STMA-Z, IDEAS (RePEc), Economists Online (RePEc), EconPapers (RePEc), Citations in Economics (RePEc) и SCOPUS (SciVerse Hub).

Contents

Preface

This edited volume contains a selection of papers that are an outgrowth of the Sixth International Conference on Game Theory and Management with a few additional contributed papers. These papers present an outlook of the current development of the theory of games and its applications to management and various domains, in particular, finance, mechanism design, environment and economics.

The International Conference on Game Theory and Management, a three day conference, was held in St. Petersburg, Russia in June 27-29, 2012. The conference was organized by Graduate School of Management St. Petersburg University in collaboration with The International Society of Dynamic Games (Russian Chapter) and Faculty of Applied Mathematics and Control Processes SPU. More than 100 participants from 27 countries had an opportunity to hear state-of-the-art presentations on a wide range of game-theoretic models, both theory and management applications.

Plenary lectures covered different areas of games and management applications. They had been delivered by Professor Michele Breton, HEC Montreal (Canada); Professor Josef Hofbauer, University of Vienna (Austria); Professor Ehud Kalai, Northwestern University (USA); Professor Sylvain Sorin, Polytechnic University, Paris (France) and Professor Sergey Aseev, Institute of Mathematics RAS (Russia).

The importance of strategic behavior in the human and social world is increasingly recognized in theory and practice. As a result, game theory has emerged as a fundamental instrument in pure and applied research. The discipline of game theory studies decision making in an interactive environment. It draws on mathematics, statistics, operations research, engineering, biology, economics, political science and other subjects. In canonical form, a game takes place when an individual pursues an objective(s) in a situation in which other individuals concurrently pursue other (possibly conflicting, possibly overlapping) objectives and in the same time the objectives cannot be reached by individual actions of one decision maker. The problem is then to determine each individual's optimal decision, how these decisions interact to produce equilibria, and the properties of such outcomes. The foundations of game theory were laid more than sixty years ago by von Neumann and Morgenstern (1944).

Theoretical research and applications in games are proceeding apace, in areas ranging from aircraft and missile control to inventory management, market development, natural resources extraction, competition policy, negotiation techniques, macroeconomic and environmental planning, capital accumulation and investment.

In all these areas, game theory is perhaps the most sophisticated and fertile paradigm applied mathematics can offer to study and analyze decision making under real world conditions. The papers presented at this Sixth International Conference on Game Theory and Management certainly reflect both the maturity and the vitality of modern day game theory and management science in general, and of dynamic games, in particular. The maturity can be seen from the sophistication of the theorems, proofs, methods and numerical algorithms contained in the most of the papers in these contributions. The vitality is manifested by the range of new ideas, new applications, the growing number of young researchers and the expanding world wide coverage of research centers and institutes from whence the contributions originated.

The contributions demonstrate that GTM2012 offers an interactive program on wide range of latest developments in game theory and management. It includes recent advances in topics with high future potential and exiting developments in classical fields.

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Editors, Leon A. Petrosyan and Nikolay A. Zenkevich

The Lexmax Rule for Bankruptcy Problems*

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Abstract This paper investigates the use of egalitarian criteria to select allocations in bankruptcy problems. In our work, we characterize the sets of Lorenz maximal elements for these problems. We show that the allocation selected by the Proportional Rule is the only allocation that belongs to all these Lorenz maximal sets. We prove that the Talmud Rule selects the lexicographic maximal element within a certain set. We introduce and analyze a new rule for bankruptcy problems that shares strong similarities with the Talmud Rule.

Keywords: Bankruptcy problems; Lorenz criterion; lexicographic criterion; Proportional Rule; Talmud Rule.
JEL classification: C79; D63; D74.

1. Introduction

A bankruptcy problem consists of a set of claimants who must divide between them an infinitely divisible good (the endowment) that is not sufficient to satisfy their claims in full. The aim of this paper is to introduce two egalitarian criteria for solving bankruptcy problems.

The use of egalitarian criteria to select outcomes from a given set has been widely analyzed in many different settings. For example, in the literature on coalitional games it is well-known that the most important solution concepts can be seen as selectors of egalitarian optimal outcomes from a certain set[1]. This approach is not entirely new in bankruptcy problems since it is known that the Talmud Rule, one of the most important sharing rules for bankruptcy problems, coincides with the nucleolus of the associated bankruptcy games and the nucleolus is a solution that selects lexicographic maximal elements from a certain set. In this paper we consider two egalitarian criteria: the Lorenz and lexicographic criteria.

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[1] See for example Arin (2007). The paper explains the relationship between the Shapley value and the least square criterion and the relationship between the nucleolus and the lexicographic criterion.

It has been noted by various authors[2] that if the set of feasible allocations of a bankruptcy problem is considered it must be concluded that the lexicographic maximal allocation coincides with the allocation selected by the sharing rule known as Constrained Equal Awards, a rule that seeks to give the same amount to each claimant whenever that amount does not exceed her/his claim[3]. Constrained Equal Losses (a rule that seeks to divide the loss equally whenever no claimant receives a negative payoff) is also a lexicographic maximizer since it selects the lexicographic maximal allocation from the set of vectors of losses. We prove that the Talmud Rule selects the lexicographic maximal allocation in the set of vectors of awards/losses considered in absolute terms. The analysis suggests the definition of a new sharing rule: the Lexmax Rule. The definition of this rule is based on a lexicographic criterion and it is a natural counterpart of the Talmud Rule, or even a kind of reverse Talmud Rule. This fact relates the new Lexmax Rule with the Reverse Talmud Rule introduced by Thomson (2007). We discuss the relationship between these rules.

Concerning the second egalitarian criterion, the Lorenz criterion [4], we characterize the sets of Lorenz maximal vectors of awards/losses. The existence of different Lorenz maximal sets is due to the fact that the vectors of awards/losses can be weighted and can be considered in real terms or absolute terms (see Subsection 2.3 for details). If the intersection of the different Lorenz sets is considered in absolute terms it is found to contain only the allocation provided by the Proportional Rule.

The rest of the paper is organized as follows: Section 2 introduces bankruptcy problems, sharing rules and egalitarian criteria. Section 3 deals with the different Lorenz maximal sets. Section 4 is devoted to the lexicographic rules – the Talmud Rule and the Lexmax Rule – and Section 5 concludes.

2. Preliminaries

2.1. Bankruptcy problems

The tuple (N, d, E) is a bankruptcy *problem* if:

a) N is a finite nonempty set.
b) $\sum\limits_{i \in N} d_i > E$.

N represents the set of *agents or claimants*, $E \in \mathbb{R}_+$ represents the amount to be divided, and $d \in \mathbb{R}_+^N$ is a vector of claims whose i-th component is d_i. Then $i \preceq j$ means that we assume $d_i \leq d_j$ and $d_1 \geq 0$. We denote by Γ the class of bankruptcy problems.

An allocation to the claimants is represented by a real valued vector $x \in \mathbb{R}^N$ that satisfies $\sum_{i \in N} x_i = E$. The i-th coordinate of the vector x denotes the allocation given to claimant i.

We say that an allocation x satisfies claim boundednes and non negativity if $d_i \geq x_i \geq 0$ for all $i \in N$.

[2] See for example Bosmans and Lauwers (2007).
[3] In Subsection 2.2 we introduce the two lexicographic criteria. The first criterion is based on the minmax principle and the second one is based on the maxmin principle.
[4] In the literature on egalitarianism it is agreed that an allocation should be maximal according to the Lorenz criterion as a minimal requirement for being called egalitarian. This fact motivates the study of Lorenz maximal allocations for bankruptcy problems.

We denote by $F(N,d,E)$ the set of allocations that satisfy claim boundednes and non negativity.
A sharing rule ϕ in a set of problems Γ is a mapping that associates a vector $\phi(N,d,E) \in F(N,d,E)$ with every problem (N,d,E) in Γ.

Some well-known sharing rules are[5]:

Constrained Equal Awards (CEA). This solution divides the endowment equally among the agents under the constraint that no claimant receives more than his\her claim. Formally:

$$CEA(N,d,E) = (min(\beta, d_i))_{i \in N}$$

where β solves the equation $\sum_{i \in N} min(\beta, d_i) = E$.

Constrained Equal Losses (CEL). This solution divides the total loss ($\sum_{i \in N} d_i - E$) equally among the agents under the constraint that no claimant receives a negative amount. Formally:

$$CEL(N,d,E) = (max(0, d_i - \beta))_{i \in N}$$

and β solves the equation $\sum_{i \in N} max(0, d_i - \beta) = E$.

The **Proportional Rule (PR)**. This solution divides the endowment among the claimants proportionally to their claims. Formally:

$$PR(N,d,E) = \beta \cdot d$$

where $\beta \geq 0$ and $\beta \cdot (\sum_{i \in N} d_i) = E$.

Some convenient, well-known properties of a rule ϕ in Γ are the following.

- ϕ satisfies **order preservation for awards and losses** if for each (N,d,E) in Γ we have that $\phi(N,d,E)$ is order preserving for awards and losses. An allocation x is order preserving for awards and losses if $d_i \leq d_j$ implies that $x_i \leq x_j$ and $d_i - x_i \leq d_j - x_j$.
- ϕ satisfies **consistency** if for any problem (N,d,E) and any $S \subset N$ it holds that $\phi_i(S, (d_i)_{i \in S}, \sum_{i \in S} \phi_i(N,d,E)) = \phi_i(N,d,E)$ for all $i \in S$.
- ϕ satisfies **half claim boundednes (HCB)** if for any $(N,d,E) \in \Gamma$ we have that either $\phi_i(N,d,E) \geq \frac{d_l}{2}$ for all $i \in N$ or $\phi_i(N,d,E) \leq \frac{d_l}{2}$ for all $i \in N$.
- ϕ satisfies **λ-claim boundednes (λ-CB)** if for any $(N,d,E) \in \Gamma$ we have that either $\phi_i(N,d,E) \geq \lambda d_l$ for all $i \in N$ or $\phi_i(N,d,E) \leq \lambda d_l$ for all $i \in N$.

HCB is discussed in Aumann and Maschler (1985). This property is satisfied by the Talmud Rule and by the Proportional Rule. The λ-CB is clearly inspired by HCB and is satisfied by the Proportional Rule for any $\lambda \in [0,1]$.

2.2. Egalitarian Criteria

For any vector $z \in \mathbb{R}^d$ we denote by $\theta(z)$ the vector that results from z by permuting the coordinates in such a way that $\theta_1(z) \leq \theta_2(z) \leq ... \leq \theta_d(z)$. Let $x, y \in \mathbb{R}^d$.

We say that the vector x *Lorenz dominates* the vector y (denoted by $x \succ_L y$) if $\sum_{i=1}^{k} \theta_i(x) \geq \sum_{i=1}^{k} \theta_i(y)$ for all $k \in \{1, 2, ..., d\}$ and if at least one of these inequalities is

[5] A long list of rules can be found in a survey by Thomson (2003).

strict. The vector x *weakly Lorenz dominates* the vector y (denoted by $x \succeq_L y$) if $\sum_{i=1}^{k} \theta_i(x) \geq \sum_{i=1}^{k} \theta_i(y)$ for all $k \in \{1, 2, ..., d\}$.

We say that the vector x *lexicographically dominates* the vector y (denoted by $x \succ_{lex} y$) if either $\theta(x) = \theta(y)$ or there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, ..., k-1\}$ and $\theta_k(x) > \theta_k(y)$.

This lexicographic criterion provides Lorenz maximal allocations.

We say that the vector x *lexmax dominates* the vector y (denoted by $x \succ_{lm} y$) if either $\theta(x) = \theta(y)$ or there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{k+1, k+2, ..., n\}$ and $\theta_k(x) < \theta_k(y)$.

The lexmax criterion provides Lorenz maximal allocations.

The last two criteria can be considered lexicographic criteria. The first one can be renamed as a maximin criterion and has been widely analyzed in many different models. The second criterion, a minimax criterion is a natural counterpart of the maximin criterion but has not received the same attention. The maximin criterion is also known as the Rawlsian criterion.

2.3. The set of awards-losses vectors

Let (N, d, E) be a problem and let x be an allocation. Each agent measures x_i in two ways. In one sense x_i measures how much he\she receives. In the other sense, $d_i - x_i$ measures how much he\she does not receive. Given the allocation x we define its associated ordered vector of awards-losses as follows:

$$x^{AL} = (x_1, ..., x_n, x_1 - d_1, ..., x_n - d_n).$$

We also use the following notation:

$$x^A = (x_1, ..., x_n) \text{ and } x^L = (x_1 - d_1, ..., x_n - d_n).$$

In this vector, awards and losses are equally weighted and equally treated. We also consider vectors where awards and losses are not equally treated. Given the allocation x we define its associated weighted vector of awards-losses as follows:

$$\lambda\text{-}x^{AL} = ((1-\lambda)x_1, ..., (1-\lambda)x_n, \lambda(x_1 - d_1), ..., \lambda(x_n - d_n))$$

where $\lambda \in [0, 1]$. Note that λ-x^{AL} with $\lambda = \frac{1}{2}$ is the vector of equal weights, which in our study is equivalent to considering x^{AL} or λ-x^{AL} with $\lambda = \frac{1}{2}$. The following set

$$|\lambda\text{-}AL(N, d, E)| = \left\{ \left|\lambda\text{-}x^{AL}\right| : x \in F(N, d, E) \right\}$$

is the set of vectors of awards-losses taken in absolute terms. Note that we use the notation λ-x^{AL} instead of λ-$x^{AL}(N, d, E)$. We consider there is no confusion, so we prefer the notation λ-x^{AL} for the sake of simplicity.

3. The Lorenz criterion

The first egalitarian criterion we consider is the Lorenz criterion. The Lorenz order is not complete and therefore by applying this criterion we do not, in general, obtain uniqueness. In this sense, the set of Lorenz maximal allocations (the set of Lorenz undominated allocations) can be seen as the maximal set of fair allocations. A

Lorenz dominated allocation is not a candidate for selection when looking for fair allocations. The set of Lorenz undominated allocations is defined as follows:

$$L(N,d,E) = \left\{ \begin{array}{c} x \in F(N,d,E); \text{there is no } y \in F(N,d,E) \\ \text{such that } y^{AL} \succ_L x^{AL} \end{array} \right\}.$$

The Lorenz maximal set coincides with the set of allocations that satisfy order preservation in both ways, awards and losses Theorem 1). Therefore, order preservation emerges as a minimal requirement for a fair allocation.

The proof of this result relies on the following fact. For two elements k and l, a vector x, and a real number $\alpha > 0$, we say that (k,l,x,α) is an *equalizing bilateral transfer* (of size α from k to l with respect to x) if

$$x_k - \alpha \geq x_l + \alpha.$$

Now, Lemma 2 of Hardy, Littlewood and Polya (1952) implies that an allocation y Lorenz dominates another allocation x only if y can be obtained from x by a finite sequence of equalizing bilateral transfers.

Theorem 1. *The Lorenz maximal set coincides with the set of all allocations that satisfy order preservation in both ways: awards and losses.*

Proof. Let $x \in F(N,d,E)$ be such that x is not order preserving for awards. Therefore, there are claimants i,j such that $d_i \geq d_j$ and $x_i < x_j$. Then it also holds that $d_i - x_i > d_j - x_j$. Consider the following allocation z :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where $\varepsilon = \min(\frac{x_j - x_i}{2}, \frac{(d_i - x_i)-(d_j - x_j)}{2})$.

It is not difficult to check that $z^{AL} \succ_L x^{AL}$ since it still holds that $z_i \leq z_j$ and $d_i - z_i \geq d_j - z_j$. The proof is similar in the case where x violates order preservation for losses.

Let x be an allocation satisfying order preservation for awards and losses. Then

$$\sum_{1 \leq i \leq n} \theta_i(x^{AL}) = E - \sum_{1 \leq i \leq n} d_i$$

since the first n elements of the vector $\theta(x^{AL})$ are the ordered losses $(x_n - d_n, ..., x_1 - d_1)$[6]. Note also that

$$\sum_{i=n+1}^{2n} \theta_i(x^{AL}) = \sum_{1 \leq i \leq n} x_i = E$$

since the last n elements of the vector $\theta(x^{AL})$ are the ordered awards $(x_1, ..., x_n)$.

[6] If there is any $x_1 < (x_i - d_i)$ we have the following contradiction:
$x_1 < (x_i - d_i) \leq (x_1 - d_1) < x_1$.

Therefore, if there is an allocation z such that $z^{AL} \succ_L x^{AL}$ should be the case that $z^L \succ_L x^L$ and $z^A \succeq_L x^A$ or $z^L \succeq_L x^L$ and $z^A \succ_L x^A$. If $z^A \succ_L x^A$ then z^A can be obtained from x^A by a finite sequence of equalizing bilateral transfers.

Now consider a vector y^A resulting from x^A after a bilateral equalizing transfer. Let i, j two claimants such that $x_i < x_j$

$$y_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where $0 < \varepsilon \leq \frac{x_j - x_i}{2}$.

It is clear that $y^A \succ_L x^A$ implies that $x^L \succ_L y^L$ and therefore y^{AL} does not Lorenz dominate x^{AL}.

A similar consideration follows for the case where we consider Lorenz domination with respect to the vector x^L. That is, if there exists an allocation y such that $y^L \succ_L x^L$ then $x^A \succ_L y^A$ and therefore y^{AL} does not Lorenz dominate x^{AL}. $\square$

Figure 1 shows the Lorenz maximal set when E moves from 0 to $d_1 + d_2$. As we know by theorem 1, figure 1 also is representing the set of all allocations that satisfy order preservation in awards and losses when E moves from 0 to $d_1 + d_2$.

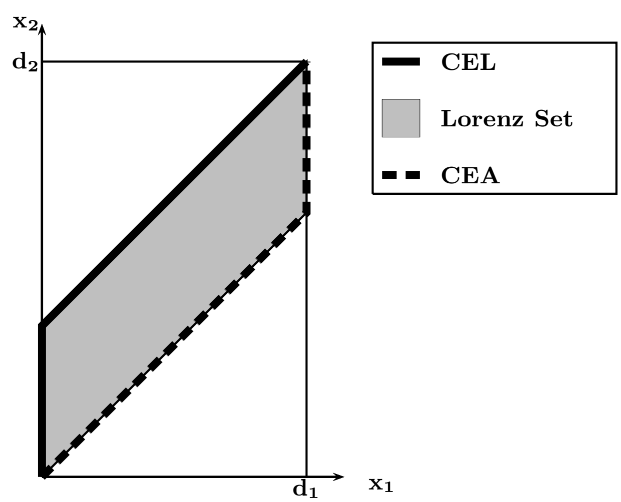


Fig. 1: *Illustration of the* ***Lorenz maximal*** *set when E moves from 0 to $d_1 + d_2$.*

The following corollary arises immediately since a convex combination of order preserving allocations is also order preserving.

Corollary 1. *The Lorenz maximal set is convex.*

Note that if an allocation x is order preserving in (N, d, E) then $(x_i)_{i \in S}$ is also order preserving in $(S, (d_i)_{i \in S}, \sum_{i \in S} x_i)$ and therefore the Lorenz set satisfies the consistency principle.

Corollary 2. *Let* $x \in L(N, d, E)$. *Then* $(x_i)_{i\in S} \in L(S, (d_i)_{i\in S}, \sum_{i\in S} x_i)$.

We also define the weighted Lorenz maximal set for $\lambda \in [0, 1]$ as follows:

$$\lambda - L(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } \lambda\text{-}y^{AL} \succ_L \lambda\text{-}x^{AL} \end{array} \right\}.$$

A direct consequence of the proof of Theorem 1 is that for $\lambda \in (0, 1)$ the weighted Lorenz maximal sets coincide. This is so because whenever $\lambda \in (0, 1)$ it is still true that $\sum_{1\le i\le n} \theta_i(\lambda\text{-}x^{AL}) = \lambda(E - \sum_{1\le i\le n} d_i)$ and $\sum_{n+1\le i\le 2n} \theta_i(\lambda - x^{AL}) = (1-\lambda)E$. Therefore the arguments of the proof can be repeated.

However it is immediately apparent that if we take $\lambda = 0$

$$L(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } y \succ_L x \end{array} \right\}$$

coincides with $CEA(N, d, E)$ and if we take $\lambda = 1$ the set

$$L(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } y^L \succ_L x^L \end{array} \right\}$$

coincides with $CEL(N, d, E)$.

The last two results were noted by Bosmans *et al.* (2007) when studying Lorenz comparisons between vectors of n elements (being n the number of claimants). Many other authors have considered Lorenz comparisons of vectors of n elements in their works. For example, this type of analysis can be found in Thomson (2007).

This analysis points out a natural question: If we consider the vector of awards-losses in absolute terms does the new Lorenz set coincide with the set of allocations that satisfy order preservation in both ways? The answer is not.

We define the new **Lorenz maximal set** as a set of Lorenz undominated allocations in the following terms:

$$L_{AT}(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } \left|y^{AL}\right| \succ_L \left|x^{AL}\right| \end{array} \right\}.$$

The new set is a subset of the Lorenz set defined above.

Theorem 2. *The set* $L_{AT}(N, d, E)$ *coincides with the set of all allocations that satisfy half claim boundednes and order preservation in both ways: awards and losses.*

Proof. Let $x \in F(N, d, E)$ be such that x is not order preserving for awards. Therefore, there are claimants i, j such that $d_i \ge d_j$ and $x_i < x_j$. Then it also holds that $d_i - x_i > d_j - x_j$. Consider the following allocation z :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where $\varepsilon = \min(\frac{x_j - x_i}{2}, \frac{(d_i - x_i) - (d_j - x_j)}{2})$.

It is not difficult to check that $\left|z^{AL}\right| \succ_L \left|x^{AL}\right|$ since it still holds that $z_i \le z_j$ and $d_i - z_i \ge d_j - z_j$. The proof is similar in the case where x violates order preservation for losses.

Let $x \in F(N, d, E)$ be such that x does not satisfy HCB. Therefore, there are claimants i, j such that $x_i < \frac{d_i}{2}$ and $x_j > \frac{d_j}{2}$. Then it also holds that $d_i - x_i > x_i$ and $x_j > d_j - x_j$. Consider the following allocation z :

$$z_l = \begin{cases} x_l + \varepsilon & \text{if } l = i \\ x_l - \varepsilon & \text{if } l = j \\ x_l & \text{otherwise} \end{cases}$$

where ε is such that still holds that $d_i - z_i \geq z_i$ and $z_j \geq d_j - z_j$.

It is not difficult to check that $\left|z^{AL}\right| \succ_L \left|x^{AL}\right|$.

Assume that $E \leq \frac{1}{2} \sum\limits_{i \in N} d_i$ and let x be an allocation satisfying HCB and order preservation in both ways. For any two claimants i, j (assuming $d_l \leq d_j$) it holds that (by HCB of x) $d_i - x_i \geq x_i$ and $d_j - x_j \geq x_j$. Since x also satisfies order preservation it also holds that $x_j \geq x_i$ and $d_j - x_j \geq d_l - x_l$.

Therefore we conclude that x_i is the minimum among the four numbers while $d_j - x_j$ is the maximum. Assume that allocation z results from a bilateral transfer made by claimant i to claimant j. That implies that $z_i < x_i$ and therefore $\left|z^{AL}\right|$ cannot Lorenz dominate $\left|x^{AL}\right|$. Assume that allocation z results from a bilateral transfer made by claimant j to claimant i. That implies that $d_j - z_j > d_j - x_j$ and therefore $\left|z^{AL}\right|$ cannot Lorenz dominate $\left|x^{AL}\right|$. The proof is almost identical if we consider $E > \frac{1}{2} \sum\limits_{i \in N} d_i$. Therefore there is no bilateral transfer between claimants allowing a new allocation that can be used to claim that x is not an element of the set $L_{AT}(N, d, E)$. □

Figure 2 illustrates the set $L_{AT}(N, d, E)$ in two-claimant problems when E moves from 0 to $d_1 + d_2$ which coincides with the set of allocations that satisfy HCB. Figure 2(a) shows the set of allocations that satisfy HCB when $\frac{d_2}{2} < d_2 - d_1$, and similarly, figure 2(b) shows the set of allocations that satisfy HCB when $\frac{d_2}{2} > d_2 - d_1$.

Following almost identical arguments as in Theorem 2, the following theorem can be proved.

Theorem 3. *The set λ-$L_{AT}(N, d, E)$ coincides with the set of all allocations that satisfy λ-claim boundednes and order preservation in both ways: awards and losses.*

The set λ-$L_{AT}(N, d, E)$ is defined as follows:

$$\lambda\text{-}L_{AT}(N, d, E) = \left\{ \begin{array}{c} x \in F(N, d, E); \text{there is no } y \in F(N, d, E) \\ \text{such that } \left|\lambda - y^{AL}\right| \succ_L \left|\lambda - x^{AL}\right| \end{array} \right\}.$$

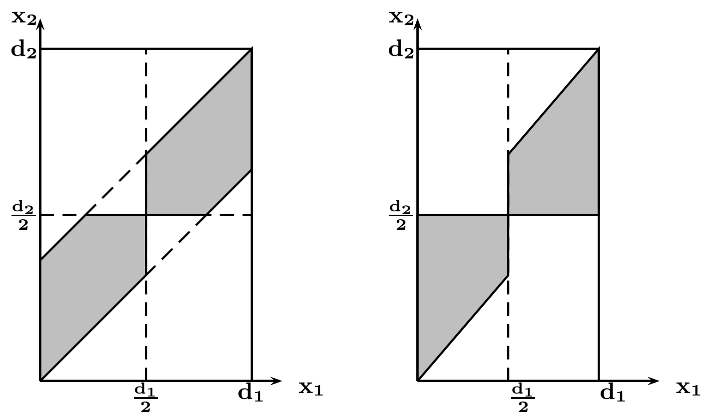
Since the Proportional Rule is the only sharing rule satisfying λ-CB for any $\lambda \in (0, 1)$ the following corollary is immediate.

Corollary 3. $\bigcap\limits_{\lambda \in (0,1)} \lambda\text{-}L_{AT}(N, d, E) = \{PR(N, d, E)\}$.

Proof. Let (N, E, d) be a bankruptcy problem and let $\lambda = \frac{E}{\sum_{n \geq l \geq 1} d_l}$. Then $E = \lambda \sum_{n \geq l \geq 1} d_l$ and therefore $\lambda\text{-}L(N, d, E) = \{\lambda d\} = \{PR(N, d, E)\}$. □

The Proportional Rule is the only rule that selects Lorenz maximal outcomes for any problem whenever awards and losses are simultaneously considered[7].

[7] Note that if we only consider awards or losses the only allocation that is Lorenz maximal is the CEA allocation or the CEL allocation.



2(a) L_{AT} when $\frac{d_2}{2} > d_2 - d_1$ 2(b) L_{AT} when $\frac{d_2}{2} < d_2 - d_1$

Fig. 2: *Illustration of the set $L_{AT}(N,d,E)$ in two-claimant problems when E moves from 0 to $d_1 + d_2$.*

4. The Lexicographic criterion

4.1. The Maximin Principle

A central rule in the literature of bankruptcy problems is the Talmud Rule introduced by Aumann and Maschler (1985). This rule explains the resolution of three numerical examples that can be found in the Talmud. For many years was an open problem what rule was behind these examples. Aumann and Maschler prove that their rule prescribes the proposals of the examples in the Talmud. They also prove that the rule coincides with the nucleolus of a TU game associated with the bankruptcy problem. Given a bankruptcy problem (N,d,E) we define its associated bankruptcy game as a TU game (N,v) where N is the set of claimants and $v(S) = \max\left\{E - \sum_{l \notin S} d_l, 0\right\}$. See O'Neill (1982).

The nucleolus (Schmeidler, 1969) selects lexicographical maximal elements in the set of vectors of satisfactions of the coalitions. We prove that the Talmud Rule is also a Lexicographic rule. First we introduce the definition of the Talmud Rule.

Let (N,d,E) be a bankruptcy problem. Then

$$T_i(N,d,E) = \begin{cases} \min\left\{\frac{d_i}{2}, \alpha\right\} & \text{if } E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2} \\ \frac{d_i}{2} + \max\left\{\frac{d_i}{2} - \alpha, 0\right\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n \geq i \geq 1} T_i(N,d,E) = E$.

This rule provides the allocation whose vector of awards-losses is the lexicographically maximal vector in the set $|AL\,(N,d,E)|$. That is,

Theorem 4. *Let (N,d,E) be a bankruptcy problem. Then*
$T(N,d,E) = \left\{x \in F(N,d,E); \left|x^{AL}\right| \succ_{lex} \left|y^{AL}\right|, \text{ for all } y \in F(N,d,E)\right\}.$

Proof. Let $z = T(N,d,E)$. We distinguish 4 cases:

a) $E \leq \frac{\sum_{n\geq l\geq 1} d_l}{2}$ and $z_i < \frac{d_i}{2}$ for all $i \in N$.

Then the first n elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{E}{n}, ..., \frac{E}{n})$ and clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

b) $E \leq \frac{\sum_{n\geq l\geq 1} d_l}{2}$ and $z_l = \frac{d_i}{2}$ for all $l \in \{1, ..., k\}$. Then the first $2k$ elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{d_1}{2}, \frac{d_1}{2}, ..., \frac{d_k}{2}, \frac{d_k}{2})$ and the next $(n-k)$ elements are $(\frac{E-\frac{1}{2}\sum_{k\geq l\geq 1} d_l}{n-k}, ..., \frac{E-\frac{1}{2}\sum_{k\geq l\geq 1} d_l}{n-k})$. Clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

c) $E > \frac{\sum_{n\geq l\geq 1} d_l}{2}$ and $d_i > z_i > \frac{d_i}{2}$ for all $i \in N$. Then the first n elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{\sum_{n\geq l\geq 1} d_l - E}{n}, ..., \frac{\sum_{n\geq l\geq 1} d_l - E}{n})$ and clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation.

d) $E > \frac{\sum_{n\geq l\geq 1} d_l}{2}$ and $z_l = \frac{d_i}{2}$ for all $l \in \{1, ..., k\}$. Then the first $2k$ elements of the vector $\theta(\left|z^{AL}\right|)$ are $(\frac{d_1}{2}, \frac{d_1}{2}, ..., \frac{d_k}{2}, \frac{d_k}{2})$ and the next $(n-k)$ elements are

$$(\frac{\frac{1}{2}\sum_{n\geq l\geq k+1} d_l - \frac{1}{2}E}{n-k}, ..., \frac{\frac{1}{2}\sum_{n\geq l\geq k+1} d_l - \frac{1}{2}E}{n-k}).$$

Clearly $\left|z^{AL}\right|$ lexicographically dominates any other vector $\left|y^{AL}\right|$ where y is an allocation. □

Weighted Talmud Rules[8] are introduced and studied by Moreno-Ternero and Villar (2006). They call this family of rules the TAL-family.

Let (N, d, E) be a problem. Then

$$\lambda\text{-}T_i(N, d, E) = \begin{cases} \min\{\lambda d_i, \alpha\} & \text{if } E \leq \lambda \sum_{n\geq l\geq 1} d_l \\ \lambda d_i + \max\{(1-\lambda)d_i - \alpha, 0\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n\geq i\geq 1} \lambda\text{-}T_i(N, d, E) = E$.

It is not difficult to check that this rule provides the allocation whose vector of awards-losses is maximal in the set $\left|AL^{\lambda}(I(N, d, E))\right|$. That is for $\lambda \in (0, 1)$ we have that

$$\lambda\text{-}T(N, d, E) = \left\{x \in F(N, d, E); \left|\lambda\text{-}x^{AL}\right| \succeq_{Lex} \left|\lambda\text{-}y^{AL}\right|, \text{ for all } y \in F(N, d, E)\right\}.$$

Given a bankruptcy problem (N, d, E) and $\lambda = \frac{E}{\sum_{n\geq l\geq 1} d_l}$ it holds that $E = \lambda \sum_{n\geq l\geq 1} d_l$ and therefore $\lambda\text{-}T_i(N, d, E) = \lambda d_i$. This fact is the proof of the following corollary.

Corollary 4. *Let (N, d, E) be a problem where $E = \lambda \sum_{n\geq l\geq 1} d_l$. Then λ-$T((N, d, E)) = PR((N, d, E))$.*

[8] The term *Weighted Talmud Rule* is introduced by Hokari and Thomson (2003). In their case the weights refer to the claimants and not to awards\losses. We keep the term since we think there is no confusion and it is more consistent with the rest of the paper.

Aumann and Maschler (1985) characterize the Talmud Rule as the unique consistent rule for bankruptcy problems (Theorem A). In their work consistency is also called CG-consistency and explained as follows:

> Intuitively, a solution is consistent if any two claimants i,j use the *contested garment principle* to divide between them the total amount $x_i + x_j$ awarded to them by the solution.

The *contested garment principle* is a solution used to solve two-claimant problems. The solution coincides with the Talmud Rule and the theorem can be interpreted as follows; the Talmud Rule is the unique solution that consistently extends to n claimant problems the *contested garment principle*.

Replacing the *contested garment principle* by the solution prescribed by a Weighted Talmud Rule in two claimant problems we can characterize this Weighted Talmud Rule as the unique rule that consistently extends to n claimant problems this solution prescribed for two claimant problems.

4.2. The Minimax Principle

This interpretation of the Talmud Rule, as a rule based in a *lexicographic maximin* criterion, suggests the definition of a new rule based on the *lexmax (lexicographic minimax)* criterion. In the literature of TU games this criterion inspires the definition of the Lexmax rule and the antinucleolus (see Arin (2007)). See also Luss (1999) for the application of the minimax principle in other models.

We call the new rule *Lexmax Rule*, and we denote it by *LM*, formally,

Definition 1. Let (N,d,E) be a bankruptcy problem. Then $LM(N,d,E) = \left\{x \in F(N,d,E); \left|x^{AL}\right| \succ_{lm} \left|y^{AL}\right|, \text{ for all } y \in F(N,d,E)\right\}$.

This rule satisfies order preservation (in both ways) and HCB since it provides allocations that belong to the set $L_{AT}(N,d,E)$. It is also quite immediately apparent that the new rule satisfies consistency.

The three facts can be used to define the following algorithm in order to compute the Lexmax Rule of a bankruptcy problem.

A procedure for computing the Lexmax Rule of a bankruptcy problem

Let (N,d,E) be a problem. In order to obtain $LM((N,d,E))$ consider the following 4 cases:

a) Let $E < \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} < d_n - CEL_n(N,d,E)$. Then

$$LM(N,d,E) = CEL(N,d,E).$$

b) Let $E < \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} \geq d_n - CEL_n(N,d,E)$. Then

$$LM_n(N,d,E) = \frac{d_n}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-1} = (N\backslash\{n\}, (d_i)_{i\in\{1,\ldots,n-1\}}, E - \frac{d_n}{2})$. If $\frac{d_n - 1}{2} < d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM(A_{n-1}) = CEL(A_{n-1}).$$

If $\frac{d_{n-1}}{2} \geq d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM_{n-1}(A_{n-1}) = \frac{d_n - 1}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-2} = \left(N \backslash \{n, n-1\}, (d_i)_{i \in \{1,\ldots,n-2\}}, E - \frac{d_n}{2} - \frac{d_{n-1}}{2}\right)$ and continue with this procedure until an allocation for all claimants is obtained.

c) Let $E \geq \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} < d_n - CEA_n(N, d, E)$. Then

$$LM(N, d, E) = CEA(N, d, E).$$

d) Let $E \geq \frac{\sum_{n \geq i \geq 1} d_i}{2}$ and $\frac{d_n}{2} \geq d_n - CEA_n(N, d, E)$. Then

$$LM_n(N, d, E) = \frac{d_n}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-1} = \left(N \backslash \{n\}, (d_i)_{i \in \{1,\ldots,n-1\}}, E - \frac{d_n}{2}\right)$. If $\frac{d_n - 1}{2} < d_{n-1} - CEA_{n-1}(A_{n-1})$ then

$$LM(A_{n-1}) = CEA(A_{n-1}).$$

If $\frac{d_{n-1}}{2} \geq d_{n-1} - CEL_{n-1}(A_{n-1})$ then

$$LM_{n-1}(A_{n-1}) = \frac{d_n - 1}{2}.$$

To obtain the allocation for the rest of the claimants consider the problem $A_{n-2} = \left(N \backslash \{n, n-1\}, (d_i)_{i \in \{1,\ldots,n-2\}}, E - \frac{d_n}{2} - \frac{d_{n-1}}{2}\right)$ and continue with this procedure until an allocation for all claimants is obtained.

In case a CEL satisfies HCB and losses are higher than awards. In case b CEL of the original problem violates HCB and therefore we fix the allocation of claimant n in order to preserve HCB. The consistency of the Lexmax Rule allows us to seek the allocation of the rest of the claimants in a new reduced problem where again losses are higher than awards. If in the new case CEL satisfies HCB this is the allocation for the rest of the claimants and otherwise we fix the allocation of claimant $n-1$ and we continue with a new reduced problem where again losses are higher than awards.

Cases c and d are the reverse of cases a and b when awards are higher than losses and the reference is CEA instead of CEL.

Figure 3 illustrates how these rules perform in two-claimant problems when E moves from 0 to $d_1 + d_2$. Figure 3(a) shows the Lexmax Rule when $\frac{d_2}{2} \leq (d_2 - d_1)$. Similarly, figure 3(b) shows the case $\frac{d_2}{2} > (d_2 - d_1)$.

Similarly, given a bankruptcy problem (N, d, E), the λ-Lexmax Rules are defined as follows:

$$\lambda\text{-}LM(N, d, E) = \left\{x \in F(N, d, E); \left|\lambda\text{-}x^{AL}\right| \succ_{lm} \left|\lambda\text{-}y^{AL}\right|, \text{ for all } y \in F(N, d, E)\right\}.$$

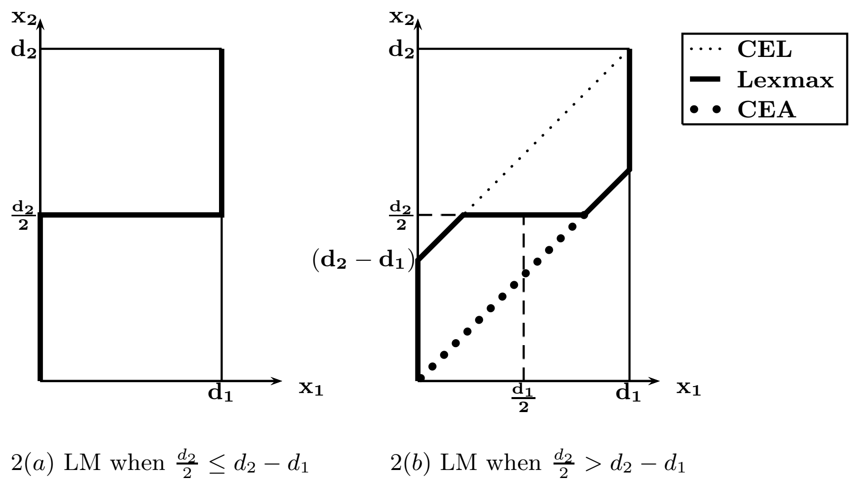


Fig. 3: *Illustration of the* ***Lexmax Rule*** *when E moves from 0 to $d_1 + d_2$.*

The computation of the Weighted Lexmax Rules results from replacing the parameter $\frac{1}{2}$ by λ in the procedure above. The procedure can be used to provide the following alternative definition of the Lexmax Rule that shares similarities with the definition of the Talmud Rule..

Let (N, d, E) be a bankruptcy problem. Then

$$LM_i(N,d,E) = \begin{cases} \max\left\{\min\left\{\frac{d_i}{2}, d_i - \alpha\right\}, 0\right\} & \text{if } E \leq \frac{\sum_{n\geq l\geq 1} d_l}{2} \\ \min\left\{\max\left\{\alpha, \frac{d_i}{2}\right\}, d_i\right\} & \text{otherwise} \end{cases}$$

where α is chosen such that $\sum_{n\geq i\geq 1} LM_i(N,d,E) = E$.

If the Estate is less than half of the total claims the Talmud Rule provides the CEA allocation whenever this allocation satisfies HCB (See Chun et al. (2001)). In the other case the Talmud Rule assigns the CEL allocation whenever this allocation satisfies HCB. The Lexmax Rule replaces CEA by CEL in the first case and CEL by CEA in the second case. Therefore the Lexmax Rule is a natural counterpart of the Talmud Rule. The following table represents different bankruptcy problems all of them with the same set of claimants and claims $((100, 200, 300))$.

E	Talmud Awards	Lexmax Awards	Talmud Losses	Lexmax Losses
100	$(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$	$(0, 0, 100)$	$(66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3})$	$(100, 200, 200)$
200	$(50, 75, 75)$	$(0, 50, 150)$	$(50, 125, 225)$	$(100, 150, 150)$
300	$(50, 100, 150)$	$(50, 100, 150)$	$(50, 100, 150)$	$(50, 100, 150)$
400	$(50, 125, 225)$	$(100, 150, 150)$	$(50, 75, 75)$	$(0, 50, 150)$
500	$(66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3})$	$(100, 200, 200)$	$(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$	$(0, 0, 100)$

The first three problems, mentioned in the Talmud, motivate the paper by Aumann and Maschler (1985).

Note that if $E < \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $T(N, d, E) \succ_L LM(N, d, E)$.

If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $LM(N, d, E) \succ_L T(N, d, E)$. The situation is reversed if we compare losses, that is, if $E < \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $(d - LM(N, d, E)) \succ_L (d - T(N, d, E))$ and if $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ then $(d - T(N, d, E)) \succ_L (d - LM(N, d, E))$.

This explanation necessarily links the new rule with a well-known rule in the literature of bankruptcy problems, the ReverseTalmud Rule, defined as follows:

$$RT_i(N, d, E) = \begin{cases} \max\left\{\frac{d_i}{2} - \alpha, 0\right\} & \text{if } E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2} \\ \frac{d_i}{2} + \min\left\{\frac{d_i}{2}, \alpha\right\} & \text{otherwise} \end{cases}$$

Arin and Benito (2010) show that the Reverse Talmud Rule is a Least Square value[9]. In the following, we explain why the two rules van be seen as *reverse Talmud rules.* The Talmud Rule allows two different interpretations:

1.- If $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEA allocation whenever the allocation satisfies HCB. If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEL allocation whomever the allocation satisfies HCB.

2.- If $E \leq \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides the CEA allocation of a new problem where the claims are half of the original claims. If $E > \frac{\sum_{n \geq l \geq 1} d_l}{2}$ the rule provides to each claimant half of his\her claim plus the CEL allocation of a new problem where the claims are half of the original claims and the Estate results $E - \frac{\sum_{n \geq l \geq 1} d_l}{2}$.

Replacing CEA by CEL and CEL by CEA the first interpretation provides the Lexmax rule. In the second interpretation the same replacement originates the Reverse Talmud rule.

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[9] That is, $RT(N, d, E) = \left\{x \in F(N, d, E); \left|x^{AL}\right| \succ_{LS} \left|y^{AL}\right|, \text{ for all } y \in F(N, d, E)\right\}$.

The vector x is said to *least square dominate* the vector y (denoted by $x \succ_{LS} y$) if $\sum_{i=1}^{d} x_i^2 \leq \sum_{i=1}^{d} y_i^2$ (assuming $x, y \in \mathbb{R}^d$).

5. Conclusions

This research can be summarized with the table below. In the table rules are linked with egalitarian criteria and sets and can be interpreted as answers to the following two questions:

1. What egalitarian criterion is used to make egalitarian comparisons between elements?
2. From what set are those elements taken?

Rule	**Criterion**	**Set**	**Weight** : λ
CEA	Lex and Lexmax	$A(N,d,E)$	
CEL	Lex and Lexmax	$L(N,d,E)$	
LM	lexmax	$\|\lambda\text{-}AL(N,d,E)\|$	$\frac{1}{2}$
T	Lex	$\|\lambda\text{-}AL(N,d,E)\|$	$\frac{1}{2}$
λ-T	Lex	$\|\lambda\text{-}AL(N,d,E)\|$	λ
PR	Lexmax and Lex	$\|\lambda\text{-}AL(N,d,E)\|$	$\frac{E}{\sum_{n\geq l\geq 1} d_l}$

The table gives a unified framework to place many different rules that have been introduced and analyzed by several authors.

The table[10] also indicates how to extend this type of solutions to other different settings. In particular, in airport problems (Littlechild, 1974) it is generally accepted that solutions must select core allocations and not merely imputations. Therefore, the search for egalitarian maximal elements should be restricted to the core of the airport problem. In other settings, other constraints may exist and solutions are required to satisfy them. This is a restriction of the set where egalitarian maximal elements are sought. Also in claim problems different constraints could be considered.

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[10] The Reverse Talmud rule and its associated Reverse λ-Talmud rules also solve the same questions. The egalitarian criterion used to generate the rules is the Least Square criterion. See Arin and Benito /2010) for details.

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Network Game of Pollution Cost Reduction

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Abstract In this paper a n-person network game theoretical model of emission reduction is considered. Each player has its own evolution of the stock of accumulated pollution. Dynamics of player $i, \quad i = 1, ..., n$ depends on emissions of players $k \in K_i$, where K_i is the set of players which are connected by arcs with player i. Nash Equilibrium is constructed. The cooperative game is considered. As optimal imputation the ES-value is supposed. The restriction on network structure to realization the irrational behavior proof condition is deduced.

Keywords: network game, Nash equilibrium, ES-value, imputation destribution procedure, irrational behavior proof condition.

1. Introduction

The public interest in environmental problems increases recently. It leads to special intergovermental agreements for reducing emissions. There may be disagreement among different parties as to the problem of allocation of costs of reducing emissions or pollution accumulations. Considerable attention is devoted to the principles of formation of agreements aimed to reduce the level of pollution, including conflict of interest parties to the agreement, as well as game-theoretic models in the field of environmental protection. One example of such models is a game-theoretic model of pollution cost reduction.

The model of pollution cost reduction is proposed in (Petrosjan and Zaccour, 2003). There is two types of costs in the model: the emission reduction cost when limiting emission to the specified level and damage cost. The players aim is to reduce their total costs.

In this paper the network game of emission reduction is considered. This model is based on the model considered in (Petrosjan and Zaccour, 2003).

2. Problem statement

Let consider network differential game $G = (P,\ L)$, where P is finite set of vertexes; L is the set of pairs $(i,\ j)$, which is named the set of arcs, where $i \in P,\ j \in P$. Let call $p \in P$ — vertexes of network, and the pair $(p,\ y) \in P$ — arc, which connect vertexes p and y.

Consider network game of emission reduction $\Gamma(I, L)$, where
I is the set of players involved in the network game, $I = \{1, 2, \ldots, n\}$.

Players of the set I are vertexes of network.

L — the set of arcs $(i,\ j) \in L, \quad i \in I,\ j \in I$.

Denote the emission of player $i,\ i = 1, 2, \ldots, n$ at time $t,\ t \in [t_0, \infty)$ as $u_i(t)$.

Denote by:
K_i is the set of players, which influence the evolution of the stock of accumulated pollution of player i; in this model K_i is the set of players, which are connected with player i with arc,
M_i is the set of players, on which the player i influences, i.e. the set of players, which have the connection with player i,
m_j — the number of players, which have the evolution of the stock of accumulated pollution, which depends on player j emissions, u_j, $|M_j| = m_j$, $M_j \neq \emptyset$, $j \in I$.

Let $x_i(t)$ be the stock of accumulated pollution of player i by time t. The evolution of the stock of accumulated pollution of player i is governed by the following differential equation:

$$\begin{aligned}
\dot{x}_i(t) &= \sum_{j \in K_i} (u_j \frac{1}{2m_j}) + \frac{u_i}{2} - \delta x_i(t), \quad m_j \neq 0, \\
\dot{x}_i(t) &= \frac{u_i}{2} - \delta x_i(t), \quad K_i = \emptyset, \\
x_i(t_0) &= x_i^0, \qquad i = 1, ..., n,
\end{aligned} \tag{1}$$

where δ denotes the natural rate of pollution absorption.

The arc $(i,\ j) \in L$ in network game of emission reduction, if the evolution of the stock of accumulated pollution of player i depends on the emissions of player j. Network is oriented, i. e. if the arc $(i,\ j) \in L$, then it doesn't follow that the arc $(j,\ i) \in L$.

Each player has its own evolution stock of accumulated pollution in the network game of emission reduction, as opposed to the model in Petrosjan and Zaccour, 2003. The evolution stock of accumulated pollution of player i can depend not only of the player i emissions, but of other players emissions, which have the connections with player i.

The game begins at time t_0 with initial state $x_0 = (x_1^0, x_2^0, ..., x_n^0)$.

Denote by $C_i(u_i)$ the emission reduction cost incurred by country i when limiting its emission to level u_i:

$$C_i(u_i(t)) = \frac{\gamma}{2}(u_i(t) - \bar{u}_i)^2,$$
$$0 \leq u_i(t) \leq \bar{u}_i, \qquad \gamma > 0.$$

Suppose that the following condition is hold:

$$\bar{u}_i \geq \frac{\pi}{\gamma(\rho + \delta)}.$$

$D_i(x_i)$ denotes its damage cost.

$$D_i(x_i) = \pi x_i(t), \qquad \pi > 0.$$

Both functions are continuously differentiable and convex, with and $C_i'(u_i) < 0$ and $D_i'(x) > 0$. Each player seeks to minimize its total cost. The payoff function of the player i is defined as:

$$K_i(x_i^0, t_0) = \int\limits_{t_0}^{\infty} e^{-\rho(t-t_0)} \left(C_i(u_i(t)) + D_i(x_i(t))\right) dt,$$

where ρ is the common social discount rate.

Example 1. Consider the example, which demonstrates the rule of constructing of evolution the stock of accumulated pollution of player i.

Four players participate in the game $I = \{1, 2, 3, 4\}$.

Player 1 influences on the player 2 only. That is the evolutions the stock of accumulated pollution of players 1 and 2 depend of emissions of player 1. First player holds a half of its own emissions and the second half it gives to player 2.

Player 2 influences on the player 3 only.

Player 3 influences on the players 2 and 4. Player 3 holds a half of its own emissions, first quarter it gives to player 2, second quarter it gives to player 4.

Player 4 influences on the players 1 and 3.

Thus we obtain the following evolutions of the stock of accumulated pollution:

$$\begin{aligned}
\dot{x}_1(t) &= \frac{u_1}{2} + \frac{u_4}{4} - \delta x_1(t), \\
\dot{x}_2(t) &= \frac{u_1}{2} + \frac{u_2}{2} + \frac{u_3}{4} - \delta x_2(t), \\
\dot{x}_3(t) &= \frac{u_2}{2} + \frac{u_3}{2} + \frac{u_4}{4} - \delta x_3(t), \\
\dot{x}_4(t) &= \frac{u_3}{4} + \frac{u_4}{2} - \delta x_4(t),
\end{aligned}$$

$$x_i(t_0) = x_i^0, \qquad i = 1, ..., 4.$$

3. Solution of the problem

In subsection 3.1 we calculate a feedback Nash equilibrium. Then in subsection 3.2 we minimize the total cost of grand coalition. The solution of the game in the form of ES-value is considered in 3.3. In subsection 3.4 the time-consistent ES-value distribution procedure is calculated. In the last subsection the irrational behavior proof condition is verified for the network game of emission reduction, when the time-consistent ES-value distribution procedure is used. The restriction on the network structure necessary for the realization the irrational behavior proof condition is deduced in the subsection 3.5.

3.1. Computation of feedback Nash equilibrium.

On the first step we compute a Nash equilibrium. To obtain a feedback Nash equilibrium, assuming differentiability of the value function, the system of Hamilton-Jacobi-Bellman equations must be satisfied. Denote by $F_i(x)$ the Bellman function of this problem. Above mentioned system is given by the following formula:

$$\rho F_i(x) = \min_{u_i}\Big\{\frac{\gamma}{2}(u_i - \bar{u}_i)^2 + \pi x_i +$$
$$+\frac{\partial F_i(x)}{\partial x_i}\Big(\sum_{j\in K_i}(u_j\frac{1}{2m_j}) + \frac{u_i}{2} - \delta x_i\Big)\Big\}, \quad i \in I, \tag{2}$$

where $x = (x_1, x_2, ..., x_n)$ — is situation in the game;.

Costs of player i in any fixed situation $x = (x_1, x_2, ..., x_n)$ depend on the stock of accumulated pollution of player i only, and it doesn't depend on the stocks of accumulated pollution of another players. So we will seek the Bellman function $F_i(x)$ in the following form:

$$F_i(x) = a_i x_i + b_i. \tag{3}$$

Differentiating the right hand side of formulas (2) with respect to u_i and equating to zero leads to:

$$u_i^N = \bar{u}_i - \frac{1}{2\gamma}\frac{\partial F_i(x)}{\partial x_i}. \tag{4}$$

Substituting u_i^N (4) and the Bellman function $F_i(x)$ (3) in Hamilton-Jacobi-Bellman equation (2) we get:

$$\rho a_i x_i + \rho b_i = \frac{1}{8\gamma}\Big[\frac{\partial(a_i x_i + b_i)}{\partial x_i}\Big]^2 + \pi x_i + \Big[\sum_{j\in K_i}(\bar{u}_j\frac{1}{2m_j}) + \frac{1}{2}\bar{u}_i -$$
$$-\frac{1}{2\gamma}\sum_{j\in K_i}(a_i\frac{1}{2m_j}) - \frac{1}{4\gamma}\frac{\partial(a_i x_i + b_i)}{\partial x_i}\Big]\frac{\partial(a_i x_i + b_i)}{\partial x_i} - \frac{\partial(a_i x_i + b_i)}{\partial x_i}\delta x_i. \tag{5}$$

Simplifying the right hand side of (5) leads to:

$$\rho a_i x_i + \rho b_i = \frac{1}{8\gamma}a_i^2 + \pi x_i + U_i^N a_i - a_i \delta x_i,$$

where

$$U_i^N = \sum_{j\in K_i}(\bar{u}_j\frac{1}{2m_j}) + \frac{1}{2}\bar{u}_i - \frac{1}{2\gamma}\sum_{j\in K_i}(a_i\frac{1}{2m_j}) - \frac{1}{4\gamma}a_i.$$

Rewrite the Nash strategies (4) in the following form:

$$u_i^N = \bar{u}_i - \frac{1}{2\gamma}a_i. \tag{6}$$

Let calculate the coefficients a_i and b_i:

$$a_i = \frac{\pi}{\rho + \delta} = a;$$

$$b_i = \frac{\pi^2}{8\rho\gamma(\rho+\delta)^2} + \frac{\pi}{\rho(\rho+\delta)}U_i^N,$$

where

$$U_i^N = \sum_{j\in K_i}(\bar{u}_j \frac{1}{2m_j}) + \frac{1}{2}\bar{u}_i - \frac{\pi}{4\gamma(\rho+\delta)}\sum_{j\in K_i}(\frac{1}{m_j}) - \frac{\pi}{4\gamma(\rho+\delta)}.$$

Substitute the coefficient a_i in the equation (6):

$$u_i^N = \bar{u}_i - \frac{\pi}{2\gamma(\rho+\delta)}. \tag{7}$$

Cost of player i in the Nash equilibrium:

$$F_i(x_i^N) = \frac{\pi}{\rho(\rho+\delta)}\Big(\frac{\pi}{8\gamma(\rho+\delta)} + U_i^N + \rho x_i^N\Big),$$

where x_i^N — noncooperative trajectory of player i.

Substituting the Nash equilibrium strategies u_i^N (7) into the differential equation (1) with initial state $x_i(t_0) = x_i^0$, we obtain the following noncooperative trajectory:

$$x_i^N = e^{-\delta(t-t_0)}x_i^0 + \frac{1}{\delta}U_i^N\big(1 - e^{-\delta(t-t_0)}\big), \quad i = 1,2,...,n.$$

3.2. Minimization the total cost of grand coalition

Minimize the total cost of the grand coalition $I = \{1,2,...,n\}$. We have following system of optimization problems:

$$\min_{u_1,u_2,\ldots,u_n}\sum_{i\in I}K_i(x_i^0,t_0) = \sum_{i\in I}\int_{t_0}^{\infty} e^{-\rho(t-t_0)}\left(C_i(u_i(t)) + \pi x_i(t)\right)dt, \tag{8}$$

subject to equation dynamics:

$$\dot{x}_i(t) = \sum_{j\in K_i}(u_j\frac{1}{2m_j}) + \frac{u_i}{2} - \delta x_i(t),$$
$$x_i(t_0) = x_i^0, \qquad i = 1,...,n.$$

Rewrite the system for dynamic programming problem (8) in the following view:

$$\min_{u_1,u_2,\ldots,u_n}\sum_{i\in I}K_i(x_i^0,t_0) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)}\left(\sum_{i\in I}(C_i(u_i(t))) + \pi\sum_{i\in I}x_i(t)\right)dt. \tag{9}$$

Denote by:

$$\bar{x} = \sum_{i\in I}x_i.$$

The minimizing functional in the right side of (9) depends only on $\bar{x}$ and it doesn't depend on x_i, $\quad i = 1,...,n$. So the minimal costs of grand coalition I depends on $\bar{x}$ and don't depend on $x_1,x_2,...,x_n$. Therefore we can consider the Bellman function as the function which depends only on $\sum_{i\in I}x_i = \bar{x}$.

The solution of the problem (9) is equivalent to the solution of the following Hamilton-Jacobi-Bellman equation:

$$\rho F(I, x_1, x_2, ..., x_n) = \min_{u_1, u_2, \ldots, u_n} \Big\{ \sum_{i=1}^{n} \big(\frac{\gamma}{2} (u_i - \bar{u}_i)^2 + \pi x_i \big) + \\ + \sum_{i=1}^{n} \frac{\partial F(I, \bar{x})}{\partial x_i} \big(U_i - \delta x_i \big) \Big\}, \tag{10}$$

where $F(I, x_1, x_2, ..., x_n)$ is the Bellman function.

Differentiating the right hand side of expression (10) subject to u_i, we get the strategies u_i^I:

$$u_i^I = \bar{u}_i - \frac{1}{2\gamma} \Big(\sum_{j \in M_i} \frac{1}{m_i} \frac{\partial F(I, x_1, x_2, ..., x_n)}{\partial x_j} + \frac{\partial F(I, x_1, x_2, ..., x_n)}{\partial x_i} \Big). \tag{11}$$

It can be shown in the usual way that the linear function $F(I, x_1, x_2, ..., x_n)$

$$F(I, x_1, x_2, ..., x_n) = a \sum_{i=1}^{n} x_i + b = a\bar{x} + b, \tag{12}$$

satisfies the equation (10). The Bellman function depends only on $\sum_{i \in I} x_i$.

By assumption,

$$F(I, x_1, x_2, ..., x_n) = F(I, \bar{x}).$$

Substitute the strategies u_i^I (11) and the Bellman function (12) in the Hamilton-Jacobi-Bellman equation:

$$\rho a \sum_{i=1}^{n} x_i + \rho b = \sum_{i=1}^{n} \frac{a^2}{2\gamma} + \pi \sum_{i=1}^{n} x_i + \\ + a \sum_{i=1}^{n} \big(\sum_{j \in K_i} (u_j^I \frac{1}{2m_j}) + \frac{1}{2} u_i^I \big) - a\delta \sum_{i=1}^{n} x_i. \tag{13}$$

Solving the equation (13) leads to the following expression for coefficients a and b:

$$a = \frac{\pi}{\rho + \delta}; \tag{14}$$

$$b = \frac{\pi}{\rho(\rho + \delta)} \Big(\sum_{i=1}^{n} \bar{u}_i - \frac{n\pi}{2\gamma(\rho + \delta)} \Big). \tag{15}$$

Taking into account (14), we get the optimal strategies of the grand coalition:

$$u_i^I = \bar{u}_i - \frac{\pi}{\gamma(\rho + \delta)}. \tag{16}$$

Substituting coefficients (14) and (15) into the formula (12) we get the minimal cost of the grand coalition as follows:

$$F(I,\bar{x}) = \frac{\pi}{\rho(\rho+\delta)}\Big(\sum_{i=1}^{n}\bar{u}_i - \frac{n\pi}{2\gamma(\rho+\delta)} + \rho\sum_{i=1}^{n}x_i^I\Big), \tag{17}$$

where x_i^I is the optimal cooperative trajectory of player $i \in I$.

Substituting the optimal strategies of the grand coalition u_i^I (16) and solving equation of dynamics (1) with initial state $x_i(t_0) = x_i^0$ we obtain the optimal cooperative trajectory of player $i \in I$:

$$x_i^I = e^{-\delta(t-t_0)}x_i^0 + \frac{1}{\delta}U_i^I\big(1 - e^{-\delta(t-t_0)}\big), \quad i = 1,2,...,n,$$

where

$$U_i^I = \sum_{j\in K_i}(\bar{u}_j\frac{1}{2m_j}) + \frac{1}{2}\bar{u}_i - \frac{\pi}{\gamma(\rho+\delta)}\Big(\sum_{j\in K_i}\frac{1}{2m_j} + \frac{1}{2}\Big).$$

The sum of Nash emissions of all players is equal to:

$$\sum_{i\in I}u_i^N = \sum_{i\in I}\bar{u}_i - \frac{n\pi}{2\gamma(\rho+\delta)}, \qquad \forall i \in I.$$

The sum of optimal emissions of players involved in grand coalition I is equal to:

$$\sum_{i\in I}u_i^I = \sum_{i\in I}\bar{u}_i - \frac{n\pi}{\gamma(\rho+\delta)}, \qquad \forall i \in I.$$

Thus the sum of emissions of all players in noncooperative case is greater than the the sum of emissions of all players in cooperative case. The more players are involved in the game the more emissions will be reduced in cooperative case as compared with noncooperative case.

3.3. The ES-value.

Definition 1. The vector

$$\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_n(t)],$$

is a *ES-value* (Driessen and Funaki, 1991), if the component of ES-value $\xi_i(t)$ is given by

$$\xi_i(t) = F_i(x_i^N) + \frac{F(I,\bar{x}) - \sum\limits_{i\in I}F_i(x_i^N)}{n}, \qquad i \in I, \tag{18}$$

where $F_i(x_i^N)$ is the costs of player i in the Nash equilibrium;
$F(I,\bar{x})$ is the the minimal cost of the grand coalition.

Let calculate ES-value in network game of emission reduction. Substitute the costs of player i in the Nash equilibrium $F_i(x_i^N)$ and minimal cooperative costs

$F(I, \bar{x})$ (17) in the equation (18). The first summand in right hand side of the equation (18) is given by:

$$\sum_{i \in I} F_i(x_i^N) = \frac{\pi}{\rho(\rho+\delta)} \Big(\sum_{i \in I} \bar{u}_i - \frac{3n\pi}{8\gamma(\rho+\delta)} + \rho \sum_{i \in I} x_i^N \Big).$$

The second summand in right hand side of the equation (18):

$$\frac{F(I, \bar{x}) - \sum\limits_{i \in I} F_i(x_i^N)}{n} =$$
$$= -\frac{\pi}{\rho(\rho+\delta)} \Big(\frac{\pi}{8\gamma(\rho+\delta)} + \rho \frac{\pi}{2\gamma\delta(\rho+\delta)} \big(1 - e^{-\delta(t-t_0)}\big) \Big).$$

Therefore the component of ES-value for the player $i, i \in I$ for the network emission reduction game is equal to:

$$\xi_i(t) = \frac{\pi}{\rho(\rho+\delta)} \Big(U_i^N + \rho \Big(e^{-\delta(t-t_0)} x_i^0 + \frac{1}{\delta} \big(\sum_{j \in K_i} (\bar{u}_j \frac{1}{2m_j}) +$$
$$+ \frac{1}{2}\bar{u}_i - \frac{\pi}{4\gamma(\rho+\delta)} \sum_{j \in K_i} \frac{1}{m_j} - \frac{3\pi}{4\gamma(\rho+\delta)} \big) \big(1 - e^{-\delta(t-t_0)} \big) \Big) \Big). \quad (19)$$

3.4. Time-consistency

Time-consistency means that if one renegotiates the agreement at any intermediate instant of time, assuming that coalitional agreement has prevailed from initial date till that instant, then one would obtain the same outcome. The notion of time-consistency was introduced by Petrosjan, 1993 and was used in problems of environmental management (Petrosjan and Zaccour, 2003).

Definition 2. The vector $\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$ is a *ES-value distribution procedure (ESDP)* (see Petrosjan, 1993) if

$$\xi_i(x_0, t_0) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \beta_i(t) dt, \quad i \in I.$$

Definition 3. The vector $\beta(t) = (\beta_1(t), \beta_2(t), \ldots, \beta_n(t))$ is a *time-consistent ESDP* (Petrosjan, 1993)if at $(x^I(t), t)$ at any $t \in [t_0, \infty)$ the following condition holds

$$\beta_i(t) = \rho \xi_i(x_i^I(t), t) - \frac{d}{dt} \xi_i(x_i^I(t), t), \quad i \in I.$$

Consider ES-value (19) that was computed in the section 3.3.

Straightforward calculations give us the following view for the time-consistent ESDP in the network game of emission reduction:

$$\beta_i(t) = \pi \Big(e^{-\delta(t-t_0)} x_i^0 + \frac{1}{\delta} \Big(\sum_{j \in K_i} (\bar{u}_j \frac{1}{2m_j}) + \frac{1}{2}\bar{u}_i - \frac{\pi}{4\gamma(\rho+\delta)} \sum_{j \in K_i} \frac{1}{m_j} -$$
$$- \frac{3\pi}{4\gamma(\rho+\delta)} \Big) \Big(1 - e^{-\delta(t-t_0)} \Big) \Big) + \frac{\pi^2}{2\gamma(\rho+\delta)^2}, \quad i \in I. \quad (20)$$

3.5. The irrational behavior proof condition

Consider the case where the cooperative scheme has proceeded up to time $t \in [t_0, +\infty)$ and some players behave irrationally leading to the dissolution of the scheme. A condition under which even if irrational behaviors appear later in the game the concerned player would still be performing better under the cooperative scheme is the irrational behavior proof condition (Yeung, 2006), which also is called the D.W.K. Yeung condition.

Consider the solution of the game in the form of ES-value. The irrational behavior proof condition for the problem of emission reduction is described as follows:

$$F_i(x_i^0) \geq \int\limits_{t_0}^{t} e^{-\rho(\tau - t_0)} \beta_i(\tau) d\tau + e^{-\rho(t-t_0)} F_i(x^I(t)), \qquad i \in I, \tag{21}$$

where $F_i(x^I(t))$ — costs of player i in the Nash equilibrium with initial state $x^I(t)$ on the optimal cooperative trajectory;
$\beta_i(\tau)$ — time-consistent ES-value distribution procedure.

Verify the realization of the irrational behavior proof condition. The left hand side of the inequality (21) is written as follows:

$$F_i(x_i^0) = \\ = \frac{\pi}{\rho(\rho+\delta)} \left(\frac{\pi}{8\gamma(\rho+\delta)} + A_i - \frac{\pi}{4\gamma(\rho+\delta)} \sum_{j \in K_i} (\frac{1}{m_j}) - \frac{\pi}{4\gamma(\rho+\delta)} + \rho x_i^0 \right). \tag{22}$$

where

$$A_i = \sum_{j \in K_i} (\bar{u}_j \frac{1}{2m_j}) + \frac{1}{2} \bar{u}_i.$$

Consider the integral in the right hand side of inequality (21). The substitution of $\beta_i(t)$ (20) leads to the following integral:

$$\int\limits_{t_0}^{t} e^{-\rho(\tau - t_0)} \beta_i(\tau) d\tau = e^{-\rho(t-t_0)} \pi \Big(-\frac{\pi}{2\rho\gamma(\rho+\delta)^2} - \frac{e^{-\delta(t-t_0)} x_0}{\rho+\delta} - \\ - \left(A_i - \frac{\pi}{\gamma(\rho+\delta)} (\sum_{j \in K_i} (\frac{1}{4m_j}) + \frac{3}{4}) \right) \frac{1}{\rho\delta} + \\ + \left(A_i - \frac{\pi}{\gamma(\rho+\delta)} (\sum_{j \in K_i} (\frac{1}{4m_j}) + \frac{3}{4}) \right) \frac{e^{-\delta(t-t_0)}}{\delta(\rho+\delta)} \Big) + \frac{\pi^2}{2\rho\gamma(\rho+\delta)^2} + \\ + \frac{\pi x_0}{\rho+\delta} + \left(A_i - \frac{\pi}{\gamma(\rho+\delta)} (\sum_{j \in K_i} (\frac{1}{4m_j}) + \frac{3}{4}) \right) \frac{\pi}{\rho(\rho+\delta)}. \tag{23}$$

The second summand in the right hand side of inequality (21) can be calculated:

$$e^{-\rho(t-t_0)}F_i(x^I(t)) = e^{-\rho(t-t_0)}\frac{\pi}{\rho(\rho+\delta)}\Big\{\frac{\pi}{8\gamma(\rho+\delta)} + A_i -$$
$$-\frac{\pi}{4\gamma(\rho+\delta)}\sum_{j\in K_i}(\frac{1}{m_j}) - \frac{\pi}{4\gamma(\rho+\delta)} +$$
$$+\rho\Big(e^{-\delta(t-t_0)}x_i^0 + \frac{1}{\delta}\big(A_i - \frac{\pi}{\gamma(\rho+\delta)}\big(\sum_{j\in K_i}(\frac{1}{2m_j}) + \frac{1}{2}\big)\big)\big(1-e^{-\delta(t-t_0)}\big)\Big)\Big\}. \quad (24)$$

The substitution of integral (23) and value of $e^{-\rho(t-t_0)}F_i(x^I(t))$, which is defined by the formula (24) into the inequality (21) leads to:

$$\int_{t_0}^{t} e^{-\rho(\tau-t_0)}\beta_i(\tau)d\tau + e^{-\rho(t-t_0)}F_i(x^I(t)) =$$
$$= e^{-\rho(t-t_0)}\left(\frac{\pi^2}{8\rho\gamma(\rho+\delta)^2} + \frac{\pi^2}{4\delta\gamma(\rho+\delta)^2} - \frac{\pi^2}{4\delta\gamma(\rho+\delta)^2}\sum_{j\in K_i}\frac{1}{m_j}\right) +$$
$$+\frac{\pi^2}{2\rho\gamma(\rho+\delta)^2} + \left(A_i - \frac{\pi}{\gamma(\rho+\delta)}\big(\sum_{j\in K_i}(\frac{1}{4m_j}) + \frac{3}{4}\big)\right)\frac{\pi}{\rho(\rho+\delta)} + x_0\frac{\pi}{\rho+\delta}. \quad (25)$$

Compare the right side (25) and the left side (22) of the formula (21). It can be shown that the inequality (21) is equivalent to the following inequality:

$$\delta\left(e^{-\rho(t-t_0)} - 1\right) + 2\rho e^{-\rho(t-t_0)}\left(1 - \sum_{j\in K_i}\frac{1}{m_j}\right) \le 0. \quad (26)$$

The inequality (26) get the following form at the moment $t = t_0$:

$$2\rho\left(1 - \sum_{j\in K_i}\frac{1}{m_j}\right) \le 0. \quad (27)$$

If the sum $\sum_{j\in K_i}\frac{1}{m_j}$ satisfies the following inequality:

$$\sum_{j\in K_i}\frac{1}{m_j} \ge 1, \quad (28)$$

the inequality (27) is satisfied. Hence the inequality (26) is satisfied at time $t = t_0$.

The first summand in the right hand side of the inequality (26) is nonpositive for all $t \in [t_0, +\infty)$. If (28) is satisfied, than the second summand in the right hand side of the inequality (26) will be nonpositive for all $t \in [t_0, +\infty)$. Therefore the inequality (26) is satisfied for all $t \in [t_0, +\infty)$, if the (28) is satisfied. It means that the following theorem is proved.

Theorem 1. *The irrational behavior proof condition is realized in the network game of emission reduction for time-consistent ES-value distribution procedure if the following restriction to the network structure is satisfied:*

$$\sum_{j \in K_i} \frac{1}{m_j} \geq 1.$$

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A Model of Coopetitive Game and the Greek Crisis

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Abstract In the present work we propose an original economic coopetitive model applied to the Greek crisis. This model is based on normal form game theory and conceived at a macro level. We aim at suggesting feasible solutions in a super-cooperative perspective for the divergent interests which drive the economic policies of the countries in the euro area.

Keywords: Greek crisis; euro area; trade imbalances; coopetition; coopetitive games; normal form games; Kalai-Pareto solutions.

1. Introduction

In this contribution we focus on the Greek crisis, because Greece, which is a EU member and a country that is part of the euro area, since the end of 2009 has entered in a deep financial and economic crisis. Although Greeces GDP reaches only 2 per cent of total GDP of the whole euro area (IMF, 2011), the Greek crisis is creating many troubles to the euro area and all over the world. The risk of insolvency of Greece, mainly due to its public finance mismanagement, has represented the extreme situation of a general sovereign debt crisis which has hit the southern countries of the eurozone (PIIGS) and that has interested the whole euro area in the last three years. The Greek economy, after its accession to the euro, has lost competitiveness, due to its generous wage increases and high domestic prices induced also by ECBs monetary policy . The lack of competitiveness has created an heavy and increasing current account imbalance. Financial aid programs have been devised to help Greece by the euro area authorities and IMF in May 2010 (EU Council, 2010) and again in July 2011 (EU Council, 2011). These financial aid programs have unfortunately proved belated and insufficient. The causes of these errors are certainly of political and institutional nature and relate to the governance of the euro area, which we do not discuss in this work. However, the success of any support program is conditioned to the capacity of Greek government to meet the fiscal adjustment targets and also by the ability of the Greek economy of triggering the growth (Darvas, Pisani-Ferry, Sapir, 2011; Schilirò, 2011). Germany, on the other hand, is the most competitive economy of the euro area, it is heavily export-oriented, in fact it is the second world's biggest exporter, with exports accounting for more than one-third of national output (IMF, 2011). Thus Germany has a large current account surplus with Greece and other euro partners; hence significant trade imbalances occur within the euro area. The main purpose of our contribution is to explore win-win solutions for Greece and Germany, adopting an appropriate game theory model in which we assume Germanys increasing demand of Greek exports. In this work we do not analyze the causes of the financial crisis in Greece and, more generally, the sovereign debt crisis of the euro area with its relevant economic, financial and institutional effects on the European Monetary Union. Rather, we concentrate on the problem of the current account imbalances of Greece providing a coopetitive model which shows the possible win-win solutions. So we look to the stability and growth of the Greek economy. Such targets, in fact, should drive the economic policy of Greece and other countries of the euro area.

Organization of the paper. The work is organized as follows:

1. section 2 examines the Greek crisis, suggesting a possible way out to reduce the intra-eurozone imbalances through coopetitive solutions within a growth path;
2. sections from 3 to 6 provides an original model of coopetitive game applied to the Eurozone context, showing the possible coopetitive solutions;
3. conclusions end up the paper.

Introduction and Section 2 of this paper are written by D. Schilirò, sections from 3 to 6 are written by D. Carfì conclusions are written by the two authors, however the whole paper is written in strict joint cooperation.

Note. Baldwin and Gros (2010, p.4) maintain that in the period 2000-2007 The one-size monetary policy plainly failed to fit all (the euro countries). Booming economic performance in Greece, Ireland, and Spain was accompanied by prices that rose much more than average. The cumulative excess inflation was 10 percentage points for Ireland, and 8 points for Greece and Spain. The asymmetric development of output and competitiveness produced massive current account imbalances. The total current account balance of Germany has been over 5 per cent of GDP in 2011 (IMF, 2011).

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2. The Greek Crisis and the coopetitive solution

The severe financial and economic crisis of Greece has revealed the weaknesses of Greek economy, particularly the mismanagement of the public finance, the difficulties of the banking sector, but above all the lack of competitiveness.

2.1. The Greek economy and the global crisis

With the outbreak of the global crisis of 2008-2009, Greece relied on state spending to drive growth, thus the country has accumulated a huge public debt, which in 2010 amounted to 328 billion euros, that is a Debt/GDP ratio equal to 142 per cent Ð according to IMF (2011) Ð and the debt situation has worsened in 2011. This has created deep concerns about fiscal sustainability of the Greek economy, whereas its financial exposition has prevented the Greek government to find capitals in the financial markets. In addition, since joining the European monetary union, Greece has lost competitiveness especially compared to France and Germany, due to the sharp increase of unit labor costs and higher domestic prices (Boone, Johnson, 2012). The austerity measures implemented by the Greek government, although insufficient, have hit hard the Greek economy, since its rate of growth has been negative in 2010 and 2011, with an unemployment rate soaring from 12. 4 per cent in 2010 to 16.5 per cent in 2011 (IMF, 2011), making the financial recovery very problematic, as Mussa (2010) had already envisaged. Furthermore, Greek exports are much less than imports, so the current account balance has been 10.45 per cent of GDP in 2010 and 8.37 per cent of GDP in 2011 (IMF, 2011). Therefore, taking for granted the need of a fiscal consolidation, the focus of economic policy of Greece should become its productive system and growth must be the major goal for the Greek economy in a medium term perspective. However, a policy-solution that implies a greater amount of exports from Greece towards the euro countries could help its re-equilibrium process.

2.2. The soundest European economy: Germany

Germany, on the other hand, is considered the soundest European economy. It is the second world's biggest exporter, its wide commercial surplus is partly originated by the exports in the euro area, that accounts just above 40 per cent of its total exports, even if this share is declining (IMF, 2011). In fact, during the last twenty years from 1991 (when the freshly unified country still traded in its own quite strong currency, the Deutsche Mark), to 2011 its export share has gradually increased vis-Ĺ-vis industrial countries, but it has also showed a changing trend, which reflects the shifting economic powers on a global scenario (see the note). Thus Germany's growth path has been driven by exports. We do not discuss in this work the factors explaining Germany's increase in export share, but we observe that its international competitiveness has been improving, with the unit labor cost which has been kept fairly constant, since wages have essentially kept pace with productivity. Consequently, the prices of the German products have been relatively cheap, favoring the export of German goods towards the euro countries, but even more towards the markets around the world, especially those of the emerging economies (China, India, Brasil, Russia). Moreover, since 2010 Germany has recovered very well from the 2008-2009 global crisis and it is growing at a higher rate than the others euro partners. Therefore we share the view that Germany in particular (but also the other surplus countries of the euro area), should contribute to overcome the Greek crisis by stimulating its demand of goods from Greece, since Germany Đ as some economists as Posen (2010) and Abadi (2010) underlined Đ has benefited from being the anchor economy for the euro area over the last 12 years.

Note. See also the article: EuropeÕs Economic Powerhouse Drifts East, on New York Times July 18, 2011, that highlights the shift of German exports and investments outside the euro area in the recent years (2006-2010).

2.3. A win-win solution for Greece and Germany

The Fiscal Compact or Fiscal Stability Treaty, the intergovernmental treaty recently signed by almost all of the member states of the European Union in March 2012 (the treaty will enter into force on 1 January 2013, if by that time 12 members of the euro area have ratified it) is probably too much focused on the budget discipline.We believe, instead, that a correct economic policy for Greece (and the other southern countries) should aim not only at adjusting government budget but also current account imbalances and, at the same time, at improving the growth path of its real economy in the medium and long term. This more complex policy , which requires a set of instruments and actions to reform the Greek economy, is probably the more suitable, although not easy to implement, for assuring a sustainable path to

Greece over time and also to contribute to the stability of whole eurozone (Schilirò, 2011). As we have just argued, Germanys relatively modest wage increases and weak domestic demand favored the export of German goods towards the euro countries and all over the world. In this context, we suggest, in accordance with Posen (2010), to look for a win-win solution (a win-win solution is the outcome of a game which is designed in a way that all participants can profit from it in one way or the other), which entails that Germany, which still represents the leading economy in Europe, should contribute to re-balance its trade surplus within the euro-area and thus ease the pressure on the southern countries of the euro area, particularly Greece. Obviously, we are aware that this is a mere hypothesis and that our framework of coopetition represents a normative model. However, we believe that a coopetitive behavior, that implies a cooperative attitude, despite the diverging interests, is the most sensible and convenient strategy that the members of the euro area should follow. A coopetitive behavior, in fact, is different form a purely cooperative attitude and it also avoid to transform the euro area into a sort of transfer union. Finally, our model does not represent a test to see whether it is convenient for Greece leave the euro or not. Therefore, we pursue our hypothesis and suggest an economic coopetive model as an innovative instrument to analyze possible outcomes to obtain a win-win solution involving Greece and Germany.

2.4. Our coopetitive model

The two strategic variables of our model are investments and exports for Greece, since this country must concentrate on them to improve the structure of production and its competitiveness, but also shift its aggregate demand towards a higher growth path in the medium term. Thus Greece should focus on innovative investments, specially investments in knowledge (Schilirò, 2010), to change and improve its production structure and to increase its production capacity and productivity. As a result of that its competitiveness will improve. These investments should be supported by the private investors and the government should make easier this process; moreover, in an open economy this innovative investments could come from abroad. An economic policy that focuses on investments and exports, instead of consumptions, will address Greece towards a sustainable growth and, consequently, its financial reputation and economic stability will also improve. On the other hand, the strategic variables of our model for Germany are private consumption and imports. While the coopetitive variable (or shared variable) in our model is represented by the export of Greek goods to Germany (or, if you like, by the import of Greek goods in Germany). Thus, the idea which is driving our model to contribute to overcome the economic crisis in Greece is based on a notion of coopetition where the cooperative aspect is very important, since both Germany and Greece belong to an economic and monetary union. Therefore, we are not considering a scenario in which Germany and Greece are competing in the same European market for the same products, rather we are assuming a situation in which Germany stimulates its domestic demand and, in doing so, will create also a larger market for products coming from abroad. In this situation Germany agrees to purchase a certain amount of goods imported from Greece, consequently Greece will increase its exports by selling more products to Germany. This shared variable, decided together by Greece and Germany, becomes the main instrumental variable of the model. The final result will be that Greece find itself in a better position, but also Germany will get an economic advantage determined by the higher growth in the two countries.

In addition, there is the important (indirect) advantage of a greater stability within the euro area. Finally, our model will provide a new set of tools based on the notion of coopetition, that could be fruitful for the setting of the euro area economic policy issues.

2.5. The coopetition in our model

The concept of coopetition was essentially devised at micro-economic level for strategic management solutions by Brandenburger and Nalebuff (1995, 1996), who suggest, given the competitive paradigm (Porter, 1985), to consider also a cooperative behavior to achieve a win-win outcome for both players. Brandenburger and Nalebuff maintains that coopetition means that Ç you have to compete and cooperate at the same time. The combination makes for a more dynamic relationship than the words competition and cooperation suggest individually (1996, pp.4-5). Therefore, coopetition becomes, in our model, a complex theoretical construct and it is the result of the interplay between competition and cooperation, since it represents the synthesis between the competitive paradigm (Porter, 1985) and the cooperative paradigm (Gulati, Nohria, Zaheer, 2000). We have already devised a coopetitive model at a macroeconomic level (Carfì, Schilirò, 2011). In this model (2011), that adopted the same variables of the present one (consumption and imports for Germany and innovative investments and exports for Greece), we have developed a coopetitive game by excluding the mutual influence of the actions (or strategies) for the two players. In other words, we excluded the dependence of the payoff functions of each player on the strategies of other players. This choice has allowed us to greatly simplify the model, secondly it has highlighted the coopetitive aspect, although at the expense of the classical feature of game theory. In the present model, instead, we continue to highlight the coopetitive strategy in its cooperative dimension, represented by the shared variable (identified in the export of Greek goods to Germany), but, in addition, we reintroduce the classical strategic interaction between the two players. Furthermore, this generalization of the model allows us to reach to competitive solutions or, better still, to a family of competitive solutions Ĺ la Nash from which to choose the win win solution. Also note that in this generalized model, competitive solutions Ĺ la Nash are not equivalent to the prisoner's dilemma solutions, because our solutions are optimal (maximum) and not minimal as in the case of the prisoner's dilemma. Therefore, our new model of coopetitive games aims at offering possible solutions to the partially divergent interests of Germany and Greece in a perspective of a cooperative attitude that should drive their policies.

3. Coopetitive games

3.1. Introduction

In this paper we develop and apply the mathematical model of a *coopetitive game* introduced by David Carfì in (Carfì and Schilirò, 2011 and Carfì, 2010). The idea of coopetitive game is already used, in a mostly intuitive and non-formalized way, in Strategic Management Studies (see for example Brandenburgher and Nalebuff).

The idea. A coopetitive game is a game in which two or more players (participants) can interact *cooperatively and non-cooperatively at the same time*. Even Brandenburger and Nalebuff, creators of coopetition, did not define, precisely, a *quantitative way to implement coopetition* in the Game Theory context.

The problem to implement the notion of coopetition in Game Theory is summarized in the following question:

- *how do, in normal form games, cooperative and non-cooperative interactions can live together simultaneously, in a Brandenburger-Nalebuff sense?*

In order to explain the above question, consider a classic two-player normal-form gain game $G = (f, >)$ - such a game is a pair in which f is a vector valued function defined on a Cartesian product $E \times F$ with values in the Euclidean plane $\mathbb{R}^2$ and $>$ is the natural strict sup-order of the Euclidean plane itself (the sup-order is indicating that the game, with payoff function f, is a gain game and not a loss game). Let E and F be the strategy sets of the two players in the game G. The two players can choose the respective strategies $x \in E$ and $y \in F$

- cooperatively (exchanging information and making binding agreements);
- not-cooperatively (not exchanging information or exchanging information but without possibility to make binding agreements).

The above two behavioral ways are mutually exclusive, at least in normal-form games:

- the two ways cannot be adopted simultaneously in the model of normal-form game (without using convex probability mixtures, but this is not the way suggested by Brandenburger and Nalebuff in their approach);
- there is no room, in the classic normal form game model, for a simultaneous (non-probabilistic) employment of the two behavioral extremes *cooperation* and *non-cooperation*.

Towards a possible solution. David Carfì (Carfì and Schilirò, 2011 and Carfì, 2010) has proposed a manner to pass this *impasse*, according to the idea of coopetition in the sense of Brandenburger and Nalebuff. In a Carfì's coopetitive game model,

- the players of the game have their respective strategy-sets (in which they can choose cooperatively or not cooperatively);
- there is a common strategy set C containing other strategies (possibly of different type with respect to those in the respective classic strategy sets) that *must be chosen cooperatively*;
- the strategy set C can also be structured as a Cartesian product (similarly to the profile strategy space of normal form games), but in any case the strategies belonging to this new set C must be chosen cooperatively.

3.2. The model for n-players

We give in the following the definition of coopetitive game proposed by Carfì in (Carfì and Schilirò, 2011 and Carfì, 2010).

Definition (of n-player coopetitive game). Let $E = (E_i)_{i=1}^n$ be a finite n-family of non-empty sets and let C be another non-empty set. We define **n-player coopetitive gain game over the strategy support** (E, C) any pair $G = (f, >)$, where f is a vector function from the Cartesian product ${}^{\times}E \times C$ (here ${}^{\times}E$ denotes the classic strategy-profile space of n-player normal form games, i.e. the Cartesian product of the family E) into the n-dimensional Euclidean space $\mathbb{R}^n$ and $>$ is the natural sup-order of this last Euclidean space. The element of the set C will be called **cooperative strategies of the game**.

A particular aspect of our coopetitive game model is that any coopetitive game G determines univocally a family of classic normal-form games and vice versa; so that any coopetitive game could be defined as a family of normal-form games. In what follows we precise this very important aspect of the model.

Definition (the family of normal-form games associated with a coopetitive game). Let $G = (f, >)$ be a coopetitive game over a strategic support (E, C). And let

$$g = (g_z)_{z \in C}$$

be the family of classic normal-form games whose member g_z is, for any cooperative strategy z in C, the normal-form game

$$G_z := (f(., z), >),$$

where the payoff function $f(., z)$ is the section

$$f(., z) : {}^{\times}E \to \mathbb{R}^n$$

of the function f, defined (as usual) by

$$f(., z)(x) = f(x, z),$$

for every point x in the strategy profile space ${}^{\times}E$. We call the family g (so defined) **family of normal-form games associated with (or determined by) the game** G and we call **normal section** of the game G any member of the family g.

We can prove this (obvious) theorem.

Theorem. *The family g of normal-form games associated with a coopetitive game G uniquely determines the game. In more rigorous and complete terms, the correspondence $G \mapsto g$ is a bijection of the space of all coopetitive games - over the strategy support (E, C) - onto the space of all families of normal form games - over the strategy support E - indexed by the set C .*

Proof. This depends totally from the fact that we have the following natural bijection between function spaces:

$$\mathcal{F}({}^{\times}E \times C, \mathbb{R}^n) \to \mathcal{F}(C, \mathcal{F}({}^{\times}E, \mathbb{R}^n)) : f \mapsto (f(., z))_{z \in C},$$

which is a classic result of theory of sets. □

Thus, the exam of a coopetitive game should be equivalent to the exam of a whole family of normal-form games (in some sense we shall specify).

In this paper we suggest how this latter examination can be conducted and what are the solutions corresponding to the main concepts of solution which are known in the literature for the classic normal-form games, in the case of two-player coopetitive games.

3.3. Two players coopetitive games

In this section we specify the definition and related concepts of two-player coopetitive games; sometimes (for completeness) we shall repeat some definitions of the preceding section.

Definition (of coopetitive game). Let E, F and C be three nonempty sets. We define **two player coopetitive gain game carried by the strategic triple** (E, F, C) any pair of the form $G = (f, >)$, where f is a function from the Cartesian product $E \times F \times C$ into the real Euclidean plane $\mathbb{R}^2$ and the binary relation $>$ is the usual sup-order of the Cartesian plane (defined component-wise, for every couple of points p and q, by $p > q$ iff $p_i > q_i$, for each index i).

Remark (coopetitive games and normal form games). The difference among a two-player normal-form (gain) game and a two player coopetitive (gain) game is the fundamental presence of the third strategy Cartesian-factor C. The presence of this third set C determines a total change of perspective with respect to the usual exam of two-player normal form games, since we now have to consider a normal form game $G(z)$, for every element z of the set C; we have, then, to study an entire ordered family of normal form games in its own totality, and we have to define a new manner to study these kind of game families.

3.4. Terminology and notation

Definitions. Let $G = (f, >)$ be a two player coopetitive gain game carried by the strategic triple (E, F, C). We will use the following terminologies:

- the function f is called the **payoff function of the game** G;
- the first component f_1 of the payoff function $f$is called **payoff function of the first player** and analogously the second component f_2 is called **payoff function of the second player**;
- the set E is said **strategy set of the first player** and the set F the **strategy set of the second player**;
- the set C is said the **cooperative (or common) strategy set of the two players**;
- the Cartesian product $E \times F \times C$ is called the (**coopetitive**) **strategy space of the game** G.

Memento. The first component f_1 of the payoff function f of a coopetitive game G is the function of the strategy space $E \times F \times C$ of the game G into the real line $\mathbb{R}$ defined by the first projection

$$f_1(x, y, z) := \mathrm{pr}_1(f(x, y, z)),$$

for every strategic triple (x, y, z) in $E \times F \times C$; in a similar fashion we proceed for the second component f_2 of the function f.

Interpretation. We have:

- two players, or better an ordered pair $(1, 2)$ of players;
- anyone of the two players has a strategy set in which to choose freely his own strategy;
- the two players can/should *cooperatively* choose strategies z in a third common strategy set C;
- the two players will choose (after the exam of the entire game G) their cooperative strategy z in order to maximize (in some sense we shall define) the vector gain function f.

3.5. Normal form games of a coopetitive game

Let G be a coopetitive game in the sense of above definitions. For any cooperative strategy z selected in the cooperative strategy space C, there is a corresponding normal form gain game

$$G_z = (p(z), >),$$

upon the strategy pair (E, F), where the payoff function $p(z)$ is the section

$$f(., z) : E \times F \to \mathbb{R}^2,$$

of the payoff function f of the coopetitive game - the section is defined, as usual, on the competitive strategy space $E \times F$, by

$$f(., z)(x, y) = f(x, y, z),$$

for every bi-strategy (x, y) in the bi-strategy space $E \times F$.

Let us formalize the concept of game-family associated with a coopetitive game.

Definition (the family associated with a coopetitive game). Let $G = (f, >)$ be a two player coopetitive gain game carried by the strategic triple (E, F, C). We naturally can associate with the game G a family $g = (g_z)_{z \in C}$ of normal-form games defined by

$$g_z := G_z = (f(., z), >),$$

for every z in C, which we shall call **the family of normal-form games associated with the coopetitive game** G.

Remark. It is clear that with any above family of normal form games

$$g = (g_z)_{z \in C},$$

with $g_z = (f(., z), >)$, we can associate:

- a family of payoff spaces

  $$(\mathrm{im} f(., z))_{z \in C},$$

  with members in the payoff universe $\mathbb{R}^2$;

– a family of Pareto maximal boundary

$$(\partial^* G_z)_{z\in C},$$

with members contained in the payoff universe $\mathbb{R}^2$;
– a family of suprema

$$(\sup G_z)_{z\in C},$$

with members belonging to the payoff universe $\mathbb{R}^2$;
– a family of Nash zones

$$(\mathcal{N}(G_z))_{z\in C};$$

with members contained in the strategy space $E \times F$;
– a family of conservative bi-values

$$v^{\#} = (v_z^{\#})_{z\in C};$$

in the payoff universe $\mathbb{R}^2$.

And so on, for every meaningful known feature of a normal form game.

Moreover, we can interpret any of the above families as *set-valued paths* in the strategy space $E \times F$ or in the payoff universe $\mathbb{R}^2$.

It is just the study of these induced families which becomes of great interest in the examination of a coopetitive game G and which will enable us to define (or suggest) the various possible solutions of a coopetitive game.

4. Coopetitive games for Greek crisis

Our first hypothesis is that Germany must stimulate the domestic demand and to re-balance its trade surplus in favor of Greece. The second hypothesis is that Greece, a country with a declining competitiveness of its products and a small export share, aims at growth by undertaking innovative investments and by increasing its exports primarily towards Germany and also towards the other euro countries.

The coopetitive model that we propose hereunder must be interpreted as normative model, in the sense that:

– *it imposes some clear a priori conditions to be respected, by binding contracts, in order to enlarge the possible outcomes of both countries;*
– *consequently, it shows appropriate win-win strategy solutions, chosen by considering both competitive and cooperative behaviors, simultaneously;*
– *finally, it proposes appropriate fair divisions of the win-win payoff solutions.*

The strategy spaces of the model are:

– the strategy set of Germany E, set of all possible consumptions of Germany, in our model, given in a conventional monetary unit; we shall assume that the strategies of Germany directly influence only Germany pay-off;

- the strategy set of Greece F, set of all possible investments of Greece, in our model, given in a conventional monetary unit (different from the above Germany monetary unit); we shall assume that the strategies of Greece directly influence only Greece pay-off;
- a shared strategy set C, whose elements are determined together by the two countries, when they choose their own respective strategies x and y, Germany and Greece. Every strategy z in C represents an amount - given in a third conventional monetary unit - of Greek exports imported into Germany, by respecting a binding contract.

Therefore, in the model, we assume that Germany and Greece define the set of coopetitive strategies.

5. The model

Main Strategic assumptions. We assume that:

- any real number x, belonging to the unit interval $U := [0, 1]$, can represent a consumption of Germany (given in an appropriate conventional monetary unit);
- any real number y, in the same unit interval U, can represent an investment of Greece (given in another appropriate conventional monetary unit);
- any real number z, again in U, can be the amount of Greek exports which is imported by Germany (given in conventional monetary unit).

In this model, we consider a linear affine mutual interaction between Germany and Greece, more adherent to the real state of the Euro-area.

Specifically, in opposition to the above first model:

- we consider an interaction between the two countries also *at the level of their non-cooperative strategies*;
- we assume that *Greece also should import (by contract) some German production*;
- we assume, that the German revenue, given by the exportations in Greece of the above production, *is absorbed by the Germany bank system* - in order to pay the Greece debts with the German bank system - so that this money does not appear in the payoff function of Germany (as possible gain) but only in the payoff function of Greece (as a loss).

Main Strategic assumptions. We assume that:

- any real number x, belonging to the interval $E := [0, 3]$, represents a possible consumption of Germany (given in an appropriate conventional monetary unit);
- any real number y, in the same interval $F := E$, represents a possible investment of Greece (given in another appropriate conventional monetary unit);
- any real number z, again in the interval $C = [0, 2]$, can be the amount of Greek exports which is imported by Germany (given in conventional monetary unit).

5.1. Payoff function of Germany

We assume that the payoff function of Germany f_1 is its Keynesian *gross domestic demand*:

- f_1 is equal to the private consumption function C_1 plus the gross investment function I_1 plus government spending (that we shall assume equal 2, constant in our interaction) plus export function X_1 minus the import function M_1, that is

$$f_1 = 2 + C_1 + I_1 + X_1 - M_1.$$

We assume that:

- the German private consumption function C_1 is the first projection of the strategic coopetitive space $S := E^2 \times C$, that is defined by

$$C_1(x, y, z) = x,$$

for every possible german consumption x in E, this because we assumed the private consumption of Germany to be the first strategic component of strategy profiles in S;
- the gross investment function I_1 is constant on the space S, and by translation we can suppose I_1 equal zero;
- the export function X_1 is defined by

$$X_1(x, y, z) = -y/3,$$

for every Greek possible investment y in innovative technology; so we assume that the export function X_1 is a strictly decreasing function with respect to the second argument;
- the import function M_1 is the third projection of the strategic space, namely

$$M_1(x, y, z) = z,$$

for every cooperative strategy $z \in 2U$, because we assume the import function M_1 depending only upon the cooperative strategy z of the coopetitive game G, our third strategic component of the strategy profiles in S.

Recap. We then assume as payoff function of Germany its Keynesian gross domestic demand f_1, which in our model is equal, at every triple (x, y, z) in the profile strategy set S, to the sum of the strategies x, $-z$ with the export function X_1, viewed as a reaction function to the Greece investments (so that f_1 is the difference of the first and third projection of the strategy profile space S plus the function export function X_1).

Concluding, the payoff function of Germany is the function f_1 of the set S into the real line $\mathbb{R}$, defined by

$$f_1(x, y, z) = 2 + x - y/3 - z,$$

for every triple (x, y, z) in the space S; where the reaction function X_1, defined from the space S into the real line $\mathbb{R}$ by

$$X_1(x, y, z) = -y/3,$$

for every possible investment y of Greece in the interval $3U$, is the export function of Germany mapping the level y of Greece investment into the level $X_1(x, y, z)$ of German export, corresponding to the Greece investment level y.

The function X_1 is a strictly decreasing function in the second argument, and this monotonicity is a relevant property of X_1 for our coopetitive model.

5.2. Payoff function of Greece

We assume that the payoff function of Greece f_2 is again its Keynesian gross domestic demand - private consumption C_2 plus gross investment I_2 plus government spending (assumed to be 2) plus exports X_2 minus imports M_2), so that

$$f_2 = 2 + C_2 + I_2 + X_2 - M_2.$$

We assume that:

- the function C_2 is irrelevant in our analysis, since we assume the Greek private consumptions independent from the choice of the strategic triple (x, y, z) in the space S; in other terms, we assume the function C_2 constant on the space S and by translation we can suppose C_2 itself equal zero;
- the function $I_2 : S \to \mathbb{R}$ is defined by

$$I_2(x, y, z) = y + nz,$$

  for every (x, y, z) in S (see above for the justification);
- the export function X_2 is the linear function defined by

$$X_2(x, y, z) = z + my,$$

  for every (x, y, z) in S (see above for the justification);
- the function M_2 is now relevant in our analysis, since we assume the import function, by *coopetitive contract with Germany*, dependent on the choice of the triple (x, y, z) in S, specifically, we assume the import function M_2 defined on the space S by

$$M_2(x, y, z) := -2x/3,$$

  so, Greece too now, must import some German product, with value $-2x/3$ for each possible German consumption x.

So, the payoff function of Greece is the linear function f_2 of the space S into the real line $\mathbb{R}$, defined by

$$f_2(x, y, z) = 2 - 2x/3 + (y + nz) + (z + my) = \\ = 2 - 2x/3 + (1 + m)y + (1 + n)z,$$

for every pair (x, y, z) in the strategic Cartesian space S.

We note that the function f_2 depends now significantly upon the strategies x in E, chosen by Germany, and that f_2 is again a linear function.

We shall assume the factors m and n non-negative and equal respectively (only for simplicity) to 0 and 1/2.

5.3. Payoff function of the game

We so have build up a coopetitive gain game with payoff function $f : S \to \mathbb{R}^2$, given by

$$f(x, y, z) = (2 + x - y/3 - z, 2 - 2x/3 + (1 + m)y + (1 + n)z) = \\ = (2, 2) + (x - y/3, -2x/3 + (1 + m)y) + z(-1, 1 + n),$$

for every (x, y, z) in $[0, 3]^2 \times [0, 2]$.

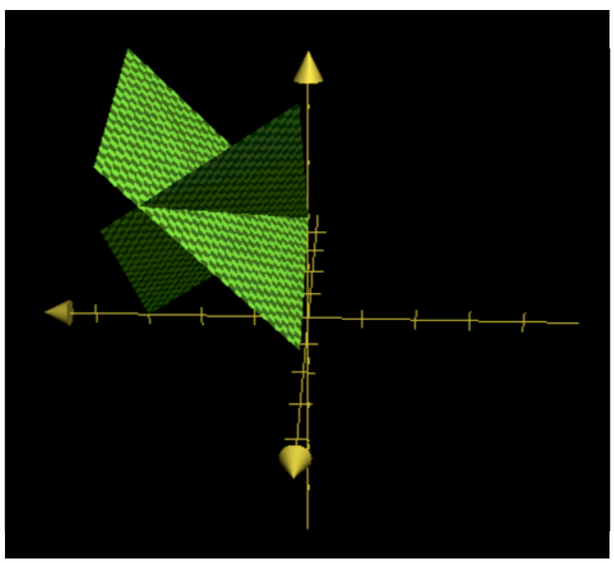

Fig. 1: 3D representation of $(f, <)$.

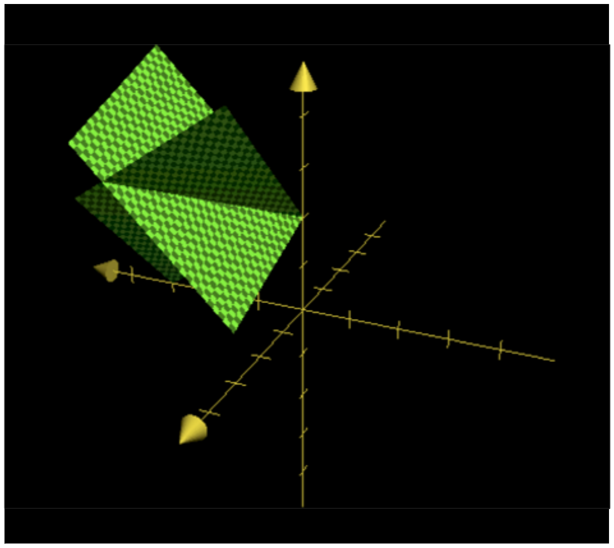

Fig. 2: 3D representation of $(f, <)$.

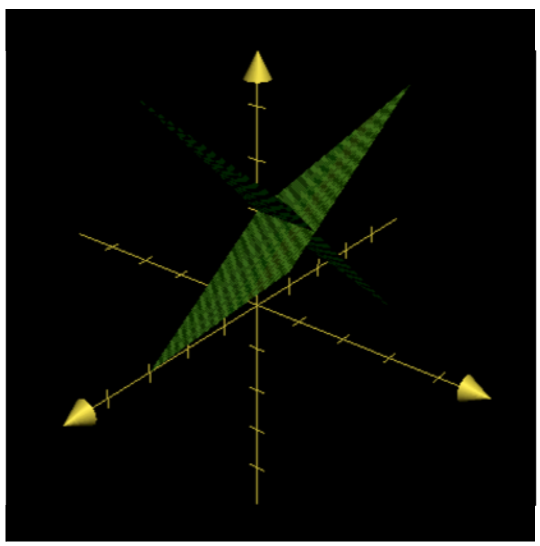

Fig. 3: 3D representation of $(f, <)$.

5.4. Study of the second game $G = (f, >)$

Note that, fixed a cooperative strategy z in $2U$, the section game $G(z) = (p(z), >)$ with payoff function $p(z)$, defined on the square E^2 by

$$p(z)(x, y) := f(x, y, z),$$

is the translation of the game $G(0)$ by the "cooperative" vector

$$v(z) = z(-1, 1 + n),$$

so that, we can study the initial game $G(0)$ and then we can translate the various informations of the game $G(0)$ by the vectors $v(z)$, to obtain the corresponding information for the game $G(z)$.

So, let us consider the initial game $G(0)$. The strategy square E^2 of $G(0)$ has vertices 0_2, $3e_1$, 3_2 and $3e_2$, where 0_2 is the origin of the plane $\mathbb{R}^2$, e_1 is the first canonical vector $(1, 0)$, 3_2 is the vectors $(3, 3)$ and e_2 is the second canonical vector.

5.5. Topological Boundary of the payoff space of G_0

In order to determine the the payoff space of the linear game it is sufficient to transform the four vertices of the strategy square (the game is an affine invertible game), the critical zone is empty.

Payoff space of the game $G(0)$. So, the payoff space of the game $G(0)$ is the transformation of the topological boundary of the strategy square, that is the parallelogram with vertices $f(0, 0)$, $f(3e_1)$, $f(3, 3)$ and $f(3e_2)$. As we show in the below figure 9.

Nash equilibria. The unique Nash equilibrium is the bistrategy $(3, 3)$. Indeed, the function f_1 is linear increasing with respect to the first argument and analogously the function f_2 is linear and increasing with respect to the second argument.

5.6. The payoff space of the coopetitive game G

The image of the payoff function f, is the union of the family of payoff spaces

$$(\mathrm{im}p_z)_{z \in C},$$

that is the convex envelope of the union of the image $p_0(E^2)$ and of its translation by the vector $v(2)$, namely the payoff space $p_2(E^2)$: the image of f is an hexagon with vertices $f(0, 0)$, $f(3e_1)$, $f(3, 3)$ and their translations by $v(2)$. As we show below.

5.7. Pareto maximal boundary of the payoff space of G

The Pareto sup-boundary of the coopetitive payoff space $f(S)$ is the union of the segments $[A', B']$, $[P', Q']$ and $[Q', C'']$, where $P' = f(3, 3, 0)$ and

$$Q' = P' + v(2).$$

Possibility of global growth. It is important to note that the absolute slopes of the segments $[A', B']$, $[P', Q']$ of the Pareto (coopetitive) boundary are strictly greater than 1. Thus the collective payoff $f_1 + f_2$ of the game is not constant on the Pareto boundary and, therefore, the game implies the possibility of a transferable utility global growth.

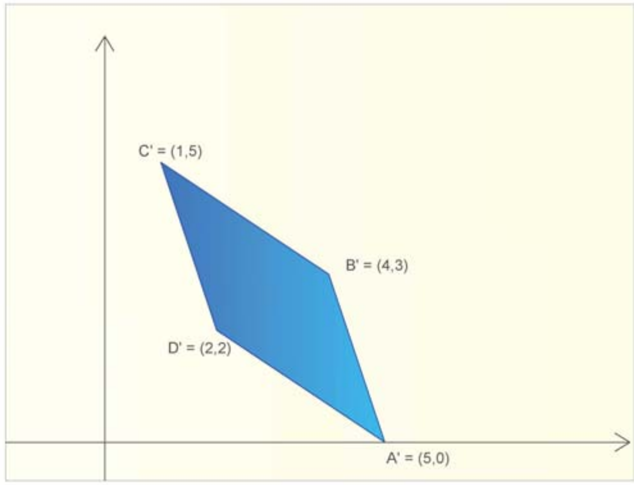

Fig. 4: Initial payoff space of the game $(f,<)$.

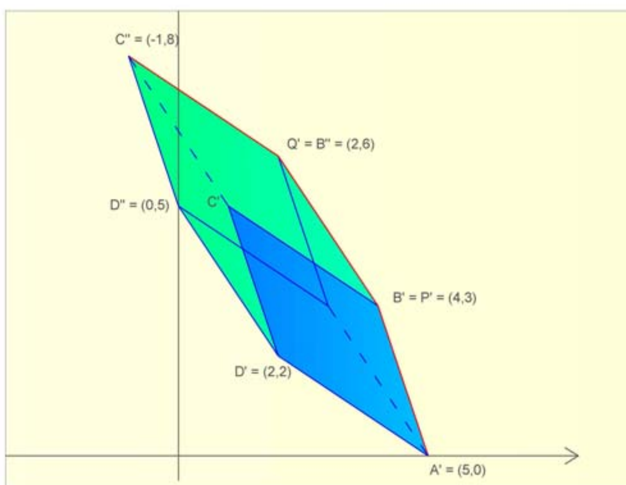

Fig. 5: Payoff space of the game $(f,<)$.

Trivial bargaining solutions. The Nash bargaining solution on the entire payoff space, with respect to the infimum of the Pareto boundary and the Kalai-Smorodinsky bargaining solution, with respect to the infimum and the supremum of the Pareto boundary, are not acceptable for Germany: they are collectively (TU) better than the Nash payoff of G_0 but they are disadvantageous for Germany (it suffers a loss!): these solutions could be thought as rebalancing solutions, but they are not realistically implementable.

5.8. Transferable utility solutions

In this coopetitive context it is more convenient to adopt a transferable utility solution, indeed:

- the point of maximum collective gain on the whole of the coopetitive payoff space is the point $Q' = (2, 6)$.

Rebalancing win-win solution relative to maximum gain for Greece in G
Thus we propose a rebalancing win-win coopetitive solution relative to maximum gain for Greece in G, as it follows (in the case $m = 0$):

1. we consider the portion s of transferable utility Pareto boundary
$$M := Q' + \mathbb{R}(1, -1),$$
obtained by intersecting M itself with the strip determined (spanned by convexifying) by the straight lines $P' + \mathbb{R}e_1$ and $C'' + \mathbb{R}e_1$, *these are the straight lines of Nash gain for Greece in the initial game* $G(0)$ *and of maximum gain for Greece in* G*, respectively.*
2. we consider the Kalai-Smorodinsky segment s' of vertices B' - Nash payoff of the game $G(0)$ - and the supremum of the segment s.
3. our best payoff rebalancing coopetitive compromise is the unique point K in the intersection of segments s and s', that is the best compromise solution of the bargaining problem $(s, (B', \sup s))$.

Figure 11 below shows the above extended Kalai-Smorodinsky solution K and the Kalai-Smorodinsky solution K' of the classic bargaining problem (M, B'). It is evident that the distribution K is a rebalancing solution in favor of Greece with respect to the classic solution K'.

Rebalancing win-win solution relative to maximum Nash gain for Greece
We propose here a more realistic rebalancing win-win coopetitive solution relative to maximum Nash gain for Greece in G, as it follows (again in the case $m = 0$):

1. we consider the portion s of transferable utility Pareto boundary
$$M := Q' + \mathbb{R}(1, -1),$$
obtained by intersecting M itself with the strip determined (spanned by convexifying) by the straight lines $P' + \mathbb{R}e_1$ and $Q' + \mathbb{R}e_1$, *these are the straight lines of Nash gain for Greece in the initial game* $G(0)$ *and of maximum Nash gain for Greece in* G*, respectively.*

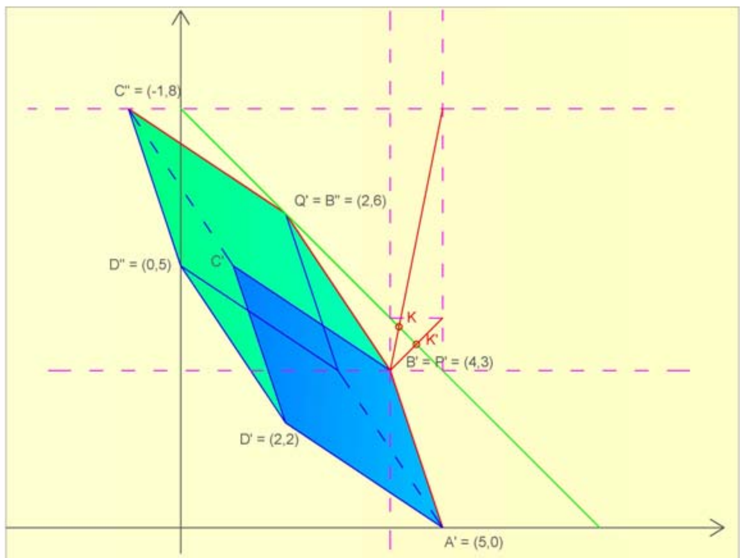

Fig. 6: Two Kalai win-win solutions of the game $(f, <)$, represented with $n = 1/2$.

2. we consider the Kalai-Smorodinsky segment s' of vertices B' - Nash payoff of the game $G(0)$ - and the supremum of the segment s.
3. our best payoff rebalancing coopetitive compromise is the unique point K in the intersection of segments s and s', that is the best compromise solution of the bargaining problem $(s, (B', \sup s))$.

Figure 12 below shows the above extended Kalai-Smorodinsky solution K and the Kalai-Smorodinsky solution K' of the classic bargaining problem (M, B'). The new distribution K is a rebalancing solution in favor of Greece, more realistic than the previous rebalancing solution.

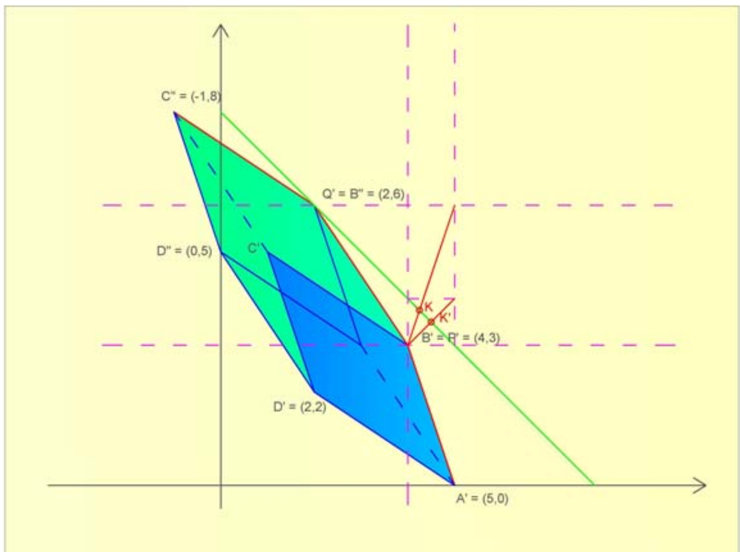

Fig. 7: Two Kalai win-win solutions of the game $(f, <)$, represented with $n = 1/2$.

5.9. Win-win solution

The payoff extended Kalai-Smorodinsky solutions K represent win-win solutions, with respect to the initial Nash gain B'. So that, as we repeatedly said, *also Germany can increase its initial profit from coopetition.*

Win-win strategy procedure. The win-win payoff K can be obtained in a **properly transferable utility coopetitive fashion**, as it follows:

- 1) the two players agree on the cooperative strategy 2 of the common set C;

- 2) the two players implement their respective Nash strategies in the game $G(2)$, so competing *à la Nash*; the unique Nash equilibrium of the game $G(2)$ is the bistrategy $(3, 3)$;
- 3) finally, they share the "social pie"

$$(f_1 + f_2)(3, 3, 2),$$

in a **transferable utility cooperative fashion** (by binding contract) according to the decomposition K.

6. Conclusions

In conclusion, we desire to stress that:

- the model of coopetitive game, provided in the present contribution, is essentially a *normative model.*
- our model of coopetition has pointed out the strategies that could bring to win-win solutions, in a **transferable utility and properly cooperative perspective**, for Greece and Germany.

In the paper, we propose:

- transferable utility and properly coopetitive solutions, which are convenient for Greece and also for Germany.
- a new extended Kalai-Smorodinsky method, appropriate to determine rebalancing partitions, for *win-win solutions*, on the transferable utility Pareto boundary of the entire coopetitive game.

The solutions offered by our coopetitive model aim at "enlarging the pie and sharing it fairly"; more specifically:

- our model is a growth model, in the sense that it suggests solutions which imply the increase of the GDP of Greece due to the actions of the variables: exports (the shared variable) and investments. It also allows to find "fair" amounts of Greek exports which Germany must cooperatively import.
- in our analytical model, the enlargement of the "pie", which is represented in figure 5 as the coopetitive payoff space $f(E \times F \times C)$, shows the set of all possible payoff shares determining reasonable (in an extended Kalai-Smorodinsky sense) win-win solutions for both Greece and Germany.

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7. Appendix: Solutions of a coopetitive game

7.1. Introduction

The two players of a coopetitive game G - according to the general economic principles of *monotonicity of preferences* and of non-satiation - should choose the cooperative strategy z in C in order that:

- the reasonable Nash equilibria of the game G_z are f-preferable than the reasonable Nash equilibria in each other game $G_{z'}$;
- the supremum of G_z is greater (in the sense of the usual order of the Cartesian plane) than the supremum of any other game $G_{z'}$;
- the Pareto maximal boundary of G_z is higher than that of any other game $G_{z'}$;
- the Nash bargaining solutions in G_z are f-preferable than those in $G_{z'}$;
- *in general, fixed a common kind of solution for any game G_z, say $S(z)$ the set of these kind of solutions for the game G_z, we can consider the problem to find all the optimal solutions (in the sense of Pareto) of the set valued path S, defined on the cooperative strategy set C. Then, we should face the problem of* ***selection of reasonable Pareto strategies*** *in the set-valued path S via proper selection methods (Nash-bargaining, Kalai-Smorodinsky and so on).*

Moreover, we shall consider the maximal Pareto boundary of the payoff space $\mathrm{im}(f)$ as an appropriate zone for the bargaining solutions.

The payoff function of a two person coopetitive game is (as in the case of normal-form game) a vector valued function with values belonging to the Cartesian plane $\mathbb{R}^2$. We note that in general the above criteria are multi-criteria and so they will generate multi-criteria optimization problems.

In this section we shall define rigorously some kind of solution, for two player coopetitive games, based on a bargaining method, namely a Kalai-Smorodinsky bargaining type. Hence, first of all, we have to precise what kind of bargaining method we are going to use.

7.2. Bargaining problems

In this paper, we shall propose and use the following original *extended* (and quite general) definition of bargaining problem and, consequently, a natural generalization of Kalai-Smorodinsky solution. In the economic literature, several examples of *extended bargaining problems* and *extended Kalai-Smorodinski solutions* are already presented. The essential root of these various extended versions of bargaining problems is the presence of *utopia points* not-directly constructed by the disagreement points and the strategy constraints. Moreover, the Kalai-type solution, of such extended bargaining problems, is always defined as a Pareto maximal point belonging to the segment joining the disagreement point with the utopia point (if any such Pareto point does exist): we shall follow the same way. In order to find suitable new win-win solutions of our realistic coopetitive economic problems, we need such new kind of versatile extensions. For what concerns the existence of our new extended Kalai solutions, for the economic problems we are facing, we remark that conditions of compactness and strict convexity will naturally hold; we remark, otherwise, that, in this paper, we are not interested in proving general or deep mathematical results, but rather to find reasonable solutions for new economic coopetitive context.

Definition (of bargaining problem). Let S be a subset of the Cartesian plane $\mathbb{R}^2$ and let a and b be two points of the plane with the following properties:

- they belong to the small interval containing S, if this interval is defined (indeed, it is well defined if and only if S is bounded and it is precisely the interval $[\inf S, \sup S]_{\leq}$);
- they are such that $a < b$;
- the intersection
$$[a, b]_{\leq} \cap \partial^* S,$$
among the interval $[a, b]_{\leq}$ with end points $a$and b (it is the set of points greater than a and less than b, **it is not** the segment $[a, b]$) and the maximal boundary of S is non-empty.

In these conditions, we call **bargaining problem on S corresponding to the pair of extreme points** (a, b), the pair

$$P = (S, (a, b)).$$

Every point in the intersection among the interval $[a, b]_{\leq}$and the Pareto maximal boundary of S is called **possible solution of the problem** P. Some time the first extreme point of a bargaining problem is called **the initial point of the problem** (or **disagreement point** or **threat point**) and the second extreme point of a bargaining problem is called **utopia point** of the problem.

In the above conditions, when S is convex, the problem P is said to be convex and for this case we can find in the literature many existence results for solutions of P enjoying prescribed properties (Kalai-Smorodinsky solutions, Nash bargaining solutions and so on ...).

Remark. Let S be a subset of the Cartesian plane $\mathbb{R}^2$ and let a and b two points of the plane belonging to the smallest interval containing S and such that $a \leq b$. Assume the Pareto maximal boundary of S be non-empty. If a and b are a lower bound and an upper bound of the maximal Pareto boundary, respectively, then the intersection

$$[a, b]_{\leq} \cap \partial^* S$$

is obviously not empty. In particular, if a and b are the extrema of S (or the extrema of the Pareto boundary $S^* = \partial^* S$) we can consider the following bargaining problem

$$P = (S, (a, b)), \ (or\ P = (S^*, (a, b)))$$

and we call this particular problem a *standard bargaining problem on* S (or *standard bargaining problem on the Pareto maximal boundary* S^*).

7.3. Kalai solution for bargaining problems

Note the following property.

Property. If $(S, (a, b))$ is a bargaining problem with $a < b$, then there is at most one point in the intersection

$$[a, b] \cap \partial^* S,$$

where $[a, b]$ is the **segment joining the two points** a and b.

Proof. Since if a point p of the segment $[a,b]$ belongs to the Pareto boundary $\partial^* S$, no other point of the segment itself can belong to Pareto boundary, since the segment is a totally ordered subset of the plane (remember that $a < b$). □

Definition (Kalai-Smorodinsky). We call **Kalai-Smorodinsky solution (or best compromise solution) of the bargaining problem** $(S,(a,b))$ the unique point of the intersection

$$[a,b] \cap \partial^* S,$$

if this intersection is non empty.

So, in the above conditions, the Kalai-Smorodinsky solution k (if it exists) enjoys the following property: there is a real r in $[0,1]$ such that

$$k = a + r(b - a),$$

or

$$k - a = r(b - a),$$

hence

$$\frac{k_2 - a_2}{k_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1},$$

if the above ratios are defined; these last equality is the *characteristic property of Kalai-Smorodinsky solutions.*

We end the subsection with the following definition.

Definition (of Pareto boundary). We call **Pareto boundary** every subset M of an ordered space which has only pairwise incomparable elements.

7.4. Nash (proper) solution of a coopetitive game

Let $N := \mathcal{N}(G)$ be the union of the Nash-zone family of a coopetitive game G, that is the union of the family $(\mathcal{N}(G_z))_{z\in C}$ of all Nash-zones of the game family $g = (g_z)_{z\in C}$ associated to the coopetitive game G. We call *Nash path of the game* G the multi-valued path

$$z \mapsto \mathcal{N}(G_z)$$

and Nash zone of G the trajectory N of the above multi-path. Let N^* be the Pareto maximal boundary of the Nash zone N. We can consider the bargaining problem

$$P_{\mathcal{N}} = (N^*, \inf(N^*), \sup(N^*)).$$

Definition. If the above bargaining problem $P_{\mathcal{N}}$ has a Kalai-Smorodinsky solution k, we say that k is the properly coopetitive solution of the coopetitive game G.

The term "properly coopetitive" is clear:

– *this solution k is determined by cooperation on the common strategy set C and to be selfish (competitive in the Nash sense) on the bi-strategy space $E \times F$.*

7.5. Bargaining solutions of a coopetitive game

It is possible, for coopetitive games, to define other kind of solutions, which are not properly coopetitive, but realistic and sometime affordable. These kind of solutions are, we can say, *super-cooperative*.

Let us show some of these kind of solutions.

Consider a coopetitive game G and

- its Pareto maximal boundary M and the corresponding pair of extrema (a_M, b_M);
- the Nash zone $\mathcal{N}(G)$ of the game in the payoff space and its extrema (a_N, b_N);
- the conservative set-value $G^\#$ (the set of all conservative values of the family g associated with the coopetitive game G) and its extrema $(a^\#, b^\#)$.

We call:

- ***Pareto compromise solution of the game*** G *the best compromise solution (K-S solution) of the problem*

$$(M, (a_M, b_M)),$$

  if this solution exists;
- ***Nash-Pareto compromise solution of the game*** G *the best compromise solution of the problem*

$$(M, (b_N, b_M))$$

  if this solution exists;
- ***conservative-Pareto compromise solution of the game*** G *the best compromise of the problem*

$$(M, (b^\#, b_M))$$

  if this solution exists.

7.6. Transferable utility solutions

Other possible compromises we suggest are the following.

Consider *the transferable utility Pareto boundary M of the coopetitive game G*, that is the set of all points p in the Euclidean plane (universe of payoffs), between the extrema of G, such that their sum

$$+(p) := p_1 + p_2$$

is equal to the maximum value of the addition $+$ of the real line $\mathbb{R}$ over the payoff space $f(E \times F \times C)$ of the game G.

Definition (TU Pareto solution). We call **transferable utility compromise solution of the coopetitive game** G the solution of any bargaining problem $(M, (a, b))$, where

- a and b are points of the smallest interval containing the payoff space of G
- b is a point strongly greater than a;
- M is the transferable utility Pareto boundary of the game G;
- the points a and b belong to different half-planes determined by M.

Note that the above fourth axiom is equivalent to require that the segment joining the points a and b intersect M.

7.7. Win-win solutions

In the applications, if the game G has a member G_0 of its family which can be considered as an "initial game" - in the sense that the pre-coopetitive situation is represented by this normal form game G_0 - the aims of our study (following the standard ideas on coopetitive interactions) are

- to "enlarge the pie";
- to obtain a win-win solution with respect to the initial situation.

So that we will choose as a threat point a in TU problem $(M, (a, b))$ the supremum of the initial game G_0.

Definition (of win-win solution). Let (G, z_0) be a **coopetitive game with an initial point**, that is a coopetitive game G with a fixed common strategy z_0 (of its common strategy set C). We call the game G_{z_0} as **the initial game of** (G, z_0). We call **win-win solution of the game** (G, z_0) any strategy profile $s = (x, y, z)$ such that the payoff of G at s is strictly greater than the supremum L of the **payoff core** of the initial game $G(z_0)$.

Remark 1. The payoff core of a normal form gain game G is the portion of the Pareto maximal boundary G^* of the game which is greater than the conservative bi-value of G.

Remark 2. From an applicative point of view, the above requirement (to be strictly greater than L) is very strong. More realistically, we can consider as win-win solutions those strategy profiles which are strictly greater than any reasonable solution of the initial game G_{z_0}.

Remark 3. Strictly speaking, a win-win solution could be not Pareto efficient: it is a situation in which the players both gain with respect to an initial condition (and this is exactly the idea we follow in the rigorous definition given above).

Remark 4. In particular, observe that, if the collective payoff function

$$^+(f) = f_1 + f_2$$

has a maximum (on the strategy profile space S) strictly greater than the collective payoff $L_1 + L_2$ at the supremum L of the payoff core of the game G_{z_0}, the portion $M(> L)$ of Transferable Utility Pareto boundary M which is greater than L is non-void and it is a segment. So that we can choose as a threat point a in our problem $(M, (a, b))$ the supremum L of the payoff core of the initial game G_0 *to obtain some compromise solution.*

Standard win-win solution. A natural choice for the utopia point b is the supremum of the portion $M_{\geq a}$ of the transferable utility Pareto boundary M which is upon (greater than) this point a:

$$M_{\geq a} = \{m \in M : m \geq a\}.$$

Non standard win-win solution. Another kind of solution can be obtained by choosing b as the supremum of the portion of M that is bounded between the minimum and maximum value of that player i that gains more in the coopetitive interaction, in the sense that

$$\max(\mathrm{pr}_i(\mathrm{im}f)) - \max(\mathrm{pr}_i(\mathrm{im}f_0)) > \max(\mathrm{pr}_{3-i}(\mathrm{im}f)) - \max(\mathrm{pr}_{3-i}(\mathrm{im}f_0)).$$

Final general remark In the development of a coopetitive game, we consider:

- a first virtual phase, in which the two players make a binding agreement on what cooperative strategy z should be selected from the cooperative set C, in order to respect their own rationality.
- then, a second virtual phase, in which the two players choose their strategies forming the profile (x, y) to implement in the game $G(z)$.

Now, in the second phase of our coopetitive game G we consider the following 4 possibilities:

1. the two players are non-cooperative in the second phase and they do or do not exchange info, but the players choose (in any case) Nash equilibrium strategies for the game $G(z)$; in this case, for some rationality reason, the two players have devised that the chosen equilibrium is the better equilibrium choice in the entire game G; we have here only one binding agreement in the entire development of the game;
2. the two players are cooperative also in the second phase and they make a binding agreement in order to choose a Pareto payoff on the coopetitive Pareto boundary; in this case we need two binding agreements in the entire development of the game;
3. the two players are cooperative also in the second phase and they make two binding agreements, in order to reach the Pareto payoff (on the coopetitive Pareto boundary) with maximum collective gain (first agreement) and to share the collective gain according to a certain subdivision (second agreement); in this case we need three binding agreements in the entire development of the game;
4. the two players are non-cooperative in the second phase (and they do or do not exchange information), the player choose (in any case) Nash equilibrium strategies; the two players have devised that the chosen equilibrium is the equilibrium with maximum collective gain and they make only one binding agreement to share the collective gain according to a certain subdivision; in this case we need two binding agreements in the entire development of the game.

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Dynamic Models of Corruption in Hierarchical Control Systems*

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Abstract Dynamic game theoretic models of corruption in two- and three-level control systems as well as optimal control problems and their applications to the optimal exploitation of bioresources and water quality control are considered. Several model examples are investigated analytically.

Keywords: corruption, hierarchical control systems, dynamic Stackelberg games, optimal control.

1. Introduction

Corruption is a social-economic phenomenon that exerts an essential negative influence to the society. The number of papers on dynamic models of corruption in hierarchical control systems is quite small. Basically they are multistage game theoretic models in which the dynamics of controlled system is not described explicitly. In one of the first papers of this class (Basu et al., 1992) a recursive setting is considered in which it is supposed that if the controller makes the collusion with the controlled person then he could be caught and should pay a bribe himself. The chain of corruption could be both finite and infinite. The authors of (Basu et al., 1992) have shown that in some conditions an increase of the probability of punishment has a greater effect in the fight with corruption than the penalty enlargement. In the paper (Olsen and Torsvik, 1998) a two-stage model of the type principal - controller - agent is considered in which the principal uses an illegal character of transactions between the controller and the agent and can get a greater payoff in a long-term period than in a short-term one. In the paper (Yang, 2005) a structural analysis of corruption in Chinese enterprises licensing is presented on the base of a repeated bargaining model. It demonstrates that once relative bargaining powers are correctly accounted then certain institutional features of the Chinese licensing system lead to bribery as a robust outcome. Exercises in comparative statics reveal that certain conventional anti-corruption measures may have counterintuitive effects. If overlapping jurisdictions are introduced, the resulting bureaucratic competition could help to fight with corruption. The model of "petty corruption" in (Lambert-Mogiliansky et al., 2007) describes the structure of bureaucratic "tracks", and the information among the participants. Entrepreneurs apply, in sequence, to a "track" of two or more bureaucrats in a prescribed order for approval of their projects. The first result establishes that in a one-shot situation no project ever

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gets approved, so a repeated interaction setting is used. In that context the trigger-strategy equilibria are characterized that minimize the social loss due to the system of bribes, and those that maximize the expected total bribe income of the bureaucrats. The results are used to shed some light on two much advocated anti-corruption policies: the single window policy and rotation of bureaucrats. In the paper (Bhattacharya and Hodler, 2010) it is studied how natural resources can feed corruption and how this effect depends on the quality of the democratic institutions. The game-theoretic model predicts that resource rents lead to an increase in corruption if the quality of the democratic institutions is relatively poor, but not otherwise. The paper (Balafoutas, 2011) investigates the role of guilt aversion for corruption in public administration. Corruption is modeled as the outcome of a psychological game played between a bureaucrat, a lobby, and the public. It is studied how the behavior of the lobby and the bureaucrat depend on perceived public beliefs, when these are constant and when they are allowed to vary over time. The papers (Blackburn and Forgues-Puccio, 2010, Cerqueti and Coppier, 2011) should be noticed separately in which the models of economic growth with corruption are investigated.

This paper logically continues the paper (Antonenko et al., 2012) where the static models of corruption were considered, and is concerned with the dynamic models in this domain. In the frame of our conception the description of corruption and the methods of struggle with it in dynamics are based on the following principles.

1. The basic modeling pattern is a hierarchical structure "principal - supervisor - agent - object" in different modifications and its investigation by means of the optimal control theory and dynamic Stackelberg games theory. The supervisor may be corrupted, while the principal is not corrupted and fights with corruption. So the elements of the above structure are bribe-fighter, bribe-taker, bribe-giver and object of impact respectively. In some cases a simplified model "bribe-taker - bribe-giver - object" (supervisor - agent - object) is used in which the non-corrupted principal is considered implicitly (parametrically).
2. The leading player of any level (principal or supervisor) uses methods of compulsion (administrative or legislative impacts) or impulsion (economic impacts) for achievement of his/her objectives. The mathematic formalization of compulsion means an impact of the leader to the set of admissible strategies of the follower (as a rule, without a feed back), and impulsion means the impact to the follower's payoff function (as a rule, with a feed back) (Ougolnitsky, 2011).
3. The cases of administrative corruption when administrative requirements or constraints are weakened for a bribe and the economic corruption when the economic ones are weakened are differentiated. The model of administrative corruption is the compulsion with a feed back on bribe, and the model of economic corruption is the impulsion with an additional feed back on bribe.
4. Corruption exists in the form of capture and extortion. In the case of capture a basic set of administrative or economic services is guaranteed while additional indulgences are provided for a bribe. In the case of extortion a bribe is required already for the basic set of services.
5. For studying corruption in hierarchical control systems with consideration of the requirements of sustainable development both descriptive and normative approaches are applicable. In the case of descriptive approach the functions of administrative

and economic corruption are given, and the main problem is to identify their parameters on statistical data. In the case of normative approach the corruption (bribery) function is found as the solution of an optimization or game theoretic problem.
6. The investigation of corruption in the system "principal - supervisor - agent - object" is possible from three points of view. If the bribery function is known then from the point of view of the agent the corruption can be described by an optimal control model. From the supervisor's point of view a hierarchical parametrical Germeyer game of the class Γ_{2t} arises which solution in the form of bribery function with a feedback on the value of bribe is known in a general form (Gorelik et al., 1991). From the point of view of the principal the problem of fight with corruption consists in seeking of such values of control parameters that the found optimal strategy of the supervisor satisfies the requirements of homeostasis for the controlled dynamic system (object).
7. It makes sense to build "genetic" series of sequentially complicated models that more and more precisely describe the real phenomena of corruption in hierarchical control systems. The principal logical pattern of this sequential complication has a form "optimal control models - dynamic hierarchical two-person games - dynamic hierarchical three-person games". With consideration of the possible modifications of the models of each type the "series" become the "genetic networks".

It is the last principle that determines the structure of the paper.

2. The system of dynamic models of corruption

Let's begin from the "genetic series" of the models of economic corruption. For the convenience of interpretation and without loss of generality we will speak about the problems of optimal exploitation of bioresources. In the case of harvesting strategy the equation of dynamics of the controlled system with initial conditions has a form

$$\dot{x} = h(x(t)) - u(t)x(t), \quad x(0) = x_0 \tag{1}$$

Here $x(t)$ - a biomass of the harvested population (for example, fish); $u(t)$ - a catch share (in the moment t); h - a function of local homogeneous population dynamics (Malthus, Verhulst-Pearl, Ricker and other types); x_0 - an initial value of the biomass.

For description of the economic corruption we suppose that the catch share is constant: $u(t) \equiv u$. Then the total fishing income is equal to aux where a is a price of the unit of biomass (the dependence on time is omitted for simplicity). Suppose that a share r of that variable (where r can be treated as a tax rate) goes to the principal (the state), and the share $1-r$ goes to the agent (a fisher). In turn, the principal gives a share p from his part as a salary (or bonus) to the supervisor (an official of the fishing control agency), and the share $1-p$ keeps to himself. At last, the agent assigns a share b of his income to the bribe for the supervisor, retaining the share $1-b$. Thus, the final income values are equal to $ar(1-p)ux$ for the principal, $a(pr + b(1-r))ux$ for the supervisor, and $a(1-b)(1-r)ux$ for the agent. Using these balance considerations, the agent's payoff functional is given in the form

$$J_A = au \int_0^{\infty} e^{-\alpha t}(1-b(t))(1-r(t))x(t)dt \to max \tag{2}$$

In the case of economic corruption the tax rate is a function of the bribe. From the point of view of the agent this function is considered as given. For example, the linear capture function has the form

$$r(b(t)) = r_0 - r_0 b(t) \tag{3}$$

Here in the left side a real tax rate is situated, the first term in the right side r_0 is its official (normative) value, and the second term means its weakening in exchange to the bribe. In particular, $r(0) = r_0,\ r(1) = 0$.

Substituting (3) in (2) we get

$$J_A = au \int_0^\infty e^{-\alpha t} \left(1 - r_0 - (1 - 2r_0)b(t) - r_0 b^2(t)\right) x(t)dt \to max \tag{4}$$

where α is a discount factor. Adding the restrictions on control values

$$0 \le b(t) \le 1 \tag{5}$$

we get an optimal control problem (1), (4)-(5). Certainly, other functions of economic corruption can be used instead of (3).

From the point of view of the supervisor the function $\tilde{r}(t) = r(b(t))$ is sought as a solution of the game Γ_{2t} betweenthe supervisor and the agent (Gorelik et al., 1991). The supervisor's payoff functional can be written in the form

$$J_S = \int_0^\infty e^{-\alpha t}((pr(t) + b(t)(1 - r(t)))aux(t) - K\mu(r_0 - r(t)))dt \to max \tag{6}$$

with restrictions on controls

$$0 \le r(t) \le r_0 \tag{7}$$

To avoid the consideration of projection the function $r(t)$ is treated as a result of the map $r(b(t))$. The second term in the integrand function in (6) represents a penalty charged to the supervisor in the case of detection of the tax arrears, where K is the penalty factor, μ - probability of detection. In general, the relations (1)-(2), (5)-(7) define a hierarchical differential two-person game of the class Γ_{2t}.

At last, the principal's payoff functional can be written as

$$J_P = \int_0^\infty e^{-\alpha t}((1 - p(t))aux(t)r(t) - M(r_0 - r(t)) - \frac{m}{r_0 - z(t)})dt \to max \tag{8}$$

It is supposed that in general the principal can use two methods of control: impulsion or compulsion with the control variables respectively

$$0 \le p(t) \le 1 \tag{9}$$

$$0 \le z(t) \le r_0 \tag{10}$$

In the second case the restrictions (7) take the form

$$z(t) \le r(t) \le r_0 \tag{11}$$

i.e. the principal can restrict from below the weakening of the tax rate for a bribe. The first term in (8) represents the principal's income, the second one means a penalty charged on him in the case of tax arrears (M is a penalty factor), and the third one are the principal's expenditures for control on the supervisor's actions (m is the respective factor).

In general, the relations (1)-(2), (5)-(10) (where instead of (7) may be (11) in the case of compulsion) define a differential hierarchical three-person game. Its reglament depends on the methods of control used by the principal. If compulsion is used then the game between the principal and the supervisor has a form Γ_{1t}, if impulsion then Γ_{1t} or Γ_{2t} depending on that whether the function $p(t)$ depends only on time or also on the supervisor's control. The game between the supervisor and the agent always has a form Γ_{2t} (Gorelik et al., 1991) because the presence of a feedback on bribe is the principal moment in the description of corruption.

Thus, on the example of bioresource exploitation a series of dynamic models of economic corruption in the form "optimal control model - dynamic hierarchical two-person game - dynamic hierarchical three-person game" is defined.

Let's define a similar series for the case of administrative corruption. Now the catch share is a function of time, and the agent chooses two control functions, namely catch share and bribe. If the linear function of administrative corruption is given as

$$s(b(t)) = s_0 + (1 - s_0)b(t) \tag{12}$$

where $s(b(t))$ is a real value of the catch quota, s_0 is an official value of the quota, $(1 - s_0)b(t)$ is its weakening in exchange for the bribe then the agent's optimal control problem can be written in the form

$$J_A = a \int_0^{\infty} e^{-\alpha t}(1 - b(t))u(t)x(t)dt \to max \tag{13}$$

$$0 \leq u(t) \leq s_0 + (1 - s_0)b(t) \tag{14}$$

$$0 \leq b(t) \leq 1 \tag{15}$$

where the equation of dynamics of the controlled system with initial conditions has also the form (1). The payoff functional of the supervisor has the form

$$J_S = \int_0^{\infty} e^{-\alpha t}(ab(t)u(t)x(t) - K\mu(s(t) - s_0))dt \to max \tag{16}$$

with restrictions on controls

$$s_0 \leq s(t) \leq 1 \tag{17}$$

Similar to (6), the second term in the integrand in (16) is a penalty charged on the supervisor in the case of detection of excess of the official catch quota. The relations (1), (13)-(17) define a differential hierarchical two-person game of the class Γ_{2t}.

At last, the principal's payoff functional has the following form (similar to (8) but without consideration of the principal's personal interests)

$$J_P = \int_0^{\infty} e^{-\alpha t}(M(s(t) - s_0) + m\frac{1 - q(t)}{q(t) - s_0})dt \to min \tag{18}$$

with restrictions on controls

$$s_0 \leq q(t) \leq 1 \tag{19}$$

If (17) is replaced by the restrictions

$$s_0 \leq s(t) \leq q(t) \tag{20}$$

which determine the compulsion by principal then the relations (1), (16), (18)-(20) define a differential hierarchical three-person game. The reglament of the game between the principal and the supervisor is Γ_{1t} , and between the supervisor and the agent Γ_{2t}. In the rest of the paper some dynamic models of economic corruption are investigated.

3. Models of optimal exploitation of bioresources considering the economic corruption

Let's consider a model of economic corruption in the form

$$J = \int_0^T e^{-\alpha t}(1 - r(b(t)) - b(t))f(u(t), x(t))dt \to max$$

$$0 \leq u(t) \leq 1 \ ; \quad 0 \leq b(t) \leq 1$$

$$\dot{x} = (1 - u(t))h(x(t)), \quad x(0) = x_0$$

where the variables have the same sense as earlier.

The functional describes profit of a fishing enterprise which may be a bribe-giver. From the profit the enterprise pays taxes and (perhaps) gives a bribe to an official of the fishing control agency. The real tax rate is a decreasing function of the bribe. If the bribe is equal to zero then the real tax rate coincides with the normative (established by the law) one. Only the case of capture is considered.

The function of economic corruption $r(b(t))$ is taken in the exponential form

$$r(b(t)) = r_0 e^{-kb(t)}$$

where the variables have the same sense as in (3); k is a bribe sensitivity. Suppose also that

$$f(u(t), x(t)) = \sqrt{u(t)x(t)}$$

Without loss of generality the price of fish biomass unit is supposed to be equal to one. Assume also for simplicity that $\alpha = 0$ (there is no discounting). The model has two control functions: the biomass of caught fish (in shares) and the bribe (also in shares). The Verhulst-Pearl model represents a natural dynamics of the fish population:

$$h(x(t)) = ax(t)(K - x(t))$$

where a is a natural increase factor, K is an environmental capacity. Thus, the following model is considered:

$$J = \int_0^T (1 - r_0 e^{-kb(t)} - b(t))\sqrt{u(t)x(t)}dt \to max$$

$$0 \leq u(t) \leq 1 \ ; \quad 0 \leq b(t) \leq 1; \ k > 0; \ t \in [0, T]$$

$$\dot{x} = (1 - u(t))ax(t)(K - x(t)), \quad x(0) = x_0$$

The optimal control problem is solved by the Pontryagin maximum principle (Grass et al., 2008). The Hamilton function has the form (the argument t is omitted):

$$H(x,u,b,\psi) = (1 - r_0 e^{-kb} - b)\sqrt{ux} + \psi(1-u)ax(K-x)$$

The conditions of maximum principle are supposed to be satisfied. Consider the first one. The derivative of H with respect to b is

$$\frac{\partial H}{\partial b} = \sqrt{ux}(r_0 k e^{-kb} - 1)$$

Equating the right side of the relation to zero and solving it with respect to b we get

$$b^* = \frac{ln(r_0 k)}{k} \tag{21}$$

Now calculate the derivative of H with respect to u:

$$\frac{\partial H}{\partial u} = \frac{(1 - r_0 e^{-kb} - b)\sqrt{x}}{2\sqrt{u}} - \psi ax(K-x)$$

Equating the derivative to zero and substituting (21) in the received expression we get:

$$u^* = \frac{(k - 1 - ln(r_0 k))^2}{4k^2\psi^2 a^2 x(K-x)^2} \tag{22}$$

Calculate the second derivatives of the Hamilton function with respect to the control variables considering (21):

$$\frac{\partial^2 H(x,u^*,b^*,\psi)}{\partial u^2} = \frac{2k^2\psi^3 a^3 x^2 (K-x)^3}{(k-1-ln(r_0 k))^2} \tag{23}$$

$$\frac{\partial^2 H(x,u^*,b^*,\psi)}{\partial b^2} = \frac{k - 1 - ln(r_0 k)}{2\psi a(K-x)} \tag{24}$$

Calculate also the mixed derivatives; due to (21) and (22) we get:

$$\frac{\partial^2 H}{\partial u \partial b} = \frac{\sqrt{x}(r_0 k/(r_0 k) - 1)}{2\sqrt{u}} \equiv 0 \tag{25}$$

The Hesse matrix has the form

$$\begin{pmatrix} -\frac{k-1-ln(r_0 k)}{2\psi a(K-x)} & 0 \\ 0 & -\frac{2k^2\psi^3 a^3 x^2 (K-x)^3}{(k-1-ln(r_0 k))^2} \end{pmatrix}$$

and it is negative definite. Thus, the sufficient condition of the maximum of Hamilton function with respect to the control variables is satisfied.

Let's analyze the expression (21). Consider the limit case $b^* = 0$ (the bribe is not profitable for the bribe-giver). It is possible when $k = 1/r_0$. In the model is supposed that the value r_0 (tax rate) is determined by the law, and the value k is determined by a fishing control agency official (bribe-taker). Given r_0 the bribe-giver chooses k such that the optimal for him value b^* takes the maximal value. It is found that the optimal for the bribe-taker value of k is defined by the formula

$k = e/r_0$. For example, if $r_0 = 0.2$ then $k \approx 13,59$. The substitution of k to (21) gives $b^* = r_0/e$. As r_0 is always positive then b^* is never equal to zero but b^* diminishes if r_0 diminishes. So, in the frame of the model the corruption cannot be eradicated completely but it can be restricted by decreasing of the tax rate.

Let's compose the expression for the conjugate variable $\psi(t)$:

$$\frac{d\psi(t)}{dt} = -\frac{(k-1-ln(r_0k))^2}{4k^2\psi a(K-x)^2} - \psi(t)a(K-2x(t)) \tag{26}$$

and define a boundary condition for (26):

$$\psi(T) = 0$$

Let's substitute (22) into the equation for the phase coordinate and transform it:

$$\dot{x}(t) = ax(t)(K-x(t)) - \frac{(k-1-ln(r_0k))^2}{4k^2\psi^2 a(K-x(t))} \tag{27}$$

The equations (26) and (27) form the system

$$\begin{cases} \dot{x}(t) = ax(t)(K-x(t)) - \frac{(k-1-ln(r_0k))^2}{4k^2\psi^2 a(K-x)}; \\ \dot{\psi} = -\frac{(k-1-ln(r_0k))^2}{4k^2\psi a(K-x)^2} - \psi(t)a(K-2x(t)) \end{cases} \tag{28}$$

with boundary conditions $\begin{cases} x(0) = x_0; \\ \psi(T) = 0 \end{cases}$

Let's solve the system by the explicit-implicit Euler method. To receive a numerical result let's give some specific values to the model parameters. Assume that $T = 3$; $r_0 = 0,2$; $x_{-3} = 8000$; $x_{-2} = 8800$; $x_{-1} = 9600$; $a = 1,14\ 10^{-5}$; $K = 104800$; $x_0 = 10385,45$; $k = 13,59$; $b = 0,07$. A common sorting method can be used for the determination of the initial value of ψ. It is appropriate because the explicit-implicit Euler method in solving (28) is stable and the value of grid function tends to zero when the number of steps grows. In this problem $\psi(0) = 0,0141$.

The function of natural dynamics of the population has the form

$$\tilde{x}(t) = \frac{x_0 K}{x_0 + (K-x_0)e^{-aKt}}$$

To receive the function $x(t)$ as an analytical expression a regression analysis is required because the values in the grid nodes are approximate. By means of the least squares method it is found that the function $x(t)$ is well approximated by the third degree polynomial

$$x(t) = -7961,28t^3 + 26486,57t^2 - 7845,41t + 13484,84$$

with determination factor $R^2 = 0,97$.

The equation (26) can be solved analytically, namely

$$\psi(t) = \sqrt{C_\psi e^{-2a(K-2x(t))t} - \frac{(k-1-\ln r_0k)^2}{4k^2a^2(K-x(t))^2(K-2x(t))}},$$

where

$$C_\psi = \frac{(k-1-ln(r_0k))^2 e^{2a(K-2x(T))T}}{4k^2a^2(K-x(t))^2(K-2x(t))}$$

The fishing enterprise profit for three years is equal to 135 conditional units.

4. Game theoretic models of economic corruption in the water quality control systems

Let's investigate fight with corruption in a dynamic three-level water resource quality control system which includes the following control levels: top (federal agency or principal - he), middle (regional agency official or supervisor - she) and bottom (enterprise or agent - he), and the controlled dynamic system (water stream or CDS).

The agent tends to maximize his production profit. In the process of production some pollutants are thrown to the CDS. It is assumed that the supervisor can change in a range the normative pollution fee tending to maximize her income. The principal must ensure a stable state of the CDS (the stability is treated in the sense of Lagrange). The principal determines which part of the fees received from the agent goes to the supervisor and which one he keeps to himself respectively. The interests of principal and supervisor are different, and the supervisor may be interested to receive bribes from the agent and to decrease the pollution fee in exchange. For the supervisor bribes are considered as a factor, together with income from the fees, in the general balance of her interests. The principal should create such conditions in which to ensure the stable state of the CDS will be profitable for the supervisor even when the corruption exists.

The principal controls the supervisor by charging penalties for the bribes. The penalty size depends on the probability of detection of the corruption and on the deviation of the real pollution fee from the normative one. The principal's control costs are considered in the model. We speak about an economic corruption because impulsion is used as the method of control (Ougolnitsky, 2011).

A case of one type of pollutants (for example, nitric) and one agent is analyzed. It is supposed that the system is in the stable state if quality standards for river water

$$0 \le B(t) \le B_{max}; \quad 0 \le t \le \varDelta \tag{29}$$

and sewage

$$\frac{W(t)(1-P(t))}{Q^0(t)} \le Q_{max}; \quad 0 \le t \le \varDelta \tag{30}$$

are satisfied, where t - time; $B(t)$ - concentration of the pollutant in the river water in the moment t ; $Q^0(t)$ - amount of sewage; $W(t)$ - number of pollutant thrown to the river before refinement; $P(t)$ - share of the pollutant eliminated from the sewage due to the refinement; $\varDelta$ - length of time period; values $B_{max},\ Q_{max}$ are given.

For description of the pollution dynamics an ordinary differential equation in the form

$$\frac{dB}{dt} = F(B(t), P(t), t) \tag{31}$$

is used where $F(B(t), P(t), t)$ is a given function.

Besides ensuring the stability of the system the principal tends to maximize his personal payoff functional

$$J_0(K(t), H(t), T(t), P(t), b(t)) = \int_0^{\varDelta} (-C_{\varPhi}(y(t)) + y(t)F(T(t))H(t)+ \tag{32}$$

$$+y(t)K(t)\mu(T_0 - T(t))b(t) - \frac{MK(t)}{T_0 - T(t)})dt \to max;\ y(t) = (1-P(t))W(t),$$

where $y(t)$ is an amount of the pollutant thrown by the agent in the river after refinement; $C_{\Phi}(y(t))$ - the principal's water quality improvement cost function ($y(t)$); $F(T(t))$ - cost of the unit of thrown pollutant; $T(t) = T(b(t))$ - a real per unit pollution fee depending on the bribe ; T_0 - the normative per unit pollution fee; $H(t)$ - a share of the fee that goes to the supervisor; μ - a given probability of the bribery detection ($0 \leq \mu \leq 1$); $K = K(t)$- a penalty function; $M = const$ - a factor of the principal's bribery control cost.

The term $C_{\Phi}(y)$ in (32) represents the principal's water refinement cost; $y(t)H(t)F(T(t))$ - an amount of the pollution fee; $y(t)K(t)\mu(T_0 - T(t))b(t)$ - an amount of penalty charged on the supervisor in the case of corruption; $MK(t)/(T_0 - T(t))$ - the principal's control cost. A maximum of the functional (32) is sought with respect to two functions $K(t)$ and $H(t)$.

The supervisor's payoff functional has the form

$$J_1(K(t), H(t), T(t), P(t), b(t)) = \int_0^{\Delta} (-C_0(y(t)) + y(t)F(T(t))(1 - H(t)) - \quad (33)$$

$$y(t)K(t)\mu(T_0 - T(t))b(t) + b(t)y(t))dt \to max$$

In (33) the term $C_0(y)$ represents the supervisor's water quality improvement cost function; $y(t)(1-H(t))F(T(t))$ - a pollution fee paid by the agent to the supervisor; $y(t)K(t)\mu(T_0 - T(t))b(t)$ - the supervisor's penalty in the case of bribery detection; $y(t) = b(t)$ - the bribe received by the supervisor if the amount of pollutant is equal to $y(t)$.

The agent's objective is to maximize his profit in the presence of corruption:

$$J_2(T(t), P(t), b(t)) = \int_0^{\Delta} (zR(\Phi(t)) - C_P(P(t))W(t) - y(t)F(T(t) - \quad (34)$$

$$b(t)y(t))dt \to max$$

Here $C_P(P)$ - the agent's per unit cost of sewage refinement; $\Phi(t)$ - production funds; $R(\Phi(t))$ - the agent's production function; $z = const$ - price of the production unit. A maximum of (34) is sought with respect to two functions: $P(t)$ and $b(t)$.

The term $zR(\Phi)$ represents the agent's profit from sale of $R(\Phi)$ - units of production; $y(t)F(T(t))$ - the pollution fee; $C_P(P(t))W(t)$ - the agent's cost of sewage refinement; $y(t)b(t)$ - the bribe value.

The problems (32) - (34) are solved with the following restrictions on the control values:
- principal

$$0 \leq H(t) \leq 1; \quad 0 \leq K(t) \leq 1; \quad 0 \leq t \leq \Delta; \quad (35)$$

- agent

$$0 \leq P(t) \leq 1 - \varepsilon; \quad 0 \leq b(t) \leq b_{max}; \quad 0 \leq t \leq \Delta; \quad (36)$$

- supervisor

$$0 \leq T(t) \leq T_0; \quad 0 \leq t \leq \Delta, \quad (37)$$

(37) where ε is determined by the agent's technological capacity; $b_{max} = const$ is the maximal feasible bribe value per unit of the pollutant.

The dynamics of production funds of the agent is described by an ordinary differential equation in the form

$$\frac{d\Phi}{dt} = -\lambda\Phi + Y; \quad \Phi(0) = \Phi_0 \ , \tag{38}$$

(38) where λ is a depreciation factor; Y - constant production investments; a constant Φ_0 is given.

Suppose that the production functions are given in the form

$$W(t) = \beta R(\Phi(t)); \quad R(\Phi(t)) = \gamma\Phi^{0.5}(t); \quad \gamma, \beta = const \tag{39}$$

The algorithm of solution of the problem (29) - (39) consists in the following steps:
1) a Germeyer game of the type Γ_{2t} (Gorelik et al., 1991) for the supervisor and the agent is considered. The value L_{2t} of guaranteed payoff of the agent if he doesn't collaborate with the supervisor is determined:

$$L_{2t} = \sup_{P,b} \inf_{T} \ J_2(T(t), P(t), b(t))$$

The supervisor's strategy which minimizes the agent's payoff functional is called her punishment strategy and denoted $T^k(t)$;
2) the optimal control problem (33), (36), (37) with an additional condition

$$L_{2t} < J_2(T(t), P(t), b(t)) \tag{40}$$

is solved. A maximum of (33) is sought with respect to three functions: $T(t), \ P(t)$ and $b(t)$. The optimal strategies depend on the principal's strategies and have the form

$$\begin{gathered} P^S(t) = P^S(K(t), H(t), t); \quad b^S(t) = b^S(K(t), H(t), t); \\ T^S(t) = T^S(K(t), H(t), t); \end{gathered} \tag{41}$$

where'$T^S(t)$ is the supervisor's reward strategy. Thus

$$T^*(K(t), H(t), t) =$$

$$\begin{cases} T^K(t) \ \ if \ \exists t_0 : \ 0 \le t_0 \le \Delta; \ P(t_0) \ne P^S(K(t_0), H(t_0), t_0) \\ \qquad or \ b(t_0) \ne b^S(K(t_0), H(t_0), t_0); \\ T^S(K(t), H(t), t) \ \ if \ \ \forall t : \ 0 \le t \le \Delta; \ P(t) = P^S(K(t), H(t), t) \\ \qquad and \ b(t) = b^S(K(t), H(t), t) \end{cases}$$

Due to the condition (40) the reward strategy is the most profitable one for the agent;
3) the functions (41) are substituted in (30) - (32). The optimal control problem (32) with additional constraints (including the phase ones) (29) - (31), (35), (38), (39) is solved. The functions which solve the problem are denoted $K^*(t)$ and $H^*(t)$.
4) the Stackelberg equilibrium in the Germeyer games Γ_{2t} (between the supervisor and the agent) and Γ_{1t} (between the principal and the supervisor) has the form

$$\{K^*(t), \ H^*(t), \ T^S(K^*(t), \ H^*(t), t), \ P^S(K^*(t), \ H^*(t), t), \ b^S(K^*(t), \ H^*(t), t)\}$$

In the general case the optimal control problems defined on the steps 2 and 3 of the algorithm are solved numerically after their digitization in time by the direct ordered sorting method.

5. Conclusion

The general principles of building the dynamic models of the fight with corruption are proposed. The "genetic series" of dynamic models of economic and administrative corruption in the form "optimal control problem - dynamic hierarchical two-person game - dynamic hierarchical three-person game" are built. Some dynamic models of economic corruption with application to the exloitation of bioresources and water quality control problems are investigated and solved. In the latter case a Stackelberg equilibrium in the Germeyer games Γ_{1t} and Γ_{2t} is used.

It is supposed in future to analyze in more details the models of economic corruption for different classes of bribery functions as well as investigate the dynamic models of administrative corruption.

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When it Pays to Think about the Competition, and When it Doesn't: Exploring Overconfidence Bias in Dynamic Games

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Abstract Despite robust behavioral research that shows a widespread bias towards overconfidence in competitive scenarios, e.g., underestimating the competitor's skill level, there is little research on the long term costs associated with this bias. We develop a theoretical framework that allows us to explore systematic long-term ramifications of opponent skill estimation bias across different competitive contexts relevant to managers. We capture these contexts with dynamic branching games that are parametrized by four features. We use Monte Carlo estimation methods to test how the expected game outcomes compare under different types biases. The results suggest that bias in evaluating an opponent's skill level is less harmful when the opponent is more skilled, and when there is greater first-mover advantage. Furthermore, they suggest that if there is any effort cost associated with making a decision, then a bias towards overestimating the opponent's skill is never advantageous, while a bias towards underestimating can be advantageous in many contexts.

Keywords: bounded rationality, overconfidence bias, heuristics.

1. Introduction

Many firms invest heavily in competitive analysis, and proverbial advice such as "never underestimate the competition" abounds in the popular business literature. Yet, little has been done to formally test this stylized wisdom from a cost-benefit perspective. There is a prevailing intuition that being caught off guard by a competitor who is more skilled than expected has negative consequences— but what are the relative consequences of the opposite mistake? How much potential surplus would be lost by routinely assuming the competition to be more skilled than it actually is? We build a framework for exploring bounded rationality in dynamic branching games and use it to compare the relative payoff ramifications of different types of biases across different competitive contexts relevant to managers.

2. Background

Traditional game theory models have generally assumed that all players are fully rational. Recent research streams have adapted models of decision making to fit the more realistic assumptions of bounded rationality, where players act rationally within the bounds of their constraints on information processing (see Narasimhan et. al., 2005).

Two primary characteristics found in bounded rationality models are that players have different skill levels (which can lead to uncertainty about the skill levels of their opponents) and that the effort of optimizing decisions is costly. Researchers have used paradigms such as level-k and cognitive hierarchy models to incorporate heterogeneous reasoning levels into game theoretic models for some time (e.g., Camerer et al., 2004; Stahl and Wilson, 1995). These models both help explain observed non-equilibrium behavior in games and estimate empirical distributions of reasoning abilities and beliefs about others' abilities. We take a different approach: knowing that error is possible (and likely unavoidable) when boundedly rational managers form beliefs about the skill levels of the opponents they encounter, we explore, from a theoretical standpoint, the payoff ramifications associated with such errors, including relative cost of effort. How much does a manager stand to lose by repeatedly overestimating or underestimating the skill levels of the different opponents he encounters? Which features of games make one type of error better than another?

Our approach is similar to that of researchers such who have compared average outcomes produced by different strategy heuristics for individual decision makers when time or effort is costly or limited (e.g., Gabaix and Laibson, 2000; Johnson and Payne, 1985; and Payne et al., 1996). These researchers employed Monte Carlo simulation techniques to approximate the mean outcomes of different heuristic strategies when applied over a large set of normal or extensive form payoff arrangements that were generated in part using random number generation to capture the inherent variation in real-world situations. In all these studies, it was concluded that simplifying heuristics can perform better than rote optimization when the decision makers are subject to some elements of bounded rationality. We build on similar conceptual and methodological foundations, but add the complexity of a second strategic player and bias surrounding comparative skill.

3. Model Framework

3.1. Game Structure

Our first goal is to define a highly generalizable game structure that captures the critical features of strategic competitive interactions faced routinely by boundedly rational managers in a variety of settings without overfitting to a particular circumstance. As such, we look for general game characteristics rather than a unique example.

We observe that when navigating complex long-term competitive relationships, managers are often faced with decisions that have not only immediate payoff ramifications, but also affect the choices (and associated payoffs) that will be available in the future. For example, a firm may be deciding between releasing a newly developed product into the market at a low price to encourage trial, or at high price in order to obtain surplus from enthusiastic early adopters (e.g., Apple releasing the initial Ipod). There are immediate profits to be gained from the decision (in terms of profits generated from initial sales) but these different moves may also have longer-term strategic consequences, based in part on the actions taken by the firm's major competitor. For example, this competitor might be developing a competing product (e.g., Microsoft releasing the Zune), and after observing how the first firm priced its product, the competitor will decide whether to release its product at a regular or a substantially discounted price. When this happens, there will be an immediate shift in profits for both firms, and also set up the first firm for a counter-response in terms of adjusting prices or promotions or investing in new development efforts. If the first firm is myopic, then it will release its product at whatever price initially maximizes sales, without considering what the competition will do in response. If the firm is more sophisticated, it will consider possible competitive responses, its own counter-response, and so on, and make its initial decision with those downstream implications in mind. Of course, there is a limit on how far out even the most sophisticated manager can think, especially when the competitive horizon is long or complex.

We use a branching tree structure to represent such real world strategic decision situations where managers repeatedly make moves and counter moves. Thus, the players alternate making decisions over time, and each node in the tree has a payoff associated with it (which represents the value of the state of affairs at that moment in time, e.g., current market share or profits). To capture different skill levels, we let players vary in the time horizon over which they can optimize from any given moment in the game.

This leads us to assume two-player alternating-move finite-horizon games of perfect and complete (within one's foresight horizon[1]) information, where for parsimony we limit our attention to constant-sum payoffs. This structure has appeal for many reasons. Dynamic games allow foresight horizon to be used as a precise measure of skill, capturing varying degrees of myopia that managers employ within their long-term competitive landscapes. Games in which information is perfect and complete (within a player's foresight window) provide an excellent platform for studying the research question at hand, because variation in outcome is caused solely by variation in skill and beliefs. Finite horizon games apply to situations in which managers are competing over a set term (for example, sales over a holiday season, performance bonuses over a fiscal year) or when players make several moves that lead to a long-term stabilization of market shares. Many competitive marketing situations are inherently constant sum (for example, employees competing over a fixed bonus pool, firms competing for market share in an inelastic market). Even if the games are not constant sum they can be re-framed as such if the payoffs are normalized to reflect relative competitive advantage. For convenience, we restrict all payoffs to the range of 0 to 1, which corresponds nicely (but not restrictively) to market share. By convention, P1 always seeks to maximize payoffs, and P2 seeks to minimize payoffs (i.e., P2 seeks to maximize one minus the payoff).

We note that our game structure assumes state values (interim payoffs) at each non-terminal node. Though ubiquitous in computer science (e.g., see Hsu, 2002 and Russell and Norvig, 2003), which as a field is inherently concerned with bounded rationality, this notion of pre-terminal state values is not widely used in the economics and management game literature. Instead, the games studied previously are generally defined in term of the final (terminal) payoffs, a paradigm appropriate for managers with unconstrained reasoning abilities. We break from this literature since we believe interim payoffs capture real-world phenomenon applicable to boundedly rational managers. Intuitively, if a manager's forsight constraint precludes him from anticipating payoffs all the way out to the end of the game, he must use something observable in the interim to guide his behavior. For example, employees competing for a year-end bonus allotment may use quarterly performance reviews as a means of

[1] i.e., players can see all payoffs that fall within their foresight horizons perfectly

evaluating standing along the way. Or, politicians campaigning for an elected office may use opinion polls as a measure of vote market share at various intervals before the election takes place. Thus, the interim payoff values we preserve in our game structure can be considered to be imperfect assessments of the strategic advantage of a state in the game (i.e., of the final outcome to which the state will lead). Of course, there are numerous ways to evaluate the worth of such interim states, and devising a sophisticated method of doing so is a skill in itself. Our correlation parameter, ρ, which will be discussed in the next section, controls the reliability of these signals in our model.

Many well-known strategy games fit this structure, including Go and Chess, which are both renowned for their strategic complexity and played in highly competitive international tournaments. Like all games of complete and perfect information, the process of solving Go or Chess is theoretically trivial—but the branching game trees are so complex that not even the best computers in the world can model the full game tree. Artificially intelligent players instead must be forward-looking to build a game tree that extends as many rounds into the future as they are capable of, and then optimize play over that limited horizon. Final scoring rules or other evaluation functions are applied to transient states of the board as a way of assessing the value of the board, even though only the configuration present when the game ends is used to tabulate final score (see Russell and Norvig, 2003).

By following in this tradition, we are able to explore player navigation in a class of games that has real-world relevence for manager but for which the curse of dimensionality precludes calculation of classically rational behavior in all but the simplest of cases. Strategic decision making under constrained foresight can be modeled in even the most complex of games, as it removes the link between game complexity and calculation complexity.

3.2. Game Parametrization

As we are interested in looking at a variety of different games that might be encountered by players in the real world, we abstract away the labels of the actions that define the circumstances of any particular game, reducing each game to a pattern of branches and payoffs (this is similar in method to the generalized decision structures used by Gabaix and Laibson, 2000; Johnson and Payne, 1985; and Payne et al., 1996) that can be populated by any number of possible payoff arrangements. Of course, real world games vary in the patterns they create: one game will have one arrangement of payoffs (based on its set of actions) and another game will have quite a different set of payoffs. Thus we create *sets* of games that include all possible games a player could encounter that fit certain useful pattern definitions. By showing the effects of bias conditions over entire game sets, we attempt to model the overall effects of such biased beliefs applied consistently over many situations with many different opponents. We allow the number of periods in the game tree to vary by length (L). For simplicity, we restrict L to even integers, so that both players have an equal number of moves. In this initial analysis, all decision nodes contain a choice between exactly two actions. However, the model can be extended in future work to allow such complexity descriptions to vary.

The payoff correlation parameter ρ is used to define the relationship between early interim payoff signals and their downstream consequences. We begin by allowing the initial period interim payoffs to be drawn randomly from some defined distribution. In our case, we use the uniform distribution $U(0, 1)$, but this can easily be modified within our framework. This initial distribution is used to capture the inherent variation of possible circumstances. The values of subsequent nodes are determined such that $v_t = \rho v_{t-1} + (1-\rho)\epsilon_t$ where v_t is the value associated with a node at period t, v_{t-1} is the value of the preceding node on the same path (parent node), ϵ_t is a draw from $U(0,1)$, and $0 \leq \rho \leq 1$. In this way, ρ captures the strength of the signal interim payoffs provide for future payoffs.

3.3. Player Parametrization

Behavioral research has shown that managers engage in a process of limited-horizon reasoning when engaging in dynamic strategic behavior (e.g., Camerer and Johnson, 2004; Johnson et al., 2002; Stahl and Wilson, 1995). Rather than performing full backward induction, players look forward a certain number of periods, and optimize only over that window, as if the game truncated there. Players vary in the number of future periods they can think though. These empirical finding drive the inclusion of such behavior in our model.

In line with extant level of thinking models, we use the parameter k_i to represent player i's skill, and specifically define k_i as *foresight horizon*: the number of periods out into the future he can think through, including the present period. Players also have beliefs about their opponent's skill level: we define parameter b_{ij} as player i's belief about k_j. A player's skill k_i and belief b_{ij} map to a player type that chooses his action at each period by considering only the subgame that starts at the current period and ends k_i periods later (even if the true game length extends beyond that horizon). In other words, at each period in which he makes a decision, player i (P_i) creates a truncated version of the game tree that starts at the current node (period t) and ends after k_i periods. From P_i's point of view, he cannot think beyond period $t+k_i-1$ and thus this player's objective is to optimize his outcome in period $t+k_i-1$ (effectively acting as if the game ends at that point, or, equivalently, that all future states that exist beyond this period have the same payoff as the their preceding state)[2]. Our motivation in this type definition is to capture the heuristic-like behavior of players under cognitive constraints.

To describe the decision rule in more detail, we break beliefs into two categories: $b_{ij} \geq k_i - 1$ (i.e., P_i believes his opponent can see out at least as far in the game as he himself can at any particular period[3]) and $b_{ij} < k_i - 1$ (P_i believes his opponent is more than one level less skilled than he himself is). When $b_{ij} \geq k_i - 1$, P_i determines his move based on full backward induction over this truncated game. This is because at each period within this truncated game, P_i believes that the player in control can see through the entire truncated game. Note that all beliefs b_{ij} that are greater than or equal to $k_i - 1$ map to the same decision algorithm for P_i. This is because a player constrained by his own foresight horizon cannot anticipate what his opponent would do outside that window, even if he believes his opponent to be looking farther (and behavioral research supports that people do behave as if the game ends at the

[2] P_i optimizes for period $t + k_i - 1$, rather than period $t + k_i$, as we have defined k_i to include the current period.

[3] The reason $k_i - 1$ rather than k_i is the critical value is due to the sequential nature of the game. If P_i is making a decision in period 1, and is a level 4 thinker, then he can "see through" to period 4, and nothing beyond that. Since his opponent P_j will not be making a move until the next period, P_j can be allotted a maximum of 3 levels from P_i's perspective (thus reaching period 4) before $P_i's$ own constraint prevents him from seeing father, even through P_j's eyes.

limit of their foresight horizon, even if they are aware that it doesn't and that other players might be thinking farther, e.g., Johnson et al., 2002). Thus, as long at as the more limited skill player knows his opponent has a greater skill level than he does (or, more precisely, not more than one level less), fine-tuning his estimate of his opponent's skill will not change his behavior. However, the higher skill level player needs to make a more precise estimate of the less skilled player's skill level.

On the other hand, when $b_{ij} < k_i - 1$, P_i must first consider the even smaller subgames he believes his opponent (P_j) will be optimizing over at each of the periods P_j controls that fall within P_i's horizon. P_i must do this first, since he believes P_j will be using different critera than the final period values of P_i's truncated game. Only then can P_i backward induct over his full foresight horizon to determine his own optimal move, given his beliefs about P_j's less sophisticated decisions. More specifically, for each state that could occur in period $t+1$ (as a result of P_i's choice in period t), P_i creates a subgame that commences with that state and extends to a length of b_{ij}. He backward inducts over each of these subgames to determine what P_j will choose if found in that state in the second period. Then for each of the potential states that could exist in period $t+1$, P_i can prune off the rejected options, as well as all the downstream branches of the tree that stem from these rejected second period options. P_i then repeats this process for each period that P_j controls within his foresight window, further pruning the tree each time. When he has determined what P_j will decide at all decision sets in the foresight window, he can backward induct over the pruned tree, and arrive at his optimal choice for the current period.

For simplicity, we define P1 to always be the more skilled player, such that $k_1 > k_2$. Furthermore, we do not consider cases in which P_i's beliefs about P_j's beliefs about P_i (i.e. P_i's belief about b_{ji}) vary enough to change the decision algorithm described above. For example, if P_i has belief $b_{ij} = 3$, then the optimal backward induction process P_i uses over the subgames of length b_{ij} (to determine what P_j will choose at a given decision set) would change if P_i believed that P_j believed that P_i was a level 1 thinker vs. a level 2 thinker (note, however, that as per the above reasoning, all beliefs of this nature that are greater than or equal to 2 map to the same full backward induction over the subgame—so it is only when $b_{ij} \geq 3$ that these higher order beliefs could affect decision processes). For simplicity, we do not consider variation in higher order beliefs, and assume that, for any opponent skill belief b_{ij}, P_i performs a full backward induction when hypothetically selecting moves from P_j's perspective.[4] We do this because there is limited room for such variation within shorter foresight horizons and experimental work suggests that people generally have quite limited levels of thinking (see, for example, Camerer et al., 2004 and Stahl and Wilson, 1995).

For simplicity, we assume that each player holds a single value for b_{ij} and applies it with certainty (i.e., instead of specifying a distribution of possible opponent beliefs, or updating beliefs over time). This assumption is in line with behavioral work that finds individuals show overconfidence about their judgments (e.g., Hoffrage, 2004), and is further justified by the limited opportunity for learning within the scope of the model. Players are not engaging in long, repeated games with familiar opponents, but rather engaging in a large variety of short, novel one-

[4] However, preliminary findings suggest that relaxing this assumption only strengthens the results found in this paper.

time games with novel opponents and only stochastic feedback. Empirical findings show that people are slow to update existing beliefs, especially amid noisy signals, and that when they do, they overweight prior beliefs (e.g., Boulding et al., 1999; Camerer and Lovallo, 1999). Furthermore, the first move, which must occur before any learning is possible, has the most influence on the game, especially when $\rho > 0$. Thus we believe our assumed model captures the general belief conditions found in many sequential games.

We also note that we use foresight horizon as our singular measure of player skill. There are, of course, many dimensions to skill when playing games in real life. For example, players may differ on their ability to accurately assess payoffs or to correctly apply backward induction. For parsimony in our manipulation, we create a level playing field on all dimensions except foresight horizon. This follows in the tradition of many analytic and level-k models, and allows a clean manipulation of skill levels. However, the model can be extended in future research to account for other such dimensions of skill—for example, by allowing players to “see" only a variably imperfect correlate of the true payoff for any given state.

Finally, we add for clarification that we assume that players do not alter their decision rules based on the value of ρ. This is because we are attempting to capture the overall effect of different bias conditions applied heuristically by managers repeatedly over many different contexts and against many different opponents. The value of ρ in a real-world setting would be difficult to observe precisely, especially under constrained reasoning. Furthermore, knowledge of ρ would only potentially change players' decision rules if players were endogenously concerned with minimizing effort cost. We assume that this is not the case and will explain our reasons for this in Section 3.5..

3.4. Bias Conditions

We begin our theoretical experiment by creating three bias conditions (B) that correspond to three “worlds" in which players all exhibit one of three types of bias in estimating their opponent's skill level, and are unaware that the bias is present.[5] We first consider the accurate ($B = A$) opponent skill estimation condition, where $b_{ij} = k_j$, to model a condition of no bias. This scenario is most similar to traditional game theoretic methods. Players have exogenously defined skill levels (where $k_1 > k_2$), and we assume that these skill levels are common knowledge.

We next consider an opponent underestimation ($B = U$) condition, corresponding to overconfidence bias. To model a slight population-wide opponent underestimation bias, we set $b_{ij} = k_j - 1$, such that each player believes his opponent to be exactly one level less skilled than is actually the case. Guided by the observation that people are often unaware of bias, we initially assume that both players think their beliefs map to true values that are common knowledge. P1 thinks that k_1 and $k_2 - 1$ are the common knowledge skill levels for P1 and P2, while P2 thinks that $k_1 - 1$ and k_2 are the levels for P1 and P2 that are common knowledge. However, we soon show in Lemma 2 that this assumption can be relaxed.

Finally, we test a condition in which players exhibit a bias towards overestimating opponent skill ($B = O$). Though such “underconfidence" does not appear to be prevalent in nature, this test provides a useful theoretical tool for understanding

[5] We later show that neither awareness of the bias nor asymmetric bias will affect player decisions or game outcome.

overconfidence. This condition is defined similarly to the U condition, except that now both players believe their opponent to be exactly one level *more* skilled than is actually the case ($b_{ij} = k_j + 1$). P1 thinks that k_1 and $k_2 + 1$ are the skill levels for P1 and P2 that are common knowledge, while P2 thinks that $k_1 + 1$ and k_2 are the skill levels for P1 and P2 that are common knowledge.

A logical implication of this framework that simplifies our analyses is that, since P2's actions are the same for all beliefs $b_{21} \geq k_2 - 1$, and since $k_1 > k_2$, then the two bias conditions in which players misestimate opponents skill level (O and U) have no effect on P2's decision process compared to the accurate estimation condition (A). Even if P1 is only one level more skilled than P2, an underestimation bias on the part of P2 will yield $b_{21} = k_1 - 1 = k_2$, which maps to the same decision in all cases as accurate estimation ($b_{21} = k_1 = k_2 + 1$) and overestimation ($b_{21} = k_1 + 1 = k_2 + 2$). It is only when the *more skilled* player has biased beliefs that the game outcome is potentially changed. This is because only the more skilled player has the opportunity (capacity) to think further out than his opponent and this foresight can result in changes in his decision process. This leads to the following lemma.

Lemma 1. *The weaker player's bias condition has no effect on game outcome*

Thus, throughout our analysis, we focus on the implications of opponent estimation error from P1's perspective. All results will hold whether or not the weaker player exhibits bias.

Lemma 1 implies that P2's bias doesn't affect his own behavior. Consequently, P1's *awareness* of P2's bias does not change P1's behavior, since P2 will behave the same in all conditions. Nor will P2's awareness of P1's bias change P2's behavior. To see this, note that when P2 is looking hypothetically through P1's eyes over his truncated subgames, trying to anticipate what P1 will do, he only has $k_2 - 1$ periods to work with—which leaves only $k_2 - 2$ periods in which he can anticipate what P1 thinks about P2. In other words, all of P2's beliefs about what P1 believes about P2 that are $\geq k_2 - 2$ map to the same decision process and the same outcome. In bias conditions U, A, and O, respectively, P1 believes k_2 to be $k_2 - 1$, k_2, and $k_2 + 1$. All of these values are $\geq k_2 - 2$, and thus even if P2 is aware of P1's bias, it does not change his behavior, due to the limitations of his own cognitive constraints.

Lemma 2. *The outcome of the game is the same whether or not players are aware of the other player's bias.*

Note that Lemmas 1 and 2 hold in our model because we are looking as small errors in opponent estimation, or slight biases. If we explored conditions in which players underestimate each others' skill to large degrees (specifically, with an underestimation bias that is $> 1 + k_1 - k_2$) then player decisions would potentially be affected. Together, these lemmas show that our model holds under assumptions that are less restrictive than we initially laid out. For logical equivalence, we need only assume that P1 is more skilled than P2 (i.e., $k_1 > k_2$)[6], that P2 believes P1 can see out at least as far as he can (i.e., $b_{21} \geq k_2 - 1$), that P1 will not underestimate P2 by more than three levels[7], and that both players are aware of all these

[6] To allow "room" for P1 to effectively overestimate P2, we focus on cases in which $k_1 > k_2 + 1$.

[7] This is the minimum lower bound over all conditions for which P2's beliefs about P1's beliefs about P2 might change P2's decision behavior.

things. In addition, to test out bias conditions, we further assume that P1 exhibits the following condition-dependent beliefs. For $B = U$, P1 underestimates P2's skill level such that $b_{12} = k_2 - 1$. For $B = A$, P1 accurately estimates P2's skill level such that $b_{12} = k_2$. And for $B = O$, P1 overestimates P2's skill level such that $b_{12} = k_2 + 1$.

3.5. Effort Cost

Under boundedly rational paradigms, it is generally accepted that there is some cost of information acquisition and processing, whether it be opportunity cost, error introduction, or sheer disutility of effort (e.g., see Shugan, 1980). With this noted, there is no consensus among researchers on how to precisely define such a cost function for decision making. Moreover, in real life these costs are likely to be highly variable across persons and contexts. Consequently, to maintain greatest external validity, we refrain from defining any specific cost functions, and assume only that such effort cost strictly increases with the amount of information processed, which can be represented through the number of game tree nodes a player generates when making a decision. This is consistent with behavioral traditions that use elementary information processes as a measure of the cost of cognitive effort (see Newell and Simon, 1972 and Payne et al., 1995). We use $\mathbb{C}(B)$ as the cost of effort exerted by P1 under bias condition B for a set k_1 and k_2 where the cost is some increasing function of nodes examined during the first move.

We also assume that players do not account for any effort cost when making decisions. Rather, players' decision rules are driven by their skill levels and beliefs, and we compute comparative effort costs post-hoc, to show the relative long-term advantages of different heuristics applied by decision makers automatically (e.g., see Stahl, 1993). This assumption is perhaps unusual in the game theory literature, especially when costs are considered purely search costs, rather than optimization costs. However, we follow in the tradition of bounded rationality paradigms and take our players' decision rules to be heuristic-like (based on empirical observation) rather than strict optimization. Thus, the players within our model apply the same decision rules (based on their own skill and their beliefs about their opponent's skill) to each decision they encounter, as a matter of course, without considering effort cost. This approach is in line with empirical and theoretical work on heuristics and biases and circumvents a general problem with optimization in bounded rationality paradigms when decision effort itself costly: the act of cost-aware decision optimization becomes impossible, as the decision of how to decide how to decide (and so forth) becomes an infinite regress where each such higher order decision exacts its own cost (e.g., see Gigerenzer and Selten, 2001).

3.6. Net Expected Outcome

We define *net expected outcome* as the gross expected outcome for a given bias condition and parameter set minus the cost of implementing the decision rule. We use $\mathbb{E}_{\mathbb{N}}(B)$ to refer to the net expected outcome for bias B over a fixed game and skill set, where $\mathbb{E}_{\mathbb{N}}(B) = \mathbb{E}(B) - \mathbb{C}(B)$. Of course, we cannot compute meaningful numeric values for $\mathbb{E}_{\mathbb{N}}(B)$ without assigning a a shape and relative scale to the cost function, which is outside the scope and motivation of this paper. However, with the one assumption that cost is a strictly increasing function of information searched, we can determine useful ordinal properties and trends. Thus, in our analysis we will look for insights relating to the the relative net expected outcomes of the three bias conditions as a function of the game parameters L and ρ and the player parameters k_1 and k_2. Figure 1 illustrates the structure of the model framework.

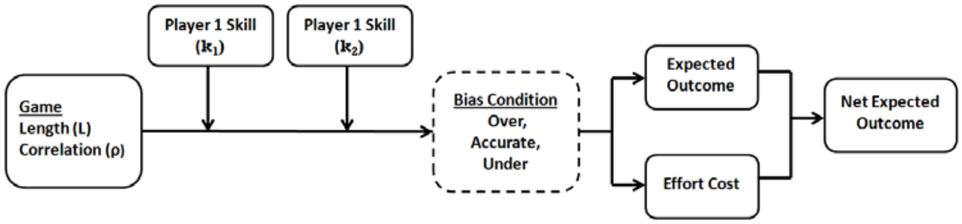

Fig. 1: Model Framework

4. Analysis

4.1. Estimating Expected Outcomes and Costs

For any specific game tree, the unique outcome of the game is entirely predictable given k_1, k_2, and bias condition B. Unlike full rationality models, the players themselves may not accurately predict final outcomes while in early periods, and may make mistakes (when foresight is incomplete or there is estimation error) that lead the game into a state they did not anticipate. However, the outcome that will be arrived at is deterministic from an outside perspective, i.e., by someone who can see the entire game tree and knows k_1, k_2, and B, and can apply the appropriate decision algorithms from the perspective of each player to arrive at the final outcome. As the outcome for any single game is of little generalizable value, we focus on estimating the expected values for the outcomes over infinite sets of possible game trees that make up game sets defined by specific values of L and ρ with player skills defined by k_1 and k_2. We use the notation $\mathbb{E}(B)$ as a shorthand for the expected outcome under bias condition $B \in \{O, A, U\}$ for a fixed parameter set.

We use Monte Carlo simulation to estimate the expected outcomes of each bias condition over a range of parameter values that we believe are reasonable within our general premise of bounded rationality. We test all parameter combinations that meet the requirements of our model that fall within the range $2 \leq k_2 \leq k_1 - 2 \leq k_1 \leq L \leq 10$ and ρ in 0.1 increments from 0 to 1 (i.e., $\rho = \{0, .1, .2, .3, ..., 1\}$).

We use the Visual Basic environment in Microsoft Excel to write a program that builds sample game trees of any specified L and ρ, where the payoffs in the first period (v_1) are drawn randomly from $U(0,1)$ using Excel's random number generator, and each child node's payoff is equal to its parent node's value plus error weighted by ρ such that $v_t = \rho v_{t-1} + (1-\rho)\epsilon_t$, where ϵ_t is a draw from $U(0,1)$. The decision rules described in Section 3.3. are programmatically applied to find the unique outcome for each generated game tree according to any specified k_1, k_2, and B.

We generate two thousand game trees for each unique combination of game parameters (L and ρ) included in the range defined above. For each tree generated, we record the outcome reached under each bias condition (B) for each combination of player skill parameters (k_1 and k_2) included in the range defined above. The mean outcome over all trials of a unique parameter set is used to approximate the expected outcome for that parameter set. For simplicity, we average the results for consecutive values of k_2 to eliminate main effects of the parity of k_2 which we do no expect to be useful in terms of managerial insights. We also count the number of nodes searched in the first turn of P1 to use as a basis for ordinal comparisons of effort cost (which as defined in Section 3.5., is considered to be some (any) strictly increasing function of information searched.

4.2. Results

Comparing the estimates for all combinations of parameter values in the range tested, we find the following results.

Result 1. *The expected outcome is greatest when P1 accurately estimates his opponent's skill level, followed by when he overestimates, followed by when he underestimates, for all otherwise fixed parameter values;* $\mathbb{E}(A) > \mathbb{E}(O) > \mathbb{E}(U)$ *for all fixed* k_1*,* k_2*,* L*, and* ρ*.*

Note that if the parameter range allowed $k_2 \geq k_1$ then the inequalities would become weak, as the search behavior (and thus outcomes) would be the same in all three conditions from P1's perspective.

Result 1 supports conventional wisdom and intuition. However, we expect that the magnitude of disparity between the raw exceptions changes in different settings. Thus, we next investigate the relationship between opponent skill and the differences in $\mathbb{E}(B)$. We first note that an increase in k_2 when $k_1 - k_2$ is held constant implies that both players are getting more skilled, while the disparity between them stays the same. On the other hand, an increase in k_2 while k_1 is held constant implies that P2 is getting more skilled, while the disparity between the players diminishes. In both cases, we find that increasing k_2 decreases the differences in $\mathbb{E}(B)$.

Result 2. *The expected outcomes of all bias conditions converge as both players become more skilled together; differences in* $\mathbb{E}(B)$ *decrease with* k_2 *for any fixed* $k_1 - k_2$*,* L*, and* ρ*.*

Result 3. *The expected outcomes of all bias conditions converge as P2 becomes more skilled relative to P1; differences in* $\mathbb{E}(B)$ *decrease with* k_2 *for any fixed* k_1*,* L*, and* ρ*.*

To explain the intuition for this result, we note that as P2 becomes more skilled, P1 has less control over the outcome, regardless of bias condition. The longer the horizon over which P2 has full foresight, the greater P2's ability to influence the outcome, which limits the range of possible outcomes available to P1. Thus the range between the worst outcome and best outcome for P1 decreases when he is playing against more skilled opponents. As a result, there is less relative payoff decrease to P1 as a result of error. This is true both as P2 becomes more skilled in absolute terms, as well as relative to P1. As P1's skill advantage over P2 decreases (i.e., as the players become more evenly matched, regardless of absolute skill level), there are fewer periods over which P1 can use his advantage—and thus fewer periods in which estimation error can detract from the potential outcome. As a result, the differences in $\mathbb{E}(B)$ decrease whenever average opponent skill increases, regardless of whether both players are getting more skilled together, or whether the disparity between them is decreasing.

As expected, we also find that the expected outcomes of each of the three bias conditions converge as ρ increases.

Result 4. *The difference between the expected outcomes of overestimation and underestimation decrease with payoff correlation for any fixed game and skill set.*

This is intuitive because increasing ρ increases the advantage generated by the first move, thus decreasing the influence P2's strategy will have on the outcome, which thereby decreases the expected payoff loss associated with opponent estimation errors.

In addition to having asymmetric effects on expected payoffs, we also fine that different types of errors in estimating opponent skill also have asymmetric implications for effort cost. When underestimating his opponent, P1 can prune off much of the game tree without ever having to generate of process the payoffs associated with those states. This results in an effort cost savings. This is true to a less extent with accurate estimation, and to an even less extent with over estimation.

Result 5. *When a player's own level of thinking is fixed, it costs him the most to overestimate his opponent, less to accurately estimate, and least to underestimate;* $\mathbb{C}(O) > \mathbb{C}(A) > \mathbb{C}(U)$ *for any fixed* k_1 *and* k_2*.*

Note that the strict inequality holds for the parameter range tested. If the the range allowed $k_2 \geq k_1$, then the inequalities would become weak, as the search behavior (and thus costs) would be the same in all three conditions from P1's perspective.

From Results 1 and 5 we know that $\mathbb{E}(O) < \mathbb{E}(A)$ and $\mathbb{C}(O) > \mathbb{C}(A)$ for all game and player sets that fit the requirements of our model. From here one can directly conclude that $\mathbb{E}(O) - \mathbb{C}(O) < \mathbb{E}(A) - \mathbb{C}(A)$. In other words, overestimation always has a lower expected net (as well as gross) return than accurate estimation, regardless of game or player parameters. In fact, the difference between the net expected outcomes is necessarily larger than the difference between gross expected outcomes. This leads us to Result 6.

Result 6. *The expected net payoff for overestimation is always strictly less than the expected net payoff of accurate estimation for any fixed parameter set:* $\mathbb{E}_{\mathbb{N}}(O) < \mathbb{E}_{\mathbb{N}}(A)$*.*

There is no such strict dominance with underestimation. From Results 1 and 5, we know $\mathbb{E}(U) < \mathbb{E}(A)$ and $\mathbb{C}(U) < \mathbb{C}(A)$, from which we cannot determine a general ordinality for $\mathbb{E}(U) - \mathbb{C}(U)$ versus $\mathbb{E}(A) - \mathbb{C}(A)$. The difference in expected net returns of under vs. accurate estimations will depend on the magnitudes of each term, which are determined by the specific parameters of a game and player set, as well as the specific cost functions used. It is not our goal in this paper to propose valid cost functions. Still it is possible to gain additional insights by considering the relationship of the parameters ρ and k_2 on the differences in $\mathbb{E}_{\mathbb{N}}(B)$.

Considering first the minimum values for k_2 and ρ, we note that, depending on the cost function, $\mathbb{E}_{\mathbb{N}}(U)$ can take any of three positions: it can be less than $\mathbb{E}_{\mathbb{N}}(O)$, it can be greater than $\mathbb{E}_{\mathbb{N}}(O)$ but less than $\mathbb{E}_{\mathbb{N}}(A)$, or it can be greater than $\mathbb{E}_{\mathbb{N}}(A)$. However, as either k_2 or ρ increases, the expectation disadvantage of U decreases relative to A and O. Regardless of where $\mathbb{E}_{\mathbb{N}}(U)$ begins relative to $\mathbb{E}_{\mathbb{N}}(O)$ and $\mathbb{E}_{\mathbb{N}}(A)$, the slope differences will cause there to be some critical value of both k_2 and ρ, above which $\mathbb{E}_{\mathbb{N}}(U)$ is the best performing condition if the trend lines are extrapolated. As we are not defining cost function scales, we cannot say if the critical value will occur within the parameter limits imposed by a player's own skill constraint. However, we can say that this becomes more likely that the cost of information acquisition and processing increases.

This brings us to Result 7.

Result 7. *Underestimation can yield the greatest net expected outcome for a parameter set. This is more likely to occur when the opponent is highly skilled, when the first mover advantage is strong, and/or when effort costs are high;* $\mathbb{E}_{\mathbb{N}}(U)$ *becomes more likely to to be higher than both* $\mathbb{E}_{\mathbb{N}}(O)$ *and* $\mathbb{E}_{\mathbb{N}}(A)$ *as* k_2*,* ρ*, and effort costs increase.*

5. Discussion

The ultimate goal of this paper is to provide an initial exploration into the question of if and when overconfidence can be beneficial to managers who make frequent complex competitive business decisions. In order to do this we needed to develop a new and general framework for analyzing boundedly rational players in "large world" (Savage, 1954) complex games. Our framework uses a branching decision tree with interim payoffs to represent a strategic game between two players where players make sequential moves over time and have limited foresight. Because our model includes a skill constraint for each player, we are able to explore branching game structures that have real-world applicability but that can be quite difficult to manage under traditional assumptions of rationality due to the curse of dimensionality. Given our interest in generalizable conclusions, we build a Monte Carlo simulation program to estimate the expected payoffs over a the distribution of possible payoff structures associated with any given game length, payoff correlation, players' skill levels, and players' beliefs about their opponent's skill level. We believe this framework could be useful to others interested in bounded rationality and branching sequential games.

Our results suggest that bias in evaluating an opponent's skill is less harmful to expected payoff when the opponent is more skilled, and when there is greater first-mover advantage. Furthermore, they suggest that if there is any effort cost associated with the making a decision, then a bias towards overestimating the opponent's skill is never advantageous, while a bias towards underestimating can be advantageous in many contexts. Thus, the overconfidence bias behavioral researchers have observed in the population may actually be helpful, rather than detrimental, as is often suggested, and we provide initial insight into when this is more likely to be the case.

Although these initial theoretical experiments begin to shed light on the relationships of interest, they have several limitations, and thus should be considered as only a start to understanding the greater relationships between skill constraints, opponent estimation errors, and outcomes. For example, the game contexts the model framework covers in this initial exploration are limited, but could, in the future, be extended to capture non-zero-sum games, games with greater complexity, and state-dependency in the parameters. In terms of players, the extant model considers only one dimension of skill (foresight horizon) and thus the results do not generalize to estimating other dimensions of competitive capabilities, such as sophistication in estimating the interim payoffs (or resources of the competing firm). We believe future research can build off this framework to address these and other limitations, thereby providing deeper and broader insights to advise managers in their real-world decisions.

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A Survey on Discrete Bidding Games with Asymmetric Information ⋆

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Abstract Repeated bidding games were introduced by De Meyer and Saley (2002) to analyze the evolution of the price system at finance markets with asymmetric information. In the paper of De Meyer and Saley arbitrary bids are allowed. It is more realistic to assume that players may assign only discrete bids proportional to a minimal currency unit. This paper represents a survey of author's results on discrete bidding games with asymmetric information.

Keywords: multistage bidding, asymmetric information, price fluctuation, random walk, repeated game, optimal strategy.

1. Introduction

1.1. Modeling financial markets by repeated games

Regular random fluctuations in stock market prices are usually explained by effects from multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley (2002) proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from the asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in the immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of an oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for risky assets (shares). The liquidation price of a share depends on a random "state of nature". Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation price of a share once and for all. Player 1 is informed on the "state of nature", but Player 2 is not. Both players know the probability of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

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In this model the uninformed Player 2 should use informed Player 1's history of moves to update his beliefs about the state of nature. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider a model where a share's liquidation price takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann and Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have the values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2). They find these values and the optimal strategies of players. As n tends to infinity, the values infinitely grow up with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

More exactly, De Meyer and Saley construct continuous time processes $\Pi^n(t)$ with $t \in [0,1]$ representing these finite random sequences and prove that, as n tends to ∞, the processes Π^n converge in law to the limit process Π expressed by means of Brownian Motion.

In De Meyer (2010) a model of a market with one risky asset and perfectly general trading mechanism was considered. For example, transactions of arbitrary amount of shares at any stage of a game and presence of non-zero bid-ask spread can be implemented by means of this mechanism. The model of De Meyer and Saley is a particular case of this general model. For this general problem the limiting properties as the number of repetitions tends to infinity were investigated. It was shown that when both players use their optimal strategies the price process (expected price of a risky asset giving the history up to a current stage) converges after proper normalization in finite dimensional distributions to a martingale adapted to the natural Brownian filtration with terminal distribution coinciding with prior distribution of the share price. This class of price evolutions was called CMMV (continuous martingales of the maximal variation). The limit of the value and "asymptotically optimal" strategy of informed player were explicitly characterized too. Rather surprisingly, it was found that all the limiting objects do not depend on particular trading mechanism. To obtain these results a breakthrough technique to analyze repeated games with incomplete information was developed. The main idea was to look at the game from the point of view of informed player and to reduce it to some martingale optimization problem, the so-called problem of the maximal variation. This approach then allowed to apply a broad variety of tools from theory of stochastic processes.

The ideas of De Meyer about reduction to the martingale optimization problem were extended by Gensbittel (2010) in his thesis to a general repeated games with incomplete information (not necessary modelling a finance market). Moreover, he considered several improvements of the general trading mechanism of De Meyer: the case of several risky assets and the non-zero sum case (the total amount of money is not conserved).

It is to be mentioned that all the results of De Meyer and Gensbittel are obtained under several assumptions on trading mechanism: invariance with respect to non-risky part of the risky asset (i.e., shift invariance) and invariance with respect to numeraire change (i.e., scale invariance). These assumptions significantly simplify the analysis because they result in very handy linear structure of a game. But both these assumptions do not reflect the properties of real bidding. Indeed, only bids proportional to the minimal currency unit are allowed in real bidding. Therefore, neither shift invariance nor scale invariance really hold.

1.2. Results on discrete bidding games with asymmetric information

De Meyer and Marino (2005), Domansky and Kreps (2005), Domansky (2007) analyze a bidding model analogous to the model of De Meyer and Moussa-Saley (2002), where market makers have to post prices within a discrete grid. it corresponds to prices proportional to a minimal currency unit . The n-stage games $G_n^m(p)$ are considered with two possible values of liquidation price, an integer $m > 0$ with probability p and 0 with probability $1-p$, and with admissible bids being integer numbers.

The works mentioned above show that, unlike the model of De Meyer and Saley, the sequence of values $V_n^m(p)$ of the games $G_n^m(p)$ is bounded from above and converges as n tends to ∞. The authors calculate its limit H^m, that is a continuous, concave, and piecewise linear function with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \ldots, m-1$, and the values at peak points $H^m(k/m) = k(m-k)/2$.

The proof in De Meyer and Marino (2005) differs in essential ways from the proof in Domansky (2007). The last proof is more concise due to exploiting a "reasonable" strategy of Player 2. In fact, this is his optimal strategy for the game with infinite number of steps.

As the sequence $V_n^m(p)$ is bounded from above, it is reasonable to consider the games $G_\infty^m(p)$ with infinite number of steps. The games $G_\infty^m(p)$ are infinitely repeated, non-discounted games with non-averaged payoffs that differs from the classical model of Aumann and Maschler (1995). Unlike the case of $n < \infty$, the existence of a value for the games $G_\infty^m(p)$ has to be proved.

In section 2 following Domansky (2007) we show that the value V_∞^m is equal to H^m and construct explicitly the optimal strategies of players. The fastest optimal strategy of Player 1 provides him the maximal possible expected gain 1/2 per step. For this strategy the posterior probabilities perform a simple random walk over the lattice l/m, $l = 0, \ldots, m$, with absorbing extreme points 0 and 1. The absorption of posterior probabilities means revealing of the true value of share by Player 2. For the initial probability k/m, the expected duration of this random walk before absorption is $k(m-k)$. The bidding terminates almost surely in a finite number of steps, and the expected number of steps is also finite. This random time of absorption is a time for disclosure of information. The game terminates naturally when the posterior expectation of liquidation price coincide with its real value.

The set of all optimal strategies of Player 1 for $G_\infty^m(p)$ consists of the described fastest strategy and its slower modifications.

The results of Domansky (2007) cannot be extended to a general transaction mechanism introduced by De Meyer (2010). As mentioned in the last paper, the discretized mechanism does not satisfy axioms of shift- and scale-invariance. Note that in practice a grid of possible bids is not shift- and scale-invariant simultaneously.

Obtaining exact solutions for games $G_n^m(p)$ with finite numbers of steps seems to be a rather hard problem because of combinatorial difficulties as this may be observed at the two simplest case: solutions for one-stage games (Sandomirskaya, Domansky, 2012) and solutions for games with three admissible bids (Kreps, 2009).

In section 3 we describe the set of peak points of value function $V_1^m(p)$ and analyze the structure of bids used in optimal strategies of both players. On the base of this analysis we develop recurrent approach to computing optimal strategies of uninformed player for any probability p. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in his optimal mixed strategy. As optimal strategy of insider equalizes the spectrum of optimal strategy of Player 2, we get Player 1' optimal strategies solving the system of difference equations arising from equalizing conditions.

In section 4 we construct the exact solutions for games $G_n^3(p)$ in the explicit form for any number of steps n. The value function $V_n^3(p)$ and the players' optimal strategies are expressed using a second-order recursive sequence.

In section 5 we show that the fastest optimal strategy of Player 1 for the infinitely repeated game $G_\infty^m(p)$ is a ε-optimal strategy of Player 1 for any finitely repeated game $G_n^m(p)$ of length n, where $\varepsilon = O(\cos^n \pi/m)$. This is not so for slower optimal strategies of Player 1 (Sandomirskaya, 2013, unpublished).

In section 6 following Domansky and Kreps (2009) we consider a model where a share liquidation price may take any integer value according to a probability distribution $\mathbf{p}$ over the one-dimensional integer lattice. Any integer bids are admissible. This n-stage model is described by a zero-sum repeated game with countable state and action spaces. The games considered in section 2 can be reduced to particular cases of these games corresponding to probability distributions with two-point supports.

We show that if the liquidation price of a share has a finite expectation, then the values of n-stage games exist. If its variance is finite, then, as n tends to ∞, the sequence of values is bounded from above and converges. The limit H is a continuous, concave, piecewise linear function with a countable number of domains of linearity.

As the sequence of n-stage game values is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with an infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$.

The optimal strategies are given in an explicit form. For constructing the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution having an integer expectation, we use the solutions for the games with two-point distributions and the symmetric representation of distributions over one-dimensional integer lattice with fixed integer mean values as convex combinations (probability mixtures) of distributions with two-point supports and with the same mean values.

The insider optimal strategy generates a random walk of posterior expectations over the one-dimensional integer lattice with absorption. The absorption may occur at any stage t if the posterior expectation of share price at this stage coincides with its prior expectation.

For any initial distribution with an integer mean value the expected duration of this random walk is equal to the variance of the liquidation price of a share. The value of infinite game is equal to the expected duration of this random walk multiplied by the constant one-step gain $1/2$ of informed Player 1.

In section 7 we consider multistage bidding models where two types of risky assets are traded between two agents that have different information on the liquidation prices of traded assets (Domansky and Kreps, 2013, submitted to RAIRO-Operation Research). These prices are random integer variables that are determined by the initial chance move according to a probability distribution $\mathbf{p}$ over the two-dimensional integer lattice that is known to both players. Player 1 is informed on the prices of both types of shares, but Player 2 is not. The bids may take any integer value.

The model of n-stage bidding is reduced to the zero-sum repeated game with lack of information on one side.

If the expectations of share prices are finite, then the value of such n-stage bidding game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if liquidation prices of both shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game.

We begin with constructing solutions for these games with distributions $\mathbf{p}$ having two- and three-point supports (elementary games). Next, using symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values (Domansky, 2013), we build the optimal strategies of Player 1 for bidding games $G_\infty(\mathbf{p})$ with arbitrary distributions $\mathbf{p}$ as convex combinations of his optimal strategies for elementary games.

The optimal strategies of Player 1 generate a random walk of transaction prices. But unlike the case of one-type assets, the symmetry of this random walk is broken at the final stages of the game.

We demonstrate that the value $V_\infty(\mathbf{p})$ is equal to the sum of values of corresponding games with one-type risky asset. Thus, the profit that Player 2 gets under simultaneous n-step bidding in comparison with separate bidding for each type of shares disappears in a game of unbounded duration.

In the bidding models considered in the sections 2-7 players propose only one price for a share at each step, i.e. bid and ask prices coincide. In a more realistic model developed in section 8 both players simultaneously propose their bid and ask prices for one share at each step of bidding. The bid-ask spread s is fixed by rules of bidding. Transaction occurs from a seller to a buyer by a bid price. The simplified model (sections 2-7) corresponds to the case $s = 0$ what is equivalent to $s = 1$ due to the price discreteness.

One-step payoff matrices of corresponding repeated games with incomplete information have more complicated structure than for the case $s = 1$ and solutions of these games are not found.

In section 8, for any integer $s > 1$ and two possible states of nature, by analogy with the zero-spread case of section 2 we construct an upper bound of value function provided by a reasonable strategy of Player 2. We construct a lower bound provided by a strategy of Player 1 that is the best strategy generating a simple random walk of price expectations. The bounds have the same form and coincide for $s = 1$ being equal to the value function of of the game under consideration.

By analogy with zero spread case (see section 6), for any integer $s > 1$ the results are generalized to the case of countable set of possible values for a share price.

As for $s > 1$ the constructed Player 1' strategy is not optimal, we conclude that the insider's optimal strategy does not generate simple random walk of price expectations and leads apparently to non-symmetric price fluctuations.

2. Bidding games with two states of nature

2.1. Bidding games of finite and infinite duration: $G_n^m(p)$ and $G_\infty^m(p)$

In this section we consider the repeated games $G_n^m(p)$ modelling the bidding with two possible random "state of nature", the state space $S = \{L, H\}$. Before bidding starts a chance move determines the "state of nature" L or H and therefore the liquidation value of a share once for all. This value is a positive integer m with probability p at the state H and 0 with probability $1-p$ at the state L. Player 1 is informed about the "state of nature", Player 2 is not. Both players know probability p. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \ldots, n$ (n may be infinite) of bidding both players simultaneously propose their prices for one share, i_t for Player 1 and j_t for Player 2. Then the pair (i_t, j_t) is announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_t > j_t$, Player 1 gets one share from Player 2 and Player 2 receives the sum of money i_t from Player 1. If $i_t < j_t$, Player 2 gets one share from Player 1 and Player 1 receives the sum l from Player 2. If $i_t = j_t$, then no transaction occurs.

The bids may take arbitrary integer numbers, but the bids $0, 1, 2, \ldots, m-1$ are efficient only. Indeed, as the minimal value of a share is 0 and the maximal value is $m > 0$, the bids $k < 0$ and $k > m-1$ are senseless and thus $k = 0, \ldots, m-1$. So the action spaces are $I = J = 0, \ldots, m-1$.

At state L, i.e. if the liquidation value of the share is equal to zero, the one-step gains of Player 1 are given with the following matrix $A^{L,m}$:

$$\begin{pmatrix} 0 & 1 & 2 & \ldots m-1 \\ -1 & 0 & 2 & \ldots m-1 \\ -2 & -2 & 0 & \ldots m-1 \\ \ldots & \ldots & \ldots & \ldots\ldots \\ -m+1 & -m+1 & -m+1 & \ldots 0 \end{pmatrix}$$

At state H, i.e. if the liquidation value of the share is equal to m, then the matrix $A^{H,m}$ of the one-step gains of Player 1 takes the form

$$\begin{pmatrix} 0 & -m+1 & -m+2 & \ldots -1 \\ m-1 & 0 & -m+2 & \ldots -1 \\ m-2 & m-2 & 0 & \ldots -1 \\ \ldots & \ldots & \ldots & \ldots\ldots \\ 1 & 1 & 1 & \ldots 0 \end{pmatrix}$$

We consider n-step games $G_n^m(p)$ with total (non-averaged) payoffs

$$K_n^m(p, \sigma, \tau) = \sum_{t=1}^{n} \mathbf{E}_{(\sigma,\tau)}[(1-p)a^{L,m}(i_t^L, j_t) + p \cdot a^{H,m}(i_t^H, j_t)]. \tag{2.1}$$

Note that at step t it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. Thus, a strategy σ for Player 1 (insider) is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state s and previous observations.

A strategy τ for uninformed Player 2 does not depend on state s and represents a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

We also consider the infinite games $G_\infty^m(p)$. For certain pairs of strategies (σ, τ), the payoff function $K_\infty^m(p, \sigma, \tau)$, given by the infinite series (2.1), may be indefinite. If we restrict the set of Player 1's admissible strategies to strategies with nonnegative one-step gains

$$\mathbf{E}_{(\sigma_1, j)}[(1-p)a^{L,m}(i^L, j) + p \cdot a^{H,m}(i^H, j)].$$

against any action j of Player 2, then the payoff function of the game $G_\infty^m(p)$ becomes completely definite (may be infinite).

Observe that Player 1 has many strategies, ensuring him a nonnegative one-step gain against any action of Player 2. In fact, any "reasonable" strategy of Player 1 should possess this property.

Remind that, due to the recursive structure of the repeated game, it is sufficient to define the first move for any prior probability p to define the whole strategy of Player 1. Further this move will be played if the current posterior probability becomes equal to p. Thus, if for any prior probability p, the first move is "reasonable", the whole strategy is "reasonable".

The games $G_n^m(p)$ with $n < \infty$, as games with a finite sets of actions, have values $V_n^m(p)$. The values $V_n^m(p)$ are positive and do not decrease, as the number of steps n increases.

2.2. Asymptotics of values $V_n^m(p)$

The next theorem provides an upper bound for the values $V_n^m(p)$.

Theorem 2.1. *The functions V_n^m are bounded from above by a function H^m that is continuous, concave, and piecewise linear with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \ldots, m-1$. It is completely determined with its values at the peak points k/m, $k = 0, \ldots, m$:*

$$H^m(k/m) = k(m-k)/2.$$

To prove this theorem, we define recursively the set of infinite "reasonable" strategies $\tau^{k,m}, k = 0, \ldots, m-1$ of Player 2, suitable for the games $G_n^m(p)$ with arbitrary n.

Definition 2.1. The first move $\tau_1^{k,m}$ is the action k. The moves $\tau_t^{k,m}$ for $t > 1$ depend on the last observed pair of actions (i_{t-1}, j_{t-1}) only:

$$\tau_t^{k,m}(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - 1, & \text{for } i_{t-1} < j_{t-1}; \\ j_{t-1}, & \text{for } i_{t-1} = j_{t-1}\ ; \\ j_{t-1} + 1, & \text{for } i_{t-1} > j_{t-1}\ . \end{cases}$$

The next theorem provides a lower bound for the values $V_n^m(p)$.

Theorem 2.2. *The following inequalities hold:*

$$L_n^m(p) \le V_n^m(p) \quad \forall p \in [0, 1],$$

where the functions L_n^m are continuous, concave, and piecewise linear on the interval $[0,1]$ with m domains of linearity $[k/m,(k+1)/m]$, $k=0,\ldots,m-1$. At the peak points k/m, the values $L_n^m(k/m)$ are given with recursive formulas

$$L_n^m(k/m)=1/2+1/2(L_{n-1}^m((k-1)/m)+L_{n-1}^m((k+1)/m)),$$

with the initial condition $L_0^m(k/m)=0$, and the boundary conditions $L_n^m(0)=L_n^m(1)=0$.

To prove this theorem, we define the strategy $\bar{\sigma}^m$ of Player 1 ensuring these lower bounds.

Definition 2.2. For the initial probability k/m, the first move of the strategy $\bar{\sigma}^m$ makes use of two actions $k-1$ and k only. These actions occur with the same total probabilities $q(k-1)=q(k)=1/2$.

The corresponding conditional posterior probabilities of the state H are

$$p^H(k-1)=(k-1)/m,$$

for the action $k-1$, and

$$p^H(k)=(k+1)/m,$$

for the action k.

Remark 2.1. These lower bounds have the same form as the upper bounds of Theorem 2.1.

Remark 2.2. As all posterior probabilities belong to the set $p=k/m, k=0,\ldots m$, these first moves define the strategy $\bar{\sigma}^m$ for the games $G_n^m(k/m)$ of arbitrary duration.

Corollary 2.1 (Asymptotics of values $V_n^m(p)$). *The following equalities hold:*

$$\lim_{n\to\infty}V_n^m(p)=H^m(p),\quad m=2,3,\ldots.$$

2.3. Solutions for the games $G_\infty^m(p)$ and random walks

As the values $V_n^m(p)$ are bounded from above on the number of steps n, the consideration of values for the games $G_\infty^m(p)$ with infinite number of steps becomes reasonable.

We restrict the set of Player 1's admissible strategies in these games to the set Σ^+ of strategies employing only the moves ensuring him a nonnegative one-step gain against any action of Player 2. Consequently, the payoff functions $K_\infty^m(p,\sigma,\tau)$ of the games $G_\infty^m(p)$ become definite (may be infinite) at all cases.

We show that the infinite game $G_\infty^m(p)$ has a value and this value is equal to $H^m(p)$.

The existence of values for these games does not follow from common considerations and has to be proved. We prove it by providing the optimal strategies explicitly.

Theorem 2.3. *The game $G_\infty^m(p)$ has a value $V_\infty^m(p)$ equal to $H^m(p)$. Both Players have optimal strategies.*

For $p = k/m, k = 1, \dots, m-1$, the optimal strategy of Player 1 is the strategy $\bar{\sigma}^m$, given by Definition 2.2. For the interior points $p \in (k/m, (k+1)/m)$, the optimal first move of Player 1 is the convex combination of the first moves corresponding to the extreme points of this interval. This optimal first move makes use of three actions $k-1, k$ and $k+1$, using them with total probabilities

$$q(k-1) = 1/2(k+1-mp), q(k) = 1/2, q(k+1) = 1/2(mp-k).$$

Corresponding posterior probabilities are

$$P(H|k-1) = (k-1)/m, \quad P(H|k) = (2k+1-mp)/m, \quad P(H|k+1) = (k+2)/m.$$

For $p \in (k/m, (k+1)/m)$, $k = 0, \dots, m-1$, the optimal strategy $\bar{\tau}^m$ of Player 2 coincides with the strategy $\tau^{k,m}$, given by Definition 2.1. For the peak points k/m, $k = 1, \dots, m-1$, any convex combination of the strategies $\tau^{k-1,m}$ and $\tau^{k,m}$ is optimal.

Corollary 2.2. *For the initial probabilities $p = l/m$, $l = 0, \dots, m$, the random sequence of posterior probabilities, generated with the optimal strategy $\bar{\sigma}^m$ of Player 1, is the elementary symmetric random walk $(\bar{p}_t^m)_{t=1}^\infty$, over the points k/m, $k = 0, \dots, m$ with the absorbing extreme points 0 and 1, i.e. the Markov chain with the transition probabilities*

$$P(k/m, (k-1)m) = P(k/m, (k+1)/m) = 1/2, \quad k = 1, \dots, m-1,$$

$$P(0,0) = P(1,1) = 1.$$

For the initial probabilities $p \neq l/m$, $l = 0, \dots, m$, the random sequence of posterior probabilities hits the set $p = k/m$, $k = 0, \dots, m$, with probability 1/2 after each step. Further it continues as the elementary symmetric random walk with the absorbing extreme points 0 and 1.

Further we consider the random process $\{c_t^m\}_{t=1}^\infty$, formed by the prices of transactions $c_t^m = \max\{x_t^m, y_t^m\}$ at sequential steps of the infinite game $G_\infty^m(p)$. We say that the transaction occurs at step t if $x_t^m \neq y_t^m$.

Theorem 2.4. *a) For each step $t = 1, 2, \dots$, the probability that transaction occurs is 1/2.*
b) For $p_t^m \in [k/m, (k+1)/m]$, under the condition that the transaction occurs at step t, the following random transaction prices occur:

$$c_t^m(p_t^m) = \begin{cases} k & \text{with probability } \textit{k+1-mp ;} \\ k+1 & \text{with probability } \textit{mp-k .} \end{cases}$$

In particular, for $p_t^m = k/m$, under the condition that the transaction occurs at step t, $c_t^m = p_t^m = k$, and the price process reproduces the random walk of posterior probabilities.
c) Player 1's one-step gain is $1/2$.

3. Solution for one-stage bidding game with incomplete information

In this section we give the solution for the one-stage bidding game $G_1^m(p)$ with arbitrary integer m and with any probability $p \in (0,1)$ of the high share price. The complete description is given in the paper of Sandomirskaia and Domansky (2012).

If the share price is zero (state L), then Player 1 posts the zero bid at the one-stage game $G_1^m(p)$ for any probability p. So the problem is to describe the optimal strategy of Player 1 for the state H and the optimal strategy of Player 2. The latter does not depend on the state of nature.

Thus, solving of the zero-sum game $G_1^m(p)$ with incomplete information is reduced to solving the game with complete information with payoff matrix

$$A^m(i,j) = \begin{cases} (1-p)j + p(m-i), & \text{for } i > j; \\ (1-p)j, & \text{for } i = j; \\ (1-p)j + p(-m+j), & \text{for } i < j, \end{cases}$$

here $i \in I$ is the bid of insider at state H, $j \in J$ is the bid of uninformed player.

We develop recurrent approach to computing optimal strategies of uninformed player for any probability p based on analysis of structure of bids used in optimal strategies of both players. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in an optimal mixed strategy.

3.1. Properties of spectra of optimal strategies.

The value $V_1^m(p)$ of the game $G_1^m(p)$ is a continuous concave piecewise linear function over $[0,1]$ with a finite number of linearity intervals. The optimal strategy of the uninformed Player 2 is constant over linearity intervals and is unique in its interiors.

Let $\mathbf{x}^m(p) = (x_0^m(p), \ldots, x_{m-1}^m(p))$ and $\mathbf{y}^m(p) = (y_0^m(p), \ldots, y_{m-1}^m(p))$ be optimal mixed strategies of Players 1 and 2 respectively for an initial probability p.

For probabilities $p \in [0, 1/m]$ and $p \in [(m-1)/m, 1]$ the game $G_1^m(p)$ has solution in pure strategies. Out of these intervals optimal strategies of Player 1 and Player 2 for the game $G_1^m(p)$ are mixed ones.

A change of the set $\operatorname{Spec} \mathbf{y}^m(p)$ (the set of positive components of the optimal strategy $\mathbf{y}^m(p)$) takes place at a point p if and only if p is a peak point of value function $V_1^m(p)$.

Consider the set $P^m = \{p_1, \ldots, p_{m-1}\}$, $0 < p_1 < \ldots < p_{m-1}$:

$$1 - p_1 = \frac{m-1}{m}, \quad 1 - p_2 = \frac{m-2}{m-1}, \quad 1 - p_k = (1 - p_{k-2})\frac{m-k}{m-k+1}.$$

and the set $Q^m = \{q_1, \ldots, q_{m-1}\}$, $1 > q_1 > \cdots > q_{m-1} = p_{m-1}$:

$$1 - q_1 = \frac{1}{m}, \quad 1 - q_2 = \frac{1}{m-1}, \qquad 1 - q_k = \frac{1 - p_{k-2}}{m-k+1}.$$

Proposition 3.1. *If $p \in P^m \cup Q^m$, then p is a peak point of value function $V_1^m(p)$ and*

$$V_1^m(p) = m \cdot p(1-p) \qquad \text{for } p \in P^m \cup Q^m.$$

Corollary 3.1. *The value of one-step bidding game with arbitrary bids being equal to $m \cdot p(1-p)$ (see De Meyer, Saley, 2002) coincides with $V_1^m(p)$ for $p \in P^m \cup Q^m$.*

Corollary 3.2. *As the set $P^m \cup Q^m$ is asymptotically everywhere dense over* $[0,1]$*, it follows that*

$$\lim_{m\to\infty} V_1^m(p)/m = p(1-p).$$

Remark 3.1. For $p < p_{m-1}$ ($p_{m-1} \approx 1/2$), the spectra of optimal strategies of both players expand as p increase until these spectra reach the bid $m-1$. For $p > p_{m-1}$ they narrow down but retaining the bid $m-1$.

Remark 3.2. For $m < 5$ there are no other peak points of $V_1^m(p)$ but the point $1/2$.

Denote by $k_1(\mathbf{x}^m(p))$ the maximal element of the set $\operatorname{Spec}\mathbf{x}^m(p)$ of positive components of strategy $\mathbf{x}^m(p)$. At a peak point p we put $k_1(\mathbf{x}^m(p))$ equal to its value to the right adjacent linearity interval.

Analogous notation $k_2(\mathbf{y}^m(p))$ for strategy $\mathbf{y}^m(p)$. The function $k_2(\mathbf{y}^m(p)) = k_2(p)$ is piece-wise constant over $[0,1]$.

Remark 3.3. If $m \geq 5$, then the function $k_2(p)$ has no jump at $p \in P^m \cup Q^m$. But the set $P^m \cup Q^m$ does not cover the set of all peak points p without a jump of $k_2(p)$.

Here we describe an ordering of two subset of peak points such that the function $k_2(p)$ has a jump at these points:

$$S^m = \{s_3, \ldots, s_{m-1}\}, \qquad p_2 < s_3 < p_3, ..., p_{m-2} < s_{m-1} < p_{m-1}.$$

At the point s_i the bid i appears at the spectrum of the optimal strategy of Player 2.

$$T^m = \{t_4, \ldots, t_{m-1}\} \qquad q_3 > t_4 > q_4, ..., q_{m-2} > t_{m-1} > q_{m-1} = p_{m-1}.$$

At the point t_{m-r}, $r = 2, \ldots, m-4$ the bid $m-r$ quits the spectrum of the optimal strategy of Player 2.

Remark 3.4. For $m = 5$ the combination $P^5 \cup Q^5 \cup S^5 \cup T^5$ coincides with the whole set of peak points $V_1^5(p)$.

Definition 3.1. We call a lacuna of a strategy spectrum the set of successive bids that player does not use in this strategy, while using greater and smaller bids with positive probability.

Note that for $m \leq 5$ there are no lacunas in the optimal strategy spectra except of either $\{1\}$ or $\{2\}$. For $m > 5$ the structure of spectra of optimal strategies is more complicated having various lacunas.

Lemma 3.1. *A spectrum of optimal strategies of any player has no lacunas such that the number of its elements is more than 1 and the first element of the spectrum after the lacuna is less than $m-1$.*

3.2. Solutions for games $G_1^m(p)$

Here we restrict ourselves to description solutions of games $G_1^m(p)$ for $p \in (0, p_{m-1})$. The solutions for the interval $(p_{m-1}, 1)$ are analogues and (not strictly speaking) mirror-like with respect to the point p_{m-1}.

We use the following numeration for the linearity intervals of value function $V_1^m(p)$:

$$I_0 = I_{1,0} = [0, p_1], \quad I_1 = I_{1,1} = I_{2,0} = [p_1, p_2] \quad \text{and} \quad I_2 = I_{2,1} = [p_2, s_3];$$

$$I_{k,0} = [s_k, p_k], \quad I_{k,1} = [p_k, s_{k+1}] \quad \text{and} \quad I_k = I_{k,0} \cup I_{k,1}, \quad k = 3, \ldots, m-1.$$

The following proposition describes the spectra of optimal strategies over intervals I_k.

Proposition 3.2. *For $p \in I_0$, Player 2 uses the bid 0. Player 1 uses the bid 1. For $p \in I_1 \cup I_2$, Player 1 uses the bids 1 and 2. For $p \in I_1 = I_{2,0}$, Player 2 uses the bids 0 and 1. For $p \in I_2 = I_{2,1}$, Player 2 uses the bids 0 and 2.*
For $p \in I_k$, $k > 2$, Player 1 uses the bids $1, 2, 3, \ldots, k$. The maximal bid of Player 2 is k.
For $p \in I_{k,0}$, $k = 3, \ldots, m-1$, Player 2 uses the bids $0, 2, 3, \ldots, k$, if the number k is odd, and the bids $0, 1, 3, \ldots, k$, if k is even.
For $p \in I_{k,1}$, $k = 3, \ldots, m-2$, Player 2 uses the bids $0, 1, 3, \ldots, k$, if the number k is odd, and the bids $0, 2, 3, \ldots, k$, if k is even.

Let $v_{k,i}^H$ and $v_{k,i}^L$ be the gains of Player 1 for the state H and for the state L corresponding to the best reply of Player 1 to the optimal strategy of Player 2 for $p \in I_{k,i}$.

The following theorem provides the recurrent description of value function $V_1^m(p)$ for any linearity domain.

Theorem 3.1. *For $p \in I_{k,i}$,*

$$V_1^m(p) = v_{k,i}^L(1-p) + v_{k,i}^H p,$$

where

$$v_{1,0}^L = 0, \quad v_{1,0}^H = m-1, \quad v_{2,0}^L = \frac{1}{m-1}, \quad v_{2,0}^H = m-2,$$

$$v_{2,1}^L = \frac{2}{m-1}, \quad v_{2,1}^H = \frac{(m-2)^2}{(m-1)},$$

and for $k = 3, \ldots, m-2$, $i = 0, 1$, payoffs $v_{k,i}^H$ and $v_{k,i}^L$ are given by the recurrent formulas

$$v_{k,i}^H = \frac{(m-k)^2}{v_{k-1,i+1}^H}, \quad v_{k,i}^L = (v_{k-1,i+1}^L - k)\left(\frac{m-k}{v_{k-1,i+1}^H}\right) + k,$$

Here $i+1$ is calculated modulo 2.

Corollary 3.3. *For any point $p \in (0,1)$ the inequality*

$$V_1^m(p) \leq m \cdot p(1-p) \tag{3.1}$$

holds. According to Proposition 3.1 for any $p \in P^m \cup Q^m$ it turns to be the equality.

Remark 3.5. As the value of one-stage bidding game with arbitrary bids is equal to $m \cdot p(1-p)$, see De Meyer, Saley (2002), the inequality (3.1) implies that this value exceeds the value of one-stage bidding game with discrete bids.

4. Solutions for games $G_n^3(p)$ and recursive sequences

The problem of solution for the n-step games $G_n^m(p)$ still remains open. The case of two admissible bids ($m = 2$) is trivial: the optimal strategy of Player 1 for any a priori probability p is to choose at the first step action 0 in the state L and action 1 in the state H, The both actions of Player 1 are "revealing" and the true price of a share is revealed by Player 2 at the first step.

At the fist step an optimal strategy of Player 2 is to post 1 for $p < 1/2$, and to post 1 for $p > 1/2$. For $p = 1/2$ any of the possible actions or any their probabilistic mixture is optimal. Thus, after the first move the insider's payoff is stabilized, and $V_n^2(p) = V_1^2(p) = \min\{p, 1-p\}$.

In this section we consider the qualitatively more complicated case of three reasonable bids 0, 1 and 2 ($m = 3$). Even the solution for the one-step game $G_1^3(p)$ is nontrivial (see the previous section).

For $m = 3$ the one-step gains of Player 1 are given with the following matrices:

$$A^L = [a_{ij}^L] = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix},$$

$$A^H = [a_{ij}^H] = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We construct the exact solutions for games $G_n^3(p)$ in the explicit form for any number of steps n. The value function $V_n^3(p)$ and the optimal players' strategies are expressed using the second-order recursive sequence δ_n, $n = 0, 1, 2, \ldots$, determined by the recurrence relations

$$\delta_{n+1} = 2(\delta_n + \delta_{n-1}), \quad \delta_0 = 0, \quad \delta_1 = 2. \tag{4.1}$$

The theory of recursive sequences can be used to obtain the analytical expression for the sequences δ_n:

$$\delta_n = \frac{(1+\sqrt{3})^n - (1-\sqrt{3})^n}{\sqrt{3}}.$$

We show that the piecewise linear continuous concave value function $V_n^3(p)$ of the game $G_n^3(p)$ has three non-smoothness points on the interval $(0, 1)$: $1/3$, $p_n \in (1/3, 2/3)$ and $2/3$, where

$$p_n = (\delta_{n-1} + \delta_n)/(\delta_{n-1} + 2\delta_n).$$

The values of the function $V_n^3(p)$ at these points are also determined using the recursive sequence δ_n.

We demonstrate that

$$V_n^3(p_n) = \max_{0 \le p \le 1} V_n^3(p),$$

i.e. the maximal payoff from private information is obtained by the insider in the case of the largest initial uncertainty of the partner which for the one-step game takes place for the prior probability of high price $p = 1/2$, and for the n-step game with three admissible bids for $p = p_n$.

The insider controls the sequence of posterior probabilities of high stock price, which are calculated with help of his strategies at the preceding steps. We show that the optimal strategy of the insider in the n-step game generates the posterior probability equal to p_{n-1} after the first step, and the posterior probability equal to p_{n-2} after the second step, etc., and finally, before the last step the probability equal to $1/2$.

The optimal first move of Player 2 for n-step game $G_n^3(p)$ is independent of the exact value of p. It depends only on the fact which linearity interval the prior probability p belongs to. The optimal move of Player 2 at the step $t = 2,' ldots, n$ depends only on the interval which the corresponding posterior probability belongs to.

When $n \to \infty$ the sequences of values $V_n^3(1/3)$, $V_n^3(p_n)$ and $V_n^3(2/3)$ converge to 1. Thus, in the limit, the non-smoothness point p_n disappears and the functions $V_n^3(p)$ converge to the value $V_\infty^3(p)$ of the game with unbounded duration $G_\infty^3(p)$ calculated in section 2.

The next theorem gives the exact formulation of the result.

Theorem 4.1. *The piecewise linear continuous value function $V_n^3(p)$ of the game $G_n^3(p)$ on the interval $[0,1]$ has three non-smoothness points: $1/3$, p_n, $2/3$. The function $V_n^3(p)$ is determined by its values at the ends of the interval $V_n^3(0) = V_n^3(1) = 0$ and the peak points:*

$$V_n^3(p) = \begin{cases} 1 - 2/3\delta_n & for\ p = 1/3, \\ 1 - 1/(2\delta_n + \delta_{n-1}) & for\ p = p_n, \\ 1 - 1/3\delta_{n-1} & for\ p = 2/3, \end{cases}$$

and

$$V_n^3(p) = \begin{cases} (3 - 2/\delta_n)p, & if\ p \in [0, 1/3], \\ (1 - 1/\delta_n)(1-p) + p, & if\ p \in [1/3, p_n], \\ (1 + 1/\delta_{n-1})(1-p) + (1 - 1/\delta_{n-1})p, & if\ p \in [p_n, 2/3], \\ (3 - 1/\delta_{n-1})(1-p), & if\ p \in [2/3, 1]. \end{cases}$$

Both players have the optimal strategies σ^{*n} è τ^{*n}, which on the four corresponding linearity intervals of the function V_2^3, enumerated by the Roman figures I, II, III, IV, have the following structure:

I. The interval $p \in [0, 1/3]$. The first move of the strategy $\sigma^{*n}(p, I)$ is

$$\sigma_1^{*n}(L, p, I) = (1 - 2p\delta_{n-1}/(1-p)\delta_n, 2p\delta_{n-1}/(1-p)\delta_n, 0), \quad \sigma_1^{*n}(H, p, I) = (0, 1, 0).$$

The first move $\tau_1^{*n}(I)$ of the strategy $\tau^{*n}(I)$ is $(1, 0, 0)$.

The continuation $\tau^{*n}(\cdot|i, I)$ of the strategy $\tau^{*n}(I)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i, I) = \begin{cases} \tau^{*(n-1)}(I), & \text{if } i = 0, \\ (\tau^{*(n-1)}(II)\delta_{n-1} + \tau^{*(n-1)}(III)\delta_{n-2})/(\delta_{n-1} + \delta_{n-2}), & \text{if } i = 1. \end{cases}$$

II. The interval $p \in [1/3, p_n]$. The first move of the strategy $\sigma^{*n}(p, II)$ is

$$\sigma_1^n(L, p, II) = (1 - \delta_{n-1}/\delta_n, \delta_{n-1}/\delta_n, 0), \quad \sigma_1^n(H, p, II) = (0, (1-p)/2p, (3p-1)/2p).$$

The first move $\tau_1^{*n}(II)$ of the strategy $\tau^{*n}(II)$ is $(1/\delta_n, 1 - 1/\delta_n, 0)$.

The continuation $\tau^{*n}(\cdot|i,II)$ of the strategy $\tau^{*n}(II)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,II)=\begin{cases}\tau^{*(n-1)}(I), & \text{if } i=0,\\ (\tau^{*(n-1)}(II)\delta_{n-1}+\tau^{*(n-1)}(III)\delta_{n-2})/(\delta_{n-1}+\delta_{n-2}), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

III. The interval $p\in[p_n,2/3]$. The first move of the strategy $\sigma^{*n}(p,III)$ is

$$\sigma_1^n(L,p,III)=((2-3p)/(1-p),(2p-1)/(1-p),0),$$

$$\sigma_1^n(H,p,III)=(0,(2p-1)\delta_n/2p\delta_{n-1},1-(2p-1)\delta_n/2p\delta_{n-1}).$$

The first move $\tau_1^{*n}(III)$ of the strategy $\tau^{*n}(III)$ is $(0,1-1/\delta_{n-1},1/\delta_{n-1})$.

The continuation $\tau^{*n}(\cdot|i,III)$ of the strategy $\tau^{*n}(III)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,III)=\begin{cases}\tau^{*(n-1)}(I), & \text{if } i=0,\\ \tau^{*(n-1)}(II), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

IV. The interval $p\in[2/3,1]$. The first move of the strategy $\sigma^{*n}(p,IV)$ is

$$\sigma_1^n(L,p,IV)=(0,1,0),\quad \sigma_1^n(p,H)=(0,(1-p)\delta_n/2p\delta_{n-1},1-(1-p)\delta_n/2p\delta_{n-1}).$$

The first move $\tau_1^{*n}(IV)$ of the strategy $\tau^{*n}(IV)$ is $(0,0,1)$.

The continuation $\tau^{*n}(\cdot|i,IV)$ of the strategy $\tau^{*n}(IV)$ after observation of the bid i is determined by the relations

$$\tau^{*n}(\cdot|i,IV)=\begin{cases}\tau^{*(n-1)}(II), & \text{if } i=1,\\ \tau^{*(n-1)}(IV), & \text{if } i=2.\end{cases}$$

5. Analysis of lower bounds for values $V_n^m(p)$ of games $G_n^m(p)$.

In section 2 we constructed the Player 1' fastest optimal strategy $\bar\sigma^m$ for the bidding game $G_\infty^m(p)$ of unlimited duration (see Definition 2.2.). The strategy σ^m provides Player 1 the maximal possible expected gain 1/2 per step. For this strategy the posterior probabilities perform a simple random walk over the grid l/m, $l=0,\dots,m$, with absorbing extreme points 0 and 1. At the random time Θ^m of absorption of posterior probabilities revealing the true share value by Player 2 occurs. For the initial probability k/m, the expected duration $\beta_\infty^m(k)=\mathbf{E}_k[\Theta^m]$ of this random walk before absorption is $k(m-k)$, where $\mathbf{E}_k$ is the expectation for the random walk starting at the point k/m.

For the n-stage game $G_n^m(p)$ the strategy $\bar\sigma^m$ ensures the Player 1' gain that does not exceed $L_n^m(p)$. The Player 1' guaranteed gain is equal to $L_n^m(p)$ if he uses the strategy $\bar\sigma^m$.

The continuous, concave, and piecewise linear lower bound $L_n^m(p)$ for value $V_n^m(p)$ at its peak points k/m is given with recursive formulas (section 2, Theorem 2.2).

In this section we obtain an explicit formula for $L_n^m(p)$, i.e. for the guaranteed gain of Player 1 in the n-stage game if he applies his optimal strategy σ^m for the game $G_\infty^m(p)$ of unlimited duration. Let $W_n^m(\sigma,\tau|p)$ be the payoff function of the game $G_n^m(p)$.

Theorem 5.1. *If Player 1 exploits the strategy σ^m in the game $G_n^m(k/m)$, then his guaranteed gain $L_n^m(k/m) = \inf_\tau W_n^m(\sigma^m, \tau|k/m)$ is given with the formula*

$$L_n^m(k/m) = \frac{(m-k)k}{2} - \varepsilon_n^m(k), \tag{5.1}$$

where

$$\varepsilon_n^m(k) = \frac{1}{2m} \sum_{l=1}^{[m/2]} \cos^n \frac{\pi(2l-1)}{m} \sin \frac{\pi k(2l-1)}{m} \mathrm{ctg} \frac{\pi(2l-1)}{2m} \left(1 + \mathrm{ctg}^2 \frac{\pi(2l-1)}{2m}\right), \tag{5.2}$$

with $[\alpha]$ being the integer part of α.

Sketch of the proof. Let $\beta_n^m(k) = \mathbf{E}_k[\Theta^m \wedge n]$ denote the average number of steps of the simple random walk of posterior probabilities starting at the point k/m in the n-stage game $G_n^m(p)$. Then the expected insider's profit is given by $W_n^m(k) = \frac{1}{2}\beta_n^m(k)$.

The recursive equations for $\beta_n^m(k)$ hold $\beta_{n+1}^m(k) = \frac{1}{2}\beta_n^m(k+1) + \frac{1}{2}\beta_n^m(k-1) + 1$ with the boundary conditions $\beta_n^m(0) = \beta_n^m(m) = 0$ and with the initial condition $\beta_0^m(k) = 0$.

The values $\beta_\infty^m(k)$ satisfy the equations $\beta_\infty^m(k) = \frac{1}{2}\beta_\infty^m(k+1) + \frac{1}{2}\beta_\infty^m(k-1) + 1$ with the boundary conditions $\beta_\infty^m(0) = \beta_\infty^m(m) = 0$.

Thus the differences $\varepsilon_n^m(k) = \frac{1}{2}\left(\beta_\infty^m(k) - \beta_n^m(k)\right)$ satisfy the homogeneous recursive equations

$$\varepsilon_{n+1}^m(k) = \frac{1}{2}\varepsilon_n^m(k+1) + \frac{1}{2}\varepsilon_n^m(k-1)$$

with the boundary conditions $\varepsilon_n^m(0) = \varepsilon_n^m(m) = 0$ and with the initial condition $\varepsilon_0^m(k) = \beta_\infty^m(k)/2$.

Solving these equations we obtain the representation (5.2) for $\varepsilon_n^m(k)$.

Corollary 5.1. *The strategy σ^m is a ε_n^m-optimal strategy of Player 1 for the finitely repeated game $G_n^m(p)$ of length n, where $\varepsilon_n^m = O(\cos^n \pi/m)$, i.e. the "error term" $\varepsilon_n^m(k)$ decreases exponentially.*

This is not so for slower optimal strategies of Player 1.

The case m=3.

For $m = 3$ the above result means

$$\varepsilon_n^3(k) = \frac{1}{2^n}, \quad k = 1, 2.$$

As the exact solutions for the n-stage games $G_n^3(p)$ are known (see section 4), we may refine the values of the "error term" estimating the difference between the value $V_n^3(p)$ and the lower bound $L_n^3(p)$, not only $(V_\infty^3(p) - L_n^3(p))$.

In section 4 the value functions $V_n^3(p)$ are expressed by means of a second-order recursive sequence. They converge to the value $V_\infty^3(p)$ of the game with unbounded duration $G_\infty^3(p)$. Using the theory of recurrent sequences it is easy to estimate function $V_n^3(p)$ at the peak points $p = 1/3$ and $p = 2/3$,

$$V_n^3(1/3) \approx 1 - \frac{2}{\sqrt{3}(1+\sqrt{3})^n}, \qquad V_n^3(2/3) \approx 1 - \frac{1+\sqrt{3}}{\sqrt{3}(1+\sqrt{3})^n},$$

and to get the refined values

$$\bar{\varepsilon}_n^3(1) = (V_n^3(1/3) - L_n^3(1/3)) \approx \frac{1}{2^n} - \frac{2}{\sqrt{3}(1+\sqrt{3})^n},$$

$$\bar{\varepsilon}_n^3(2) = (V_n^3(2/3) - L_n^3(2/3)) \approx \frac{1}{2^n} - \frac{1+\sqrt{3}}{\sqrt{3}(1+\sqrt{3})^n}.$$

So for sufficiently large n the optimal strategy of the insider for the bidding game of infinite duration is a rather good approximation of his optimal strategy for the n-stage game.

6. Bidding games $G_n(\mathbf{p})$ and $G_\infty(\mathbf{p})$ with countable state space

In this section we consider the model where any integer non-negative bids are admissible and the liquidation price of a share $C_{\mathbf{p}}$ may take any nonnegative integer values $k = 0, 1, 2, \ldots$ according to a probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$.

At stage 0 a chance move determines the liquidation value of a share for the whole period of bidding n according to the probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ over the one-dimensional integer lattice, $S = \mathbb{Z}_+$. Structure of information and trading mechanism are the same as in section 2 for the case of two possible states of nature.

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information of Player 2 and with countable state space $S = \mathbb{Z}_+$ and with countable action spaces $I = \mathbb{Z}_+$ and $J = \mathbb{Z}_+$. One-step gains of Player 1 are given with the matrices $A^s = [a^s(i,j)]_{i \in I, j \in J}$, $s \in S$,

$$a^s(i,j) = \begin{cases} j - s, & \text{for } i < j \ ; \\ 0, & \text{for } i = j \ ; \\ -i + s, & \text{for } i > j \ . \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a^s(i_t, j_t).$$

This description is common knowledge to both Players. The games $G_n^m(p)$ considered in section 2 represent particular cases of these games corresponding to probability distributions with two-point supports, $p_0 = 1 - p$ and $p_m = p$.

Theorem 6.1. *If the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}]$, then the values $V_n(\mathbf{p})$ of n-stage games $G_n(\mathbf{p})$ exist The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps n increases.*

The theorem follows from the fact that for this case the payoff of game $G_n(\mathbf{p})$ can be approximated by payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support.

6.1. Upper bound for values $V_n(\mathbf{p})$

If the variance $\mathbf{D}[C_{\mathbf{p}}]$ is infinite, then, as n tends to ∞, the sequence $V_n(\mathbf{p})$ diverges.

The next theorem demonstrates that on the contrary, if the variance $\mathbf{D}[C_{\mathbf{p}}]$ is finite, then, as n tends to ∞, the sequence of values $V_n(\mathbf{p})$ of the games $G_n(\mathbf{p})$ is bounded from above.

Theorem 6.2. *For* $\mathbf{p}$ *such that* $\mathbf{D}[C_{\mathbf{p}}] < \infty$*, the values* $V_n(\mathbf{p})$ *are bounded from above by a continuous, concave, and piecewise linear function* $H(\mathbf{p})$*. Its domains of linearity are*

$$L(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] \in [k, k+1]\}, \quad k = 0, 1, \ldots.$$

Its domains of non-smoothness are

$$\Theta(k) = \{\mathbf{p} : \mathbf{E}[\mathbf{p}] = k\}.$$

The equality holds

$$H(\mathbf{p}) = (\mathbf{D}[\mathbf{p}] - \alpha(\mathbf{p})(1 - \alpha(\mathbf{p})))/2, \tag{6.1}$$

where $\alpha(\mathbf{p}) = \mathbf{E}[\mathbf{p}] - ent[\mathbf{E}[\mathbf{p}]]$ *and* $ent[x]$*,* $x \in R^1$ *is the integer part of* x*.*

The result is provided by a "reasonable" strategy of Player 2. The strategy is analogous to his optimal strategy for two-states game $G_n^m(p)$ (see section 2): at the first move Player 2 posts $ent[\mathbf{E}[\mathbf{p}]]$ and then his moves depend on the last observed pair of actions only.

6.2. Solutions for games $G_\infty(\mathbf{p})$ with arbitrary p

As the sequence $V_n(\mathbf{p})$ is bounded from above, it is reasonable to consider the games $G_\infty(\mathbf{p})$ with infinite number of steps. We show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$. We get solutions for these games in the explicit form.

The optimal strategy of Player 2 is his "reasonable" strategy mentioned above. We construct the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution having an integer expectation on the base of the solutions for the games with two-point distributions obtained in section 2. The result is due to the symmetric representation of distributions over the one-dimensional integer lattice with fixed integer mean values as convex combinations (probability mixture) of distributions with two-point supports and with the same mean values (see, e.g. Obloy, 2004).

Symmetric representation of distributions over the one-dimensional integer lattice. Let $\mathbf{p}$ be a probability distribution over the set of integers $\mathbb{Z}^1$ with mean value equal to an integer r. Then

$$\mathbf{p} = p_r \cdot \delta^r + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p_{r+t}} p_{r-l} p_{r+k} \cdot \mathbf{p}_{r+k,r-l}^r, \tag{6.2}$$

where $\mathbf{p}_{r+k,r-l}^r$ is the probability distribution with the two-point support $r-l, r+k$ and with mean value equal to r.

We treat coefficients

$$\mathbf{P}_{\mathbf{p}}(\mathbf{p}_{r+k,r-l}^r) = \frac{k+l}{\sum_{t=1}^{\infty} t \cdot p_{r+t}} p_{r-l} p_{r+k}$$

of decomposition (6.2) as probabilities of corresponding distributions with two-point supports $(r+k)$, $(r-l)$ in this probability mixture.

Given one point z (equal to $r+k$ or to $r-l$) in the support of two-point distribution, the conditional probability of complementary point ($r-l$ or $r+k$) may be calculated

$$\mathbf{P}_{\mathbf{p}}(r+k|r-l) = \frac{k \cdot p_{r+k}}{\sum_{t=1}^{\infty} t \cdot p_{r+t}}, \quad \mathbf{P}_{\mathbf{p}}(r-l|r+k) = \frac{l \cdot p_{r-l}}{\sum_{t=1}^{\infty} t \cdot p_{r+t}}. \tag{6.3}$$

Player 1' optimal strategy σ^*. We construct Player 1' optimal strategy σ^* for the game $G_\infty(\mathbf{p})$ making use of the obtained decomposition for the initial distribution $\mathbf{p}$ with mean value equal to an integer r (the prior expectation of share price is r).
a) If the state chosen by chance move is r, then Player 1 stops the game (Player 1' informational advantage disappears).
b) If chance move chooses $z = r+k$ (or $z = r-l$), where k, l are integer positive numbers, then Player 1 chooses a point $z_2 = r-l$ (or $z_2 = r+k$) by means of lottery with probabilities (6.3) and plays his optimal strategy for the state z in the two-point game $G(\mathbf{p}^r_{r+k,r-l})$ (see section 2).

The described optimal strategy of Player 1 generates a symmetric random walk of posterior mathematical expectations of liquidation price with absorption. The absorption may occur at any stage if the posterior expectation of share price at this stage coincides with its prior expectation. If the liquidation price chosen by the chance move coincides with its prior expectation, then the absorption occurs at the first stage. Note that it is impossible for two-point support distributions.

The expected duration of this random walk is equal to the initial variance of liquidation price. The guaranteed total gain of Player 1 (the value of the game) is equal to this expected duration multiplied with the fixed gain per step.

7. Repeated games with asymmetric information modeling financial markets with two risky assets

In this section we consider multistage bidding models where two types of risky assets are traded. Two players with opposite interests have money and two types of shares. The liquidation prices of both share types may take any integer values x and y. At stage 0 a chance move determines the "state of nature" s and therefore the liquidation prices of shares (s^1, s^2) for the whole period of bidding n according to the probability distribution $\mathbf{p}$ over the two-dimensional integer lattice known to both Players. Player 1 is informed about the result of chance move z, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each step of bidding both players simultaneously make their integer bids, i.e. they post their prices for each type of shares. The player who posts the larger price for a share of a given type buys one share of this type from his opponent at this price. Any integer bids are admissible. Players aim to maximize the values of their final portfolios, calculated as money plus obtained shares evaluated by their liquidation prices.

The described model of n-stage bidding is reduced to the zero-sum repeated game $G_n(\mathbf{p})$ with lack of information on one side and with two-dimensional one-step actions with components corresponding to bids for each type of assets. The countable state space is $S = \mathbb{Z}^2$ and the countable action spaces are $I = \mathbb{Z}^2$ and $J = \mathbb{Z}^2$. The one-step gain $a(s, i, j)$ of Player 1 corresponding to the state $s = (s^1, s^2)$ and the actions $i = (i^1, i^2)$ and $j = (j^1, j^2)$ is given with the sum

$\sum_{e=1}^{2} a^e(s^e, i^e, j^e)$, where

$$a^e(s^e, i^e, j^e) = \begin{cases} j^e - s^e, & \text{for } i^e < j^e; \\ 0, & \text{for } i^e = j^e; \\ -i^e + s^e, & \text{for } i^e > j^e. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a(s, i_t, j_t),$$

where s is the result of a chance move. This description is a common knowledge of both Players.

It is easy to show that if the expectations of share prices are finite, then the value of such n-stage bidding game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

We show that, if liquidation prices of both shares have finite variances, then the value $V_n(\mathbf{p})$ of n-stage bidding games does not exceed the function $H(\mathbf{p})$ which is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game $G_\infty(\mathbf{p})$. We give the solutions for these games with arbitrary probability distributions over the two-dimensional integer lattice with finite component variances.

Both players have optimal strategies. The optimal strategy for Player 2 is a direct combination of his optimal strategies for the games with one-type of risky asset (see section 6).

We begin with constructing Player 1' optimal strategies for games $G_\infty(\mathbf{p})$ with distributions $\mathbf{p}$ having two- and three-point supports – elementary games. Next, using symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values (Domansky, 2013), we build the optimal strategies of Player 1 for bidding games $G_\infty(\mathbf{p})$ with arbitrary distributions $\mathbf{p}$ as convex combinations of his optimal strategies for elementary games.

The optimal strategy of Player 1 generates a random walk of transaction prices. But unlike the case of one-type assets, the symmetry of this random walk is broken at the final stages of the game.

We show that this game terminates naturally when the posterior expectations of both liquidation prices come close enough to their real values. We demonstrate that the value $V_\infty(\mathbf{p})$ coincides with $H(\mathbf{p})$. So it is equal to the sum of values of corresponding games with one-type risky asset. Thus, the profit that Player 2 gets under simultaneous n-step bidding in comparison with separate bidding for each type of shares disappears in a game of unbounded duration.

7.1. Solutions for games $G_\infty(\mathbf{p})$ with p having two-point supports

For games $G_\infty(\mathbf{p})$ with the support of distribution $\mathbf{p}$ containing two states, we show that the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$.

To construct optimal strategies σ^* of Player 1 for games $G_\infty(\mathbf{p})$ with two states we use the results for games with one-type assets and with two states. But the fastest optimal strategy of Player 1 described in section 2 is not sufficient for this purpose. We use Player 1' slower optimal strategies.

Without loss of generality we assume that one of support points is $(0, 0)$. Thus there are two states $0 = (0, 0)$ and $z = (x, y)$, where x and y are integers and $x > 0$. The distribution $\mathbf{p}$ can be depicted with a scalar parameter $p \in [0, 1]$ being the probability of state z. For definiteness set $y > 0$.

The strategy σ^* of Player 1 generates an asymmetric random walk of posterior probabilities by adjacent points of the irregular lattice

$$Lat(x, y) = \{k/x, k = 0, \ldots, x\} \cup \{l/y, l = 0, \ldots, y\}$$

formed with those probabilities where at least one of the price expectations has an integer value. The probabilities of jumps provide martingale characteristics of posterior probabilities and with absorption at extreme points 0 and 1.

7.2. Solutions for games $G_\infty(\mathbf{p})$ with p having three-point supports

We construct optimal strategies σ^* of Player 1 that ensure $H(\mathbf{p})$ for games $G_\infty(\mathbf{p})$ with three states $z_1, z_2, z_3 \in \mathbf{Z}^2$.

Denote $\triangle(z_1, z_2, z_3)$ the triangle spanned across the support points of distribution. A distribution $\mathbf{p}$ with the support z_1, z_2, z_3 is uniquely determined with a vector $w = (u, v) \in \triangle(z_1, z_2, z_3)$ of expectations of coordinates (the barycenter of distribution $\mathbf{p}$). Denote it $\mathbf{p}^w_{z_1,z_2,z_3}$.

For $\mathbf{p}^w_{z_1,z_2,z_3}$ the first step of optimal strategy σ^* may efficiently use the actions $(u-1, v-1)$, $(u, v-1)$, $(u-1, v)$ and (u, v). With the help of these actions Player 1 can perform moves such that the modulus of difference between posterior expectations of each coordinate and its initial expectation is not more than one.

There are several types of optimal first moves of Player 1, in particular, the first moves σ_1^{NE-SW} (north-east – south-west), σ_1^{NW-SE}, and their probabilistic mixtures. Denote $e = (1, 1), \bar{e} = (1, -1)$. The first move σ_1^{NE-SW} exploits only two actions $w - e$ and w with posterior expectations $w - b \cdot e$ and $w + a \cdot e$. The first move σ_1^{NW-SE} makes use of actions $(u - 1, v)$ and $(u, v - 1)$ with posterior expectations $w - b\bar{e}$ and $w + a\bar{e}$.

The martingale of posterior expectations generated by the optimal strategy of Player 1 for the game $G_\infty(\mathbf{p}^w_{z_1,z_2,z_3})$ represents a symmetric random walk over points of integer lattice lying within the triangle $\triangle(z_1, z_2, z_3)$.

The symmetry is broken at the moment that the walk hits the triangle boundary. From this moment, the game turns into one of games with distributions having two-point supports.

7.3. Solutions for games $G_\infty(\mathbf{p})$ with arbitrary p

We construct Player 1's optimal strategy for the game $G_\infty(\mathbf{p})$ with an arbitrary distribution $\mathbf{p}$ having an integer expectation vector (k, l), as a convex combination (a probability mixture) of his optimal strategies for games with distributions having not more than three-point supports and the same expectation vector (k, l).

To realize the idea we use symmetric representations of probability distributions over the two-dimensional plane with given mean values as convex combinations of elementary distributions – distributions with supports containing not more than three points and with the same mean values (Domansky, 2013).

This decomposition is a generalization of the analogous decomposition of one-dimensional distributions into a convex combination of distributions with no more than two-point supports and with the same expectation that was used in section 6 for constructing solutions for bidding games with a one-type risky asset.

The coefficient at an elementary distribution may be regarded as its probability in this probability mixture. Given one point z in the support of elementary distribution, the conditional probability of any elementary distribution having z in its support may be calculated. Then we obtain the conditional probability $\mathbf{P_p}(2|z)$ of elementary two-point support distributions and the conditional probability $\mathbf{P_p}(3|z)$ of elementary three-point support distributions. For constructing the optimal Player 1' strategy we use also conditional probabilities $\mathbf{P_p}(z_2|z,2)$ of a complementary point z_2 for the two-point support (z, z_2) and conditional probabilities $\mathbf{P_p}(z_2, z_3|z, 3)$ of complementary points z_2, z_3 for the three-point support (z, z_2, z_3).

The optimal strategy of Player 1 is given by the following algorithm:

1. If the state $z = (x, y)$ chosen by chance move coincides with the price expectation vector, $(x, y) = (k, l)$, then Player 1 stops the game. In this case he cannot receive any profit from his informational advantage.

2. If not, $z = (x, y) \neq (k, l)$, then Player 1' optimal strategy is constructed with help of a two-stage lottery.

a) To choose between two-point and three-point distributions Player 1 realizes the Bernoulli trial with probabilities $\mathbf{P_p}(2|z)$ and $\mathbf{P_p}(3|z)$.

b) If two-point distributions are chosen, then Player 1 plays his optimal strategy in a game with two-point support (z, z_2) choosing a complementary point z_2 by means of the lottery with conditional probabilities $\mathbf{P_p}(z_2|z, 2)$.

If three-point distributions are chosen, then Player 1 plays his optimal strategy in a game with three-point support (z, z_2, z_3) choosing two complementary points by means of the lottery with conditional probabilities $\mathbf{P_p}(z_2, z_3|z, 3)$.

8. Bidding models with non-zero bid-ask spread

We generalize bidding models with one-type risky assets investigated in the previous sections where players proposed only one price for a share at each step, i.e. bid and ask prices coincide. Here we drop this restriction. We assume that at each step of bidding both players simultaneously propose their bid and ask prices for one share. The bid-ask spread s is fixed by rules of bidding. Transaction occurs from seller to buyer by bid price. The simplified model (sections 2-7) corresponds to the case $s = 0$ what is equivalent to $s = 1$ due to the price discreteness.

The model is reduced to a repeated game with incomplete information. Depending on bid-ask spread s one-step payoff matrices for these games have more complicated structure to compare with the case $s = 1$.

As for the zero bid-ask spread models we start with the case of two possible states of nature (two possible values for a share price). We generalize the results of section 2 for multistage games: we construct the upper and lower bounds for the values of n-stage games as $n \to \infty$. The bounds coincide for $s = 1$.

We generalize the developed in section 3 recursive approach to solutions of one-stage bidding games (see Sandomirskaya, 2012). The spectrum structure of optimal strategies becomes more complicated as lacunas longer than in the case $s = 1$ appear. The idea of equalizing insider's spectrum and obtaining recurrent relations on weights in the Player 2's optimal strategy remain applicable, however the difficulties concerned with explicit weight representation increase enormously.

Here we go to the case of two-point state of nature and generalize the results of sections 2 for bidding games with bid-ask spread. After this we make necessary comment on how to extend results for two-point state of nature to the case of countable one.

8.1. The model of bidding with two possible values for a share price

As for the zero bid-ask case we start with bidding games with two states of nature: the state m (integer positive) with probability p and the state 0 with probability $1-p$. In this model any integer bids are admissible. For the sake of simplicity we assume that $m \text{mod} s = 0$. A chance move and an information structure of its outcome are the same as for models with zero bid-ask spread.

At each subsequent stage $t = 1, \ldots, n$ of bidding both players simultaneously propose their integer bid prices and integer ask prices for one share. The bid-ask spread s is fixed by rules of bidding. It is the same for both players. Denote i_t a bid price for Player 1 at stage t and j_t a bid price for Player 2 at stage t. Then $i_t + s$ and $j_t + s$ are ask prices for Player 1 and for Player 2 at stage t.

At stage t transaction of one share occurs if and only if an ask price of one player does not exceed a bid price of his opponent, i.e. either $i_t + s \leq j_t$, or $j_t + s \leq i_t$. If so, then a player-buyer gets one share from his opponent-seller according to his (buyer) bid price.: if $i_t + s \leq j_t$, then at stage t Player 2 buys one share from Player 1 for the price j_t; if $j_t + s \leq i_t$, then at stage t Player 1 buys one share from Player 2 for the price i_t. Thus, at stage t there is no transactions if and only if $|i_t - j_t| < s$.

This n-stage model with the bid-ask spread equal to s is described by a zero-sum repeated game $G_n^{m,s}(p)$ with incomplete information of Player 2 and with countable state and action spaces. The corresponding games $G_n^{m,s}(p)$ are given by the two matrices of one-step payoffs.

$$a^{L,m,s}(i,j) = \begin{cases} -i, & \text{if } i \geq j + s, \\ 0, & \text{if } |i - j| < s, \\ j, & \text{if } j \geq x + s, \end{cases}$$

$$a^{H,m,s}(i,j) = \begin{cases} m - i, & \text{if } i \geq j + s, \\ 0, & \text{if } |i - j| < s, \\ -m + j, & \text{if } j \geq i + s. \end{cases}$$

For $s = 0$, zero elements of the matrices appear at the principal diagonal only. For $s > 1$, zero elements fill a "band of s-range" along the principal diagonal. For $s > 1$ the more complicated structure of payoff matrices makes an analysis of games $G_n^{m,s}(p)$ more difficult.

8.2. Upper and lower bounds for the game value $V_n^{m,s}(p)$

Following the guideline of section 2 we get upper and lower bounds for for value function $V_n^{m,s}(p)$ provided by a "reasonable" strategy of Player 2 and a "reasonable" strategy of Player 1.

Upper bound for $V_n^{m,s}(p)$.

Theorem 8.1. *For any number of steps n functions $V_n^{m,s}$ are bounded from above by a function $H^{m,s}$ that is continuous, concave, and piecewise linear with m/s linearity domains $[sk/m, s(k+1)/m]$, $k = 0, 1, \dots, m/s-1$. The function $H^{m,s}$ is completely determined with the values at its peak points $p_k = sk/m$, $k = 0, 1, \dots, m/s$:*

$$H^{m,s}(p_k) = \frac{m^2}{2s} p_k(1 - p_k). \tag{8.1}$$

To prove the theorem we construct the following "reasonable" strategy $\tau^{m,s}$ of Player 2 that is an analogue of his optimal strategy in the game of infinite duration with $s = 1$ (see section 2).

For the initial probability $p \in \left[\frac{sk}{m}, \frac{s(k+1)}{m}\right)$ the first move of Player 2 strategy $\tau^{m,s}$ is to propose the bid price sk. Then at step t, $t = 2, 3, \dots$, Player 2 shifts his bid price by s upwards or downwards depending on the insider's bid at the previous step:

$$\tau_t^{m,s}(i_{t-1}, j_{t-1}) = \begin{cases} j_{t-1} - s, & \text{if } i_{t-1} \le j_{t-1} - s; \\ j_{t-1}, & \text{if } |i_{t-1} - j_{t-1}| < s \\ j_{t-1} + s, & \text{if } i_{t-1} \ge j_{t-1} + s; \end{cases}$$

As the values $V_n^{m,s}$ are bounded from above as $n \to \infty$, the consideration of games with infinite number of steps becomes reasonable.
Lower bound for $V_\infty(p)$.

Theorem 8.2. *The function $V_\infty^{m,s}$ is bounded from below by a function $L^{m,s}$ that is continuous, concave, and piecewise linear with m/s linearity domains $[sk/m, s(k+1)/m]$, $k = 0, 1, \dots, m/s-1$. The function $L^{m,s}$ has the following values at the peak points $p_k = sk/m$, $k = 0, 1, \dots, m/s$:*

$$L^{m,s}(p_k) = V_1(s) \frac{m^2}{s^2} p_k(1 - p_k). \tag{8.2}$$

Value $V_1(s)$ is a guaranteed insider's gain per step, explicit formula will be given a few below.

Remark 8.1. The obtained upper and lower bounds have the same form.

Sketch of the proof for Theorem 8.2. As for the case $s = 1$ the Player 1' optimal strategy in the game of infinite duration generates the simple random walk (SRW) on the lattice of posterior probabilities of share prices, for the case $s > 1$ it is natural to investigate the class Σ^{SRW} of strategies with SRW-property on the lattice $\left\{\frac{sk}{m} \mid k = 0, ..m/s\right\}$ corresponding to the case $s > 1$.

Below we construct the best strategy in the class Σ^{SRW} and show that this strategy provides the result of Theorem 8.2.

To determine this strategy we use the following notation,

$$g(d) = \frac{1}{s} + \frac{1}{s-1} + \ldots + \frac{1}{s-d},$$

$$d^* = \max\{d \mid g(d) \le 1\},$$

$$\varepsilon^* = 1 - g(d^*).$$

For probability $p_k = sk/m$ the first move of insider's strategy $\sigma^{k,m,s}$ is to mix bid prices $\{sk - 2s\}$ and $\{sk, sk+1, \ldots, sk+d^*, sk+d^*+1\}$ in accordance with total probabilities

$$\sigma_1^{k,m,s}(sk - 2s|H) = \tfrac{1}{2},$$
$$\sigma_1^{k,m,s}(sk + d|H) = \tfrac{1}{2(s-d)}, \quad d = 0, 1, \ldots, d^*,$$
$$\sigma_1^{k,m,s}(sk + d^* + 1|H) = \tfrac{1}{2}\varepsilon^*.$$

Conditional probabilities of these bids are calculated so that corresponding posterior probabilities of high share price will be the following

$$p(i = sk - 2s) = s(k-1)/m = p_{k-1},$$
$$p(i = sk + d) = s(k+1)/m = p_{k+1}, \quad d = 0, 1, \ldots, d^*, d^* + 1.$$

At the next step insider must apply the same strategy, but for the posterior probability calculated at the previous step.

This strategy generates the simple random walk over the lattice sk/m with absorption at extreme points, insider's profit per step being equal to $V(1)$ given by

$$V_1(s) = \frac{1}{2}\left(d^* + 1 + \varepsilon^*(s - d^* - 1)\right). \tag{8.3}$$

It is the best strategy in the class Σ^{SRW}.

Remark 8.2. For the minimal nontrivial case $s = 2$ the constructed "reasonable" strategy of insider is not his optimal strategy for the game of infinite duration.

Therefore, we conclude that the insider's optimal strategy does not generate simple random walk of price expectations and leads apparently to non-symmetric price fluctuations.

Relationships between upper and lower bounds. For the case $s = 1$, the obtained upper and lower bounds coincide and give the value function of bidding game of unlimited duration at its peak points $p_k = k/m$:

$$H^{m,1}(p_k) = L^{m,1}(p_k) = \frac{m^2}{2} p_k(1 - p_k) = V_\infty^{m,1}(p_k).$$

For the case of minimal nontrivial bid-ask spread $s = 2$ the following equality holds at the points $p_k = 2k/m$,

$$L^{m,2}(p_k) = 3/4 H^{m,2}(p_k).$$

As $s \to \infty$ the ratio between L and H decreases and in the limit yields

$$L^{m,s}(p_k) \approx 0,63 \cdot H^{m,s}(p_k).$$

As shown above, bid-ask spread plays a role of regulator for transaction activity on stock market. As bid-ask spread increases transactions occur less frequently and expected insider's profit falls at least by s times to compare with the model without spread.

Generalization of the model with non-zero bid-ask spread to the case of countable set of possible values for a share price The results above are generalized to the case of countable set of possible values for a share price. We analyze the model where this price can take values on the lattice $sk,\ k \in \mathbb{Z}$ by analogy with section 6. The principal idea is to represent distributions on the integer lattice with given first moment as convex combinations (probability mixtures) of two-point distributions with the same first moments. It is shown that upper and lower bounds obtained above preserve their form with replacement of the term $sk(m - sk)$ by the variance $\mathbf{D}(\mathbf{p})$ for distributions with mean values $\mathbf{E}(\mathbf{p}) = sk$, $k \in \mathbb{Z}$. We construct the insider's strategies for these games as probability mixtures of strategies for two-point games implementing a preliminary additional lottery for the choice of two-point distribution.

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Playability Properties in Games of Deterrence and Evolution in the Replicator Dynamics

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Abstract Since the seminal work of John Maynard Smith (1982), a vast literature has developed on evolution analysis through game theoretic tools. Among the most popular evolutionary systems is the Replicator Dynamics, based in its classical version on the combination between a standard non cooperative matrix game and a dynamic system which evolution depends on the payoffs of the interacting species.
Despite its weaknesses, in particular the fact that it does not take into account emergence and development of species that did not initially exist, the Replicator Dynamics has the advantage of proposing a relatively simple model that analyzes and tests some core features of Darwinian evolution.
Nevertheless, the simplicity of the model reaches its limits when one needs to predict accurately the conditions for reaching evolutionary stability. The reason for it is quite obvious: it stems from the possible difficulties to find an analytical solution to the system of equations modelling the Replicator Dynamics.
An alternative approach has been developed, based on matrix games of a different kind, called Games of Deterrence. Matrix Games of Deterrence are qualitative binary games in which selection of strategic pairs results for each player in only two possible outcomes: acceptable (noted 1) and unacceptable (noted 0). It has been shown (Rudnianski, 1991) that each matrix Game of Deterrence can be associated in a one to one relation with a system of equations called the playability system, the solutions of which determine the playability properties of the players' strategies.
Likewise, it has been shown (Ellison and Rudnianski, 2009) that one could derive evolutionary stability properties of the Replicator Dynamics from the solutions of the playability system associated with a symmetric matrix Game of Deterrence on which the Replicator Dynamics is based.
Thus, it has been established that (Ellison and Rudnianski, 2009):

- To each symmetric solution of the playability system corresponds an evolutionarily stable equilibrium set (ESES)
- If a strategy is not playable in every solution of the playability system, the proportion of the corresponding species in the Replicator Dynamics vanishes with time in every solution of the dynamic system

Keywords: evolutionary games, Games of Deterrence, playability, Replicator Dynamics, species, strategies.

Based on these results, the proposed paper will first extend the analysis already undertaken and propose new results in terms of relations between the solutions of the Game of Deterrence playability system and the solutions of the dynamic system.

The paper will then provide a method for systematically modelling standard matrix games as Games of Deterrence, allowing the previous results to be extended to any standard matrix game. In particular, in certain situations where the standard methods for analyzing dynamic systems do not work, the above bridging between standard games and Games of Deterrence will enable to determine the systems' asymptotic behaviour.

More precisely, in a first part, after having briefly recalled the definition of the Replicator Dynamics, the paper will recall the definitions and basic properties of Games of Deterrence.

A second part will distinguish between three categories of strategies in the Game of Deterrence under consideration, and will associate specific evolutionary properties with each one.

The third and last part will then develop an algorithm associating a Game of Deterrence with any standard quantitative symmetric matrix game in a way that will enable to generalize the method to the analysis of quantitative evolutionary games.

1. Replicator Dynamics and Games of Deterrence

1.1. Replicator Dynamics

The Replicator Dynamics is a classical dynamic system describing the evolution of a population broken down into several species. The outcome of the interaction between two individuals is given by a symmetric matrix game G.
Moreover, if $\theta = (\theta_1, \theta_2, ..., \theta_n)$ represents the population's profile (i.e. θ_i is the proportion of species i in the population), then the Replicator Dynamics associated with G is the dynamical system $D(G)$ defined by $\theta_i' = \theta_i(u_i - u_T)$
where:

- $u_i = \sum\limits_k \theta_k u_{ik}$ where u_{ik} represents the payoff of species i when interacting with species k
- $u_T = \sum\limits_i \theta_i u_i$

u_i defines the fitness of species i, and it then stems straightforwardly from the above system of differential equations that the evolution of the proportion of a species i in the population depends on the relative fitness of i with respect to the average fitness of the entire population.

The above classical representation of the Replicator Dynamics is equivalent to the following:

- Let Θ be the space of population profiles
- Let f be a vector field on Θ such that $\theta' = f(\theta)$ with $f_i(\theta) = \theta_i(u_i - u_T)$

An equilibrium of the Replicator Dynamics is then defined as a fixed point of f.

In the following, we will always consider that all species are present in the initial state, i.e. $\forall i \in \{1, ..., n\}, \theta_i(0) \neq 0$

1.2. Games of Deterrence basic properties

Games of Deterrence consider only two possible states of the world:

- Those which are acceptable for the player under consideration (noted 1)
- Those which are unacceptable for that same player (noted 0)

Given that the players' objective is to be in an acceptable state of the world, Games of Deterrence analyze the strategies' playability.

For the sake of simplicity, in the following we shall only consider matrix games, but the definitions that will be introduced extend straightforwardly to N-player games.
Let E and R be two players with respective strategic sets S_E (card $S_E = n$) and S_R (card $S_R = p$).

We shall consider finite bi-matrix games (S_E, S_R, U, V) in normal form where possible outcomes are taken from the set $\{0, 1\}$. More precisely, for any strategic pair $(i, k) \in S_E \times S_R, u_{ik}$ and v_{ik} define the outcomes for player E and R respectively.

A strategy i of E is said to be *safe* iff $\forall k \in S_R, u_{ik} = 1$.
A non-safe strategy is said to be *dangerous*.

Let $J_E(i)$ be an index called *index of positive playability*, such that:
If i is safe then $J_E(i) = 1$
If not, $J_E(i) = (1 - j_E)(1 - j_R) \prod\limits_{k \in S_R} [1 - J_R(k)(1 - u_{ik})]$
With $j_E = \prod\limits_{i \in S_E} (1 - J_E(i))$; and $j_R = \prod\limits_{k \in S_R} (1 - J_R(k))$
If $J_E(i) = 1$, strategy $i \in S_E$ is said to be *positively playable*.
If there are no positively playable strategies in S_E, that is if $j_E = 1$, all strategies $i \in S_E$ are said to be *playable by default*.

Similar definitions apply by analogy to strategies k of S_R.

A strategy in $S_E \cup S_R$ is *playable* iff it is either positively playable or playable by default.
The system P of all equations of $J_E(i), i \in S_E$, $J_R(k), k \in S_R$, j_E and j_R is called *the playability system* of the game.

$\{0, 1\}^{n+p+2}$ is called the *playability set* of P

The playability system P may be considered as a dynamic system $J = \hat{f}(J)$ on the playability set.

A *solution* of the matrix Game of Deterrence is an element of the playability set which is a solution of P.

It has been shown in (Rudnianski, 1991) that any matrix Game of Deterrence has at least one solution, and that in the general case, there is no uniqueness of the solution.

Given a strategic pair $(i, k) \in S_E \times S_R$, i is said to be a deterrent strategy vis-à-vis k iff the three following conditions apply:

- i is playable
- $v_{ik} = 0$
- $\exists k' \in S_R : J_R(k') = 1$

It has been shown (Rudnianski, 1991) that a strategy $k \in S_R$ is playable iff there is no strategy $i \in S_E$ deterrent vis-à-vis k. Thus, the study of deterrence properties amounts to analyzing the playability properties of the strategies.

A *symmetric Game of Deterrence* is a Game of Deterrence (S_E, S_R, U, V) such that $S_E = S_R$ and $U = V^t$ (i.e. $\forall i, k, u_{ik} = v_{ki}$)

In the case of symmetric games, the strategic set will be noted S.
A *symmetric solution* is a solution in which $\forall i \in S, J_E(i) = J_R(i)$

It has been shown (Ellison and Rudnianski, 2009) that in a symmetric Game of Deterrence, $j_E = j_R$

1.3. Deterrence and evolution

It has been shown (Ellison and Rudnianski, 2009) that for a symmetric Game of Deterrence G with playability system P and Replicator Dynamics $D(G)$, if:

- P has a symmetric solution for which no strategy is playable by default
- at $t = 0$, the proportion of each positively playable strategy is greater than the sum of the proportions of the non-playable strategies,

then, whatever the initial profile:

- The proportion of each non-playable strategy decreases exponentially towards zero
- The proportion of each playable strategy has a non-zero limit

This result can be interpreted as follows: each symmetric solution of the playability system is associated with an Evolutionarily Stable Equilibrium Set of the Replicator Dynamics, i.e. the union of the attraction basins of the equilibria is a neighbourhood of the equilibrium set.

2. Further properties of evolutionary Games of Deterrence

2.1. Equivalent strategies and evolution

Definition 1. Two strategies i and j are equivalent if $\forall k \in S, u_{ik} = u_{jk}$

Lemma 1. *If i and j are equivalent, then:*

- $\frac{\theta_i}{\theta_j}$ *is constant in every solution of the Replicator Dynamics*
- i *and* j *have the same playability in every solution of the playability system*

Proof. Since strategies i and j are equivalent, $u_i = u_j$
hence $(ln\frac{\theta_i}{\theta_j})' = (ln\theta_i)' - (ln\theta_j)' = (u_i - u_T) - (u_j - u_T) = 0$

Definition 2. Given a subset X of the strategic set S, let $i, k \in S$,
k is said to be X-dominant vis-à-vis i if $\forall l \in X, u_{il} \leq u_{kl}$.
Likewise, i and k are said to be X-equivalent if i is X-dominant vis-à-vis k and k is X-dominant vis-à-vis i.

X-dominance is a reflexive and transitive relation.

2.2. Categorization of playability system solutions

Let G be a symmetric Game of Deterrence with playability system P.
Let Ψ be a function which associates with any given solution σ of P a partition (A, B, C) of the strategic set S of G such that:

- $A = \{i \in S |\ i$ is positively playable for both players$\}$
- $B = \{i \in S |\ i$ is either positively playable for exactly one player or playable by default for both players$\}$
- $C = \{i \in S |\ i$ is non-playable for both players$\}$

Proposition 1. *If a partition (A, B, C) of S verifies:*

$$\left.\begin{array}{l} i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C) \\ i \in C \Leftrightarrow \exists k \in A : u_{ik} = 0 \end{array}\right\} (C1)$$

then $(A, B, C) \in Im\Psi$

Conversely if $(A, B, C) \in Im\Psi$, then (A, B, C) verifies:

$$\left.\begin{array}{l} i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C) \\ \exists k \in A : u_{ik} = 0 \Rightarrow i \in C \end{array}\right\} (C2)$$

Proof. Let (A, B, C) be a partition of S verifying $(C1)$

-if $A \neq \emptyset$,
Let us consider the following element of the playability set defined by:

- $\forall i \in A, J_E(i) = J_R(i) = 1$
- $\forall i \in B, J_E(i) = 1$ and $J_R(i) = 0$
- $\forall i \in C, J_E(i) = J_R(i) = 0$
- $j_E = j_R = 0$

Let us now verify that this element is a solution of P:
It stems from $(C1)$ that:
$\forall i \in A, (1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 1$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 1$
$\forall i \in C, (1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 0$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 0$

It also stems from $(C1)$ that $\forall i \in B, \exists k \in B : u_{ik} = 0$.
Indeed, if $i \in B, i \notin A$ and $i \notin C$, so $\exists k \notin A \cup C : u_{ik} = 0$

Hence $\forall i \in B$,
$(1-j_E)(1-j_R)\prod_{k\in S}(1-J_R(k)(1-u_{ik})) = 1$ and $(1-j_E)(1-j_R)\prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 0$

Also $\prod_{k\in S}(1-J_R(k)) = 0$ and $\prod_{k\in S}(1-J_E(k)) = 0$
The chosen values indeed define a solution σ of P, and $(A,B,C) = \Psi(\sigma)$

-if $A = \emptyset$,
it stems from the second part of $(C1)$ that $C = \emptyset$
Hence $B = S$

Also, it stems from the first part of $(C1)$ that no strategy in S is safe.
Therefore, there is a solution σ_0 of P in which all strategies are playable by default, and $(A,B,C) = (\emptyset, S, \emptyset) = \Psi(\sigma_0)$

Let τ be a solution of P and $(A,B,C) = \Psi(\tau)$,
-if $j_E = j_R = 1$ in τ,
then $(A,B,C) = (\emptyset, S, \emptyset)$
and since no strategy is safe, $\forall i \in S, \exists k \in S : u_{ik} = 0$
Hence (A,B,C) verifies $(C1)$.

-if $j_E = j_R = 0$,
$A = \{i \in S | J_E(i) = J_R(i) = 1\} = \{i \in S | \prod_{k\in S}(1-J_R(k)(1-u_{ik})) = \prod_{k\in S}(1-J_E(k)(1-u_{ik})) = 1\}$
$= \{i \in S | u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0\} = \{i \in S | u_{ik} = 0 \Rightarrow k \in C\}$

similarly, $C = \{i \in S | \exists k \in A : u_{ik} = 0\}$
Hence (A,B,C) verifies $(C1)$.

Let σ be a solution of P and $(A,B,C) = \Psi(\sigma)$
let $i \in S$,

$i \in A \Leftrightarrow J_E(i) = J_R(i) = 1$
$\Leftrightarrow i$ is safe or $(1-j_E)(1-j_R)\prod_{k\in S}[1-J_R(k)(1-u_{ik})] = (1-j_E)(1-j_R)\prod_{k\in S}[1-J_E(k)(1-u_{ik})] = 1$

$\Leftrightarrow i$ is safe or ($j_e = j_R = 0$ and ($u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0$))

Yet i is safe $\Rightarrow$ ($j_e = j_R = 0$ and ($u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0$))
so $i \in A \Leftrightarrow$ ($j_e = j_R = 0$ and ($u_{ik} = 0 \Rightarrow J_E(k) = J_R(k) = 0$))
$i \in A \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C)$

If $\exists k \in A : u_{ik} = 0$,
then k is deterrent vis-à-vis i for both players.
Hence $i \in C$

2.3. Categorization of the solutions of the Replicator Dynamics

Let G be a symmetric Game of Deterrence and $D(G)$ its Replicator Dynamics.
Let Γ be a function which associates with any given solution σ of $D(G)$ a partition (A', B', C') of the strategic set S of G such that:

- $A' = \{i \in S |\ \theta_i$ does not have a zero limit$\}$
- $B' = \{i \in S |\ lim\ \theta_i = 0$ and $\theta(i)$ is not integrable$\}$
- $C' = \{i \in S |\ \theta_i$ is integrable$\}$

Proposition 2. *If a solution σ of $D(G)$ verifies $\int_0^\infty 1 - u_T < \infty$, then $(A', B', C') = \Gamma(\sigma)$ verifies:*

$$\left.\begin{array}{l} A' \neq \emptyset \\ i \in A' \Leftrightarrow (u_{ik} = 0 \Rightarrow k \in C') \\ \exists k \notin C' : k \textit{ is } (A' \cup B')\textit{-dominant vis-à-vis } i \textit{ and } u_{ik} < u_{kk} \Rightarrow i \in C' \\ \exists k \in C' : k \textit{ is } (A' \cup B')\textit{-dominant vis-à-vis } i \Rightarrow i \in C' \end{array}\right\} (C3)$$

Proof. Let σ be a solution of $D(G)$ such that $\int_0^\infty 1 - u_T < \infty$, and let $(A', B', C') = \Gamma(\sigma)$.

$A' \neq \emptyset$ because $\sum\limits_{i \in S} \theta_i = 1$

Let $i \in S$,
$\frac{\theta_i'}{\theta_i} = u_i - u_T = (1 - u_T) - (1 - u_i)$
hence $\theta_i(t) = \theta_i(0) e^{\int_0^t 1 - u_T} e^{-\int_0^t 1 - u_i}$

$\theta_i(0) e^{\int_0^t 1 - u_T}$ has a non-zero finite limit,
and $e^{-\int_0^t 1 - u_i}$ has a finite limit, since it is positive and decreasing
so θ_i has a limit.
This being true for all $i \in S$, the solution σ converges towards an equilibrium.

Also $lim\ \theta_i = 0 \Leftrightarrow lim \int_0^t 1 - u_i = +\infty$
$1 - u_i = 1 - \sum\limits_{k \in S} \theta_k u_{ik} = \sum\limits_{k \in S} \theta_k (1 - u_{ik}) = \sum\limits_{k | u_{ik} = 0} \theta_k$
hence $lim\ \theta_i = 0 \Leftrightarrow \exists k \in S : u_{ik} = 0$ and θ_k is not integrable
$i \in A' \Leftrightarrow (\forall k \in S, u_{ik} = 0 \Rightarrow k \in C')$

Let $i, k \in S$ such that k is $(A' \cup B')$-dominant vis-à-vis i,

let $\theta_{C'} = \sum_{c \in C'} \theta_c$,
By definition of C', $\theta_{C'}$ is integrable.

$u_i - u_k = \sum_{l \in S} \theta_l(u_{il} - u_{kl}) = \sum_{l \in C'} \theta_l(u_{il} - u_{kl}) + \sum_{l \notin C'} \theta_l(u_{il} - u_{kl}) \leq \theta_{C'}$
$\frac{\theta_i}{\theta_k}(t) = \frac{\theta_i}{\theta_k}(0)e^{\int_0^t u_i - u_k} \leq \frac{\theta_i}{\theta_k}(0)e^{\int_0^t \theta_{C'}} \leq \frac{\theta_i}{\theta_k}(0)e^{\int_0^\infty \theta_{C'}} < +\infty$
$\frac{\theta_i}{\theta_k}$ is upper-bounded.
Hence, if $k \in C'$, then $i \in C'$

Now if $k \notin C'$ and $u_{ik} < u_{kk}$,
$u_i - u_k \leq \theta_{C'} + (u_{ik} - u_{kk})\theta_k = \theta_{C'} - \theta_k$
so $(\frac{\theta_i}{\theta_k})' = \frac{\theta_i}{\theta_k}(u_i - u_k) \leq \frac{\theta_i}{\theta_k}(\theta_{C'} - \theta_k) = \frac{\theta_i}{\theta_k}\theta_{C'} - \theta_i$
$0 \leq \frac{\theta_i}{\theta_k}(t) \leq \frac{\theta_i}{\theta_k}(0) + \int_0^t \frac{\theta_i}{\theta_k}\theta_{C'} - \int_0^t \theta_i$
hence $\int_0^t \theta_i \leq \frac{\theta_i}{\theta_k}(0) + \int_0^t \frac{\theta_i}{\theta_k}\theta_{C'}$

Since $\frac{\theta_i}{\theta_k}$ is upper-bounded and $\theta_{C'}$ is integrable, $\frac{\theta_i}{\theta_k}\theta_{C'}$ is integrable
hence θ_i is integrable, and $i \in C'$

Corollary 1. *For any solution σ of $D(G)$, let $\Gamma_\sigma : S \to \{A', B', C'\}$ be such that $\forall i \in S, i \in \Gamma_\sigma(i)$ in the partition $\Gamma(\sigma)$. Let us equip the set $\{A', B', C'\}$ with the alphebetical order: $A' \geq B' \geq C'$.*
Let $(i, k) \in S^2$. If k is $(A' \cup B')$-dominant vis-à-vis i, then $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$
Also if i and k are $(A' \cup B')$-equivalent, then $\Gamma_\sigma(i) = \Gamma_\sigma(k)$

Proof. Let k be $(A' \cup B')$-dominant vis-à-vis i,
If $k \in C'$, then it stems from proposition 2 that $i \in C'$

If $k \in B'$, then $k \notin A'$, hence $\exists l \notin C' : u_{kl} = 0$
and since $u_{il} \leq u_{kl}$, $u_{il} = 0$
whence $i \notin A'$

If $k \in A'$, then $A' \geq \Gamma_\sigma(i)$
Hence $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$

If i and k are $(A' \cup B')$-equivalent,
then $\Gamma_\sigma(k) \geq \Gamma_\sigma(i)$ and $\Gamma_\sigma(i) \geq \Gamma_\sigma(k)$
Hence $\Gamma_\sigma(i) = \Gamma_\sigma(k)$

3. Bridging binary and quantitative games

In a first part, the present section will proceed to a classical analysis of the Replicator Dynamics associated with an elementary example of 2x2 standard game. In a second part, an alternative approach based on the transformation of the standard game into a Game of Deterrence will be developed. The third part will generalize the new approach, which will be applied in the fourth part to a case which the standard approach cannot solve comprehensively.

3.1. Example 1: the standard approach

Let us consider the following symmetric matrix game G in which $0 < a < 1$:

G

	i	k
i	$(1,1)$	$(1,a)$
k	$(a,1)$	$(0,0)$

Let $\theta = (\theta_i, \theta_k) \in \Theta$ be the profile of the population.

The average payoffs of the two species are:
$u_i = 1$
$u_k = a\theta_i$
and $u_T = \theta_i + a\theta_i\theta_k$

Hence $\theta' = f(\theta) = (\theta_i(1 - \theta_i - a\theta_i\theta_k), \theta_k(a\theta_i - \theta_i - a\theta_i\theta_k)$

It can be seen by the classical analysis of the Replicator Dynamics that in every solution of $D(G)$, θ_k decreases exponentially, leading to the equilibrium $\theta = (1, 0)$. Indeed, in this simple example, the classical approach enables to completely determine the trajectories, and the equilibria.

3.2. Alternative approach

Let us now introduce the following alternative approach the rationale of which will be justified later.

A possible interpretation of player Column receiving payoff a when the strategic pair (i, k) is selected, is that species i can be divided into two sub-species i_1 and i_2, such that player Column, when playing species k, gets a payoff of 1 against species i_1, and 0 against species i_2, provided that the proportion in species i of i_1 and i_2 is given by $(a, 1 - a)$.

This in turn implies that the dynamics associated with G may be considered equivalent to the dynamics of the following game G' when the ratio of the two sub-species equals $\frac{a}{1-a}$.

G'

	i_1	i_2	k
i_1	$(1,1)$	$(1,1)$	$(1,1)$
i_2	$(1,1)$	$(1,1)$	$(1,0)$
k	$(1,1)$	$(0,1)$	$(0,0)$

Let $\zeta = (\zeta_{i_1}, \zeta_{i_2}, \zeta_k)$ be the profile of the population.

The average payoffs of the three species are:
$v_{i_1} = 1$
$v_{i_2} = 1$
$v_k = \zeta_{i_1}$
and $v_T = \zeta_{i_1} + \zeta_{i_2} + \zeta_{i_1}\zeta_k$

Hence the Replicator Dynamics $\zeta' = g(\zeta)$ is such that:
$\zeta'_{i_1} = \zeta_{i_1}(1 - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$
$\zeta'_{i_2} = \zeta_{i_2}(1 - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$
$\zeta'_k = \zeta_k(\zeta_{i_1} - \zeta_{i_1} - \zeta_{i_2} - \zeta_{i_1}\zeta_k)$

As it stems from the matrix of G' that strategies i_1 and i_2 are equivalent, $\frac{\zeta_{i_1}}{\zeta_{i_2}}$ is constant (lemma 1).

Let H be the subset of the set of profiles of $D(G')$ such that $(1-a)\zeta_{i_1} = a\zeta_{i_2}$.
Since the ratio is constant, H is stable under the dynamics $D(G')$.
Let us then denote by $D_H(G')$ the restriction of $D(G')$ to H

let us then define the splitting maps h and $\tilde{h}$ as follows:
$h : \Theta \to H$
$(\theta_i, \theta_k) \mapsto (a\theta_i, (1-a)\theta_i, \theta_k)$

and $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^3$
$(x, y) \mapsto (ax, (1-a)x, y)$

It can be easily seen from the above that $\tilde{h} \circ f = g \circ h$

h generates the breakdown of species i into i_1 and i_2 on the set of profiles, while $\tilde{h}$ does the same on the tangent space of Θ

This relation translates in terms of flows as follows:
Let ϕ_f^t and ϕ_g^t be the flows associated with f and g.
$h \circ \phi_f^t(\theta) = h(\theta + \int_0^t f(\theta)) = h(\theta) + \int_0^t \tilde{h} \circ f(\theta) = h(\theta) + \int_0^t g \circ h(\theta) = \phi_g^t(h(\theta))$
$h \circ \phi_f^t = \phi_g^t \circ h$

Hence, since h is bijective, $D(G)$ and $D_H(G')$ are topologically conjugate.
In other words, the dynamics of G is equivalent to the dynamics of G' restricted to H.

The playability system P' of G' has a unique solution in which strategies i_1 and i_2 are positively playable while k is not playable for both players. Indeed, strategies i_1 and i_2 are safe and i_2 is deterrent vis-à-vis k.

It then stems from (Ellison and Rudnianski, 2009) that whatever the initial profile:
ζ_{1_1} and ζ_{1_2} have a non-zero limit
ζ_2 has a zero limit

Since f and $g|_H$ are topologically conjugate, whatever the initial profile $\theta(0)$ in G:
θ_1 has a limit equal to 1
θ_2 has a zero-limit

These conclusions match exactly those drawn from the standard approach.

3.3. Generalization

Let $\widetilde{G}$ be a standard symmetric matrix game,
Let $M = max\ u_{ik}$ and $m = min\ u_{ik}$
Through replacing all the payoffs u_{ik} by their images via the affinity $x \mapsto \frac{x-m}{M-m}$, we obtain a game G with payoffs comprised between 0 and 1.

It is well known (Weibull, 1995) that the Replicator Dynamics is invariant under positive affine transformation of payoffs. In this case, it is accelerated by a factor $\frac{1}{M-m}$. If $\tilde{f}$ and f denote the vector fields of $D(\widetilde{G})$ and $D(G)$ respectively, the associated flows satisfy the following relation:
$\phi^t_{\tilde{f}} = \phi^{(M-m)t}_f$

Proposition 3. *Given a standard symmetric game G with payoffs comprised between 0 and 1, there is a binary symmetric matrix game G' and a subset H of its set of profiles such that the restriction $D_H(G')$ of $D(G')$ to H and $D(G)$ are topologically conjugate.*

Proof. This demonstration will use an algorithmic construction of the game G'.
Let G be a standard symmetric matrix game with strategic set $S = \{1, ..., n\}$

Let $i \in S$,
let $p = card(\{u_{ki}, k \in S\} \cup \{0, 1\}) - 1$,
let $(a_0, ..., a_p)$ be such that:
$0 = a_0 < a_1 < ... < a_p = 1$ and $\{a_0, ..., a_p\} = \{u_{ki}, k \in S\} \cup \{0, 1\}$

Let G_i be the game obtained from G by replacing strategy i with p equivalent strategies $i_1, ..., i_p$ and by setting the following payoffs:
$v_{kl} = u_{kl}$, for $k, l \in S - \{i\}$
$v_{i_m l} = u_{il}$, for $1 \leq m \leq p, l \in S - \{i\}$
$v_{ki_m} = 1$ if $m \leq r$, where r is such that $u_{ki} = a_r$; and $v_{ki_m} = 0$ otherwise, for $k \in S - \{i\}, 1 \leq m \leq p$
$v_{i_m i_{m'}} = 1$ if $m' \leq r$, where r is such that $u_{ii} = a_r$; and $v_{i_m i_{m'}} = 0$ otherwise, for $1 \leq m, m' \leq p$

Let H_i be the subset of the set of profiles Θ_i of G_i defined by the following equations:
$\forall 1 \leq m \leq p, \theta_{i_m} = (a_m - a_{m-1}) \sum\limits_{m'=1}^{p} \theta_{i_{m'}}$

The strategies $i_1, ..., i_p$ are equivalent.
Hence, it stems from lemma 1 that H_i is stable under the dynamics $D(G_i)$

Let h_i be the splitting map:
$h_i : \Theta \to H_i$
$\theta \mapsto (\theta_1, ..., \theta_{i-1}, (a_1 - a_0)\theta_i, ..., (a_p - a_{p-1})\theta_i, \theta_{i+1}, ..., \theta_n)$

and $\tilde{h}_i : \mathbb{R}^n \to \mathbb{R}^{n+p-1}$
$(x_1, ..., x_n) \mapsto (x_1, ..., x_{i-1}, (a_1 - a_0)x_i, ..., (a_p - a_{p-1})x_i, x_{i+1}, ..., x_n)$

Let $\theta \in \Theta$ and $k \in S - \{i\}$,
$v_k(h_i(\theta)) = v_k(\theta_1, ..., \theta_{i-1}, (a_1 - a_0)\theta_i, ..., (a_p - a_{p-1})\theta_i, \theta_{i+1}, ..., \theta_n)$
$= \sum\limits_{l \neq i} \theta_l v_{kl} + \sum\limits_{m=1}^{p} (a_m - a_{m-1})\theta_i v_{ki_m}$
$= \sum\limits_{l \neq i} \theta_l u_{kl} + \sum\limits_{m=1}^{r} (a_m - a_{m-1})\theta_i$ where r is such that $a_r = u_{ki}$
$= \sum\limits_{l \neq i} \theta_l u_{kl} + a_r\theta_i$
$= u_k(\theta)$
hence $\forall k \in S - \{i\}, v_k \circ h_i = u_k$

Similarly, for $k \in \{i_1, ..., i_p\}, v_k \circ h_i = u_i$

hence, by linearity $v_T \circ h_i = u_T$
and if f and f_i denote the vector fields of the Replicator Dynamics of G and G_i respectively,
$\tilde{h}_i \circ f = f_i \circ h_i$

Hence the flows are conjugate via h_i, i.e. $h_i \circ \phi_f^t = \phi_{f_i}^t \circ h_i$

And since h_i is a one-to-one correspondance between Θ and H_i,
$D(G)$ and $D_{H_i}(G_i)$ are topologically conjugate via h_i.

Also, $\{v_{ki_m}, k \in S - \{i\} \cup \{i_1, ..., i_p\}, 1 \leq m \leq p\} \subset \{0, 1\}$
Hence, the splitting of species i reduces by 1 the number of species which, when selected by one player, may generate a non-binary payoff for the other player, unless strategy i already verifies that property, in which case the algorithm does not modify the game.

Let $G' = G_{1_{2_{\dots n}}}$ be the game obtained from G by successively applying the above transformation for each strategy of S,
and let H be the corresponding subset of the set of profiles Θ' of G',

G' is a binary matrix game and $D(G)$ and $D_H(G')$ are topologically conjugate.

Consequence: the asymptotic properties of G can be analyzed through G' and its playability system.

As the algorithm is applied to G, each strategy is split into up to n equivalent strategies. Hence, G' may have up to n^2 strategies which can be grouped into n

sets of equivalent strategies. Now, it is generally useful to reduce the size of the playabily system. In the case of G', the fact that equivalent strategies have the same playability in every solution (cf. lemma 1) allows us to reduce the playability system. Indeed:

Proposition 4. *Let G' be a symmetric Game of Deterrence with strategic set $S' = \{1, ..., i-1,$
$i_1, ..., i_p, i+1, ..., n\}$, where $i_1, ..., i_p$ are equivalent strategies, and let G'' be the game obtained from G' by replacing strategies $i_1, ..., i_p$ by a strategy i_0 and by setting:*
$w_{kl} = v_{kl}, \forall k, l \neq i_1, ..., i_p$
$w_{i_0 k} = v_{i_1 k} \forall k \neq i_1, ..., i_p$
$w_{k i_0} = \prod_{m=1}^{p} v_{k i_m}, \forall k \neq i_1, ..., i_p$
$w_{i_0 i_0} = \prod_{m=1}^{p} v_{i_1 i_m}$
*Let $H' = \{(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p),$
$J_R(i+1), ..., J_R(n), j_E, j_R) | J_E(i_1) = ... = J_E(i_p)$ and $J_R(i_1) = ... = J_R(i_p)\} \subset \{0,1\}^{2n+2p}$,*

let P' and P'' be the playability systems associated with G' and G'' respectively,

H' is stable under P', and the restriction $P'_{H'}$ of P' to H' is topologically conjugate to P''.

Proof. Let $\hat{f} : \{0,1\}^{2n+2p} \to \{0,1\}^{2n+2p}$ and $\hat{\hat{f}} : \{0,1\}^{2n+2} \to \{0,1\}^{2n+2}$ be the playability systems P' and P'' respectively.

Since, $i_1, ..., i_p$ are equivalent, the components of $\hat{f}$ corresponding to $J_E(i_1), ..., J_E(i_p)$ are equal, as are those corresponding to $J_R(i_1), ..., J_R(i_p)$.
Hence H' is stable under $\hat{f}$. (In fact, $Im\hat{f} \subset H'$.)
So P' can be restricted to H'.

Let $h_i : H' \to \{0,1\}^{2n+2}$ be such that:
$h_i : (J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R) \mapsto (J_E(1), ..., J_E(i-1), J_E(i_1), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1),$
$J_R(i_1), J_R(i+1), ..., J_R(n), j_E, j_R)$
h_i is a bijection.

In order to prove the topological conjugacy, we must verify that $h_i \circ \hat{f}|_{H'} = \hat{\hat{f}} \circ h_i$

Let $(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R) \in H'$, let $k \neq i_1, ..., i_p$,
It stems from the construction of G'' that k is safe in G'' iff it is safe in G'
So if k is safe, the components of $h_i \circ \hat{f}|_{H'}$ and $\hat{\hat{f}} \circ h_i$ corresponding to $J_E(k)$ and $J_R(k)$ are all equal to 1.

Similarly, i_0 is safe iff $i_1, ..., i_p$ are all safe.
Let us now suppose that strategy k is dangerous.
The component of $\hat{\hat{f}}(J_E(1), ..., J_E(i-1), J_E(i_1), J_E(i+1), ..., J_E(n), J_R(1), ..., J_R(i-1), J_R(i_1),$
$J_R(i+1), ..., J_R(n), j_E, j_R)$ corresponding to $J_E(k)$ is:

$$(1-j_E)(1-j_R)\prod_{l\neq i_0}(1-J_R(l)(1-w_{kl}))\times(1-J_R(i_1)(1-w_{ki_0}))$$
$$=(1-j_E)(1-j_R)\prod_{l\neq i_0}(1-J_R(l)(1-v_{kl}))\times(1-J_R(i_1)(1-\prod_{m=1}^{p}v_{ki_m}))$$
$$=(1-j_E)(1-j_R)\prod_{l\neq i_0}(1-J_R(l)(1-v_{kl}))\times\prod_{m=1}^{p}1-J_R(i_1)(1-vki_m)$$
$$=(1-j_E)(1-j_R)\prod_{l\in S'}(1-J_R(l)(1-v_{kl})),$$

which is exactly the same component of $h_i\circ\hat{f}(J_E(1), ..., J_E(i-1), J_E(i_1), ..., J_E(i_p),$
$J_E(i+1), ..., J_E(n),$
$J_R(1), ..., J_R(i-1), J_R(i_1), ..., J_R(i_p), J_R(i+1), ..., J_R(n), j_E, j_R)$

Similarly, all other components match.
Hence $P'_{H'}$ and P'' are topologically conjugate.

Corollary 2. *Let G' be a symmetric Game of Deterrence with a strategic set containing several subsets of equivalent strategies.*
Let G'' be the game obtained by replacing each subset of equivalent strategies by a single strategy as in proposition 4.
Let H' be the subset of the playability set of elements such that any two equivalent strategies have the same playability for both players.
Then, using the notations of proposition 4, $P'_{H'}$ and and P'' are topologically conjugate.

Proof. The result stems straightforwardly from the application of proposition 4 to each subset of equivalent strategies.

Remark 1: Since $Im\hat{f}\subset H'$, all the solutions of the playability system P' are in H', and restricting P' to H' does not reduce the number of solutions. Thus, solving P'' is equivalent to solving P'.

Remark 2: The above simplification of the playability system also works in the case of non symmetric Games of Deterrence, when either player E or player R has equivalent strategies.

Remark 3: Let G be a symmetric game with payoffs comprised between 0 and 1, and let G'' be the game obtained by first transforming G into G' as in proposition 3, then transforming G' into G'' as in proposiiton 4. If G does not have equivalent strategies in its strategic set, then the strategic set of G'' contains the same number of strategies as that of G. Indeed, each strategy is first replaced by a set of equivalent strategies, which is in turn replaced by a single strategy. If there are equivalent strategies in the strategic set of G, we will choose not to regroup those strategies when building G'', so as to maintain the number of strategies.

3.4. Example 2

Let us consider the following example deriving from the one developped in (Ellison and Rudnianski, 2009), in which individuals may adopt one of three possible behaviours:

- A: aggressive
- D: defensive
- N: neutral

Furthermore, let us assume that:

- when two individuals of the same type interact, the outcome for each one is 1, which means that an aggressive individual will not try to attack another aggressive individual (maybe because of the fear of the outcome)
- a defensive type, when encoutering an aggressive individual, will respond by inflicting damages, represented by a payoff $0 \leq x < 1$ for the aggressor, and will get a 0
- when meeting a defensive or a neutral type, the defensive type does not attack, and the outcome pair is $(1, 1)$
- a neutral type never responds agressively, and receives a payoff $0 \leq y < 1$ when attacked.

G

	A	D	N
A	$(1,1)$	$(x,0)$	$(1,y)$
D	$(0,x)$	$(1,1)$	$(1,1)$
N	$(y,1)$	$(1,1)$	$(1,1)$

It has been shown (Ellison and Rudnianski, 2009) that in the extreme case where $x = y = 0$, the profile $(1, 0, 0)$, which corresponds to the whole population being aggressive, is an evolutionarily stable equilibrium, and the set of profiles $\{(0, t, 1 - t), 0 < t < 1\}$, which are not individually evolutionarily stable, is an evolutionarily stable equilibrium set.

Let us now consider the case where $0 < x, y < 1$.

Let G' and G'' be the following matrix games:

G'

	A_1	A_2	D_1	D_2	N
A_1	$(1,1)$	$(1,1)$	$(1,0)$	$(0,0)$	$(1,1)$
A_2	$(1,1)$	$(1,1)$	$(1,0)$	$(0,0)$	$(1,0)$
D_1	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(1,1)$
D_2	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(1,1)$
N	$(1,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(1,1)$

G''

	A_0	D_0	N
A_0	$(1,1)$	$(0,0)$	$(1,0)$
D_0	$(0,0)$	$(1,1)$	$(1,1)$
N	$(0,1)$	$(1,1)$	$(1,1)$

Let H be the set of profiles in G' such that $(1-x)\theta_{A_1} = x\theta_{A_2}$ and $(1-y)\theta_{D_1} = y\theta_{D_2}$. By proposition 3, $D(G)$ and $D_H(G')$ are topologically conjugate.

Let H' be the subset of the playability set of G' comprised of elements such that A_1 and A_2 on one hand, and D_1 and D_2 on the other hand, have the same playability for both players.
By proposition 4, $P'_{H'}$ and P'' are topologically conjugate.

It can be easily seen from the matrix of G'' that P'' has three solutions:

- $(1,0,0,1,0,0,0,0)$ (A_0 is positively playable while D_0 and N are not playable for both players)
- $(0,1,1,0,1,1,0,0)$ (D_0 and N are positively playable while A_0 is not playable for both players)
- $(0,0,0,0,0,0,1,1)$ (all the strategies are playable by default for both players)

Hence, it stems from the topological conjugacy that P' also has three solutions:

- $(1,1,0,0,0,1,1,0,0,0,0,0)$ (A_1 and A_2 are positively playable while D_1, D_2 and N are not playable)
- $(0,0,1,1,1,0,0,1,1,1,0,0)$ (D_1, D_2 and N are positively playable while A_1 and A_2 are not playable)
- $(0,0,0,0,0,0,0,0,0,0,1,1)$ (all the strategies are playable by default for both players)

The first two of these solutions satisfy the conditions described in section 1.3. Hence $D(G')$ has two evolutionarily stable equilibrium sets:

- $ESES_1 = \{(t, 1-t, 0, 0, 0), 0 < t < 1\}$ where only species A_1 and A_2 remain
- $ESES_2 = \{(0, 0, t_1t_2, (1-t_1)t_2, 1-t_2), 0 < t_1, t_2 < 1\}$ where only species D_1,D_2 and N remain

Hence $ESES_1 \cap H$ and $ESES_2 \cap H$ are asymptotically stable equilibrium sets in $D_H(G')$
$ESES_1 \cap H = \{(x, 1-x, 0, 0, 0)\}$ and $ASES_2 \cap H = \{(0, 0, yt_2, (1-y)t_2, 1-t_2), 0 < t_2 < 1\}$

Now $D_H(G')$ is topologically equivalent to $D(G)$,
hence $(1,0,0)$ is an evolutionarily stable equilibium and $\{(0, t, 1-t), 0 < t < 1\}$ is an evolutionarily stable equilibrium set in $D(G)$.

The results previously established for the game G in the case where $x = y = 0$ have been extended to all $0 < x, y < 1$. The bridging between binary and quantitative games allows us to establish asymptotic properties of evolutionary quantitative games via playability properties of associated Games of Deterrence.

Also, if a solution σ of $D(G)$ tends towards the equilibrium $(1,0,0)$, then θ_D and θ_N decrease exponentially. So $\Gamma(\sigma) = (\{A\}, \emptyset, \{D, N\})$.

And if σ tends towards $\{(0,t,1-t), 0<t<1\}$, then θ_A decreases exponentially. So $\Gamma(\sigma) = (\{D,N\}, \emptyset, \{A\})$.

It can be easily seen from the matrix of G that these two partitions are the only ones which verify condition $(C3)$. In this case, $Im\Gamma$ is exactly the set of partitions of S which verify $(C3)$.

3.5. Shortcut

Proposition 5. *Let $\widetilde{G}$ be a symmetric matrix game.*
Let M and m be the maximal and minimal payoffs in $\widetilde{G}$.
Let G be the game obtained by applying the affinity $x \mapsto \frac{x-m}{M-m}$ to all the payoffs of $\widetilde{G}$. Let G' be defined as in proposition 3, and G'' as in proposition 4.
Then G'' is the game obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$.

Proof. Using the previous notations (u_{ik}, v_{ik} and w_{ik} represent the payoffs in the games G, G' and G'' respectively), we have:

$$w_{ki_0} = \prod_{m=1}^{p} v_{ki_m}, \forall k \neq i_1, ..., i_p$$

$$w_{i_0i_0} = \prod_{m=1}^{p} v_{i_1i_m}$$

and:
$v_{ki_m} = 1$ if $m \leq r$, where r is such that $u_{ki} = a_r$; and $v_{ki_m} = 0$ otherwise, for $k \in S - \{i\}, 1 \leq m \leq p$
$v_{i_mi_{m'}} = 1$ if $m' \leq r$, where r is such that $u_{ii} = a_r$; and $v_{i_mi_{m'}} = 0$ otherwise, for $1 \leq m, m' \leq p$

Hence:
$w_{ki_0} = 1$ if $u_{ki} = 1$ and $w_{ki_0} = 0$ otherwise
$w_{i_0i_0} = 1$ if $u_{ii} = 1$ and $w_{i_0i_0} = 0$ otherwise

As payoff 1 in game G is the image of payoff M in game $\widetilde{G}$, it follows that G'' is obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$.

Proposition 6. *Let $\widetilde{G}$ be a symmetric matrix game, and let G'' be the game obtained by replacing the maximum payoff by 1 and all other payoffs by 0 in the matrix of $\widetilde{G}$. Let σ be a solution of $D(G)$. If:*

- *the playability system P'' of G'' has a symmetric solution for which no strategy is playable by default*
- *σ is such that at $t=0$, the proportion of each strategy of $\widetilde{G}$ corresponding to a positively playable strategy in G'' is greater than the sum of the proportions of the strategies of $\widetilde{G}$ corresponding to non-playable strategies in G'',*

then:

- *The proportion of each strategy of $\widetilde{G}$ corresponding to a non-playable strategy in G'' decreases exponentially towards zero*

- *The proportion of each strategy of $\widetilde{G}$ corresponding to a playable strategy in G'' has a non-zero limit*

Proof. Let M and m be the maximal and minimal payoffs in $\widetilde{G}$.
Let G be the game obtained by applying the affinity $x \mapsto \frac{x-m}{M-m}$ to all the payoffs of $\widetilde{G}$. Let G' and H be defined as in proposition 3, and G'' and H' as in proposition 4.

P'' is topologically conjugate to $P'_{H'}$, so the symmetric solution of P'' is conjugate to a solution τ of P', which is also symmetric.
By applying the result of section 1.3 to τ, we obtain that if at $t = 0$, the proportion of each strategy which is positively playable in τ is greater than the sum of the proportions of the non-playable strategies, then the proportion of each positively playable strategy has a non-zero limit, and the proportion of each non-playable strategy decreases exponentially towards zero.

Then, the conclusions about $\widetilde{G}$ follow from the topological conjugacy between $D(G)$ and $D_H(G')$ and the invariance by affine transformation linking $D(\widetilde{G})$ and $D(G)$.

4. Conclusion

Starting from a symmetric quantitative game $\widetilde{G}$, we have established the following construction:

$$\widetilde{G} \longrightarrow G \longrightarrow G' \longrightarrow G''$$

such that:

- the payoffs of G are comprised between 0 and 1 and $\phi^t_{\tilde{f}} = \phi_f^{(M-m)t}$
- G' is binary and $D_H(G')$ and $D(G)$ are topologically conjugate
- G'' has the same size as $\widetilde{G}$ and $P'_{H'}$ and P'' are topologically conjugate

Now, G'' can be constructed directly from $\widetilde{G}$ without computing G and G'.

The results obtained in the previous sections thus enable to:

1. overcome the possible difficulties of solving analytically the Replicator Dynamics
2. establish asymptotic properties of solutions of the Replicator Dynamics associated with any standard symmetric matrix game
3. bridge standard quantitative games with Games of Deterrence, thus paving the way for a treatment of optimality issues through acceptability analysis.

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Consistency to the Values for Games in Generalized Characteristic Function Form

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Abstract In the classical game space, Evans (1996) introduced a procedure, such that the solution of a game determined endogenously as the expected outcome of a reduction of the game to a two-person bargaining problem, is just the Shapley value. This approach is not suitable for games in generalized characteristic function form, in which the order of players entering into the game affects the worth of coalitions. Based on Evans's approach, in this paper we propose a new procedure which induces the generalized Shapley value defined by Sanchez and Bergantinos (1997). Moreover, this generalized procedure can be adapted to characterize the class of values satisfying efficiency, linearity and symmetry, for games in generalized characteristic function form.

Keywords: cooperative game, orders, value, consistency, procedure

1. Introduction

In the classical case, a cooperative game with transferable utility, or shortly, a TU game, is an ordered pair $\langle N, v\rangle$, where N is a nonempty, finite set of players, and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. An element $i \in N$ is called a player, and a subset $S \subseteq N$ is called a coalition. The associated real number $v(S)$ is called the worth of coalition S. The size of coalition S is denoted by $|S|$, or shortly by s if no ambiguity arises. Particularly $|N|$, or equivalently n denotes the size of the grand coalition N. We denote by $\mathcal{G}_N$ the set of all cooperative TU games with player set N and by $\mathcal{G}$ the space of all cooperative TU games with arbitrary player set. A value $\phi = (\phi_i)_{i\in N}$ on $\mathcal{G}$ is a mapping which assigns to every TU game $\langle N, v\rangle$ exactly one element $\phi(N, v) \in \mathbb{R}^N$. One of the most well-known values is the Shapley value (Shapley, 1953a).

Evans (1996) introduced the following procedure: given an n-player cooperative game and a feasible "wage" n-vector, suppose that the players in a cooperative game are randomly split into two coalitions, each with a randomly chosen leader; the two leaders bargain bilaterally and each pays, out of his share, a wage to each member of his coalition as specified by the wage vector. More precisely for an arbitrary cooperative game $\langle N, v\rangle$ in $\mathcal{G}$, the following procedures are done sequentially:

(A) the players in the grand coalition N are randomly split into two coalitions, say S and $N \setminus S$ ($S \neq \emptyset, N$);
(B) each coalition generates randomly a leader, say leader i represents S and leader j represents $N \setminus S$, $i \in S$, $j \in N \setminus S$;
(C) player i and j play a two-person bargaining process based on coalition S and $N \setminus S$ separately. The rule is that each leader pays to each member of his coalition, an certain part of what he gets in the two-person bargaining process.

A value is said to be *consistent* with the above procedure if it is equal to the expected payoff. Under such consistency condition, Evans proved that the Shapley value is the unique solution, if all randomly chosen processes are with respect to the uniform distribution, and the two-person bargaining result is *standard* according to Hart and Mas-Colell (1989). Remind that a value ϕ on $\mathcal{G}$ is standard for two-person game, means for an arbitrary two-person game $\langle \{i,j\}, v\rangle$,

$$\phi_k(\{i,j\},v) = v(\{k\}) + \frac{1}{2}\left(v(\{i,j\}) - v(\{i\}) - v(\{j\})\right) \quad \text{for } k \in \{i,j\}. \qquad (1.1)$$

The above results are derived in the classical game space, where the characteristic function assigns to each group of players a fixed single number, regardless of how players are ordered in the group. However, in modeling some economic situations or some special relationships among players, the earning of a group of players may depend not only on its members, but also on the sequential ordering of players joining the game. So a better approximation to some real life situation is to consider games where the so-called characteristic function is defined on all possible orders for coalitions of players. Such generalized model is introduced first by Nowak and Radzik (1994). They redefined the efficiency, null player property and strong monotonicity in this new game space, and axiomatized an adapted Shapley value by using two groups of redefined properties. The first group of properties contains efficiency, null player property and additivity, and the uniqueness proof follows the approach given by Shapley (1953a). The second group of properties are efficiency and strong monotonicity, and the proof proceeds according to the one given by Young (1985). The symmetry property compared to the classical case, was included in the definition of the null player in the null player property as well as the marginal contribution in the strong monotonicity. Sanchez and Bergantinos (1997) found the inaccuracy of the symmetry property, and gave more "suitable" definitions for the null player, symmetric player and marginal contribution, in the new game space. In this way, a new Shapley value was characterized, by using these new defined properties. Later such Shapley value was generalized to games with a priori unions in the same way that Owen (1977) did for the classical Shapley value, by Sanchez and Bergantinos (1999), who also characterized the weighted Shapley value in the new game space (Bergantinos and Sanchez, 2001), based on the results in the classical game space given by Shapley (1953b) as well as Kalai and Samet (1987).

In our point of view, the properties used in Sanchez and Bergantinos' papers are more fair and attractive, since it considered all possibilities (positions) how a single player joining into a coalition. We follow the notation given by Sanchez and Bergantinos (1997).

For any subset $S \subseteq N$, denote by $H(S)$ the set of all orders of players in S. The element $S' \in H(S)$ is called an *ordered coalition*. For notational convenience, we use S to represent a general coalition with size s regardless of order and $S' \in H(S)$ to represent an ordered coalition with the same player set. Note that $H(\emptyset) = \emptyset$ as well as $H(\{i\}) = \{\{i\}\}$ for all $i \in N$. Denote by Ω the set of all ordered coalitions, that is,

$$\Omega = \{S' | S' \in H(S), S \subseteq N, S \neq \emptyset\}.$$

Obviously, the total number of ordered coalitions in Ω equals

$$m := \sum_{s=1}^{n} s! C_n^s \quad \text{where} \quad C_n^s = \binom{n}{s} = \tfrac{s!(n-s)!}{n!} \quad \text{for all } 1 \leq s \leq n. \qquad (1.2)$$

Definition 1.1. A game in generalized characteristic function form, or a generalized game is an ordered pair $\langle N, v\rangle$, where N is a non-empty, finite set of players and $v : \Omega \to \mathbb{R}$ is a generalized characteristic function that assigns to each $S' \in \Omega$, the real-valued worth $v(S')$ as the utility obtained by players in S according to the order S', such that $v(\emptyset) = 0$.

Denote by $\mathcal{G}'_N$ the set of all generalized cooperative games with player set N, and $\mathcal{G}'$ the set all generalized cooperative games with arbitrary player set. A value ϕ on $\mathcal{G}'_N$ is a mapping assigning exactly one element $(\phi_i(N,v))_{i\in N} \in \mathbb{R}^N$ to every $v \in \mathcal{G}'_N$. The following definition will play an important role in our solution theory for generalized TU games.

Definition 1.2. Let $S' \in H(S), S \subsetneqq N$ be given. A set T' is called an extension of S' of size $t, t > s$ if a set of $t - s$ players in $N \setminus S$ is inserted among the players of S' in such a way that the players in S appear in T' in the same order as in S'. We denote by $V(S')$ the set of all extensions of S'.

As a special case we define an extension $T' = (S', i^h)$ with $i \notin S$, $t = s + 1$ as follows. Given player $i \in N$, coalition $S \subseteq N \setminus \{i\}$ of size s, ordered coalition $S' \in H(S)$, and height $h \in \{1, 2, \ldots, s+1\}$, then (S', i^h) denotes the $(s+1)$-person ordered coalition with player i inserted in the h-th position, that is, if $S' = (i_1, \ldots, i_s)$, then $(S', i^1) = (i, i_1, \ldots, i_s)$; $(S', i^{s+1}) = (i_1, \ldots, i_s, i)$; and $(S', i^h) = (i_1, \ldots, i_{h-1}, i, i_h, \ldots, i_s)$ for all $2 \le h \le s$.

Definition 1.3. (Sanchez and Bergantinos, 1997) For any generalized TU game $\langle N, v\rangle$, the generalized Shapley value $Sh'(N,v) = (Sh'_i(N,v))_{i\in N}$ is given by

$$Sh'_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} \frac{p^n_s}{(s+1)!} \sum_{S'\in H(S)} \sum_{h=1}^{s+1} \left[v(S', i^h) - v(S')\right] \quad \text{for all } i \in N.$$

We can rewrite the equation above in view of the extension (see Definition 1.2) in the following way:

$$Sh'_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} p^n_s \sum_{S'\in H(S)} (s!)^{-1} \sum_{\substack{T'\in V(S'), T'\ni i,\\ t=s+1}} \frac{v(T') - v(S')}{s+1}. \tag{1.3}$$

Shapley (1953a) introduced p^n_s as the classical probability measure over the collection of (unordered) coalitions not containing any fixed player. The difference compared with the classical case is that in this new setting, any player $i \in N$ has $(s+1)$ ways to join any ordered coalition S' of size s, $S' \in H(S)$, $S \subseteq N \setminus \{i\}$, yielding various marginal contributions $v(T') - v(S')$ for all $T' \in V(S')$ containing player i, of size $t = s + 1$. The expected payoff to any player i with respect to the underlying classical probability measure is obtained through averaging over all the player's marginal contributions as well as over all $s!$ possible ordered coalitions with player set S.

Definition 1.4. (Nowak and Radzik, 1994; Sanchez and Bergantinos, 1997) A value ϕ on $\mathcal{G}'$ satisfies the

(i) efficiency, if

$$\sum_{i \in N} \phi_i(N, v) = \frac{1}{n!} \sum_{N' \in H(N)} v(N') \quad \text{for all generalized game } \langle N, v \rangle; \tag{1.4}$$

(ii) symmetry, if $\phi_i(N, v) = \phi_j(N, v)$ for all symmetric players i and j. Two players $i, j \in N$ are symmetric in $\langle N, v \rangle$ if for every ordered coalition S' such that $S' \not\ni i, j$, we have that $v(S', i^h) = v(S', j^h)$ for all $h \in \{1, 2, \ldots, s+1\}$;

(iii) null player property, if $\phi_i(N, v) = 0$ for every generalized game $\langle N, v \rangle$, and every null player $i \in N$. Player i is called a null player in $\langle N, v \rangle$ if for every ordered coalition S' not containing i, we have that $v(S', i^h) = v(S')$ for every $h \in \{1, 2, \ldots, s+1\}$.

Denote by $\bar{v}(N)$ the average sum of the worths for all permutations $N' \in H(N)$, i.e.,

$$\bar{v}(N) = \frac{1}{n!} \sum_{N' \in H(N)} v(N'),$$

then the efficiency condition (1.4) is equivalent to $\sum_{i \in N} \phi_i(N, v) = \bar{v}(N)$. Sanchez and Bergantinos (1997) proved that the Shapley value is the unique value on $\mathcal{G}'$ satisfying efficiency, additivity, symmetry and null player property.

Definition 1.5. Let $S' \in H(S), S \subseteq N$ be given. A set T' is called an restriction[1,2] of S' if $T' \in H(T)$, $T \subseteq S$, and the order of players in T' is in accordance with that in S'. We denote by $R(S')$ the set of all restrictions of S'.

Although Evans's procedure proceeds well on the classical game space $\mathcal{G}$, it is not suitable to characterize a solution on the generalized game space $\mathcal{G}'$. The problem is that, when players are randomly split into two coalitions, there is no order information about the two subcoalitions, so the leader does not know what he actually owns to bargain with his opponent. In Section 2 we will give a generalized procedure, based on Evans's approach, to characterize the generalized Shapley value of form (1.3). In Section 3, the procedure we derived in section 2 is modified to characterize a class of values satisfying efficiency, linearity and symmetry on the generalized game space $\mathcal{G}'$.

2. Generalization of Evans's procedure

Following Evans's procedure, we assume that for a set of fixed players, each player has the same probability to be chosen as a leader in all possible permutations of the

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[1] Sanchez and Bergantinos (1997) use the notation $T' = S'/T'$ to express the restriction T' of S'. We change the notation to avoid the possible confusion with the set-minus sign "\".

[2] In order to explain such 'restriction set', we introduce the notion of predecessors and successors. Consider an arbitrary ordered coalition $S' \in \Omega$, $S' = \{i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_s\}$. For any $k \in \{2, \ldots, s\}$, denote the predecessors of i_k according to S' by $pre(i_k, S')$. For any $k \in \{1, \ldots, s-1\}$, denote the successors of i_k according to S' by $suc(i_k, S')$. Then $pre(i_k, S') = \{i_1, \ldots i_{k-1}\}$ as well as $suc(i_k, S') = \{i_{k+1}, \ldots i_s\}$ hold. For any two player $i, j \in T'$ where $T' \in H(T), T \subseteq N$, the restriction set $T' \in R(S')$ means $T \subseteq S$, and if $i \in pre(j, S')$ then $i \in pre(j, T')$, or if $i \in suc(j, S')$ then $i \in suc(j, T')$.

set of players, i.e. for any $S \subseteq N$, $S', S'' \in H(S)$, the probability of i to be chosen as a leader in S' is the same as that in S'' for all $i \in S$. Remind the problem lying in Evans's procedure for the generalized case is the lack of order information for the two partitioned coalitions in (A). In order to fix the orders of the two coalitions in the two-person bargaining process, we first fix one permutation $N' \in H(N)$ with some probability, then a partition $\{S', N' \setminus S'\}$ can be chosen based on N' where $S', N' \setminus S' \in R(N')$, $S' \in H(S)$, $S \subsetneq N$ and $S \neq \emptyset$.

Let $\theta : \mathcal{G}' \to \mathbb{R}^2$ be the payoff of the two-person bargaining process between S' and $N' \setminus S'$, say S' gets $\theta_{S'}^{N'}(v)$ and $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$. According to Evans's procedure, the leader of each ordered coalition is then obliged to pay to each member of his coalition a prespecified feasible allocation $x = (x_i)_{i \in N} \in \mathbb{R}^N$. If i is chosen as the leader of S', then what he gets is

$$\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k.$$

Similarly if j is the leader of $N' \setminus S'$ then he gets

$$\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k.$$

Denote by f the probability distribution that determines the choice of the permutation N', the partition $\{S', N' \setminus S'\}$, and which two players are the leader of S' and $N' \setminus S'$ respectively. Given the triple (f, θ, x), denote by $E_f\,(\Pi_i | \theta, x)$ the expected payoff to player i. We generalize in the following the consistency concept defined by Evans:

Definition 2.1. Given a pair (f, θ), a feasible payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ satisfies Evans's consistency with respect to (f, θ) if $x_i = E_f\,(\Pi_i | \theta, x)$ for $i \in N$.

We assume that the distribution f is uniform. Then the whole procedure under the uniform distribution can be described as follows:

(i) Choose a permutation N' from the set $H(N)$ with probability $1/n!$;

(ii) Choose the size of the first coalition S' with each possible size $\{1, 2, \ldots, n-1\}$ being equally likely, hence with probability $1/(n-1)$. Suppose s is the chosen size;

(iii) Choose an ordered coalition S' of size s in $R(N')$. Since the positions of players in N' are all fixed, we only need to fix s players with probability $1/C_n^s$. Once S' is fixed, its complement $N' \setminus S'$ according to N' is also fixed;

(iv) Choose a leader i from S' (already fixed in (iii)) with probability $1/s$, and a leader j from its complement $N' \setminus S'$ (already fixed in (iii)) with probability $1/(n-s)$;

(v) Leader i and j play a two-person bargaining game based on coalition S' and $N' \setminus S'$ respectively. Coalition S' gets $\theta_{S'}^{N'}(v)$ while $N' \setminus S'$ gets $\theta_{N' \setminus S'}^{N'}(v)$;

(vi) Leader i gets $\theta_{S'}^{N'}(v) - \sum_{k \in S \setminus \{i\}} x_k$ after assigning each of his member x_k for all $k \in S' \setminus \{i\}$, meanwhile leader j gets $\theta_{N' \setminus S'}^{N'}(v) - \sum_{k \in N \setminus (S \cup \{j\})} x_k$ after assigning each of his member x_k for all $k \in N \setminus (S \cup \{j\})$.

According to the above procedure the probability that player i will find himself leader of coalition S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{1}{s} \cdot 2,$$

and the probability of being a follower in S' according to N' is

$$\frac{1}{n!} \cdot \frac{1}{n-1} \cdot \frac{1}{C_n^s} \cdot \frac{s-1}{s} \cdot 2.$$

Player i could be either in the first coalition or in the second one, hence we add the factor 2 in the probability. Now everything is well-defined except for θ in the generalized case.

In contrast to the standard two-person bargaining solution (1.1), we give the following definition:

Definition 2.2. For any two-person generalized game $\langle\{i,j\},v\rangle$, the generalized standard bargaining solution $\phi : \mathcal{G}' \to \mathbb{R}^2$ is defined by

$$\phi_k(\{i,j\},v) = v(\{k\}) + \frac{1}{2}\left(\bar{v}(\{i,j\}) - v(\{i\}) - v(\{j\})\right) \quad \text{for } k \in \{i,j\}.$$

Clearly ϕ satisfies the efficiency condition (1.4). Hence the solution θ of the two-person bargaining process between S' and $N' \setminus S'$ in game $\langle N,v\rangle$ is

$$\begin{aligned} \theta_{S'}^{N'}(v) &= v(S') + \frac{1}{2}\left(\bar{v}(N) - v(S') - v(N' \setminus S')\right); \\ \theta_{N'\setminus S'}^{N'}(v) &= v(N' \setminus S') + \frac{1}{2}\left(\bar{v}(N) - v(N' \setminus S') - v(S')\right). \end{aligned} \tag{2.5}$$

Lemma 2.3. *The generalized Shapley value of form (1.3) for any generalized game $\langle N,v\rangle$ is equivalent to:*

$$Sh_i'(N,v) = \sum_{\substack{S'\in\Omega,\\ S'\ni i}} \frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!} - \frac{v(S'\setminus\{i\})}{(s-1)!}\right) \quad \textit{for all } i \in N. \tag{2.6}$$

The proof of this lemma can be found in the appendix.

Theorem 2.4. *A feasible payoff vector $x \in \mathbb{R}^N$ is consistent with (f,θ) for the generalized game $\langle N,v\rangle$ if and only if x is the generalized Shapley value of form (1.3).*

Proof. According to the procedure, player i's expected payoff x_i is

$$x_i = \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\left[\frac{1}{s}\left(\theta_{S'}^{N'}(v) - \sum_{k\in S\setminus\{i\}} x_k\right) + \frac{s-1}{s}x_i\right]. \tag{2.7}$$

We first show that x satisfies the efficiency condition (1.4):

$$\begin{aligned}
\sum_{i\in N} x_i &= \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ s\neq n,0}} \sum_{i\in S} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\left[\frac{1}{s}\left(\theta_{S'}^{N'}(v)-x(S)\right)+x_i\right] \\
&= \sum_{N'\in H(N)} \frac{1}{n!} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ s\neq n,0}} \frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\theta_{S'}^{N'}(v) \\
&= \sum_{N'\in H(N)} \frac{1}{n!}\sum_{s=1}^{n-1} C_n^s\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{2}\cdot\bar{v}(N) = \bar{v}(N).
\end{aligned}$$

Note that (2.7) is equivalent to

$$0 = \sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\left(\theta_{S'}^{N'}(v)-x(S)\right), \tag{2.8}$$

since

$$\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s} = \sum_{N'\in H(N)}\sum_{s=1}^{n-1} C_{n-1}^{s-1}\cdot 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s} = 1.$$

We now simplify the formula for x_i given by (2.8). Note that $x(S) = x_i + x(S\setminus\{i\})$. Then the coefficient of x_i on the right hand side of (2.8) is

$$-\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s} = -\sum_{N'\in H(N)}\sum_{s=1}^{n-1} C_{n-1}^{s-1}\cdot 2\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s} = -\frac{2}{n},$$

while the item concerning $x(S\setminus\{i\})$ on the right hand side of (2.8) is

$$\begin{aligned}
&-\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\cdot x(S'\setminus\{i\}) \\
&= -\sum_{N'\in H(N)} \sum_{j\in N\setminus\{i\}} x_j \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i,j, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s} \\
&= -\sum_{N'\in H(N)} \sum_{j\in N\setminus\{i\}} x_j \sum_{s=2}^{n-1} C_{n-2}^{s-2}\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s} \\
&= -\frac{n-2}{n(n-1)}\sum_{j\in N\setminus\{i\}} x_j = -\frac{n-2}{n(n-1)}\left(\bar{v}(N)-x_i\right).
\end{aligned}$$

The latter equation is because of the efficiency of x. The only thing that is not treated yet on the right hand side of (2.8) is

$$\begin{aligned}
&\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot 2\cdot\frac{1}{s}\cdot\theta_{S'}^{N'} \\
= &\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S') \qquad (2.9)\\
- &\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(N'\setminus S') \qquad (2.10)\\
+ &\sum_{N'\in H(N)} \sum_{\substack{S'\in\Omega, S'\in R(N'),\\ S'\ni i, |S'|\neq n}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot \bar{v}(N). \qquad (2.11)
\end{aligned}$$

In is easy to derive that the result of (2.11) is $\bar{v}(N)/n$. By changing the order of summations, (2.9) is equivalent to

$$\begin{aligned}
\sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \sum_{\substack{N'\in H(N),\\ N'\in V(S')}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S') &= \sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \frac{n!}{s!}\cdot\frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^s}\cdot\frac{1}{s}\cdot v(S') \\
&= \frac{1}{n-1}\sum_{\substack{S'\in\Omega, S'\ni i,\\ s\neq n}} \frac{(s-1)!(n-s)!}{n!}\cdot\frac{1}{s!}\cdot v(S').
\end{aligned}$$

Let $T' = N'\setminus S'$, then (2.10) is equivalent to

$$\begin{aligned}
&-\sum_{N'\in H(N)} \sum_{\substack{T'\in\Omega, T'\in R(N'),\\ T'\not\ni i, |T'|\neq 0}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^{n-t}}\cdot\frac{1}{n-t}\cdot v(T') \\
=&-\sum_{\substack{T'\in\Omega,\\ T'\not\ni i}} \sum_{\substack{N'\in H(N),\\ N'\in V(T')}} \frac{1}{n!}\cdot\frac{1}{n-1}\cdot\frac{1}{C_n^{n-t}}\cdot\frac{1}{n-t}\cdot v(T') \\
=&-\frac{1}{n-1}\sum_{\substack{T'\in\Omega,\\ T'\ni i}} \frac{(t-1)!(n-t)!}{n!}\cdot\frac{1}{(t-1)!}\cdot v(T'\setminus\{i\}).
\end{aligned}$$

In summary we derive that

$$x_i = \sum_{\substack{S'\in\Omega,\\ S'\ni i}} \frac{(s-1)!(n-s)!}{n!}\cdot\left(\frac{v(S')}{s!} - \frac{v(S'\setminus\{i\})}{(s-1)!}\right). \qquad (2.12)$$

Then by Lemma 2.3, we have $x = Sh'(N, v)$ for all generalized game $\langle N, v\rangle$.

In fact Theorem 2.4 can be restated as follows, where κ (which was defined above as the "uniform" distribution over two-configuration for the given game $\langle N, v\rangle$) is to be understood now as a function from games to such probability distributions.

Corollary 2.5. *The generalized Shapley value is the unique value on $\mathcal{G}'$ that is both consistent with κ and standard on two-person games.*

3. Consistency to the class of generalized values

Remind in the classical game space, a value ϕ on $\mathcal{G}$ is said to satisfy

(i) efficiency, if $\sum_{i\in N} \phi_i(N,v) = v(N)$ for all $v \in \mathcal{G}_N$;
(ii) linearity, if $\phi(N, a\cdot v + b\cdot w) = a\cdot\phi(N,v) + b\cdot\phi(N,w)$ for all $a,b\in\mathbb{R}$, all $v,w\in\mathcal{G}_N$;
(iii) symmetry, if $\phi_i(N,v) = \phi_j(N,v)$ for all symmetric players i and j in game $v\in\mathcal{G}_N$. Players i and j are called symmetric players in $\langle N,v\rangle$ if $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S\subseteq N\setminus\{i,j\}$.

The ELS value is denoted to the class of values on $\mathcal{G}_N$ satisfying efficiency, linearity and symmetry. Since additivity can be deduced from linearity, and additivity is equivalent to linearity for continuous values, the Shapley value clearly belongs to this class of values. The ELS value was firstly characterized by Ruiz, Valenciano and Zarzuelo (1998). Later Driessen (2002; 2010) gave the following characterization for the ELS value:

Theorem 3.1. *(Driessen and Radzik, 2002; Driessen, 2010) A value Φ on $\mathcal{G}_N$ satisfies the efficiency, linearity and symmetry if and only if there exists a (unique) collection of constants $\mathcal{B} = \{b_s^n \mid n\in\mathbb{N}\setminus\{0,1\}, s=1,2,\dots,n\}$ with $b_n^n = 1$ such that, for every n-person game $\langle N,v\rangle$ with at least two players,*

$$\Phi_i(N,v) = \sum_{S\subseteq N\setminus\{i\}} p_s^n\cdot\left(b_{s+1}^n\cdot v(S\cup\{i\}) - b_s^n\cdot v(S)\right) \quad \textit{for all } i\in N. \tag{3.13}$$

Whenever $b_s^n = 1$ for all $s\in\{1,2,\dots,n\}$, the expression on the right hand of (3.13) reduces to the Shapley value payoff of player i in the n-person game $\langle N,v\rangle$ itself. We now generalize the ELS value on the classical game space $\mathcal{G}$ to the generalized game space $\mathcal{G}'$ by considering all possible orders of coalitions.

Theorem 3.2. *There is a unique value $\Phi' : \mathcal{G}'_N \to \mathbb{R}^N$ satisfying the generalized efficiency, linearity and the generalized symmetry, such that for all $v\in\mathcal{G}'_N$ and all $i\in N$,*

$$\Phi'_i(N,v) = \sum_{\substack{S\subseteq N,\\ S\ni i}} \left(p_{s-1}^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)} v(S') - \sum_{\substack{S\subseteq N,\\ S\not\ni i}} \left(p_s^n\cdot b_s^n\right)\cdot\frac{1}{s!}\sum_{S'\in H(S)} v(S). \tag{3.14}$$

The proof of this theorem is postponed to the appendix. Remind the procedure we used in the latter section to characterize the generalized Shapley value on $\mathcal{G}'$. If we change the standard two-person bargaining solution (2.5) by

$$\eta_{S'}^{N'}(v) = b_s^n\cdot v(S') + \frac{1}{2}\left(b_n^n\cdot\bar{v}(N) - b_s^n\cdot v(S') - b_{n-s}^n\cdot v(N'\setminus S')\right);$$
$$\eta_{N'\setminus S'}^{N'}(v) = b_{n-s}^n\cdot v(N'\setminus S') + \frac{1}{2}\left(b_n^n\cdot\bar{v}(N) - b_{n-s}^n\cdot v(N'\setminus S') - b_s^n\cdot v(S')\right),$$

then by a similar arguments as in the proof of Theorem 2.4, we can derive the following result:

Theorem 3.3. *A feasible payoff vector $x\in\mathbb{R}^N$ is consistent with (f,η) for the generalized game $\langle N,v\rangle$ if and only if x is the generalized ELS value of form (3.14).*

4. Conclusions

In this paper all characterizations are done in the generalized game space. The difference compared with the classical game space is that, the order of players entering into the game influences the worth of coalitions. So for a fixed set of players, different permutations of this set may take different worths, which makes the characterization more complicated.

In the classical game space, Evans (1996) introduced an approach, such that the solution of the game determined endogenously as the expected outcome of a reduction of the game to a two-person bargaining problem, is just the Shapley value. However this approach is not suitable for the generalized games. So we modify Evans's approach in the following way: for any generalized game $\langle N, v\rangle$, firstly choose one permutation $N' \in H(N)$, secondly choose two subcoalitions S' and $N' \setminus S'$ according to N', and then choose two leaders from these two subcoalitions separately. The two leaders play a two-person bargaining game and promise to give the left players some part of his earning. We prove if all the choosing processes are under uniform distribution, and the standard solution on two-person games is used, then the expectation under the procedure is the generalized Shapley value. This also means, the generalized Shapley value can be axiomatized by Evans's consistency and the standardness on two-person games.

The class of values satisfying efficiency, linearity and symmetry on the generalized game space is well-defined. By a simple change to the standard two-person bargaining solution, the procedure we used to characterize the generalized Shapley value can be further used to characterize the class of values.

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Appendix A: Proof of Lemma 2.3

Proof. We will show the value defined by (2.6) satisfies additivity, together with efficiency, symmetry, null player property (see Definition 1.4), since Sanchez and Bergantinos (1997) proved the Shapley value in Definition 1.3 is the unique value on $\mathcal{G}'$ satisfying these four properties. Additivity is clear. Denote by ϕ the value defined by (2.6), then

$$\begin{aligned}
\sum_{i\in N}\phi_i(N,v) &= \sum_{i\in N}\sum_{\substack{S'\in\Omega,\\ S'\ni i}} \frac{(s-1)!(n-s)!}{n!}\left(\frac{v(S')}{s!}-\frac{v(S'\setminus\{i\})}{(s-1)!}\right)\\
&= \sum_{S'\in\Omega}\sum_{i\in S}\frac{(s-1)!(n-s)!}{n!}\frac{v(S')}{s!} - \sum_{\substack{S'\in\Omega,\\ s\neq n}}\sum_{i\notin S}\frac{s!(n-s-1)!}{n!}\frac{v(S')}{s!}\\
&= \sum_{S'\in\Omega}\frac{s!(n-s)!}{n!}\frac{v(S')}{s!} - \sum_{\substack{S'\in\Omega,\\ s\neq n}}\frac{s!(n-s)!}{n!}\frac{v(S')}{s!} = \bar{v}(N).
\end{aligned}$$

This proves the efficiency. Now suppose player i is a null player in $\langle N, v\rangle$, that is, $v(S', i^h) = v(S')$ for all $S' \in \Omega, S' \not\ni i$, $h \in \{1, 2, \ldots, s+1\}$, then we have

$\phi_i(N,v) = 0$ since

$$\sum_{\substack{S' \in \Omega,\\ S' \ni i}} v(S') = \sum_{\substack{S' \in \Omega,\\ S' \ni i}} s \cdot v(S' \setminus \{i\}).$$

In order to explain the above equality, we consider a coalition $S \subseteq N$, $S \ni i$. Fix $S' \backslash \{i\} \in H(S \backslash \{i\})$, then $(S' \backslash \{i\}, i^h)$, $h \in \{1, 2, \ldots, s\}$ results s different $S' \in H(S)$. This proves the null player property. To prove the symmetry, consider two symmetric players $i, j \in N$, $i \neq j$, that is, $v(S', i^h) = v(S', j^h)$ for all $S' \in \Omega, S' \not\ni i, j$, $h \in \{1, 2, \ldots, s+1\}$. We can rewrite (2.6) in the following way:

$$\begin{aligned}
\phi_i(N,v) &= \left(\sum_{\substack{S' \in \Omega,\\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega,\\ S' \ni i, S' \not\ni j}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{i\})}{(s-1)!} \right) \\
&= \left(\sum_{\substack{S' \in \Omega,\\ S' \ni i,j}} + \sum_{\substack{S' \in \Omega,\\ S' \ni j, S' \not\ni i}} \right) \frac{(s-1)!(n-s)!}{n!} \left(\frac{v(S')}{s!} - \frac{v(S' \setminus \{j\})}{(s-1)!} \right) = \phi_j(N,v).
\end{aligned}$$

This proves the symmetry.

Appendix B: Proof of Theorem 3.2

Proof. Linearity is clear. Suppose the pair $i, j \in N$ are symmetric players. Then $v(S', i^h) = v(S', j^h)$ for all $S' \in \Omega$, $S' \not\ni i, j$, $h \in \{1, 2, \ldots, s+1\}$ gives

$$\sum_{\substack{S \subseteq N,\\ S \ni i, S \not\ni j}} \sum_{S' \in H(S)} v(S') = \sum_{\substack{S \subseteq N,\\ S \ni j, S \not\ni i}} \sum_{S' \in H(S)} v(S').$$

Hence

$$\Phi'_i(N,v) - \Phi'_j(N,v) = \left(\sum_{\substack{S \subseteq N,\\ S \ni i, S \not\ni j}} - \sum_{\substack{S \subseteq N,\\ S \ni j, S \not\ni i}} \right) (p^n_{s-1} + p^n_s) \cdot b^n_s \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') = 0.$$

This proves the generalized symmetry. Next we show that Φ' satisfies the generalized efficiency: for any $v \in \mathcal{G}'$,

$$\begin{aligned}
\sum_{i \in N} \Phi'_i(N,v) =& \sum_{i \in N} \left(\sum_{\substack{S \subseteq N,\\ S \ni i}} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{\substack{S \subseteq N,\\ S \not\ni i}} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \right) \\
=& \sum_{\substack{S \subseteq N\\ S \neq \emptyset}} \left(\sum_{i \in S} (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{i \notin S} (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \right) \\
=& \sum_{\substack{S \subseteq N\\ S \neq \emptyset}} s \cdot (p^n_{s-1} \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S') - \sum_{\substack{S \subsetneq N\\ S \neq \emptyset}} (n-s) \cdot (p^n_s \cdot b^n_s) \cdot \frac{1}{s!} \sum_{S' \in H(S)} v(S) \\
=& \frac{1}{n!} \sum_{N' \in H(N)} v(N').
\end{aligned}$$

This completes the sufficient proof. Now we show the uniqueness. Suppose there is another value ϕ on $\mathcal{G}'$ satisfying the generalized efficiency, linearity and the generalize symmetry. With every ordered coalition $T' \in \Omega$, $T' \neq \emptyset$, there is an associated zero-one game $\langle N, e_{T'} \rangle$ defined by $e_{T'}(T') = 1$ and $e_{T'}(S') = 0$ for all $S' \neq T'$, $S' \in \Omega$. Since $v(S') = \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot e_{T'}(S')$ for all $S' \in \Omega$, all $v \in \mathcal{G}'_N$, by linearity we have

$$\phi_i(N, v) = \phi_i(N, \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot e_{T'}) = \sum_{T \subseteq N} \sum_{T' \in H(T)} v(T') \cdot \phi_i(N, e_{T'}),$$

for all $i \in N$. Next we determine $\phi_i(N, e_{T'})$. Fix the coalition $T' \in \Omega$, by symmetry we know that players in T' as well as players outside T' get the fixed payoff respectively, which only depend on the size of T'. Then by efficiency, (3.14) is derived.

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Analysing the Folk Theorem for Linked Repeated Games

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Abstract We deal with the linkage of infinitely repeated games. Results are obtained by analysing the relations between the feasible individually rational payoff regions of the isolated games and the linked game. In fact we have to handle geometric problems related to Minkowski sums, intersections and Pareto boundaries of convex sets.

Key words: asymmetries, convex set, feasible individually rational payoff region, Folk theorem, full cooperation, linking, Minkowski sum, Pareto boundary, tensor game.

1. Introduction

There has developed an interest in the theory and applications of linking, also called 'interconnection'. The basic idea is the following. Consider a group of decision makers who are simultaneously involved in several different real world problems (issues). The standard approach is to consider the decision making process for each problem in isolation. In practice, however, the decision making process with respect to one problem is usually influenced by the decision making processes with respect to the other problems (spill-over effects or links). Discarding the links among the issues and analyzing the decision process on each issue separately rather than in a multi-issue decision making context is likely to lead to biased outcomes. Particularly, a single issue approach ignores the possibility that if the issues have compensating asymmetries of similar magnitudes, an exchange of concessions may allow and enhance cooperation which extends beyond cooperation in the single issue context.

Some well-known real world examples of linking are the negotiations 'on land for peace' between Israel and Palestina and the deal on WTO membership and participation in the Kyoto agreement between the EU and Russia. In the economics literature the notion of linking has been applied in the context of multimarket behaviour in oligopolistic markets (see e.g. Bernheim and Whinston, 1990; Spagnolo, 1999) and of international environmental problems (see e.g. Folmer et al., 1993; Botteon and Carraro, 1998; Carraro and Siniscalco, 1999; Finus, 2001).

A game theoretical framework for the linking of repeated games was developed by Folmer et al. (1993) and by Folmer and von Mouche (1994). In Folmer and von Mouche (2000) the following themes for linking of discounted infinitely repeated games were suggested:

– linking may sustain more cooperation;[1]
– linking may eliminate social welfare losses;
– linking may bring Pareto improvements;
– linking may facilitate cooperation.

We observe that 'may' is used here to indicate that the characteristics of linking of repeated games mentioned do no hold unconditionally but depend on the particular nature of the problem at hand. However, to our best knowledge, the conditions under which these characteristics hold have not yet been thoroughly analysed which is a major omission in the light of the practical and theoretical relevance of linking. Admittedly, some results about the conditions under which the characteristics of more cooperation and Pareto improvements hold can be found in Ragland (1995) and Just and Netanyahu (2000). However, these results are limited in scope because the settings in these publications concern the special case of linking of two repeated 2×2-bimatrix games.

The main purpose of the article[2] is to identify classes of isolated stages games for which the themes 'linking may sustain more cooperation' and 'linking may bring Pareto improvements' materialize or not; special attention is paid to the role of asymmetries. As these themes refer to properties of subgame perfect Nash equilibria of the linked and isolated games, Folk theorems, and in particular feasible individually rational payoff regions, come into the picture. In fact we formalize the two themes in terms of these regions and analyse how these regions for the isolated games relate to that of the linked game. Our results apply to the linking of an arbitrary finite number of discounted infinitely repeated games with an arbitrary finite number of (the same) players.

From a mathematical point of view analysing the two themes concerns the handling of two geometric problems. As these problems are in their own interesting and make sense without their game theoretic motivation, we organize the article as follows. In Section 2 we introduce notations and present some useful general results about Minkowski sums, normal cones and Pareto boundaries with which we shall handle the two geometric problems. The material in this section may have some interest in its own, especially as we cannot give good references for it in the literature. In Section 3 we state and analyse the two geometric problems in their pure form. Next, in Section 4 we show how the results in Section 3 induce results for the two themes for linked repeated games.

2. Convexity and Geometry

For the whole article we fix positive numbers m, n and write

$$N := \{1, \ldots, n\}, \ \ M := \{1, \ldots, m\}.$$

For $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ we write $\mathbf{a} \geq \mathbf{b}$ if $a_i \geq b_i$ for all $i \in N$. We write $\mathbf{a} > \mathbf{b}$ if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. And we write $\mathbf{a} \gg \mathbf{b}$ if $a_i > b_i$ for all $i \in N$.

[1] This is the counterpart of the theme 'repetition enables cooperation' for repeated games. 'More' is relative to the single issue case.

[2] The article concerns an improved version of Folmer and von Mouche (2007) and deals with a research question proposed in Folmer and von Mouche(2000).

The relation $\geq$ on $\mathbb{R}^n$ is a partial order. For $A \subseteq \mathbb{R}^n$ its *(strong) Pareto boundary*

$$\mathrm{P}(A)$$

is defined as the set of maximal elements of A, i.e. as the set of elements $\mathbf{a}$ of A for which there does not exist $\mathbf{c} \in A$ with $\mathbf{c} > \mathbf{a}$. And for $A \subseteq \mathbb{R}^n$ its *weak Pareto boundary*

$$\mathrm{P}_w(A)$$

is defined as the set of elements $\mathbf{a}$ of A for which there does not exist $\mathbf{c} \in A$ with $\mathbf{c} \gg \mathbf{a}$. Of course, $\mathrm{P}(A) \subseteq \mathrm{P}_w(A)$.

Proposition 1. *Let A be a compact subset of $\mathbb{R}^n$. For every $\mathbf{a} \in A$ there exists $\mathbf{b} \in \mathrm{P}(A)$ with $\mathbf{b} \geq \mathbf{a}$.* $\diamond$

Proof. $Z := \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} \geq \mathbf{a}\}$ is closed. This implies that $Z \cap A$ is compact. As $\mathbf{a} \in Z \cap A$ w have $Z \cap A \neq \emptyset$ and therefore also $\mathrm{P}(Z \cap A) \neq \emptyset$. Take $\mathbf{b} \in \mathrm{P}(Z \cap A)$. Then $\mathbf{b} \in Z \cap A \subseteq Z$, so $\mathbf{b} \geq \mathbf{a}$. Now we prove by contradiction that $\mathbf{b} \in \mathrm{P}(A)$. So suppose there would exist $\mathbf{c} \in A$ with $\mathbf{c} > \mathbf{b}$. Then we had $\mathbf{c} > \mathbf{b} \geq \mathbf{a}$, so $\mathbf{c} \in Z \cap A$ and $\mathbf{c} > \mathbf{b}$, which is a contradiction with $\mathbf{b} \in \mathrm{P}(Z \cap A)$. Q.E.D.

Proposition 2. *Let $B, C \subseteq \mathbb{R}^n$. Suppose for no $\mathbf{c} \in C$ there exists $\mathbf{d} \in \mathbb{R}^n \setminus C$ with $\mathbf{d} > \mathbf{c}$. Then $\mathrm{P}(B \cap C) = \mathrm{P}(B) \cap C$.* $\diamond$

Proof. '$\subseteq$': by contradiction. So suppose $\mathbf{a} \in \mathrm{P}(B \cap C)$ and $\mathbf{a} \notin \mathrm{P}(B) \cap C$. As $\mathbf{a} \in B \cap C \subseteq C$, it follows that $\mathbf{a} \notin \mathrm{P}(B)$. As $\mathbf{a} \in B$, there is $\mathbf{b} \in B$ with $\mathbf{b} > \mathbf{a}$. As $\mathbf{a} \in \mathrm{P}(B \cap C)$, it follows that $\mathbf{b} \notin B \cap C$. Thus $\mathbf{b} \in \mathbb{R}^n \setminus C$, $\mathbf{a} \in C$ and $\mathbf{b} > \mathbf{a}$, which is a contradiction.

'$\supseteq$': suppose $\mathbf{d} \in \mathrm{P}(B) \cap C$. So $\mathbf{d} \in B \cap C$. If we would have $\mathbf{a} \in B \cap C$ such that $\mathbf{a} > \mathbf{d}$, then, noting that $\mathbf{a} \in B$ and $\mathbf{d} \in B$, we would have a contradiction. Q.E.D.

Denote the set of permutations of N by

$$S_n.$$

For $\pi \in S_n$, the mapping $T_\pi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_\pi(x_1, \ldots, x_n) \ := \ (x_{\pi(1)}, \ldots, x_{\pi(n)})$$

is a linear isomorphism. We have

$$T_{\pi_2} \circ T_{\pi_1} = T_{\pi_1 \circ \pi_2}, \ \ T_{\mathrm{id}} = \mathrm{id}, \ (T_\pi)^{-1} = T_{\pi^{-1}}.$$

We call $A \subseteq \mathbb{R}^n$ *permutation-symmetric* if $T_\pi(A) = A$ for all permutations $\pi \in S_n$. So each subset of $\mathbb{R}$ is permutation symmetric.

Define the function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{C}(\mathbf{a}) \ := \ \sum_{l \in N} a_l$$

and for a subset A of $\mathbb{R}^n$, denoting by $\mathcal{C} \upharpoonright A$ the restriction of the function $\mathcal{C}$ to A,

$$S(A) \ := \ \mathrm{argmax}(C \upharpoonright A), \ \ s(A) \ := \ \sup(\mathcal{C} \upharpoonright A). \tag{1}$$

The following simple properties hold:

$$s(\mathrm{Conv}(A)) = s(A) \text{ and } S(\mathrm{Conv}(A)) = \mathrm{Conv}(S(A)); \tag{2}$$

$$\text{for all } \pi \in S_n : \quad s(T_\pi(A)) = s(A) \text{ and } S(T_\pi(A)) = T_\pi(S(A)). \tag{3}$$

Closedness (boundedness) of A implies closedness (boundedness) of $S(A)$. And, with Weierstrass' theorem,

$$A \text{ non-empty and compact } \;\Rightarrow S(A) \text{ non-empty and compact.} \tag{4}$$

The sets $S(A), \mathrm{P}(A), \mathrm{P}_w(A)$ are subsets of the topological boundary ∂A of A:

$$S(A) \subseteq \mathrm{P}(A) \subseteq \mathrm{P}_w(A) \subseteq \partial A.$$

So, by (4), $\mathrm{P}(A) \neq \emptyset$ if A is non-empty and compact.

Definition 1. Let A_k $(k \in M)$ be non-empty subsets of $\mathbb{R}^n$ and $\mathbf{a} \in A = \sum_{k \in M} A_k$.[3] We call $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}) \in A_1 \times \cdots \times A_m$ a *decomposition* of $\mathbf{a}$ if $\mathbf{a} = \sum_{k \in M} \mathbf{a}^{(k)}$. $\diamond$

For this situation:

Proposition 3. *For every* $\mathbf{p} \in \mathbb{R}^n$ *and* $\mathbf{a} \in A$

$$\mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{a} \; (\mathbf{z} \in A) \;\Leftrightarrow\; \mathbf{p} \cdot \mathbf{z}^{(k)} \leq \mathbf{p} \cdot \mathbf{a}^{(k)} \; (k \in M,\; \mathbf{z}^{(k)} \in A_k). \;\diamond$$

Proof. '$\Rightarrow$': by contradiction, suppose there exists k and $\mathbf{z}^{(k)}$ such that $\mathbf{p} \cdot \mathbf{z}^{(k)} > \mathbf{p} \cdot \mathbf{a}^{(k)}$. Then $\mathbf{b} := \mathbf{z}^{(k)} + \sum_{l \in M \setminus \{k\}} \mathbf{a}^{(l)} \in A$ and $\mathbf{p} \cdot \mathbf{b} > \mathbf{p} \cdot \mathbf{a}$, which is a contradiction.

'$\Leftarrow$': suppose $\mathbf{z} \in A$. Let $(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)})$ be a decomposition of $\mathbf{z}$. By assumption $\mathbf{p} \cdot \mathbf{z}^{(k)} \leq \mathbf{p} \cdot \mathbf{a}^{(k)}$ $(k \in M)$. Summing over $k \in M$ gives $\mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{a}$. Q.E.D.

Proposition 4. *Let* A_k $(k \in M)$ *be subsets of* $\mathbb{R}^n$ *and* $A = \sum_{k \in M} A_k$.

1. *If* $A_k \neq \emptyset$ $(k \in M)$, *then* $s(A) = \sum_{k \in M} s(A_k)$.
2. $S(A) = \sum_{k \in M} S(A_k)$. $\diamond$

Proof. 1. As $A_k \neq \emptyset$ $(k \in M)$ we obtain

$$s(\sum_k A_k) = \sup(\mathcal{C}(\sum_k A_k)) = \sup(\sum_k \mathcal{C}(A_k)) = \sum_k \sup(\mathcal{C}_k(A_k)) = \sum_k s(A_k).$$

2. In case there is an k with $A_k = \emptyset$, the desired result holds. Now suppose $A_k \neq \emptyset$ $(k \in M)$. Taking $\mathbf{p} = (1, 1, \ldots, 1)$ in Proposition 3 gives for a decomposition $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ of $\mathbf{a} \in A$: $\mathbf{a} \in S(A) \;\Leftrightarrow\; \mathbf{a}^{(k)} \in S(A_k)$ $(k \in M)$, i.e. the desired result. Q.E.D.

Proposition 5. *Let* A_k $(k \in M)$ *be subsets of* $\mathbb{R}^n$ *and* $A = \sum_{k \in M} A_k$.

1. *Suppose every* A_k *is non-empty. For every decomposition* $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ *of* $\mathbf{a} \in A$ *it holds that* $\mathbf{a} \in \mathrm{P}(A) \;\Rightarrow\; \mathbf{a}^{(k)} \in \mathrm{P}(A_k)$ $(k \in M)$.
2. $\mathrm{P}(A) \subseteq \sum_{k \in M} \mathrm{P}(A_k)$. $\diamond$

[3] The sum here is a Minkowski sum.

Proof. 1. By contradiction, suppose $\mathbf{a} \in \mathrm{P}(A)$ and there exists l such that $\mathbf{b}^{(l)} \in A_l$ and $\mathbf{b}^{(l)} > \mathbf{a}^{(l)}$. With $y := \mathbf{b}^{(l)} + \sum_{k \in M \setminus \{l\}} \mathbf{a}^{(k)} \in A$ one has $\mathbf{y} = \mathbf{a} + (\mathbf{b}^{(l)} - \mathbf{a}^{(l)}) > \mathbf{a}$, a contradiction.

2. This follows from part 1. Q.E.D.

In general, the inclusion in Proposition 5(2) is not an equality. Here is a special case where equality holds:

Proposition 6. *If $m = 2$ and A_1 or A_2 has a maximiser, then* $\mathrm{P}(A_1 + A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2)$. $\diamond$

Proof. We may assume that A_2 has a maximiser, say $\mathbf{b}$. So we have

$$\mathbf{y} \leq \mathbf{b} \; (\mathbf{y} \in A_2) \tag{5}$$

This implies $P(A_2) = \{\mathbf{b}\}$. By Proposition 5(2) only '$\supseteq$' remains to be proved. This we do by contradiction. So suppose $\mathbf{c} \in \mathrm{P}(A_1) + \mathrm{P}(A_2)$, but $\mathbf{c} \notin \mathrm{P}(A_1 + A_2)$. Let $\mathbf{a} \in \mathrm{P}(A_1)$ such that $\mathbf{c} = \mathbf{a} + \mathbf{b}$. As $\mathbf{c} \in A_1 + A_2$ and $\mathbf{c} \notin \mathrm{P}(A_1 + A_2)$, there is $\mathbf{d} \in A_1 + A_2$ with $\mathbf{d} > \mathbf{c}$. Let $\mathbf{a}' \in A_1$ and $\mathbf{b}' \in A_2$ such that $\mathbf{d} = \mathbf{a}' + \mathbf{b}'$. Then, by (5), $\mathbf{a}' > \mathbf{a} + (\mathbf{b} - \mathbf{b}') \geq \mathbf{a}$, so $\mathbf{a}' > \mathbf{a}$. But $\mathbf{a} \in \mathrm{P}(A_1)$, a contradiction. Q.E.D.

Let A be a non-empty subset of $\mathbb{R}^n$ and $\mathbf{z} \in \overline{A}$, i.e. $\mathbf{z}$ is an element of the topological closure of A. Then

$$N_A(\mathbf{z}) := \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{d} \cdot (\mathbf{a} - \mathbf{z}) \leq 0 \text{ for all } \mathbf{a} \in A\}.$$

$N_A(\mathbf{z})$ is a convex cone and is called the *normal cone* of A in $\mathbf{z}$. Moreover, we define for $\mathbf{z} \in \overline{A}$ the *positive normal cone* of A in $\mathbf{z}$ as

$$N_A^+(\mathbf{z}) := \{\mathbf{d} \in N_A(\mathbf{z}) \mid \mathbf{d} > \mathbf{0}\}.$$

Note that $\mathbf{0} \in N_A(\mathbf{z})$, but that $N_A^+(\mathbf{z})$ may be empty.

Proposition 7. *Let A_k $(k \in M)$ be non-empty subsets of $\mathbb{R}^n$, $A = \sum_{k \in M} A_k$ and $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)})$ a decomposition of $\mathbf{a} \in A$. Then $N_A(\mathbf{a}) = \cap_{k \in M} N_{A_k}(\mathbf{a}^{(k)})$ and $N_A^+(\mathbf{a}) = \cap_{k \in M} N_{A_k}^+(\mathbf{a}^{(k)})$.* $\diamond$

Proof. We prove the first statement; then the second holds too.

$\subseteq$: suppose $\mathbf{d} \in N_A(\mathbf{a})$. So $\mathbf{d} \cdot \mathbf{z} \leq \mathbf{d} \cdot \mathbf{a}$ $(\mathbf{z} \in A)$. Proposition 3 implies $\mathbf{d} \cdot \mathbf{z}^{(k)} \leq \mathbf{d} \cdot \mathbf{a}^{(k)}$ $(k \in M,\ \mathbf{z}^{(k)} \in A_k)$. Thus $\mathbf{d} \in N_{A_k}(\mathbf{a}_k)$ $(k \in M)$.

$\supseteq$: suppose $\mathbf{d} \in \cap_{k \in M} N_{A_k}(\mathbf{a}^{(k)})$. So $\mathbf{d} \cdot \mathbf{z}^{(k)} \leq \mathbf{d} \cdot \mathbf{a}^{(k)}$ $(k \in M, \mathbf{z}^{(k)} \in A_k)$. Proposition 3 implies $\mathbf{d} \cdot \mathbf{z} \leq \mathbf{d} \cdot \mathbf{a}$ $(\mathbf{z} \in A)$. Thus $\mathbf{d} \in N_A(\mathbf{a})$. Q.E.D.

Proposition 8. *Let A be a non-empty convex subset of $\mathbb{R}^n$. Then $\mathbf{z} \in \mathrm{P}_w(A) \Rightarrow N_A^+(\mathbf{z}) \neq \emptyset$.* $\diamond$

Proof. Define $B := \{\mathbf{b} \in \mathbb{R}^n \mid \mathbf{b} \geq \mathbf{z}\}$. For $\mathring{B}$, i.e. for the topological interior of B one has $\mathring{B} = \{\mathbf{b} \in \mathbb{R}^n \mid \mathbf{b} \gg \mathbf{z}\}$ and thus $\mathring{B} \cap A = \emptyset$. The sets $\mathring{B}$ and A are convex, non-empty and disjoint. Using a separation theorem, there exists an affine hyperplane that separates A and $\mathring{B}$. Therefore there exists $\mathbf{d} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{d} \cdot \mathbf{a} \leq \mathbf{d} \cdot \mathbf{b}$ $(\mathbf{a} \in A,\ \mathbf{b} \in \mathring{B})$. Even now

$$\mathbf{d} \cdot \mathbf{a} \leq \mathbf{d} \cdot \mathbf{b} \; (\mathbf{a} \in A, \mathbf{b} \in B). \tag{6}$$

With $\mathbf{b} = \mathbf{z}$ it follows that $\mathbf{d} \cdot \mathbf{a} \le \mathbf{d} \cdot \mathbf{z}$ ($\mathbf{a} \in A$). Now we prove by contradiction that $\mathbf{d} > 0$. So (remembering that $\mathbf{d} \neq \mathbf{0}$) suppose $d_i < 0$ for some i. For $\mathbf{b} \in B$ defined by $b_j := z_j$ ($j \neq i$) and $b_i := c$ where $c \ge a_i$, we have

$$\mathbf{d} \cdot \mathbf{b} = \sum_{j \in N \setminus \{i\}}^{n} d_j z_j + d_i c.$$

For c large enough $\mathbf{d} \cdot \mathbf{b} \le \mathbf{d} \cdot \mathbf{z}$, which is a contradiction with (6). Q.E.D.

Proposition 9. *For non-empty subsets A and B of $\mathbb{R}^n$ with $A \subseteq B$ one has: B compact and* $\mathrm{P}(B) \subseteq A \Rightarrow N_B^+(\mathbf{z}) = N_A^+(\mathbf{z})$ $(\mathbf{z} \in \overline{A})$. $\diamond$

Proof. Because $A \subseteq B$ one has $N_B^+(\mathbf{z}) \subseteq N_A^+(\mathbf{z})$. By contradiction we prove that $N_B^+(\mathbf{z}) \supseteq N_A^+(\mathbf{z})$. So suppose $\mathbf{d} \in N_A^+(\mathbf{z}) \setminus N_B^+(\mathbf{z})$. Now $(\mathbf{w} - \mathbf{z}) \cdot \mathbf{d} \le 0$ for all $\mathbf{w} \in A$, but not for all $\mathbf{z} \in B$. This implies that there is a $\mathbf{w} \in B \setminus A$ such that $\mathbf{d} \cdot (\mathbf{w} - \mathbf{z}) > 0$. As B is compact, there is, by Proposition 1, $\mathbf{b} \in \mathrm{P}(B)$ such that $\mathbf{b} \ge \mathbf{w}$. As $\mathbf{d} > \mathbf{0}$, also $\mathbf{d} \cdot (\mathbf{b} - \mathbf{z}) > 0$. So $\mathbf{b} \notin A$. But $\mathbf{b} \in \mathrm{P}(B) \subseteq A$, which is a contradiction. Q.E.D.

3. Two Geometric Problems

3.1. Stating the Problems

In this section fix non-empty subsets $U_1, \ldots, U_m$ of $\mathbb{R}^n$ and define, denoting by $\mathbb{R}^n_+$ the closed positive octant of $\mathbb{R}^n$,

$$F_k := \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+ \ (k \in M), \quad F := \sum_{k \in M} F_k,$$

$$U := \sum_{k \in M} U_k, \quad F_\star := \mathrm{Conv}(U) \cap \mathbb{R}^n_+.$$

Note that $\mathrm{Conv}(U) = \sum_{k \in M} \mathrm{Conv}(U_k)$. It is easy to see (also see Proposition 10(1)) that

$$F \subseteq F_\star. \tag{7}$$

Problem 1. Provide interesting conditions under which $F \subset F_\star$.

Our second problem deals with the following object

$$\mathrm{EXP} = \{\mathbf{v} \in \mathrm{P}(F) \mid \text{ there exists } \mathbf{w} \in F_\star \text{ with } \mathbf{w} \gg \mathbf{v}\}.$$

We refer to the elements of EXP as *expansion points* of PB(F). Note that So

$$F = F_\star \ \Rightarrow \ \mathrm{EXP} = \emptyset. \tag{8}$$

Of course, also $P(F) = \emptyset$ implies that $\mathrm{EXP} = \emptyset$. But $\mathrm{EXP} = \emptyset$ is also possible if $P(F) \neq \emptyset$ and $F \subset F_\star$ as Figure 4 below shows.

Problem 2. Provide interesting conditions under which $\mathrm{EXP} = \emptyset$, under which $\emptyset \subset \mathrm{EXP} \subset \mathrm{P}(F)$ and under which $\emptyset \subset \mathrm{EXP} = \mathrm{P}(F)$.

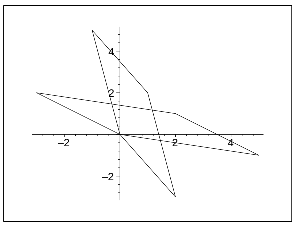
Now we illustrate these two problems (in case $m = n = 2$) with some figures.[4]

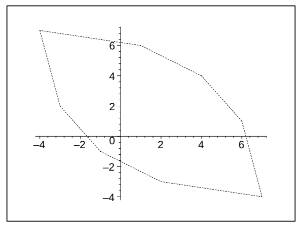
Remark. (1) In all these figures every $\mathrm{Conv}(U_k)$ are polygons. This makes that F_k $(k \in m)$, F, U and $F_\star$ are polygons.

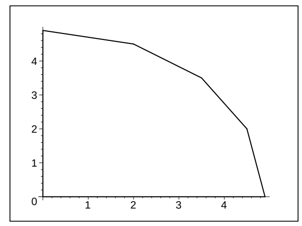
Figure 1 relates to

$$U_1 = \{(2,1), (-3,2), (5,-1), (0,0)\}, \;\; U_2 = \{(1,2), (-1,5), (2,-3), (0,0)\}.$$

Figure 1 (and also Figures 2 – 5) are to be interpreted as follows. Four polygons are drawn: the sets $\mathrm{Conv}(U_1)$ and $\mathrm{Conv}(U_2)$, the Minkowski sum of these two sets and the set $F = F_1 + F_2$ being the boldfaced polygon. These four polygons are respectively drawn in the following three figures:







We note that in the case of Figure 1

$$U_1 + U_2 = \{(3,3), (1,6), (-2,4), (-4,7), (4,-2), (2,1), (-1,-1), (-3,2),$$
$$(6,1), (4,4), (1,2), (-1,5), (7,-4), (5,-1), (2,-3), (0,0)\}.$$

Figure 2 relates to

$$U_1 = \{(0,2), (3,1), (-3,0), (0,0)\}, \;\; U_2 = \{(0,1), (1,1/2), (-2,0), (0,0)\}.$$

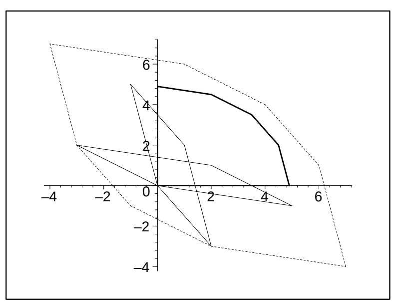


Fig. 1: $F \subset F_\star$ and $\emptyset \subset \mathrm{EXP} = P(F)$. (Neither U_1 nor U_2 is permutation-symmetric).

Figure 3 relates to

$$U_1 = \{(7,1), (-3,3), (10,-2), (0,0)\}, \;\; U_2 = \{(1,7), (-2,10), (3,-3), (0,0)\}.$$

Figure 4 relates to

$$U_1 = \{(2,2), (-2,4), (4,-2), (0,0)\}, \;\; U_2 = \{(2,2), (-1,1), (1,-1), (0,0)\}.$$

[4] Figures 1, 3, 4, 5 are taken from Folmer and von Mouche (2000).

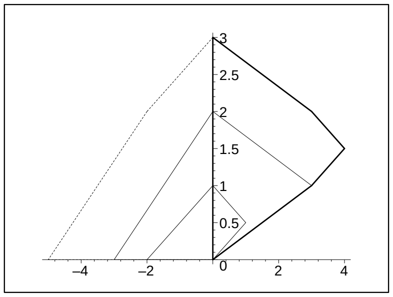


Fig. 2: $F = F_\star$ and $\mathrm{EXP} = \emptyset$. (Neither U_1 nor U_2 is permutation-symmetric).

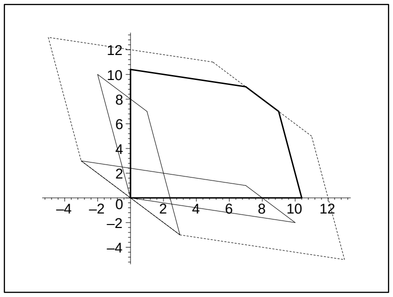


Fig. 3: $F \subset F_\star$ and $\emptyset \subset \mathrm{EXP} \subset P(F)$ (U_1 and U_2 are not permutation-symmetric).

Figure 5 relates to

$$U_1 = \{(2,2),\ (-2,10),\ (10,-2),\ (0,0)\},\ \ U_2 = \{(3,3),\ (-3,4),\ (4,-3),\ (0,0)\}.$$

3.2. On Problem 1

Now let us return to Problem 1.

Proposition 10. *1. $F \subseteq F_\star$. And $F = F_\star$ if and only if $\sum_{k \in M}(\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+) \supseteq \mathbb{R}^n_+ \cap \sum_{k\in M} \mathrm{Conv}(U_k)$.*
2. If $m = 1$, then $F = F_\star$. $\diamond$

Proof. 1. $F = \sum_{k\in M}(\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+) \subseteq \sum_{k\in M} \mathbb{R}^n_+ \cap \sum_{k\in M} \mathrm{Conv}(U_k) = \mathbb{R}^n_+ \cap \sum_{k\in M} \mathrm{Conv}(U_k) = F_\star$.
2. $F = F_1 = \mathrm{Conv}(U_1) \cap \mathbb{R}^n_+ = \mathrm{Conv}(U) \cap \mathbb{R}^n_+ = F_\star$. Q.E.D.

Of course, because of (7), if $F_\star = \emptyset$, then $F = F_\star$ holds. The next proposition identifies two little bit less trivial cases for this to hold:

Proposition 11. *Each of the following conditions is sufficient for $F = F_\star$ to hold.*

1. *There exist $r_k > 0$ $(k \in M)$ and $c \in \mathbb{R}^n$ such that $U_k = r_k(U_1 + c)$ $(k \in M)$.*
2. *$U_k \subseteq \mathbb{R}^n_+$ $(k \in M)$.* $\diamond$

Proof. 1. We have $\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+ = \mathrm{Conv}(r_k(U_1 + c)) \cap \mathbb{R}^n_+ = r_k\, \mathrm{Conv}(U_1 + c) \cap r_k\mathbb{R}^n_+ = r_k(\mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+)$; here the last equality holds as $r_k \neq 0$. This implies, with $r := \sum_k r_k$ and with sums on $k \in M$

$$\sum(\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+) = \sum r_k(\mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+) = r(\mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+);$$

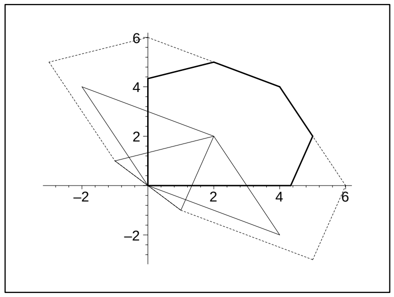


Fig. 4: $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$ (U_1 and U_2 are permutation-symmetric).

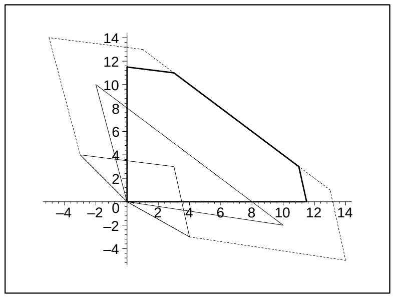


Fig. 5: $F \subset F_\star$ and $\emptyset \subset \mathrm{EXP} \subset P(F)$ (U_1 and U_2 are permutation-symmetric).

here the last equality holds as $\mathbb{R}^n_+ \cap \mathrm{Conv}(U_1 + c)$ is convex and the r_k are non-negative. Further

$$r(\mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+) = r\ \mathrm{Conv}(U_1 + c) \cap r\ \mathbb{R}^n_+ = r\ \mathrm{Conv}(U_1 + c) \cap \mathbb{R}^n_+$$

$$= \mathbb{R}^n_+ \cap \sum (r_k\ \mathrm{Conv}(U_1 + c)) = \mathbb{R}^n_+ \cap \sum \mathrm{Conv}(U_k).$$

So the proof is complete by Proposition 10(1).

2. Using $U_k \subseteq \mathbb{R}^n_+$ and $\sum_k \mathrm{Conv}(U_k) \subseteq \mathbb{R}^n_+$ we obtain $\sum_k (\mathrm{Conv}(U_k) \cap \mathbb{R}^n_+) = \sum_k \mathrm{Conv}(U_k) = \mathrm{Conv}(\sum_k U_k) = \mathrm{Conv}(\sum_k U_k) \cap \mathbb{R}^n_+ = \mathbb{R}^n_+ \cap \sum_k \mathrm{Conv}(U_k)$. Q.E.D.

Figure 2 shows that there are situations with $F = F_\star$ that are not covered by Proposition 11. In all other figures $F \subset F_\star$ holds. Theorem 1 below gives our main result for $F \subset F_\star$ to hold. This theorem is based on the following principle:

Proposition 12. *Suppose there exists $l \in M$ such that* $\mathrm{Conv}(S(U_l)) \cap \mathbb{R}^n_+ = \emptyset$ *and* $S(U) \cap \mathbb{R}^n_+ \neq \emptyset$*, then $F \subset F_\star$.* $\diamond$

Proof. We shall prove that $S(U) \cap \mathbb{R}^n_+ \subseteq F_\star \setminus F$ (and then the desired result follows). So fix $\mathbf{b} \in S(u) \cap \mathbb{R}^n_+$. Of course, $\mathbf{b} \in F_\star$. Now we shall prove by contradiction that $\mathbf{b} \notin F$. So suppose $\mathbf{b} \in F = \sum_k F_k$. Take $\mathbf{h}^k \in \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+$ such that $\mathbf{b} = \sum_k \mathbf{h}^k$. Using (2), we have for every $k \in M$

$$\sum_j h^k_j \leq s(\mathrm{Conv}U_k) = s(U_k). \tag{9}$$

Because $\mathbf{h}^l \in \mathbb{R}^n_+$ it follows that $\mathbf{h}^l \notin \mathrm{Conv}(S(U_l))$ and so $\mathbf{h}^l \in \mathrm{Conv}(U_l) \setminus \mathrm{Conv}(S(U_l))$. By virtue of (2) we have $\mathrm{Conv}(S(U_l)) = S(\mathrm{Conv}(U_l))$ and so $\mathbf{h}^l \in$

$\mathrm{Conv}(U_l) \setminus S(\mathrm{Conv}(U_l))$. Therefore, in (9) we have a strict inequality for $k = l$. Because $\mathbf{b} \in S(U)$, one has $\sum_j b_j = s(U)$. With Proposition 4 it follows that $s(U) = \sum_k s(U_k) > \sum_k \sum_j F_j^k = \sum_j \sum_k F_j^k = \sum_j b_j = s(U)$, which is a contradiction. Q.E.D.

Now we shall identify a more concrete situation (i.e. in terms of the U_k) that satisfies this principle. In order to do so we introduce some notions in the following two definitions.

Definition 2. Let A_k $(k \in M)$ be subsets of $\mathbb{R}^n$. The sets A_k $(k \in M)$ have *compensating asymmetries of exactly the same magnitude* if $m = n$ and there are $\pi_k \in S_n$ $(k \in M)$ with $\pi_1 = \mathrm{Id}$ such that

$$\{\pi_1(j), \ldots, \pi_n(j)\} = N \ (j \in N) \text{ and } A_k = T_{\pi_k}(A_1) \ (k \in M). \ \diamond$$

Remarks. (2) If $m = n = 1$, then A_1 has compensating asymmetries of exactly the same magnitude.

(3) If at least one A_k is permutation-symmetric, then A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude if and only if all A_k are identical.

(4) If A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude, their Minkowski sum A is not necessarily permutation-symmetric as the following example shows;[5] but it is if $m = 2$ as Proposition 13(4) shows.

Example 1. Let $m = n = 3$, $A_1 = \{(3,0,1),(0,2,4)\}$ and (using cycle notations) $\pi_1 = \mathrm{id}$, $\pi_2 = (132)$, $\pi_3 = (123)$. So $A_2 = \pi_2(A_1) = \{(1,3,0),(4,0,2)\}$, $A_3 = \pi_3(A_1) = \{(0,1,3),(2,4,0)\}$. The sets A_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude and

$$A = \{(4,4,4),\ (6,7,1),\ (7,1,6),\ (9,4,3),\ (1,6,7),(3,9,4),(4,3,9),(6,6,6)\}. \ \diamond$$

Proposition 13. *Suppose A_k $(k \in M)$ are subsets of $\mathbb{R}^n$ that have compensating asymmetries of exactly the same magnitude. Let $A = \sum_{k \in M} A_k$ and $l \in M$.*

1. *$S(A_l) \neq \emptyset \Leftrightarrow S(A) \neq \emptyset$. And $\#S(A_l) = 1 \Leftrightarrow \#S(A) = 1$.*
2. *If $S(A_l) \neq \emptyset$, then $(s(A_l), \ldots, s(A_l)) \in S(A)$.*
3. *$s(A) = ns(A_l)$.*
4. *If $m = 2$, then A is permutation-symmetric.* $\diamond$

Proof. It is easy to see that we may suppose $l = 1$.

Let π_k $(k \in M)$ be as in Definition 2. By Proposition 4(2) and (3)

$$S(A) = \sum_{k \in M} T_{\pi_k}(S(A_1)). \tag{10}$$

1. By (10).

2. Let $\mathbf{a} \in S(A_1)$. By (10), $\mathbf{b} := \sum_k T_{\pi_k}(\mathbf{a}) \in S(A)$. For $i \in N$ we have $b_i = \sum_k a_{\pi_k(i)} = \sum_k a_k$. Thus $b_1 = \cdots = b_n$. As $nb_1 = s(A) = ns(A_1)$ it follows that $\mathbf{b} = (s(A_1), \ldots, s(A_1)) \in S(A)$.

[5] Indeed, there $T_{(12)}A \neq A$.

3. This holds if $A_1 = \emptyset$. Now suppose $A_1 \neq \emptyset$. By Proposition 4, $s(A) = s(\sum_k T_{\pi_k}(A_1)) = \sum_k s(T_{\pi_k}(A_1)) = \sum_k s(A_1) = ns(A_1)$.

4. Let $\pi \in S_n$. We shall prove that $T_\pi(A) = A$. Well, $T_\pi(A) = T_\pi(A_1 + T_{\pi_2}(A_1)) = T_\pi(A_1) + T_\pi(T_{\pi_2}(A_1)) = T_\pi(A_1) + (T_{\pi_2 \circ \pi})(A_1)$. As $S_2 = \{\pi_1, \pi_2\}$, we obtain $T_\pi(A) = A_1 + T_{\pi_2}(A_1) = A$. Q.E.D.

The notion in the following definition is taken from Folmer and von Mouche (2000).

Definition 3. Let X be a subset of $\mathbb{R}^n$. For $j \in N$, X has a *j-defect* if $y_j < 0$ for all $\mathbf{y} \in S(X)$. And X has a *defect* if it has a j-defect for some j. $\diamond$

Proposition 14. *Let X be a subset of $\mathbb{R}^n$ with a defect.*

1. *If X has a j-defect and $\pi \in S_n$, then $T_\pi(X)$ has a $\pi^{-1}(j)$-defect.*
2. *If $S(X) \neq \emptyset$ and $X \cap \mathbb{R}^n_+ \neq \emptyset$, then X is not permutation-symmetric.*
3. $\mathrm{Conv}(S(X)) \cap \mathbb{R}^n_+ = \emptyset$. $\diamond$

Proof. We suppose that X has a j-defect.

1. Suppose $\mathbf{b} \in S(T_\pi(X))$. By (3), $\mathbf{b} \in T_\pi(S(X))$. Take $\mathbf{a} \in S(X)$ such that $\mathbf{b} = T_\pi(\mathbf{a})$. So $b_{\pi^{-1}(j)} = a_j$. Using that X has a j-defect, we see that $b_{\pi^{-1}(j)} < 0$.

2. By contradiction, suppose X is permutation-symmetric. By part 1, for each $\pi \in S_n$ the set $T_\pi(X)$ has a $\pi^{-1}(j)$-defect. As $T_\pi(X) = X$, the set X has an i-defect for every $i \in N$. Take $\mathbf{y} \in S(X)$. Now $y_i < 0$ $(i \in N)$. Let $\mathbf{w} \in X \cap \mathbb{R}^n_+$. Then one has $\sum_{j=1}^n w_j \geq 0 > \sum_{i=1}^n y_i$, a contradiction with $\mathbf{y} \in S(X)$.

3. With $I_j := \{\mathbf{a} \in \mathbb{R}^n \mid a_j < 0\}$, X having a j-defect is equivalent with $S(X) \subseteq I_j$. As I_j is convex, this in turn is equivalent with $\mathrm{Conv}(S(X)) \subseteq I_j$. As $I_j \cap \mathbb{R}^n_+ = \emptyset$, it follows that $\mathrm{Conv}(S(X)) \cap \mathbb{R}^n_+ = \emptyset$. Q.E.D.

Proposition 15. *Suppose U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude. Let $l \in M$.*

1. *$[S(U_l) \neq \emptyset$ and $s(U_l) \geq 0] \Leftrightarrow S(U) \cap \mathbb{R}^n_+ \neq \emptyset$.*
2. *$[S(U_l) \neq \emptyset$ and $s(U_l) > 0] \Leftrightarrow S(U) \cap \mathbb{R}^n_{++} \neq \emptyset$.* $\diamond$

Proof. 1. '$\Leftarrow$': so $S(U) \neq \emptyset$. By Proposition 4, $S(U_l) \neq \emptyset$. Take $\mathbf{u} \in S(U) \cap \mathbb{R}^n_+$. Then $s(U) = \mathcal{C}(\mathbf{u}) \geq 0$. Proposition 13(3) implies $s(U_l) \geq 0$.

'$\Rightarrow$': with Proposition 13(2), $(s(U_l), \ldots, s(U_l)) \in S(U) \cap \mathbb{R}^n_+$.

2. Analogous to part 1. Q.E.D.

Theorem 1. *Suppose U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude, $S(U_1) \neq \emptyset$ and $s(U_1) \geq 0$.*[6]

1. (a) *$S(U) \cap \mathbb{R}^n_+ \neq \emptyset$, so U does not have a defect.*
   (b) *if U_1 has a defect, then $F \subset F_\star$.*
2. *Suppose $U_1 \cap \mathbb{R}^n_+ \neq \emptyset$. Fix $\mathbf{n} \in U_1 \cap \mathbb{R}^n_+$ and $\mathbf{y} \in S(U_1)$.*
   (a) *No U_k is permutation-symmetric.*
   (b) *With*[7] *$\mathbf{a} := \sum_k T_{\pi_k}(\mathbf{n})$ and $\mathbf{b} := \sum_k T_{\pi_k}(\mathbf{y})$ we have $\mathbf{a} \in U \cap \mathbb{R}^n_+$, $\mathbf{b} \in S(U)$ and $\mathbf{b} \gg \mathbf{a}$.*

[6] This implies that $n \neq 1$ and therefore also that $m \neq 1$.

[7] Here π_k $(k \in M)$ are as in Definition 2.

(c) $S(U) \cap \mathbb{R}^n_{++} \neq \emptyset$. $\diamond$

Proof. 1a. By Proposition 15(1).

1b. By the principle (i.e. Proposition 12). It applies by virtue of Proposition 14(3) and part 1a.

2. a. By Proposition 14(2), U_1 is not permutation symmetric. Now further apply also the first part of this proposition.

b. Note that $T_{\pi_k}(\mathbf{n}), T_{\pi_k}(\mathbf{y}) \in U_k$ $(k \in M)$. Also $T_{\pi_k}(\mathbf{n}) \in \mathbb{R}^n_+$ $(k \in M)$. So

$$\mathbf{a} = \sum_k T_{\pi_k}(\mathbf{n}) \in U \cap \mathbb{R}^n_+, \;\; \mathbf{b} = \sum_k T_{\pi_k}(\mathbf{y}) \in U.$$

By (3), $T_{\pi_k}(\mathbf{y}) \in S(T_{\pi_k}(U_1)) = S(U_k)$ $(k \in M)$. By Proposition 4, $\mathbf{b} \in S(U)$. For $i \in N$ we have $a_i = \sum_k n_{\pi_k(i)} = \sum_k n_k$ and $b_i = \sum_k y_{\pi_k(i)} = \sum_k y_k$. So $a_1 = a_2 = \cdots = a_n =: a$ and $b_1 = b_2 = \cdots = b_n =: b$ follows. As U_1 has a defect and $\mathbf{n} \in \mathbb{R}^n_+$, $\mathbf{n} \notin S(U_1)$ holds. Proposition 3 now implies that $\mathbf{a} \notin S(U)$. It follows that $na < nb$. Therefore $a < b$ which implies that $\mathbf{b} \gg \mathbf{a}$.

c. By part 2b. Q.E.D.

Theorem 1(1b) explains $F \subset F_\star$ in Figure 1. In this figure also the assumptions of Theorem 1(2) and therefore also its conclusions hold.

Although in Theorem 1(2) no U_k is permutation symmetric, we observe from Figures 4 and 5 that $F \subset F_\star$ is compatible with every U_k permutation-symmetric.

The next result generalises Theorem 1(1): indeed, there in case $s(U_1) \geq 0$ it is possible to take $W_k = U_k$ $(k \in M)$ and $v^{(k)} = T_{\pi_k}(\mathbf{y})$ $(k \in M)$.

Theorem 2. *Suppose* U_k $(k \in M)$ *have compensating asymmetries of exactly the same magnitude and* $S(U_1) \neq \emptyset$*. Fix* $\mathbf{y} \in S(U_1)$*. Suppose* $W_1, \ldots, W_n$ *are subsets of* $\mathbb{R}^n$ *such that for every* $k \in M$ *there exists* $\mathbf{v}^{(k)} \in S(W_k)$ *such that*

$$v_i^{(k)} \geq y_{\pi_k(i)} - \frac{s(U_1)}{n} \; (i \in N). \tag{11}$$

Let $W := \sum_k W_k$,

1. *(a)* $S(W) \cap \mathbb{R}^n_+ \neq \emptyset$*, so* W *does not have a defect;*
   (b) if some W_k *has a defect, then* $F \subset F_\star$*.*
2. *if the inequalities in (11) are strict, then* $S(W) \cap \mathbb{R}^n_{++} \neq \emptyset$*.* $\diamond$

Proof. 1a. Let $\mathbf{v} := \sum_k \mathbf{v}^{(k)}$. By Proposition 4(2), $\mathbf{v} \in S(W)$. For $i \in N$ we have

$$v_i = \sum_k v_i^{(k)} \geq \sum_k (y_{\pi_k(i)} - \frac{s(U_1)}{n}) = \sum_k y_k - s(U_1) = s(U_1) - s(U_1) = 0.$$

Thus also $\mathbf{v} \in \mathbb{R}^n_+$.

1b. By the principle. It applies by virtue of Proposition 14(3) and part 1a.

2. Analogous to part 1a. Q.E.D.

Figure 3 shows that there are situations where the U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude where $F \subset F_\star$ holds that are not covered by Theorem 1.

3.3. On Problem 2

Now let us return to problem 2.

Proposition 16. *1.* $\mathrm{P}(F_\star) = \mathrm{P}(\mathrm{Conv}(U)) \cap \mathbb{R}^n_+$ *and* $\mathrm{P}(F_k) = \mathrm{P}(\mathrm{Conv}(U_k)) \cap \mathbb{R}^n_+$ *$(k \in M)$.*
2. $\mathrm{EXP} = \mathrm{P}(F) \setminus \mathrm{P}_w(F_\star)$. $\diamond$

Proof. 1. By Proposition 2.

2. '$\subseteq$': suppose $\mathbf{u} \in \mathrm{EXP}$. Then $\mathbf{u} \in \mathrm{P}(F)$ and there exists $\mathbf{w} \in F_\star$ such that $\mathbf{w} \gg \mathbf{u}$. By (7), $\mathbf{u} \in F_\star$. Therefore $\mathbf{u} \notin \mathrm{P}_w(F_\star)$.

'$\supseteq$': suppose $\mathbf{u} \in \mathrm{P}(F) \setminus \mathrm{P}_w(F_\star)$. By (7), $\mathbf{u} \in F_\star$. As $\mathbf{u} \notin \mathrm{P}_w(F_\star)$, there is an $\mathbf{w} \in F_\star$ with $\mathbf{w} \gg \mathbf{u}$. Thus $\mathbf{u} \in \mathrm{EXP}$. Q.E.D.

Theorem 1(1b) also explains $F \subset F_\star$ in the following example and shows that $\mathrm{EXP} = \emptyset$ can hold under the general assumptions of Theorem 1.

Example 2. $m = n = 2$, $U_1 = \{(-1,1),(-1,-2)\}$, $U_2 = \{(1,-1),(-2,-1)\}$. Now $U = \{(0,0),(-3,0),(0,-3),(-3,-3)\}$, $F_1 = \emptyset$, $F_2 = \{-1\} \times [0,1]$, $F = \emptyset$, $F_\star = \{(0,0)\}$, $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$. $\diamond$

Proposition 17. *If* $\mathbf{a} \in \mathrm{P}(F)$*, then* $\mathbf{a} \in \mathrm{EXP} \Leftrightarrow N^+_{\mathrm{Conv}(U)}(\mathbf{a}) = \emptyset$. $\diamond$

Proof. '$\Rightarrow$': let $\mathbf{c} \in F_\star$ such that $\mathbf{c} \gg \mathbf{a}$. For all $\mathbf{d} > \mathbf{0}$ one has $\mathbf{d} \cdot (\mathbf{c} - \mathbf{a}) > 0$. Because $\mathbf{c} \in \mathrm{Conv}(U)$, it follows that $\mathbf{d} \notin N^+_{\mathrm{Conv}(U)}(\mathbf{a})$.

'$\Leftarrow$': by Proposition 8, $\mathbf{a} \notin \mathrm{P}_w(\mathrm{Conv}(U))$. So there exists $\mathbf{c} \in \mathrm{Conv}(U)$ with $\mathbf{c} \gg \mathbf{a}$. Since $\mathbf{a} \in \mathbb{R}^n_+$, also $\mathbf{c} \in \mathbb{R}^n_+$. This implies $\mathbf{c} \in F_\star$. Thus $\mathbf{a} \in \mathrm{EXP}$. Q.E.D.

We have already seen that if $F = F_\star$ holds, then $\mathrm{EXP} = \emptyset$. A natural question now is whether $F \subset F_\star$ implies that $\mathrm{EXP} = \emptyset$. The answer is 'no' as Figure 3 shows. Proposition 11(2) implies that the condition $U_k \subseteq \mathbb{R}^n_+$ $(k \in M)$ is sufficient for $\mathrm{EXP} = \emptyset$ to hold. This condition is quite strong. In the next proposition, which also explains $\mathrm{EXP} = \emptyset$ in Figure 4, there are more interesting conditions.

Proposition 18. *If, in case $m = 2$,* $\mathrm{P}(\mathrm{Conv}(U)) \subseteq \mathbb{R}^n_+$ *and* $\mathrm{Conv}(U_1)$ *or* $\mathrm{Conv}(U_2)$ *has a maximiser which belongs to* $\mathbb{R}^n_+$*, then*

1. $\mathrm{P}(F_\star) \supseteq \mathrm{P}(F)$*;*
2. $\mathrm{EXP} = \emptyset$. $\diamond$

Proof. 1. We may assume that $\mathrm{Conv}(U_2)$ has a maximiser, say $\mathbf{b}$. This implies $P(\mathrm{Conv}(U_2)) = \{\mathbf{b}\}$. As $\mathbf{b} \in \mathbb{R}^n_+$, we have $\mathbf{b} \in F_2$. This implies that $\mathbf{b}$ also is a maximiser of F_2 and therefore $P(F_2) = \{\mathbf{b}\}$. Now with Proposition 16(1) and Proposition 6

$$\mathrm{P}(F_\star) = \mathrm{P}(\mathrm{Conv}(U)) \cap \mathbb{R}^n_+ = \mathrm{P}(\mathrm{Conv}(U)) = \mathrm{P}(\mathrm{Conv}(U_1) + \mathrm{Conv}(U_2))$$

$$= \mathrm{P}(\mathrm{Conv}(U_1)) + \mathrm{P}(\mathrm{Conv}(U_2)) \supseteq \mathrm{P}(\mathrm{Conv}(U_1)) \cap \mathbb{R}^n_+ + \mathrm{P}(\mathrm{Conv}(U_2))$$

$$= \mathrm{P}(F_1) + \mathrm{P}(F_2) = \mathrm{P}(F_1 + F_2) = \mathrm{P}(F).$$

2. By part 1 and Proposition 16(2). Q.E.D.

<u>Remark</u>. (5) Figure 4 shows that the general conditions of Proposition 18 are compatible with $F \subset F_\star, \mathrm{EXP} = \emptyset$ and $\mathrm{P}(F_\star) \subset \mathrm{P}(F)$.

The next theorem explains $\mathrm{EXP} = \emptyset$ in Figure 2.

Theorem 3. *Suppose* $\mathrm{Conv}(U_k)$ $(k \in M)$ *are compact. If* $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathbb{R}^n_+$ $(k \in M)$*, then* $\mathrm{EXP} = \emptyset$. $\diamond$

Proof. According to Proposition 17 the proof is complete if we can prove that $N^+_{\mathrm{Conv}(U)}(\mathbf{z}) \neq \emptyset$ for all $\mathbf{z} \in \mathrm{P}(F)$. So suppose $\mathbf{z} \in \mathrm{P}(F) = \mathrm{P}(\sum_k F_k)$. By Proposition 8 one has $N^+_{\sum_k F_k}(\mathbf{z}) \neq \emptyset$. As $\mathbf{z} \in \sum_k F_k$, there exists $\mathbf{z}^{(k)} \in F_k$ $(k \in M)$ such that $\mathbf{z} = \sum_k \mathbf{z}^{(k)}$. With Proposition 7 one obtains

$$\emptyset \neq N^+_{\sum_k F_k}(\mathbf{z}) = \cap_k N^+_{F_k}(\mathbf{z}).$$

By assumption $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathbb{R}^n_+$ for all k. Therefore $\mathrm{P}(\mathrm{Conv}(U_k)) \subseteq \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+ = F_k$. So we can apply Proposition 9 with $A = F_k$, $B = \mathrm{Conv}(U_k)$ and $\mathbf{z} = \mathbf{z}^{(k)}$ and get

$$N^+_{\mathrm{Conv}(U_k)}(\mathbf{z}^{(k)}) = N^+_{F_k}(\mathbf{z}^{(k)}) \ (k \in M)$$

and therefore $\cap_k N^+_{\mathrm{Conv}(U_k)}(\mathbf{z}) \neq \emptyset$. Applying Proposition 7, $N^+_{\mathrm{Conv}(U)}(\mathbf{z}) \neq \emptyset$ follows. Q.E.D.

Note that in Figure 2 even $F = F_\star$ holds. However, under the conditions of Theorem 3, $F \subset F_\star$ may hold as the following example shows.

Example 3. In case $m = 3, n = 1$, $U_1 = \{-1, 1\}$, $U_2 = \{2\}$, $U_3 = \{3\}$ one has $F = [5, 6], F_\star = [4, 6]$. Thus $F \subset F_\star$ and $\mathrm{EXP} = \emptyset$. $\diamond$

The above results partially solve Problem 2.

4. Application to Linked Repeated Games

4.1. Games in strategic form

Consider a game in strategic form Γ among n players. That is, for each *player* $i \in N = \{1, \ldots, n\}$ we have a non-empty (*action*) set X^i and a real-valued (*payoff*) function f^i on the set of *action profiles* $\mathbf{X} := X^1 \times \cdots \times X^n$. For $\mathbf{x} \in \mathbf{X}$, $\mathbf{f}(\mathbf{x}) := (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x}))$ is called the *payoff vector* at $\mathbf{x}$ and $f^i(\mathbf{x})$ is called the *payoff* of player i at $\mathbf{x}$. We call

$$B := \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}\}$$

the *set of basic payoff vectors*. Its convex hull $\mathrm{Conv}(B)$ is called the *feasible set*. The minimax payoff of player i is defined by

$$\overline{v}^i := \inf_{\mathbf{z} \in X^1 \times \cdots \times X^{i-1} \times X^{i+1} \times \cdots \times X^n} \sup_{x^i \in X^i} f^i(z^1, \ldots, z^{i-1}, x^i, z^{i+1}, \ldots, z^n).$$

An element $\mathbf{w}$ of $\mathbb{R}^n$ is called *individually rational* if $w^i \geq v^i$ $(i \in N)$ and *strictly individually rational* if $w^i > v^i$ $(i \in N)$

We call the game *regular* if each payoff function is bounded and each player has minimax payoff 0.[8]

A *Nash equilibrium* $\mathbf{e}$ is an action profile with the property that for every $i \in N$ the function $f^i(e^1, \dots, e^{i-1}, \cdot, e^{i+1}, \dots, e^n)$ has e_i as maximiser. Payoff vectors at Nash equilibria are individually rational. An action profile that maximises the total payoff function $\sum_{i \in N} f^i$ is called *fully cooperative*. Denoting the set of fully cooperative strategy profiles by Y we have (in terms of (1)

$$S(B) = \mathbf{f}(Y), \tag{12}$$

So sufficient for Y to be non-empty is that B is compact.

For $\pi \in S_n$, i.e. a permutation of N, the game in strategic form $\pi(\Gamma)$ (called a *permuted game* of Γ) is defined as the game in strategic form where the action set Z^i of player i is $X^{\pi(i)}$ and his payoff function h^i is given by $h^i(z^1, \dots, z^n) = f^{\pi(i)}(z^{\pi^{-1}(1)}, \dots, z^{\pi^{-1}(n)})$. Note that

$$\text{the set of basic payoff vectors of } \pi(\Gamma) \text{ equals } T_\pi(B). \tag{13}$$

The game Γ is called *symmetric* if each player has the same action set and if for every $\pi \in S_n$ one has $\Gamma = \pi(\Gamma)$. If Γ is symmetric, then $T_\pi(B) = B$ for all $\pi \in S_n$, i.e. (see section 2) B is permutation-symmetric.

4.2. Repeated games

A repeated game is specified by a game in strategic form Γ, called the *stage game*, a number T (positive or $+\infty$) and a number $\delta \in [0, 1]$. Such a game simply will be denoted by

$$< \Gamma > .$$

T is called the *number of repetitions* and δ is called a *discount factor*.[9] When $T = \infty$, we always suppose to avoid convergence problems that $\delta < 1$ and that payoff functions are bounded. Itself $< \Gamma >$ is a game in strategic form with player set N where the action set of player i now is called his *strategy set*, denoted by $[X^i]$, and defined as the collection of sequences of mappings $\sigma^i = (\sigma^i_t)_{0 \leq t < T}$ with $\sigma^i_t : \prod_{\tau=0}^{t-1} \mathbf{X} \to X^i$. And the payoff function of player i in $< \Gamma >$ is the function $[f^i] : [X^1] \times \cdots \times [X^n] \to \mathbb{R}$ defined by

$$[f^i](\boldsymbol{\sigma}) := \sum_{t=0}^{T-1} \delta^t f^i(\mathbf{a}_t(\boldsymbol{\sigma})),$$

where $a^j_t(\boldsymbol{\sigma}) \in X^j$ $(0 \leq t < T)$, called *outcome path* for player j, inductively is defined by $a^j_0(\boldsymbol{\sigma}) := \sigma^j_0$ and $a^j_t(\boldsymbol{\sigma}) := \sigma^j_t(\mathbf{a}_0(\boldsymbol{\sigma}), \mathbf{a}_1(\boldsymbol{\sigma}), \dots, \mathbf{a}_{t-1}(\boldsymbol{\sigma}))$ $(1 \leq t < T)$.

For a regular game in strategic form Γ the intersection of $\mathbb{R}^n_+$ and its feasible set is an important object. One calls it the *feasible individually rational payoff region* of the game. The feasible individually rational payoff region plays an important role in Folk theorems which relate to the geometric structure of the set of (average)

[8] Note that for a regular game in strategic form it is possible that its feasible set does not contain $\mathbf{0}$. Indeed, this for example holds for the regular bimatrix game $\begin{pmatrix} -2;\ 2 & 0;\ -4 \\ 1;\ -3 & -2;\ 0 \end{pmatrix}$.

[9] Notice that in our setting a discount factor is player independent.

subgame perfect Nash equilibrium payoff vectors for infinitely repeated games $< \Gamma >$. For the purpose of this paper it is not necessary to go into the details of the Folk theorems.[10] For this, we refer to, for example, Benoît and Krishna (1996).

4.3. Direct sum games

Consider games in strategic form ${}_1\Gamma, \ldots, {}_m\Gamma$ with (the same) n players. We refer to them as *isolated stage games*. $M = \{1, \ldots, m\}$ is the set of *issues*. Denote, for $k \in M$, by

$$U_k$$

the set of basic payoff vectors of ${}_k\Gamma$. So $U_k \subseteq \mathbb{R}^n$. Let, for $k \in M$ and $j \in N$, ${}_kX^j$ be the action set and ${}_kf^j$ the payoff function of player j in ${}_k\Gamma$. Define for each $k \in M$

$${}_k\mathbf{X} \ := \ {}_kX^1 \times \cdots \times {}_kX^n$$

and for each player $j \in N$

$${}_*X^j \ := \ {}_1X^j \times \cdots \times {}_mX^j.$$

Moreover, define the mapping $\Psi : {}_1\mathbf{X} \times \cdots \times {}_m\mathbf{X} \to {}_*X^1 \times \cdots \times {}_*X^n$ by

$$\Psi({}_1\mathbf{x}, \cdots, {}_m\mathbf{x}) \ := \ ({}_*x^1, \ldots, {}_*x^n).$$

Ψ is called the *canonical mapping*. Note that the canonical mapping is a bijection. The *trade-off direct sum game* $(\oplus\Gamma)_\alpha$ is defined as the game in strategic form where player j has action set ${}_*X^j$ and his payoff function is given by[11]

$$f^j_\alpha({}_*x^1, \ldots, {}_*x^n) \ := \ \sum_{k \in M} {}_kf^j({}_kx^1, \ldots, {}_kx^n).$$

(In the case of two bimatrix games $(\oplus\Gamma)_\alpha$ is the tensor sum of the individual bimatrix games.) The set of basic payoffs vectors U of $(\oplus\Gamma)_\alpha$ equals the Minkowski sum of the U_k:

$$U = \sum_{k \in M} U_k.$$

Let, for $k \in M$, ${}_kE$ be the set of Nash equilibria of ${}_k\Gamma$, ${}_kY$ the set of fully cooperative action profiles of ${}_k\Gamma$. And let E_α be the set of Nash equilibria of $(\oplus\Gamma)_\alpha$ and Y_α the set of fully cooperative action profiles of $(\oplus\Gamma)_\alpha$. It can be shown that (see Folmer and von Mouche (1994))

$$\Psi({}_1E \times \cdots \times {}_mE) = E_\alpha, \tag{14}$$

$$\Psi({}_1Y \times \cdots \times {}_mY) = Y_\alpha \tag{15}$$

and also that regularity of each ${}_k\Gamma$ implies regularity of $(\oplus\Gamma)_\alpha$. In this case the feasible individually rational payoff region of ${}_k\Gamma$ is

$$F_k \ := \ \mathrm{Conv}(U_k) \cap \mathbb{R}^n_+$$

[10] Especially one has to specify the types of strategies.

[11] The α refers to the fact that in this formula the payoffs of the isolated games are added (with weights 1).

and the feasible individually rational payoff region of $(\oplus\Gamma)_\alpha$ is

$$F_\star = \mathrm{Conv}(U) \cap \mathbb{R}^n_+.$$

Finally, define the *aggregated feasible individually rational payoff region* as

$$F := \sum_{k \in M} F_k.$$

4.4. Tensor games

Let ${}_1\Gamma, \ldots, {}_m\Gamma$ be regular isolated stage games with (the same) n players and consider the infinitely repeated games $< {}_k\Gamma >$ $(k \in M)$.[12] Linking of the (isolated) repeated games $< {}_k\Gamma >$ $(k \in M)$ is done by combining them into a repeated game $(\otimes\Gamma)_\alpha$, a so-called *trade-off tensor game*. Formally $(\otimes\Gamma)_\alpha$ just is the infinitely repeated game with $(\oplus\Gamma)_\alpha$ as stage game. In Folmer et al. (1993) it is shown that Nash equilibria for each repeated game $< {}_k\Gamma >$ lead in a canonical way to a Nash equilibrium for the trade-off tensor game $(\otimes\Gamma)_\alpha$.[13] In general, the trade-off tensor game also has other (subgame perfect) Nash equilibria. Folk theorems are useful for investigating these equilibria. In fact, the effects of linking can be studied by comparing the sets F and $F_\star$. This has been done in Section 3. All the results there, in particular $F \subseteq F_\star$, apply. The five figures in Section 3 are compatible with the following regular games. Below we shall discuss game theoretic pendants of the results in Section 3.

Figure 1: ${}_1\Gamma = \begin{pmatrix} 2;\ 1 & -3;\ 2 \\ 5;\ -1 & 0;\ 0 \end{pmatrix}$, $\quad {}_2\Gamma = \begin{pmatrix} 1;\ 2 & -1;\ 5 \\ 2;\ -3 & 0;\ 0 \end{pmatrix}$.

Figure 2: ${}_1\Gamma = \begin{pmatrix} 0;\ 2 & 3;\ 1 \\ -3;\ 0 & 0; 0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 0;\ 1 & 1;\ 0.5 \\ -2;\ 0 & 0;\ 0 \end{pmatrix}$.

Figure 3: ${}_1\Gamma = \begin{pmatrix} 7;\ 1 & -3;\ 3 \\ 10;\ -2 & 0; 0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 1;\ 7 & -2;\ 10 \\ 3;\ -3 & 0;\ 0 \end{pmatrix}$.

Figure 4: ${}_1\Gamma = \begin{pmatrix} 2;\ 2 & -2;\ 4 \\ 4;\ -2 & 0; 0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 2;\ 2 & -1;\ 1 \\ 1;\ -1 & 0;\ 0 \end{pmatrix}$.

Figure 5: ${}_1\Gamma = \begin{pmatrix} 2;\ 2 & -2;\ 10 \\ 10;\ -2 & 0; 0 \end{pmatrix}$, ${}_2\Gamma = \begin{pmatrix} 3;\ 3 & -3;\ 4 \\ 4;\ -3 & 0;\ 0 \end{pmatrix}$.

A strict inclusion $F \subset F_\star$ (see Problem 1 in Section 3) can be interpreted as 'linking sustains more cooperation'. And EXP $\neq \emptyset$, i.e. the existence of an expansion point of the Pareto boundary PB(F) (see Problem 2 in Section 3), can be interpreted as 'Linking brings Pareto improvements'. So in this way we now have formalized for tensor games the themes 'linking may sustain more cooperation' and 'linking may bring Pareto improvements' from the introduction.

The results in Section 3 now can be formulate in terms of the above game theoretic situation. (8) implies that in the case linking brings Pareto improvements, it also sustains more cooperation. The reverse does not hold in general. Proposition 11 leads in an obvious way to two classes of isolated stage games for which linking does

[12] It is is implicitly assumed that in each of them the periods are the same and the discount factors are the same.

[13] It is straightforward to show that this statement remains valid if one replaces 'Nash equilibrium' by 'subgame perfect Nash equilibrium'.

not sustain more cooperation. The next theorem is the game theoretic pendant of Theorem 1 and is a formalisation of the basic idea that an exchange of concessions may enhance cooperation if the issues have compensating asymmetries of similar magnitude.

Theorem 4. *Consider isolated regular stage games* ${}_1\Gamma, \ldots, {}_m\Gamma$ *with* $m = n$ *players for which there are* $\pi_k \in S_n$ $(k \in M)$ *with* $\pi_1 = \mathrm{Id}$ *such that* $\{\pi_1(j), \ldots, \pi_n(j)\} = N$ $(j \in N)$ *and* ${}_k\Gamma = T_{\pi_k}({}_1\Gamma)$ $(k \in M)$. *Also suppose the basic payoff set* U_1 *is compact. Suppose* ${}_1\Gamma$ *has a Nash equilibrium and* U_1 *has a defect.*

1. *Then linking sustains more cooperation.*
2. *The game* $(\oplus\Gamma)_\alpha$ *has a Nash equilibrium* $\mathbf{e}$ *and a fully cooperative action profile* $\mathbf{y}$*, with strictly individually payoff vector, which is an unanimous Pareto improvement of* $\mathbf{e}$.[14] $\diamond$

Proof. Let $\mathbf{n}$ be a Nash equilibrium of ${}_1\Gamma$. As ${}_1\mathbf{f}(\mathbf{n})$ is individually rational, we have ${}_1\mathbf{f}(\mathbf{n}) \in U_1 \cap \mathbb{R}^n_+$. So $s(U_1) \geq 0$. As U_1 is compact, $S(U_1) \neq \emptyset$. By (12), $S(U_1) = {}_1\mathbf{f}({}_1Y)$. Fix $\mathbf{r} \in {}_1Y$. So ${}_1\mathbf{f}(\mathbf{r}) \in S(U_1)$.

1. (13) implies that U_k $(k \in M)$ have compensating asymmetries of exactly the same magnitude. Now apply Theorem 1(1b).

2. Now ${}_k\mathbf{x} := (n^{\pi_k(1)}, \ldots, n^{\pi_k(n)}) \in {}_kE$ $(k \in M)$. By (14), $\mathbf{e} := \Psi({}_1\mathbf{x}, \ldots, {}_m\mathbf{x}) \in E_\alpha$, i.e. $\mathbf{e}$ is a nash equilibrium of $(\oplus\Gamma)_\alpha$. Also ${}_k\mathbf{z} := (r^{\pi_k(1)}, \ldots, r^{\pi_k(n)}) \in {}_kY$ $(k \in M)$. By (15), $\mathbf{y} := \Psi({}_1\mathbf{z}, \ldots, {}_m\mathbf{z}) \in Y_\alpha$, i.e. $\mathbf{y}$ is a fully cooperative action profile of $(\oplus\Gamma)_\alpha$.

The payoff vector at $\mathbf{e}$ equals $\mathbf{a} := \sum_{k \in M} T_{\pi_k}({}_1\mathbf{f}(\mathbf{n}))$.[15] And that at $\mathbf{y}$ equals $\mathbf{b} := \sum_{k \in M} T_{\pi_k}({}_1\mathbf{f}(\mathbf{y}))$. Now apply Theorem 1(2b,2c). Q.E.D.

With Theorem 2 we have studied how far can one deviate in Theorem 1 from the situation of (exact) permuted games. In doing so, we have made more precise the above 'similar magnitude'. Concerning Pareto improvements, we identified in Proposition 18 and Theorem 3 classes where linking does not bring Pareto improvements. We also showed with Figure 5 that in the case all isolated stage game are symmetric (but not identical), more cooperation and even Pareto improvements are possible.

We note that the above isolated stage games related to Figures 1, 3 and 5 are prisoners' dilemma games.[16] Concerning this we mention that sufficient for the condition 'Suppose ${}_1\Gamma$ has a Nash equilibrium and U_1 has a defect' in Theorem 4 to hold is that ${}_1\Gamma$ is a 2×2-bimatrix prisoners' dilemma game with a *unique* fully cooperative action profile.[17]

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[14] I.e. $f^j_\alpha(\mathbf{y}) > f^j_\alpha(\mathbf{e})$ $(j \in N)$.

[15] Indeed: $\mathbf{a} = (\sum_k {}_kf^1({}_k\mathbf{x}), \ldots, \sum_k {}_kf^n({}_k\mathbf{x})) = \sum_k({}_kf^1({}_k\mathbf{x}), \ldots, {}_kf^n({}_k\mathbf{x})) = \sum_k({}_1f^{\pi_k(1)}(\mathbf{n}), \ldots, {}_1f^{\pi_k(n)}(\mathbf{n})) = \sum_k T_{\pi_k}({}_1f^1(\mathbf{n}), \ldots, {}_1f^n(\mathbf{n})) = \sum_k T_{\pi_k}({}_1\mathbf{f}(\mathbf{n}))$.

[16] We call a game in strategic form a *prisoners' dilemma game* if every player $i \in N$ has a strictly dominant action (i.e. a unique action that gives player i for every choice of actions of the other players a maximal payoff) and the unique Nash equilibrium is in the weak sense Pareto-inefficient (i.e. there exists an action profile in which every payoff is higher than in the equilibrium).

[17] Indeed, for this situation ${}_1\Gamma$ has a Nash equilibrium and a defect. The existence of a defect follows from the fact that for every 2×2-bimatrix prisoners' dilemma game for each player his payoff at the unique Nash equilibrium equals his minimax payoff 0.

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Service Quality's Effects on the Selection of a Partner Airline in the Formation of Airline Alliances*

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Abstract Airline alliance has become a prominent feature in the competitive airline industry. However, most research in this field focuses on the revenue management or pricing mechanism, rather than the initial intent of an airline alliance: providing a network of connectivity and convenience for international passengers and convenient marketing branding to facilitate travelers making inter-airline codeshare connections within countries. The main concern in this paper is how an airline's service quality might affect the selection of its partner airline during the formation of airline alliances. The main contribution is to show the strategic effects of the service quality on the proposed complementary airline alliances following a three-stage analysis framework, where the pre-alliance industry of the potential alliance members can either be monopoly or duopoly. We find that an airline will cooperate with the one which has the same service quality level if the pre-alliance service quality distribution of the airlines in the whole market differs greatly, while it tends to choose the one with similar (either higher or lower) service quality level as its partner if the distribution is approximately uniform.

Keywords: airline alliances, service quality, three-stage analysis framework.

1. Introduction

An airline alliance is an agreement between two or more airlines to cooperate on a substantial level (e.g., codeshare flights, ticketing systems, maintenance facilities, ground handling personnel, check-in and boarding staff, and etc.) to provide a network of convenient and seamless connectivity for passengers. At present, most major airlines belong to one of the three big airline alliances: *Star Alliance*, *Oneworld*, and *SkyTeam*. One of the fundamental building blocks of an airline alliance is the codeshare flights. Codeshare is an aviation business agreement where two or more airlines share the same flight. A seat purchased from one airline's ticketing system is actually operated by its partner airline under a different flight number or code. Take three big Asian airlines of Star Alliance as an example, passengers' demand from Tokyo (NRT) to Beijing (PEK) can be satisfied either by a direct flight under ANA (NH), or an optional transit flight with the first leg Tokyo (NRT) to Seoul (ICN) by Asiana Airlines (OZ), and the second leg Seoul (ICN) to Beijing (PEK)

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by Air China (CA). Under codeshare agreement, this interline product is marketed by both Asiana Airlines and Air China, and generates profit for both carriers.

Airline alliances can be categorized from different aspects, i.e., commercial or strategic, passenger or cargo, and etc. From the competitiveness of the pre-alliance market, it can be classified as parallel or complementary (Park, 1997). A parallel alliance refers to collaboration between two or more airlines competing on the same route. The pre-alliance market is duopoly or oligopoly. The complementary alliance refers to the case where two airlines link up their existent networks providing an interline service to the passengers, where the pre-alliance market might be monopoly. In reality, two airlines might form both a parallel and a complementary alliance. The example of Asiana Airlines (OZ) and Air China (CA) mentioned above is complementary, while in fact the second leg is usually under a codeshare flight by Asiana Airlines (OZ) and Air China (CA), in this case, the two airlines are parallel from Seoul (ICN) to Beijing (PEK). In this paper, we focus on the complementary alliance, and leave parallel alliance as a future extension.

The existing literature on airline alliance is quite sparse and limited. One classical research is to provide hypothesis and reasons under which hub-and-spoke networks are equilibrium structures, i.e., Hendricks et al., 1997; Berry et al., 2006; Aguirregabiria and Ho, 2012. Recent trend is focusing on the revenue management and sharing aspect, i.e., Vinod, 2005; Chen et al., 2010; Wright et al., 2010. However, little attention has been paid to the initial intent to form an airline alliance: better service quality for passengers. The delivery of high service quality is essential for airlines' survival and competitiveness. One of the distinguishing features of our paper is to discuss about the service quality's effects on the formation of airline alliances.

Service quality is a consumer's overall impression of the relative inferiority and superiority of the organization and its services (Bitner and Hubbert, 1994). Airline service quality is different from services in other industries, comprising tangible and intangible attributes, i.e., seat pitch and size, in-flight service, service frequency, on time performance, and etc. It can be evaluated by the five-star quality rating system. The idea that the service quality has an important effect on the selection of alliance partners came from the member airlines' service quality rating data for the three big alliances[1]:

Table 1: Service quality rating data.

Rating	*Star Alliance* (28 members)	*Oneworld* (12 members)	*SkyTeam* (19 members)	Other
5-Star	40%	20%	0%	40%
4-Star	29.03%	12.9%	12.9%	45.17%
3-Star	13.93%	4.91%	9.01%	72.15%
2-Star	0%	0%	0%	100%
1-Star	0%	0%	0%	100%

The rest of the industry belongs to none of the airline alliances above. We can see that these three main alliances do not accept airlines rating lower than

[1] Data source: IATA, as of June 2012

3 as its member, and the average service quality rate of Star Alliance is obviously higher than that of the other two alliances, which indicates that the service quality of an airline affects to some extent on the alliance formation. Airlines with high service quality tend to cooperate with each other. As to the literature we know, this is the first paper analyzing its effect on the selection of alliance partners by our proposed three-stage analysis framework, namely pre-alliance equilibria analysis, alliance equilibria analysis and criteria verification.

Colonques and Fillol, 2005 analyzes the profitability of two alliances from the pricing aspect. Their model is less general because of the specific assumption of monopoly pre-alliance market. Another feature of our paper is a general network topology allowing for both monopoly and duopoly pre-alliance market. The analysis and conclusion for oligopoly pre-alliance market is similar but a little bit complicated compared to that of the duopoly one, which is an important extension to pursue in the future.

The rest of our paper is organized as follows: in Section 2, we describe our general network model. Section 3 exposes the three-stage analysis framework for three types of pre-alliance market: Monopoly–Monopoly, Monopoly–Duopoly, and Duopoly–Duopoly. The optimal strategy for each airline is discussed in Section 4, and our concluding remarks and extensions are presented in Section 5.

2. General network model

2.1. Network

We consider a simple network with 3 airports A, B, and C. There is direct flight(s) between airport A and B, also B and C, but no direct flight between A and C. Passengers wishing to fly from A to C (or C to A) have to transit once in airport B. The airline industry of A–B, and B–C can either be monopoly or duopoly, then three types of basic pre-alliance markets are formed as below:

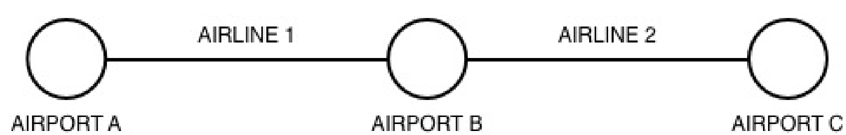

Fig. 1: Monopoly–Monopoly

There are two airlines in the Monopoly–Monopoly case, where each airline owns monopoly power in their respective market.

For the Monopoly–Duopoly case, the market of airline 1 is monopoly, while airlines 2 and 3 are competing on the same route B–C.

Finally for the Duopoly–Duopoly case, airlines 1 and 4, and airlines 2 and 3 are competing on the same route, respectively.

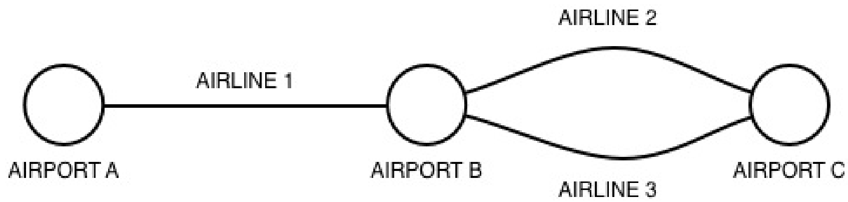

Fig. 2: Monopoly–Duopoly

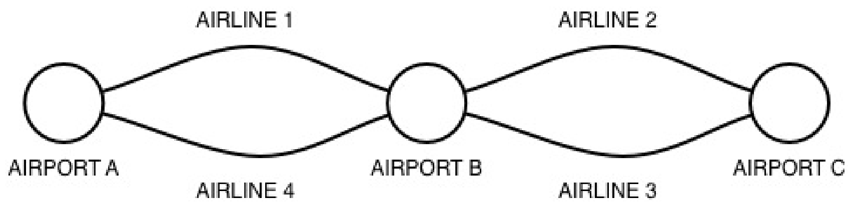

Fig. 3: Duopoly–Duopoly

2.2. Notation

We denote by $\mathcal{A}$ the set of airlines, which we index by $i = 1, 2, 3, 4$ in the analysis. Some notation that we will use to model the structure of the alliance is shown as below:

-d_i: passengers' demand for airline i.

-d_{ij}: the pre-alliance passengers' demand for market A–C, where i is the airline of market A–B, and j of market B–C.

-$d^{a_{ij}}$: passengers' demand for alliance $i - j$ if airlines i and j form an alliance. It does not include the demand for each airline's self-operated market. We assume that each airline's strategy and demand in their respective individual market are not affected by the decision of the alliance. The superscript a is used to denote quantities associated with *Alliance*.

-C_i: the overall operational cost for airline i.

-Π_i: the pre-alliance profit of airline i. Π_i^* denotes the equilibrium profit.

-$\Pi^{a_{ij}}$: the joint profit of alliance $i-j$ if airlines i and j form an alliance, including the profit generated in each airline's self-operated market. $\Pi^{a_{ij}*}$ denotes the equilibrium joint profit.

-$\Pi_i^{a_{ij}*}$,$\Pi_j^{a_{ij}*}$: the profit allocated to airline i, j respectively, if airlines i and j form an alliance.

-p_i: the fare charged for passengers by airline i.

-$p^{a_{ij}}$: the fare decided and charged jointly by alliance $i - j$ if airlines i and j form an alliance.

-$q_i = q_g$ or q_b: the service quality of airline i, which is assumed to be either q_g or q_b in this paper. The subscript g and b are used to denote "good" and "bad" service quality, respectively.

-m: a positive parameter that measures the market size.

-γ^d: a positive parameter in the demand function which measures the effect of service quality on the demand. The superscript d is used to denote the quantity associated to *Demand*. Assuming identical passengers, this effect does not differ among airlines.

-γ_i^c: a positive parameter in the cost function which measures the effect of service quality on the cost of airline i. The superscript c is used to denote quantities associated to *Cost*. We assume that $\gamma_i^c = \gamma_g^c$ if $q_i = q_g$, and $\gamma_i^c = \gamma_b^c$ if $q_i = q_b$.

θ: a positive parameter measuring the improvement of the alliance service quality over two individually operated airlines, assuming $\theta \in (1/2, 1)$.

-$\beta_i^{a_{ij}}$, $\beta_j^{a_{ij}}$: the fraction of the joint profit $\Pi^{a_{ij}*}$ collected by airline i, j respectively, if airlines i and j form an alliance, where $\beta_i^{a_{ij}} + \beta_j^{a_{ij}} = 1$. We denote by $\mathcal{R}$ the profit allocation rule, and $\mathcal{R}_p$ the proportional rule.

The rest of the notation will be introduced in the corresponding sections.

2.3. Demand and cost function

Definition 1. The demand function for airline i is linear as follows:

Monopoly market:

$$d_i = m - p_i + \gamma^d q_i \tag{1}$$

Duopoly market of airlines i and k:

$$d_i = m - p_i + p_k + \gamma^d q_i - \gamma^d q_k \tag{2}$$

In the monopoly market, the demand of airline i is decreasing with the fare it charged for passengers, and increasing with its service quality. In the duopoly market, the demand of airline i is increasing with the fare its rival k charged, and decreasing with the rival's service quality. For simplicity, we assume linear demand functions and the parameter measuring the effect of price is assumed to be 1.

Definition 2. The pre-alliance demand function of passengers between airport A and C is:

$$d_{ij} = m - (p_i + p_j) + \gamma^d (\frac{q_i + q_j}{2}) \tag{3}$$

where airline i operates route A–B, and airline j operates route B–C.

Before forming any alliance, the perceived service quality for passengers between A and C is assumed to be the average service quality of the two airlines.

Definition 3. The demand function for alliance $i - j$ is:

$$d^{a_{ij}} = m - p^{a_{ij}} + \gamma^d \theta (q_i + q_j) \tag{4}$$

where $\theta \in (1/2, 1)$.

If airlines i and j form an alliance, the perceived service quality for passengers of market A–C is higher than that before forming an alliance, for reasons like no necessity of luggage claim during transit, faster mileage accumulation, and etc. Thus we assume $\theta \in (1/2, 1)$.

Definition 4. The cost function for airline i is:

$$C_i = \gamma_i^c q_i \tag{5}$$

The alliance formation cost is neglectable compared to the operational cost, i.e., the integration of the ticketing system, share of check-in and boarding staff, and etc. It is assumed to be 0 in this paper.

2.4. Assumptions about profit allocation

In general, the proration scheme $\mathcal{R}$ used by the alliance will influence both the overall profit of the alliance and the allocated profit to each airline. We assume that the ultimate aim of each airline is to maximize its own profit. It is reasonable to assume that airlines are seeking a strategy that increases the joint profit by forming an alliance. The profit allocation mechanism is actually a bargaining problem, which is left as a future extension work. In this paper, we assume that the proportional rule $\mathcal{R}_p$ has already been chosen by the alliance, and primarily focus on examining how the service quality affects the selection of a partner airline.

Definition 5. The proportional rule $\mathcal{R}_p$ is defined as:

$$\begin{aligned} \beta_i^{a_{ij}} &= \frac{\mathit{\Pi}_i{}^*}{\mathit{\Pi}_i{}^* + \mathit{\Pi}_j{}^*} \\ \beta_j^{a_{ij}} &= \frac{\mathit{\Pi}_j{}^*}{\mathit{\Pi}_i{}^* + \mathit{\Pi}_j{}^*} \end{aligned} \tag{6}$$

2.5. Decision criteria

The fundamental questions faced by airline i with service quality q_i are:

-Whether to cooperate with another airline to form an alliance.

-If yes, which airline should be chosen as the partner.

For the first question, airline i will form an alliance with airline j only if the cooperation is to bring more profit for i than that of the pre-alliance equilibria. Both collective and individual rationality should be satisfied.

Definition 6. Collective rationality. For two airlines i and j, they are to form an alliance only if the joint profit of the alliance is more than the sum of their pre-alliance profit.

$$\mathit{\Pi}^{a_{ij}*} > \mathit{\Pi}_i^* + \mathit{\Pi}_j^* \tag{7}$$

Definition 7. Individual rationality. For two airlines i and j, they are to form an alliance only if the alliance profit allocated to each of them is more than that of their respective pre-alliance profit.

$$\begin{aligned} \mathit{\Pi}_i^{a_{ij}*} &> \mathit{\Pi}_i^* \\ \mathit{\Pi}_j^{a_{ij}*} &> \mathit{\Pi}_j^* \end{aligned} \tag{8}$$

However, as the proportional rule $\mathcal{R}_p$ is assumed to be adopted as the proration scheme in this paper, these two criteria coincide with each other. Only the collective rationality is to be checked in the following analysis.

For the second question, if airline i has two options, namely airlines j and k, it will select the one which brings more profit to itself as the partner. The stability of each proposed formation should be checked, and the more stable alliance will be formed.

Definition 8. Stability. For airline i with two potential partner airlines j and k, the stability of alliance $i - j$ is higher than that of alliance $i - k$ if and only if

$$\mathit{\Pi}_i^{a_{ij}*} > \mathit{\Pi}_i^{a_{ik}*} \tag{9}$$

3. Analysis: a three-stage framework

As mentioned above, we proceed to the analysis for the equilibria of three types of pre-alliance market: Monopoly–Monopoly, Monopoly–Duopoly, and Duopoly–Duopoly by our proposed three-stage framework.

3.1. Monopoly–Monopoly

For the pre-alliance Monopoly–Monopoly situation, airlines 1 and 2 both own monopoly power for the route A–B and B–C, respectively. From the service quality's perspective, each airline's rate could either be q_g or q_b, thus three cases will be analyzed:

-Case 1: $q_1 = q_g, q_2 = q_g$
-Case 2: $q_1 = q_b, q_2 = q_b$
-Case 3: $q_1 = q_g, q_2 = q_b$

It is easy to estimate that the equilibria of the first two cases are the same. Let us first give the analysis for the alliance of two airlines with high service quality.

Case 1: $\boldsymbol{q_1 = q_g, q_2 = q_g}$

Pre-alliance equilibria. We start by defining the total profit for airline i:

$$\Pi_i = p_i(d_i + d_{12}) - C_i \tag{10}$$

where d_{12} is defined by equation (3).

By differentiation, we get:

$$\Pi_1^* = \Pi_2^* = \frac{8}{25}(m + \gamma^d q_g)^2 - \gamma_g^c q_g \tag{11}$$

Alliance equilibria. If airlines 1 and 2 form an alliance, the total profit that the alliance might receive is:

$$\Pi^{a_{12}} = p_1 d_1 + p_2 d_2 + p^{a_{12}} d^{a_{12}} - C_1 - C_2 \tag{12}$$

where $d^{a_{12}}$ is defined by equation (4).

We get the following result:

$$\Pi^{a_{12}*} = \frac{12}{25}(m + \gamma^d q_g)^2 + (\frac{m}{2} + \theta\gamma^d q_g)^2 - 2\gamma_g^c q_g \tag{13}$$

The proportional rule $\mathcal{R}_p$ is applied to make the profit allocation, where $\beta_1^{a_{12}} = \beta_2^{a_{12}} = 1/2$. It yields,

$$\Pi_1^{a_{12}*} = \Pi_2^{a_{12}*} = \frac{\Pi^{a_{12}*}}{2} \tag{14}$$

Criteria verification. The Monopoly–Monopoly case is the simplest one in which neither of the airlines has an optional potential partner. Hence only the collective rationality needs to be verified. Straightforward calclulation shows that

$$\Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0 \tag{15}$$

is satisfied. This cooperation is to bring more profit for both airlines.

For **case 2** and **case 3**, following the same three-stage analysis framework, the collective rationality can be verified and we get the same conclusion.

3.2. Monopoly–Duopoly

We consider the pre-alliance Monopoly–Duopoly network, in which airlines 2 and 3 are competing in the B–C market, while airline 1 still enjoys the monopoly power as in the previous section. For a passenger of market A–C, there are two options:

- A–B by airline 1, B–C by airline 2.
- A–B by airline 1, B–C by airline 3.

These two options are assumed to be competitive with each other no matter for the pre-alliance market, or the re-formed market if airline 1 cooperates with another airline. In the Monopoly–Duopoly setting, where airlines 2 and 3 differ in service quality, which one is to be selected as airline 1's partner becomes our main concern. Note that there are 8 possible combinations here, only two representative cases will be analyzed:

-Case 1: $q_1 = q_g, q_2 = q_g, q_3 = q_b$
-Case 2: $q_1 = q_b, q_2 = q_g, q_3 = q_b$

Let us first discuss the case when 1 and 2 are airlines with high service quality, while 3 with low service quality.

Case 1: $\boldsymbol{q_1 = q_g, q_2 = q_g, q_3 = q_b}$

Pre-alliance equilibria. Passengers' demand for market A–C is defined as:

$$\begin{aligned} d_{12} &= m - (p_1 + p_2) + (p_1 + p_3) + \gamma^d(\frac{q_1 + q_2}{2}) - \gamma^d(\frac{q_1 + q_3}{2}) \\ d_{13} &= m - (p_1 + p_3) + (p_1 + p_2) + \gamma^d(\frac{q_1 + q_3}{2}) - \gamma^d(\frac{q_1 + q_2}{2}) \end{aligned} \tag{16}$$

The definition above suggests that before any alliance is formed, the fare and service quality of airlines 2 and 3 interactively affect A–C passengers' choice.

The total profit for each airline is defined as:

$$\begin{aligned} \Pi_1 &= p_1(d_1 + d_{12} + d_{13}) - C_1 \\ \Pi_2 &= p_2(d_2 + d_{12}) - C_2 \\ \Pi_3 &= p_3(d_3 + d_{13}) - C_3 \end{aligned} \tag{17}$$

The equilibria solutions by differentiation are:

$$\begin{aligned} \Pi_1^* &= \frac{1}{4}(3m + \gamma^d q_g)^2 - \gamma_g^c q_g \\ \Pi_2^* &= 2(m + \frac{1}{4}\gamma^d(q_g - q_b))^2 - \gamma_g^c q_g \\ \Pi_3^* &= 2(m - \frac{1}{4}\gamma^d(q_g - q_b))^2 - \gamma_b^c q_b \end{aligned} \tag{18}$$

Alliance equilibria. If airlines 1 and 2 form an alliance, passengers' demand for market A–C will be:

$$d^{a_{12}} = m - p^{a_{12}} + (p_1 + p_3) + \gamma^d\theta(q_1 + q_2) - \gamma^d(\frac{q_1 + q_3}{2}) \tag{19}$$

The journey of two tickets issued by airlines 1 and 3 separately is still a competitive option for alliance 1-2. The total profit of alliance 1-2 is defined the same as in equation (12) and we can get $\Pi^{a_{12}*}$, the maximum alliance profit. Applying the proportional rule $\mathcal{R}_p$, the profit allocated to each airline under the cooperation scheme of 1-2 is:

$$\begin{aligned} \Pi_1^{a_{12}*} &= \beta_1^{a_{12}} \Pi^{a_{12}*} \\ \Pi_2^{a_{12}*} &= \Pi^{a_{12}*} - \Pi_1^{a_{12}*} \end{aligned} \tag{20}$$

The calculation under the cooperation scheme of 1-3 can be done similarly.

Criteria verification. Let us verify the collective rationality first, assume $q_b = \alpha q_g$, where $\alpha \in (0, 1)$:

$$\Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0 \tag{21}$$

is satisfied if and only if

$$q_g \in (\omega_{m-d}^{a_{12}}(m, \gamma^d, \theta, \alpha), +\infty) \tag{22}$$

where $\omega_{m-d}^{a_{12}}(m, \gamma^d, \theta, \alpha) \in R^+$. It indicates that an airline will consider forming an alliance with another if and only if its service quality reaches a certain level, i.e., low accident rate. Otherwise, it is difficult for any other airline to accept it as a partner. Also the airline itself is focusing on improving its service quality and rarely has spare capital to invest in alliance formation. For the stability of formation,

$$\Pi_1^{a_{12}*} - \Pi_1^{a_{13}*} > 0 \tag{23}$$

is satisfied if and only if

$$\alpha \in (0, \upsilon_{m-d}(m, \gamma^d, q_g)) \tag{24}$$

where $\upsilon_{m-d}(m, \gamma^d, q_g) \in (0, 1)$ and is close to 1. The alliance structure of 1-2 is more stable than that of 1-3, and vice versa if $q_g \in (\omega_{m-d}^{a_{13}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon_{m-d}(m, \gamma^d, q_g), 1)$, 1-3 is more stable.The conclusion of **case 2** is opposite to that of **case 1**.

3.3. Duopoly–Duopoly

In this section, we consider the network with four airlines shown in Fig. 3, where the service quality of the two airlines competing on the same route differs as in the previous section. This topology represents a typical situation of the airlines in or to-be-in the three big airline alliances. Before examining the specific strategy to be adopted by the three-stage analysis framework, we describe a simple example of two big airlines in Taiwan: EVA Air (BR) and China Airlines (CI). The network coverage of the two airlines is nearly the same. In other words, they are competing nearly on each route. China Airlines joined *SkyTeam* in 2011, and EVA Air is to join *Star Alliance* later in 2013. As is known that China Airlines has records of many incidents and accidents since its formation, and was announced as the one with worst safety record among 60 international airlines by *Jet Airliner Crash Data Evaluation Centre (JACDEC)* in January, 2013. On the contrary, Eva Air has not had any aircraft losses or passenger fatalities in its operational history. From the perspective of the most important factor of service quality, safety, China Airlines' rate definitely cannot exceed that of EVA Air. Referring the three big airline alliances' service quality rating data, *Star Alliance* is doing better than *SkyTeam* as well. The analysis in this section can also be viewed as providing a theoretical support for the member selection criteria by the three big airline alliances. Let's take *Star Alliance* as an airline with high service quality, *SkyTeam* as one with low service quality, and start the analysis from the pre-alliance equilibria.

The representative case: $q_1 = q_g, q_2 = q_g, q_3 = q_b, q_4 = q_b$

Pre-alliance equilibria. A–C Passengers' demand for the first option is defined as:

$$\begin{aligned} d_{12} = m &- (p_1+p_2) + (p_1+p_3) + (p_4+p_2) + (p_4+p_3) \\ &+ \gamma^d(\frac{q_1+q_2}{2}) - \gamma^d(\frac{q_1+q_3}{2}) - \gamma^d(\frac{q_4+q_2}{2}) - \gamma^d(\frac{q_4+q_3}{2}) \end{aligned} \tag{25}$$

d_{13}, d_{42}, and d_{43} can be defined similarly as d_{12}.

The pre-alliance profit for airline 1 is:

$$\Pi_1 = p_1(d_1 + d_{12} + d_{13}) - C_1 \tag{26}$$

Π_2, Π_3, and Π_4 can be defined respectively as well. We use $\Pi_1^*, \Pi_2^*, \Pi_3^*$ and Π_4^* to denote the equilibria solutions. The calculation is simple, and we are not to present the long results here.

Alliance equilibria. If the alliance structure is 1-2 (high-high) and 4-3 (low-low), passengers' demand for market A–C will become:

$$\begin{aligned} d^{a_{12}} &= m - p^{a_{12}} + p^{a_{43}} + \gamma^d\theta(q_1+q_2) - \gamma^d\theta(q_4+q_3) \\ d^{a_{43}} &= m - p^{a_{43}} + p^{a_{12}} + \gamma^d\theta(q_4+q_3) - \gamma^d\theta(q_1+q_2) \end{aligned} \tag{27}$$

It is reasonable assuming passengers will not choose the option constituted by two airlines from different alliances. The alliance profit is defined the same as in equation (12). For alliance structure of 1-3 and 4-2, follow the same pattern above to make the definitions. By assuming $q_b = \alpha q_g, \gamma_b^c = \alpha\gamma_g^c$, where $\alpha \in (0,1)$, we can get the equilibria solutions of $\Pi^{a_{12}*}, \Pi^{a_{43}*}, \Pi^{a_{13}*}$ and $\Pi^{a_{42}*}$. Applying the proportional rule $\mathcal{R}_p$, the profit allocated to each airline under different cooperation schemes can be denoted as $\Pi_1^{a_{12}*}, \Pi_2^{a_{12}*}, \Pi_4^{a_{43}*}, \Pi_3^{a_{43}*}, \Pi_1^{a_{13}*}, \Pi_3^{a_{13}*}, \Pi_4^{a_{42}*}$ and $\Pi_2^{a_{42}*}$.

Criteria verification. Let us verify the collective rationality first:

$$\begin{aligned} \Pi^{a_{12}*} - \Pi_1^* - \Pi_2^* > 0 \\ \Pi^{a_{43}*} - \Pi_4^* - \Pi_3^* > 0 \end{aligned} \tag{28}$$

are satisfied if and only if

$$q_g \in (\omega_{d-d}^{a_{12}-a_{43}}(m, \gamma^d, \theta, \alpha), +\infty) \tag{29}$$

where $\omega_{d-d}^{a_{12}-a_{43}}(m, \gamma^d, \theta, \alpha) \in R^+$. For the stability of formation,

$$\begin{aligned} \Pi_1^{a_{12}*} - \Pi_1^{a_{13}*} > 0 \\ \Pi_2^{a_{12}*} - \Pi_2^{a_{42}*} > 0 \\ \Pi_3^{a_{43}*} - \Pi_3^{a_{13}*} > 0 \\ \Pi_4^{a_{43}*} - \Pi_4^{a_{42}*} > 0 \end{aligned} \tag{30}$$

are satisfied if and only if

$$\alpha \in (0, \upsilon_{d-d}(m, \gamma^d, q_g)) \tag{31}$$

where $\upsilon_{d-d}(m, \gamma^d, q_g) \in (0, 1)$ and is close to 1. The alliance structure of 1-2 and 4-3 is more stable than that of 1-3 and 4-2, and vice versa the structure of 1-3 and 4-2 is more stable if

$$\begin{aligned} &q_g \in (\omega_{d-d}^{a_{13}-a_{42}}(m, \gamma^d, \theta, \alpha), +\infty) \\ &\alpha \in (\upsilon_{d-d}(m, \gamma^d, q_g), 1) \end{aligned} \tag{32}$$

where $\omega_{d-d}^{a_{13}-a_{42}}(m, \gamma^d, \theta, \alpha) \in R^+$.

4. The optimal strategy

Proposition 1. *For a pre-alliance Monopoly–Monopoly network consisted of airlines i and j, for any $q_i, q_j \in R^+$, assuming the proration scheme $\mathcal{R}$ is proportional, then*

$$\begin{aligned} &\Pi_i^{a_{ij}*} > \Pi_i^* \\ &\Pi_j^{a_{ij}*} > \Pi_j^* \end{aligned} \tag{33}$$

The optimal strategy of the two airlines is cooperation with each other.

This proposition indicates that for a Monopoly–Monopoly market, the cooperation will always bring more profit for each of its member, mainly due to the extension of network coverage for each airline, and demand increment because of the more convenient service during transit.

Proposition 2. *For a pre-alliance Monopoly–Duopoly network consisted of airlines i, j and k, in which airline i's market is monopoly, for any $q_k = \alpha q_i = \alpha q_j$, where $\alpha \in (0, 1)$, assuming the profit allocation rule $\mathcal{R}$ is proportional, then if $q_i = q_j \in (\omega_{m-d}^{a_{ij}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (0, \upsilon_{m-d}(m, \gamma^d, q_g))$*

$$\begin{aligned} &\Pi_i^{a_{ij}*} > \Pi_i^* \\ &\Pi_i^{a_{ij}*} > \Pi_i^{a_{ik}*} \end{aligned} \tag{34}$$

Airline i's optimal strategy is to select airline j as its partner in the alliance formation. Vice versa, if $q_i = q_j \in (\omega_{m-d}^{a_{ik}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon_{m-d}(m, \gamma^d, q_g), 1)$, the equilibrium alliance structure should be $i - k$.

If the pre-alliance service quality distribution differs greatly, the airline in the monopoly market will choose the one with the same service quality level as its partner, while if the distribution is approximately uniform, a combination of service quality and price competitiveness tends to be formed.

Proposition 3. *For a pre-alliance Duopoly–Duopoly network consisted of airlines i, j, k and l, in which airlines i and l, airlines j and k each form a duopoly market, for any $q_k = q_l = \alpha q_i = \alpha q_j$, where $\alpha \in (0, 1)$, assuming the proration scheme $\mathcal{R}$ is proportional, then if $q_i = q_j \in (\omega_{d-d}^{a_{ij}-a_{lk}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (0, \upsilon(m, \gamma^d, q_g))$*

$$\begin{aligned} &\Pi_i^{a_{ij}*} > \Pi_i^{a_{ik}*} \\ &\Pi_j^{a_{ij}*} > \Pi_j^{a_{lj}*} \\ &\Pi_k^{a_{lk}*} > \Pi_k^{a_{ik}*} \\ &\Pi_l^{a_{lk}*} > \Pi_l^{a_{lj}*} \end{aligned} \tag{35}$$

The equilibrium alliance structure should be $i - j$, and $k - l$. Vice versa, if $q_i = q_j \in (\omega_{d-d}^{a_{ik}-a_{lj}}(m, \gamma^d, \theta, \alpha), +\infty)$, and $\alpha \in (\upsilon(m, \gamma^d, q_g), 1)$, the equilibrium alliance structure should be $i - k$, and $l - j$.

This conclusion is intuitive. If the difference between airlines with high service quality and low service quality is large, airlines tend to form an alliance with another with the same service quality level. An airline with high service quality will not accept one with poor service quality to degrade itself too much. Whereas if the difference is relatively small, an airline with high service quality tends to select the one with price competitiveness as its partner, even if this kind of cooperation might reduce the overall rate of service quality a little bit.

5. Concluding remarks and extensions

Airline alliances are selling increasing numbers of interline products. The service quality rating data for the three big airline alliances suggests the need to understand the impact of service quality during the alliance formation. This paper is the first to propose a framework studying service quality's effects on the selection of a partner airline. In particular, we model the optimal strategy decision process by a three-stage analysis framework. In the first stage, analyze the pre-alliance equilibria that each airline manages its own market in a non-cooperative fashion so as to maximize its expected profit. In the second stage, analyze the alliance equilibria under different cooperation schemes assuming a particular profit allocation rule. In the third stage, verify the collective rationality and stability to finalize the decision process.

The three main insights can be corroborated by airlines of the three big alliances, i.e., China Airlines and Eva Air. Basically airlines prefer to play with the one with the same service quality level. When the service quality of the airlines in the whole market does not differ too much with each other, the trend becomes a combination of service quality and price competitiveness. Of course this conclusion is more or less depending on the assumption of the demand functions.

In studying the effects of service quality, we find that the optimal strategy of an airline is, to some extent, sensitive to the particular profit allocation rule. We assume for simultaneous move in this paper. If we assign airline i, who owns monopoly power in its pre-alliance market in the Monopoly–Duopoly section, with the privilege to move first, the conclusion deviates from proposition 2 such that equilibrium alliance structure is always $i - j$.

An important feature of this study is the more general network topology. It suggests an extension of the oligopoly pre-alliance market, which is more close to the real situation. This is more complicated compared to the analysis of the duopoly market, in this respect, our results can be viewed as the first step to understand how airlines with different service quality will act assuming a particular allocation scheme.

Another aspect of the model deserves some attention is the profit allocation rule. In the first stage, the proportional rule is assumed to be applied in our analysis. For the future research, the application of strong Nash equilibrium or the equilibrium of bargaining game is an important extension to pursue.

Finally, in our model only complementary alliance is considered, the real situation is the coexistence of complementary and parallel alliances among partner airlines. Such a scheme, however, requires more factors, i.e., the fleet size, the capacity, service frequency and etc., to be included in the model for analysis. Describing service quality's strategic effects under the coexistence scheme will also be an interesting area of further study.

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Coalitional Model of Decision-Making over the Set of Projects with Different Preferences of Players

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Abstract Let be N the set of players and M the set of projects. The coalitional model of decision-making over the set of projects is formalized as family of games with different fixed coalitional partitions for each project that required the adoption of a positive or negative decision by each of the players. The players' strategies are decisions about each of the project. Players can form coalitions in order to obtain higher income. Thus, for each project a coalitional game is defined. In each coalitional game it is required to find in some sense optimal solution. Solving successively each of the coalitional games, we get the set of optimal n-tuples for all coalitional games. It is required to find a compromise solution for the choice of a project, i. e. it is required to find a compromise coalitional partition. As an optimality principles are accepted generalized PMS-vector (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006) and its modifications, and compromise solution (Malafeyev, 2001). The proposed paper is the generalization of the paper "Static Model of Decision-making over the Set of Coalitional Partitions" (Grigorieva, 2012) for the case when the preferences of players are different.

Keywords: coalitional game, PMS-vector, compromise solution.

1. Introduction

The set of agents N and the set of projects M are given. Each agent fixed his participation or not participation in the project by one or zero choice. The participation in the project is connected with incomes or losses which the agents wants to maximize or minimize. Agents may form coalitions. This gives us an optimization problem which can be modeled as game. This problem we call static coalitional model of decision-making.

Denote players by $i \in N$ and the projects by $j \in M$. The family M of different noncooperative games are considered. In each game G_j, $j \in M$ the player i has two strategies accept or reject the project. The payoff of player in each game is determined by strategies chosen by all players in this game G_j. As it was mentioned before the players can form coalitions to increase the payoffs. In each game G_j coalitional partition is formed and the problem is to find the optimal strategies for coalitions and the allocation of coalitional payoff between the members of coalition. The games $G_1, \ldots, G_m$ are solved by using the PMS-vector first introduced in (Petrosjan and Mamkina, 2006) and its modifications (Grigorieva and Mamkina, 2009).

Then having the solutions of games G_j, $j = \overline{1,\, m}$ the new optimality principle – "the compromise solution" is proposed to select the best projects $j^* \in M$. The problem is illustrated in three players case.

2. Statement of the problem

Consider the following problem. Suppose

- $N = \{1\,, \ldots\,, n\}$ is the set of players;
- $M = \{1\,, \ldots\,, m\}$ is the set of projects, which require making positive or negative decision by each of the n players;
- $X_i^j = \{0\,;\ 1\}$ is the set of pure strategies x_i^j of player $i\,,\ i = \overline{1,\ n}$. The strategy x_i^j can take the following values: $x_i^j = 0$ as a negative decision for the some project j and $x_i^j = 1$ as a positive decision;
- $l_i = 2$ is the number of pure strategies of player i for all j;
- x^j is the n-tuple of pure strategies chosen by the players;
- $X^j = \prod\limits_{i=\overline{1\,,\,n}} X_i^j$ is the set of n-tuples;
- $\mu_i^j = \left(\xi_i^{0,\,j}\,,\ \xi_i^{1,\,j}\right),\, \xi_i^{0,\,j} + \xi_i^{1,\,j} = 1,\, \xi_i^{0,\,j},\, \xi_i^{1,\,j} \geq 0$, is the mixed strategy of player i, where $\xi_i^{0,\,j}$ is the probability of making negative decision by the player i for some project j, and $\xi_i^{1,\,j}$ is the probability of making positive decision correspondingly;
- M_i^j is the set of mixed strategies of i-th player;
- μ^j is the n-tuple of mixed strategies chosen by players for some project j;
- $\mathrm{M}^j = \prod\limits_{i=\overline{1,\,n}} \mathrm{M}_i^j$ is the set of n-tuples in mixed strategies for some project j;
- $K_i^j\left(x^j\right)\ :\ \ X^j \to R^1$ is the payoff function defined over the set X^j for each player $i\,,\ i = \overline{1,\ n}\,$, and for some project j.

Thus, for a fixed project j we have noncooperative n-person game $G^j\left(x^j\right)$:

$$G^j\left(x^j\right) = \left\langle N,\ \left\{X_i^j\right\}_{i=\overline{1\,,\,n}},\ \left\{K_i^j\left(x^j\right)\right\}_{i=\overline{1\,,\,n}\,,\,x^j\in X^j} \right\rangle\,. \tag{1}$$

Now suppose a coalitional partitions Σ^j of the set N is defined for all $j = \overline{1\,,\,m}$:

$$\Sigma^j = \left\{S_1^j,\, \ldots,\, S_l^j\right\}\,,\ \ l \leq n\,,\ \ n = |N|\ \,,\ S_k^j \cap S_q^j = \emptyset\ \ \forall\ \ k \neq q,\ \bigcup_{k=1}^{l} S_k^j = N\,.$$

Then we have m simultaneous l-person coalitional games $G_j\left(x_{\Sigma^j}^j\right),\ j = \overline{1\,,\,m}\,$, in normal form associated with the game $G^j\left(x^j\right)$:

$$G_j\left(x_{\Sigma^j}^j\right) = \left\langle N,\ \left\{\tilde{X}_{S_k^j}^j\right\}_{k=\overline{1\,,\,l}\,,\ S_k^j\in\Sigma^j},\ \left\{\tilde{H}_{S_k^j}^j\left(x_{\Sigma^j}^j\right)\right\}_{k=\overline{1\,,\,l}\,,\ S_k^j\in\Sigma^j} \right\rangle\,,\ \ j = \overline{1\,,\,m}\,. \tag{2}$$

Here for all $j = \overline{1\,,\,m}$:

- $\tilde{x}_{S_k^j}^j = \left\{x_i^j\right\}_{i\in S_k^j}$ is the l-tuple of strategies of players from coalition $S_k^j\,,\ k = \overline{1,\ l}$;

- $\tilde{X}^j_{S^j_k} = \prod\limits_{i \in S^j_k} X^j_i$ is the set of strategies $\tilde{x}^j_{S^j_k}$ of coalition S^j_k, $k = \overline{1,\ l}$, i. e. Cartesian product of the sets of players' strategies, which are included into coalition S^j_k;
- $x^j_{\Sigma^j} = \left(\tilde{x}^j_{S^j_1}, \ \ldots, \ \tilde{x}^j_{S^j_l}\right) \in \tilde{X}^j$, $\tilde{x}^j_{S^j_k} \in \tilde{X}^j_{S^j_k}$, $k = \overline{1,\, l}$ is the l-tuple of strategies of all coalitions;
- $\tilde{X}^j = \prod\limits_{k=\overline{1,\, l}} \tilde{X}^j_{S^j_k}$ is the set of l-tuples in the game $G_j\left(x^j_{\Sigma^j}\right)$;
- $l^j_{S^j_k} = \left|\tilde{X}^j_{S^j_k}\right| = \prod\limits_{i \in S^j_k} l_i$ is the number of pure strategies of coalition S^j_k;
- $l^j_{\Sigma^j} = \prod\limits_{k=\overline{1,l}} l^j_{S^j_k}$ is the number of l-tuples in pure strategies in the game $G_j\left(x^j_{\Sigma^j}\right)$.
- $\tilde{\mathrm{M}}^j_{S^j_k}$ is the set of mixed strategies $\tilde{\mu}^j_{S^j_k}$ of the coalition S^j_k, $k = \overline{1,\ l}$;
- $\tilde{\mu}^j_{S^j_k} = \left(\tilde{\mu}^{1,\,j}_{S^j_k}, \ldots, \tilde{\mu}^{l_{S^j_k},\,j}_{S^j_k}\right)$, $\tilde{\mu}^{\xi,\,j}_{S^j_k} \geq 0$, $\xi = \overline{1,\, l_{S^j_k}}$, $\sum\limits_{\xi=1}^{l_{S^j_k}} \tilde{\mu}^{\xi,\,j}_{S^j_k} = 1$, is the mixed strategy, that is the set of mixed strategies of players from coalition S^j_k, $k = \overline{1,\ l}$;
- $\mu^j_{\Sigma^j} = \left(\tilde{\mu}^j_{S^j_1}, \ \ldots, \ \tilde{\mu}^j_{S^j_l}\right) \in \tilde{\mathrm{M}}^j$, $\tilde{\mu}^j_{S^j_k} \in \tilde{\mathrm{M}}^j_{S^j_k}$, $k = \overline{1,\, l}$, is the l-tuple of mixed strategies;
- $\tilde{\mathrm{M}}^j = \prod\limits_{k=\overline{1,\, l}} \tilde{\mathrm{M}}^j_{S^j_k}$ is the set of l-tuples in mixed strategies.

From the definition of strategy $\tilde{x}^j_{S^j_k}$ of coalition S^j_k it follows that $x^j_{\Sigma^j} = \left(\tilde{x}^j_{S^j_1}, \ \ldots, \ \tilde{x}^j_{S^j_l}\right)$ and $x^j = \left(x^j_1, \ \ldots, \ x^j_n\right)$ are the same n-tuples in the games $G^j\left(x^j\right)$ and $G_j\left(x^j_{\Sigma^j}\right)$. However it does not mean that $\mu^j = \mu^j_{\Sigma^j}$.

Payoff function $\tilde{H}^j_{S^j_k} : \ \tilde{X}^j \to R^1$ of coalition S^j_k for the fixed projects j, $j = \overline{1,\ m}$, and for the coalitional partition Σ^j is defined under condition that:

$$\tilde{H}^j_{S^j_k}\left(x^j_{\Sigma^j}\right) \geq H^j_{S^j_k}\left(x^j_{\Sigma^j}\right) = \sum_{i \in S^j_k} K^j_i\left(x^j\right), \ k = \overline{1\,,\, l}\,, \ \ j = \overline{1\,,\, m}\,, \ \ S^j_k \in \Sigma^j\,, \tag{3}$$

where $K^j_i\left(x\right)$, $i \in S^j_k$, is the payoff function of player i in the n-tuple $x^j_{\Sigma^j}$.

Definition 1. A set of m coalitional l-person games defined by (2) is called *static coalitional model of decision-making.*

Definition 2. *Solution of the static coalitional model of decision-making in pure strategies* is $x^{*,\,j^*}_{\Sigma^{j^*}}$, that is Nash equilibrium (NE) in pure strategies in l-person game $G_{j^*}\left(x^{j^*}_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 3.2).

Definition 3. *Solution of the static coalitional model of decision-making in mixed strategies* is $\mu^{*,\,j^*}_{\Sigma^{j^*}}$, that is Nash equilibrium (NE) in a mixed strategies in l-person game $G_{j^*}\left(\mu^{j^*}_{\Sigma^{j^*}}\right)$, with the coalitional partition Σ^{j^*}, where coalitional partition Σ^{j^*} is the compromise coalitional partition (see 3.2).

Generalized PMS-vector is used as the coalitional imputation (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006).

3. Algorithm for solving the problem

3.1. Algorithm of constructing the generalized PMS-vector in a coalitional game.

Remind the algorithm of constructing the generalized PMS-vector in a coalitional game (Grigorieva and Mamkina, 2009; Petrosjan and Mamkina, 2006).

1. Calculate the values of payoff $\tilde{H}^j_{S^j_k}\left(x^j_{\Sigma^j}\right)$ for all coalitions $S^j_k \in \Sigma^j$, $k = \overline{1, l}$, for coalitional game $G_j\left(x^j_{\Sigma^j}\right)$ by using formula (3).

2. Find NE (Nash, 1951) $x^{*,j}_{\Sigma^j}$ or $\mu^{*,j}_{\Sigma^j}$ (one or more) in the game $G_j\left(x^j_{\Sigma^j}\right)$. The payoff vector of coalitions in NE in mixed strategies is equal to $E\left(\mu^{*,j}_{\Sigma^j}\right) = \left\{v\left(S^j_k\right)\right\}_{k=\overline{1,l}}$.

Payoff of coalition S^j_k in NE in mixed strategies is computed by formula

$$v\left(S^j_k\right) = \sum_{\tau=1}^{l^j_{\Sigma^j}} p_{\tau,j}\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right), k = \overline{1,l},$$

where

- $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$ is the payoff function of coalition S^j_k;
- $p_{\tau,j} = \prod_{k=\overline{1,l}} \tilde{\mu}^{\xi_k,j}_{S^j_k}$, $\xi_k = \overline{1, l^j_{S^j_k}}$, $\tau = \overline{1, l^j_{\Sigma^j}}$, is probability of realization $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$.

The value $\tilde{H}^j_{\tau,S^j_k}\left(x^j_{\Sigma^j}\right)$ is random variable. There could be many l-tuples of NE in the game, therefore, $v\left(S^j_1\right)$, ..., $v\left(S^j_l\right)$, are not uniquely defined.

The payoff of each coalition in NE $E\left(\mu^{*,j}_{\Sigma^j}\right)$ is allocated according to the Shapley value (Shapley, 1953) $Sh\left(S_k\right) = \left(Sh\left(S^j_k : 1\right), \ldots, Sh\left(S^j_k : s\right)\right)$:

$$Sh\left(S^j_k : i\right) = \sum_{\substack{S' \subset S^j_k \\ S' \ni i}} \frac{(s'-1)!\,(s-s')!}{s!}\left[v\left(S'\right) - v\left(S' \setminus \{i\}\right)\right] \quad \forall\, i = \overline{1, s}, \tag{4}$$

where $s = \left|S^j_k\right|$ $(s' = |S'|)$ is the number of elements in sets S^j_k (S'), and $v\left(S'\right)$ are the maximal guaranteed payoffs for $S' \subset S_k$.

Moreover

$$v\left(S^j_k\right) = \sum_{i=1}^{s} Sh\left(S^j_k : i\right).$$

Then PMS-vector in the NE in mixed strategies $\mu^{*,j}_{\Sigma^j}$ in the game $G_j\left(x^j_{\Sigma^j}\right)$ is defined as

$$\mathrm{PMS}^j\left(\mu^{*,j}_{\Sigma^j}\right) = \left(\mathrm{PMS}^j_1\left(\mu^{*,j}_{\Sigma^j}\right), \ldots, \mathrm{PMS}^j_n\left(\mu^{*,j}_{\Sigma^j}\right)\right),$$

where

$$\mathrm{PMS}^j_i\left(\mu^{*,j}_{\Sigma^j}\right) = Sh\left(S^j_k : i\right),\ i \in S^j_k,\ k = \overline{1,l}.$$

3.2. Algorithm for finding the set of compromise solutions.

We also remind the algorithm for finding a set of compromise solutions (Malafeyev, 2001; p.18).

$$C_{\mathrm{PMS}}(M) = \arg\min_j \max_i \left\{ \max_j \mathrm{PMS}_i^j - \mathrm{PMS}_i^j \right\}.$$

Step 1. Construct the ideal vector $R = (R_1, \ldots, R_n)$, where $R_i = \mathrm{PMS}_i^{j^*} = \max_j \mathrm{PMS}_i^j$ is the maximal value of payoff function of player i in NE on the set M, and j is the number of project $j \in M$:

$$\begin{pmatrix} \mathrm{PMS}_1^1 & \ldots & \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ \mathrm{PMS}_1^m & \ldots & \mathrm{PMS}_n^m \end{pmatrix}$$
$$\begin{matrix} \downarrow & \ldots & \downarrow \\ \mathrm{PMS}_1^{j_1^*} & \ldots & \mathrm{PMS}_n^{j_n^*} \end{matrix}$$

Step 2. For each j find deviation of payoff function values for other players from the maximal value, that is $\Delta_i^j = R_i - \mathrm{PMS}_i^j$, $i = \overline{1, n}$:

$$\Delta = \begin{pmatrix} R_1 - \mathrm{PMS}_1^1 & \ldots & R_n - \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ R_1 - \mathrm{PMS}_1^m & \ldots & R_n - \mathrm{PMS}_n^m \end{pmatrix}.$$

Step 3. From the found deviations Δ_i^j for each j select the maximal deviation $\Delta_{i_j^*}^j = \max_i \Delta_i^j$ for all players i:

$$\begin{pmatrix} R_1 - \mathrm{PMS}_1^1 & \ldots & R_n - \mathrm{PMS}_n^1 \\ \ldots & \ldots & \ldots \\ R_1 - \mathrm{PMS}_1^m & \ldots & R_n - \mathrm{PMS}_n^m \end{pmatrix} = \begin{pmatrix} \Delta_1^1 & \ldots & \Delta_n^1 \\ \ldots & \ldots & \ldots \\ \Delta_1^m & \ldots & \Delta_n^m \end{pmatrix} \begin{matrix} \to \Delta_{i_1^*}^1 \\ \ldots. \\ \to \Delta_{i_m^*}^m \end{matrix}$$

Step 4. Choose the minimal deviation for all j from all maximal deviations among all players i $\Delta_{i_{j^*}^*}^{j^*} = \min_j \Delta_{i_j^*}^j = \min_j \max_i \Delta_i^j$.

The project $j^* \in C_{\mathrm{PMS}}(M)$, on which the minimum is reached is a compromise solution of the game $G_j\left(x_{\Sigma^j}^j\right)$ for all players.

3.3. Algorithm for solving the static coalitional model of decision-making.

We have an algorithm for solving the problem.

1. Fix a j, $j = \overline{1, m}$.

2. Find the NE $\mu_{\Sigma^j}^{*,j}$ in the coalitional game $G_j\left(x_{\Sigma^j}^j\right)$ and find allocation in NE, that is $\mathrm{PMS}^j\left(\mu_{\Sigma^j}^{*,j}\right)$.

3. Repeat iterations 1-2 for all other j, $j = \overline{1, m}$.

4. Find compromise solution j^*, that is $j^* \in C_{\mathrm{PMS}}(M)$.

4. Example

Consider the set $M = \{j\}_{j=\overline{1,3}}$ and the set $N = \{I_1, I_2, I_3\}$ of three players, each having 2 strategies in noncooperative game $G^j(x)$: $x_i = 1$ is "yes" and $x_i = 0$ is "no" for all $i = \overline{1,3}$. The payoff's functions of players in the game $G^j(x)$ are determined by tables 1, 3, 5.

Table 1: The payoffs of players in the coalitional game $G_1(x_{\Sigma^1})$ with coalitional partition $\Sigma^1 = \{\{I_1, I_2\}, I_3\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_1, I_2\}$
1	1	1	4	2	1	6
1	1	0	1	2	1	3
1	0	1	3	1	5	4
1	0	0	5	1	3	6
0	1	1	5	3	1	8
0	1	0	1	2	2	3
0	0	1	0	4	2	4
0	0	0	0	4	3	4

1. Compose and solve the coalitional game $G_1(x_{\Sigma^1})$, $\Sigma^1 = \{\{I_1, I_2\}, I_3\}$, i. e. find NE in mixed strategies in the game:

$$\begin{array}{rlll} & & \eta = 3/7 & 1-\eta = 4/7 \\ & & 1 & 0 \\ 0 & (1,\,1) & [6,\,1] & [3,\,1] \\ 0 & (0,\,0) & [4,\,2] & [4,\,3] \\ \xi = 1/3 & (1,\,0) & [4,\,5] & [6,\,3] \\ 1-\xi = 2/3 & (0,\,1) & [8,\,1] & [3,\,2]\,. \end{array}$$

It's clear, that first matrix row is dominated by the last one and the second is dominated by third. One can easily calculate NE and we have

$$y = \left(3/7\ 4/7\right),\ x = \left(0\ 0\ 1/3\ 2/3\right).$$

Then the probabilities of payoff realizations (coalitions $S = \{I_1, I_2\}$ and $N \backslash S = \{I_3\}$ in mixed strategies (in NE)) are as follows:

$$\begin{array}{lll} & \eta_1 & \eta_2 \\ \xi_1 & 0 & 0 \\ \xi_2 & 0 & 0 \\ \xi_3 & 1/7 & 4/21 \\ \xi_4 & 2/7 & 8/21 \end{array}.$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E(x,\,y) = \frac{1}{7}[4,\,5] + \frac{2}{7}[8,\,1] + \frac{4}{21}[6,\,3] + \frac{8}{21}[3,\,2] = \left[\frac{36}{7},\,\frac{7}{3}\right] = \left[5\frac{1}{7},\,2\frac{1}{3}\right].$$

In the table 2 pure strategies of coalition $N \backslash S$ and its mixed strategy y are given horizontally at the right side. Pure strategies of coalition S and its mixed strategy

Table 2: The maximal guaranteed payoffs of players I_1 and I_2.

	Mathematical Expectation		x	The strategies of $N\backslash S$, the payoffs of S and $N\backslash S$; y	0.43 +1	0.57 +0
	2.286	2.000	0	$-(1, 1)$	$(4, 2)$	$(1, 2)$
	4.143	1.000	0.33	$+(1, 0)$	$(3, 1)$	$(5, 1)$
	2.714	2.429	0.67	$+(0, 1)$	$(5, 3)$	$(1, 2)$
	0.000	4.000	0	$-(0, 0)$	$(0, 4)$	$(0, 4)$
	$v(I_1)$	$v(I_2)$				
min 1	2.286	2.000				
min 2	0.000	1.000				
max	**2.286**	**2.000**				

x are given vertically. Inside the table players' payoffs from the coalition S and players' payoffs from the coalition $N\backslash S$ are given at the right side.

Allocate the game's Nash value in mixed strategies according to Shapley's value (4):

$$Sh_1 = v(I_1) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right],$$
$$Sh_2 = v(I_2) + \tfrac{1}{2}\left[v(I_1, I_2) - v(I_2) - v(I_1)\right].$$

Find the maximal guaranteed payoffs $v(I_1)$ and $v(I_2)$ of players I_1 and I_2. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \begin{pmatrix} 3/7 & 4/7 \end{pmatrix}.$$

Denote mathematical expectations of players' payoffs from coalition S when mixed NE strategies are used by coalition $N\backslash S$ by $E_{S(i,j)}(\bar{y}), i, j = \overline{1,2}$. In the table 2 the mathematical expectations are located at the left, and values are obtained by using the following formulas:

$$\begin{aligned}
E_{S(1,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 2 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{2}{7};\ 2;\ 1\tfrac{4}{7}\right);\\
E_{S(1,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 5;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 3\right) = \left(4\tfrac{1}{7};\ 1;\ 3\tfrac{6}{7}\right);\\
E_{S(2,1)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 5 + \tfrac{4}{7}\cdot 1;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2;\ \tfrac{3}{7}\cdot 1 + \tfrac{4}{7}\cdot 2\right) = \left(2\tfrac{5}{7};\ 2\tfrac{3}{7};\ 1\tfrac{4}{7}\right);\\
E_{S(2,2)}(\bar{y}) &= \left(\tfrac{3}{7}\cdot 0 + \tfrac{4}{7}\cdot 0;\ \tfrac{3}{7}\cdot 4 + \tfrac{4}{7}\cdot 4;\ \tfrac{3}{7}\cdot 3 + \tfrac{4}{7}\cdot 2\right) = \left(0;\ 4;\ 2\tfrac{3}{7}\right).
\end{aligned}$$

Third element here is mathematical expectation of payoffs of the player I_3 (see table 1 too).

Then, look at the table 1 or table 2,

$$\left.\begin{aligned}
&\min H_1(x_1 = 1, x_2, \bar{y}) = \min\left\{2\tfrac{2}{7};\ 4\tfrac{1}{7}\right\} = 2\tfrac{2}{7};\\
&\min H_1(x_1 = 0, x_2, \bar{y}) = \min\left\{2\tfrac{5}{7};\ 0\right\} = 0;
\end{aligned}\right|\ v(I_1) = \max\left\{2\tfrac{2}{7};\ 0\right\} = 2\tfrac{2}{7};$$
$$\left.\begin{aligned}
&\min H_2(x_1, x_2 = 1, \bar{y}) = \min\left\{2;\ 2\tfrac{3}{7}\right\} = 2;\\
&\min H_2(x_1, x_2 = 0, \bar{y}) = \min\{1;\ 4\} = 1;
\end{aligned}\right|\ v(I_2) = \max\{2;\ 1\} = 2.$$

Thus, maxmin payoff for player I_1 is $v(I_1) = 2\frac{2}{7}$ and for player I_2 is $v(I_2) = 2$. Hence,

$$Sh_1(\bar{y}) = v(I_1) + \tfrac{1}{2}\left(5\tfrac{1}{7} - v(I_1) - v(I_2)\right) = 2\tfrac{2}{7} + \tfrac{1}{2}\left(5\tfrac{1}{7} - 2\tfrac{2}{7} - 2\right) = 2\tfrac{5}{7};$$
$$Sh_2(\bar{y}) = 2 + \tfrac{3}{7} = 2\tfrac{3}{7}.$$

Thus, PMS-vector is equal to

$$\mathrm{PMS}_1 = 2\frac{5}{7};\ \ \mathrm{PMS}_2 = 2\frac{3}{7};\ \ \mathrm{PMS}_3 = 2\frac{1}{3}\,.$$

Table 3: The payoffs of players in the coalitional game $G^2\left(x_{\Sigma^2}\right)$ with coalitional partition $\Sigma^2 = \left\{\left\{I_1,\, I_3\right\},\, I_2\right\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_1,\, I_3\}$
1	1	1	4	2	1	5
1	1	0	1	1	2	3
1	0	1	3	1	5	8
1	0	0	5	1	3	8
0	1	1	5	3	1	6
0	1	0	1	3	2	3
0	0	1	0	4	3	3
0	0	0	0	3	2	2

2. Compose and solve the coalitional game $G^2\left(x_{\Sigma^2}\right),\ \Sigma^2 = \left\{\left\{I_1,\, I_3\right\},\, I_2\right\}$, i. e. find NE in mixed strategies:

$$\begin{array}{cccc}
 & & \eta = 5/6 & 1-\eta = 1/6 \\
 & & 1 & 0 \\
\xi = 1/2 & (1,\,1) & [5,\,2] & [8,\,1] \\
0 & (0,\,0) & [3,\,3] & [2,\,3] \\
0 & (1,\,0) & [3,\,1] & [8,\,1] \\
1-\xi = 1/2 & (0,\,1) & [6,\,3] & [3,\,4]\,.
\end{array}$$

It's clear, that second and third matrix rows are dominated by the first. One can easily calculate NE and we have

$$y = \left(5/6\ 1/6\right),\ x = \left(1/2\ 0\ 0\ 1/2\right)\,.$$

The Nash value of the game in mixed strategies is calculated by formula:

$$E\left(x,\,y\right) = \frac{5}{12}\left[5,\,2\right] + \frac{1}{12}\left[8,\,1\right] + \frac{5}{12}\left[6,\,3\right] + \frac{1}{12}\left[3,\,4\right] = \left[\frac{66}{12},\,\frac{30}{12}\right] = \left[5\frac{1}{2},\,2\frac{1}{2}\right]\,.$$

Find the maximal guaranteed payoffs $v\left(I_1\right)$ and $v\left(I_3\right)$ of players I_1 and I_3. For this purpose fix a NE strategy of a third player as

$$\bar{y} = \left(5/6\ 1/6\right)\,.$$

Then maxmin payoff for player I_1 is $v\left(I_1\right) = 1.68$ and for player I_3 is $v\left(I_3\right) = 2$ (see table 4). Allocate the game's Nash value in mixed strategies $E_1\left(x,\,y\right) = 5.5$ according to Shapley's value (4):

$$\begin{array}{l}
Sh_1 = v\left(I_1\right) + \frac{1}{2}\left[v\left(I_1, I_3\right) - v\left(I_1\right) - v\left(I_3\right)\right] = 2.59\,, \\
Sh_3 = v\left(I_3\right) + \frac{1}{2}\left[v\left(I_1, I_3\right) - v\left(I_1\right) - v\left(I_3\right)\right] = 2.91\,.
\end{array}$$

Table 4: The maximal guaranteed payoffs of players I_1 and I_3.

Mathematical Expectation			The strategies of $N\backslash S$, the payoffs of S and $N\backslash S$			
			x	y	0.5 +1	0.5 +0
	3.83	1.68	0.5	$-(1,1)$	$(4,\ 1)$	$(3,\ 5)$
	1.68	2.17	0	$+(1,0)$	$(1,\ 2)$	$(5,\ 3)$
	4.15	1.34	0.5	$+(0,1)$	$(5,\ 1)$	$(0,\ 3)$
	0.83	2.00	0	$-(0,0)$	$(1,\ 2)$	$(0,\ 2)$
	$v(I_1)$	$v(I_3)$				
min 1	1.68	1.34				
min 2	0.83	2.00				
max	**1.68**	**2.00**				

Table 5: The payoffs of players in the coalitional game $G^3(x_{\Sigma^3})$ with coalitional partition $\Sigma^3 = \{\{I_2, I_3\}, I_1\}$.

The strategies			The payoffs			The payoffs of coalition
I_1	I_2	I_3	I_1	I_2	I_3	$\{I_2, I_3\}$
1	1	1	4	2	1	3
1	1	0	0	2	2	4
1	0	1	3	1	5	6
1	0	0	3	1	3	4
0	1	1	4	3	1	4
0	1	0	1	2	2	4
0	0	1	0	4	3	7
0	0	0	0	4	2	6

Thus, PMS-vector in mixed strategies is equal to

$$\mathrm{PMS}_1 = 2.59;\ \ \mathrm{PMS}_2 = 2.5;\ \mathrm{PMS}_3 = 2.91.$$

3. Compose and solve the coalitional game $G^3(x_{\Sigma^3})$, $\Sigma^3 = \{\{I_2, I_3\}, I_1\}$, i. e. find NE in mixed strategies in the game:

$$\begin{array}{ccc} & \eta = 1 & 1-\eta = 0 \\ & 1 & 0 \\ 0 \quad (1,\,1) & [3,\,4] & [4,\,4] \\ 0 \quad (0,\,0) & [4,\,3] & [6,\,0] \\ 0 \quad (1,\,0) & [4,\,0] & [4,\,1] \\ \xi = 1 \quad (0,\,1) & [6,\,3] & [7,\,0]\,. \end{array}$$

The first three matrix rows are dominated by the last. Then second column is dominated by the first. Hence we have

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix},\ x = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Nash value of the game equals:

$$E(x,\,y) = [6,\ 3]\,.$$

Find the maximal guaranteed payoffs of players I_2 and I_3. Fix a NE strategy of a first player as

$$\bar{y} = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then

$$\left.\begin{array}{l} \min H_2(\bar{y},\, x_2 = 1,\, x_3) = \min\{2;\, 2\} = 2\,; \\ \min H_2(\bar{y},\, x_2 = 0,\, x_3) = \min\{1;\, 1\} = 1; \\ \min H_3(\bar{y},\, x_2,\, x_3 = 1) = \min\{1;\, 5\} = 1\,; \\ \min H_3(\bar{y},\, x_2,\, x_3 = 0) = \min\{2;\, 3\} = 2; \end{array}\right| \begin{array}{l} v(I_2) = \max\{2;\, 1\} = 2; \\ \\ v(I_3) = \max\{1;\, 2\} = 2. \end{array}$$

Allocate the game's Nash value in mixed strategies $E_1(x,\, y) = 6$ according to Shapley's value (4):

$$\begin{array}{l} Sh_2 = v(I_2) + \frac{1}{2}\left[v(I_2, I_3) - v(I_2) - v(I_3)\right] = 3\,, \\ Sh_3 = v(I_3) + \frac{1}{2}\left[v(I_2, I_3) - v(I_2) - v(I_3)\right] = 3\,. \end{array}$$

Thus, PMS-vector in pure strategies is equal:

$$\mathrm{PMS}_1 = \mathrm{PMS}_2 = \mathrm{PMS}_3 = 3\,.$$

Present the obtained solution in the table 6.

Table 6: Payoffs of players in NE for various cases of the coalitional partition of players.

Project	Coalitional partitions	The n-tuple of NE $(I_1,\, I_2,\, I_3)$	Probability of realization NE	Payoffs of players in NE
1	$\Sigma^1 = \{\{I_1,\, I_2\}\,\{I_3\}\}$	$((1,\, 0)\,,\, 1)$ $((1,\, 0)\,,\, 0)$ $((0,\, 1)\,,\, 1)$ $((0,\, 1)\,,\, 0)$	1/7 4/21 2/7 8/21	$((2.71,\, 2.43)\,,\, 2.33)$
2	$\Sigma^2 = \{\{I_1,\, I_3\}\,\{I_2\}\}$	$(1,\, (1)\,,\, 1)$ $(1,\, (0)\,,\, 1)$ $(0,\, (1)\,,\, 1)$ $(0,\, (0)\,,\, 1)$	5/12 1/12 5/12 1/12	$(2.59,\, (2.5)\,,\, 2.91)$
3	$\Sigma^3 = \{\{I_2,\, I_3\}\,\{I_1\}\}$	$(1,\, (0,\, 1))$	1	$(3,\, (3,\, 3))$

Applying the algorithm for finding a compromise solution, we get the set of compromise coalitional partitions (table 7). Therefore, compromise imputation is

Table 7: The set of compromise coalitional partitions.

	I_1	I_2	I_3		I_1	I_2	I_3	
$\Sigma^1 = \{\{I_1,\, I_2\}\,\{I_3\}\}$	2.71	2.43	2.33	$\Delta\{\{I_1,\, I_2\}\,\{I_3\}\}$	0.29	0.57	0.67	0.67
$\Sigma^2 = \{\{I_1,\, I_3\}\,\{I_2\}\}$	2.59	2.5	2.91	$\Delta\{\{I_1,\, I_3\}\,\{I_2\}\}$	0.41	0.5	0.09	0.5
$\Sigma^3 = \{\{I_2,\, I_3\}\,\{I_1\}\}$	3	3	3	$\Delta\{\{I_2,\, I_3\}\,\{I_1\}\}$	0	0	0	**0**
R	**3**	**3**	**3**					

PMS-vector in coalitional game with the coalition partition Σ^3 in NE $(1\,,\ (0\,,\ 1))$ in pure strategies with payoffs $(3\,,\ (3\,,\ 3))$.

Moreover, in situation, for example, $(1\,,\ (0\,,\ 1))$ the first and third players give a positive decision for corresponding project. In other words, if the first and third players give a positive decision for corresponding project, and the second does not, then payoff of players will be optimal in terms of corresponding coalitional interaction.

5. Conclusion

A static coalitional model of decision-making over the set of projects with different preferences of players and algorithm for finding optimal solution are constructed in this paper, and numerical example are presented.

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Decision Making Procedure in Optimal Control Problem for the SIR Model

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Abstract In this work we join on classical SIR model to describe influenza epidemic in urban population with procedure of making decision. We suppose that agent in urban population makes a choice: whether or not to participate in vaccination company. Each decision involve different costs and indirectly influence on the population state. We formulated an optimal control problem to study the optimal behavior during epidemic period and vaccination company. All theoretical results are also supported by the numerical simulations.

Keywords: SIR model, vaccination problem, evolutionary games, optimal control, epidemic process.

1. Introduction

Originally Susceptible-Infected-Recovered model and its modification describe a fast spreading process, such as influenza epidemic or other forms of respiratory viral diseases, circulated in urban population. Total population is divided into three subgroups: Susceptible, Infected and Recovered. Susceptible is group where people are not infected, Infected is a group of people having the disease, and Recovered is group, where all members have immunity to the disease. Human population meets influenza epidemic almost every year then SIR model is very actual and can be used in social and economic applications.

During the years many medical methods such as preventive measures, intensive treatment, etc. were developed to protect entire population during annual epidemics. Hence preventive measures or medical treatment can be considered as an external influence to the development of epidemics and can be used as control parameters in the model. Since vaccination is one of the most effective method to protect population from the annual epidemics then we chose is as an control parameter.

However vaccination can not be absolutely effective and moreover such as it was proofed in the previous research total vaccination is very expensive and usually do not apply to protect population against a flu epidemics. Hence we can establish a new problem such as vaccination problem.

We assume that vaccination company occurs before the seasonal epidemic begins, because it is necessary take into account regulation immune system of individual after vaccination, because failing health after vaccination not allow to resist against another viruses. Unfortunately flu vaccines are effective only for one season owing to mutation of pathogens and waning immunity. We suppose that influenza epidemic continues until there are no more newly infected individuals.

In this model we assume that before epidemic period each agent of population may choose a behavior: participate or not in vaccination company. All decision provoke corresponding costs and influence on the future agent's incomes. If agent of population chooses the application of vaccination then he should pay it costs and estimate the consequences if vaccine is not effective. Hence in current work we extend classical Susceptible-Infected-Recovered (SIR) model with the procedure of making decisions. We reformulate original model in terms of optimal control and couple it with the process of making decisions.

1.1. Related Works

Recent literature has seen a large amount of interest in using optimal control and game-theoretic methods to study disease control of influenza for public health. First, this research problem was refereed in (Kermack and Kendrick, 1927), where an Susceptible-Infected-Recovered model has been proposed to study the epidemic spread in a homogeneous population. It provides a deterministic dynamical system model as the mean field approximation of the underlying stochastic evolution of the host subpopulations. In (Behncke, 2000) and (Kolesin and Zhitkova, 2004), many variants of optimal control models of SIR-epidemics are investigated for the application of medical vaccination and health promotion campaigns. In the paper (Fu et al., 2010) the vaccination problem is considered from the individual agents' point of view.

Also epidemic models can be applied to the different fields of human activity, for instance in (Khouzani et al., 2010; Khouzani et al., 2011), optimal control methods have been used to study the class of epidemic models in mobile wireless networks, and Pontryagin's maximum principle is used to quantify the damage that the malware can inflict on the network by deploying optimum decision rules.

Different from the work done in the past, in current work population agents based on the available information make a decision whether or not they participate in vaccination company. Their choices are included in the SIR model and as a result we receive the optimal control strategy (intensity of vaccination) that depends on the decision procedure.

2. Model

We use Susceptible-Infected-Recovered model to describe epidemiological process in urban population with following assumption that each agent in population allows to participate in vaccination company or refuses it. In current model vaccination company establish the influence to the population and hence we can consider it as control parameter in the model. Then, at time t, n_s, n_I, n_R correspond to fractions of the population who are susceptible, infected and for all t, condition $N = n_s+n_I+n_R$ is justified. Define

$$S(t) = \frac{n_S}{N},\ I(t) = \frac{n_{I_1}}{N},\ R(t) = \frac{n_R}{N},\ (R(t) = 1 - S(t) - I(t))$$

as portions of the susceptible, the infected and the recovered in the population.

And in addition to the above the model is formulated as follows (Khatri, 2003; Kermack and Kendrick, 1927):

$$\begin{aligned} \frac{dS}{dt} &= -\delta SI - u; \\ \frac{dI}{dt} &= \delta SI - \sigma I; \end{aligned} \tag{1}$$

here transmission rate from state S to I is

$$\delta = \delta_0 m(\frac{n_I}{N}) = \delta_0 m I, \tag{2}$$

where value δ_0 is a transmissibility of disease, m is a number of contacts per time unit, and parameter $\sigma = \frac{1}{T}$ which is intensive rate of transition from infected to recovered and it is in inverse proportion to the average duration of the disease. Here $R = 1 - S - I$, variable $u(t) \in (0, 1)$ is control parameter which is interpreted as the intensity of vaccination in agents per day.

2.1. Objective Function

In this work we will minimize aggregated cost in time interval $[0, T]$, hence at any given t following costs exist in the system: $f_i(I(t))$ these are individual's treatment costs, which are non-decreasing and twice-differentiable, convex functions, such as $f_i(0) = 0,\ f_i(I(t)) > 0,\ i = \overline{1, N}$ for $I(t) > 0$; functions $l_i(R(t))$ are agent's benefit rate, which arise when infected agent becomes recovered, $l_i(R(t))$ is non-decreasing and differentiable function and $l(0) = 0$; functions $h_i(u(t))$ describe vaccination costs that help to reduce epidemic spreading, $h_i(u(t))$ is twice-differentiable and increasing function in $u_i(t)$ such as $h_i(0) = 0,\ h_i(x) > 0,\ i = \overline{1, N}$ when $u > 0$. Hence costs function for i-th agent in population is:

$$J_i = f_i(I(t)) - l_i(R(t)) + h_i(u(t)). \tag{3}$$

Therefore aggregated system costs is:

$$J = \int_0^T \sum_{i=1}^{N} (f_i(I(t)) - l_i(R(t)) + h_i(u(t)))dt. \tag{4}$$

2.2. Making decision procedure

In current section we present a procedure of making decisions that influence to the epidemic process in urban population. Previous researches have proofed that vaccination company as a preventive measure is very effective and allows to reduce the quantity of infected in entire population. However each agent in population have a possibilities to estimate his own profit of participation in vaccination company. Agent can take into account the vaccination cost, feasible complications after vaccination and also he can estimate the herd immunity. We suppose that the last circumstance does not presume than an agent necessarily knows the exact information, he can evaluate the average number of his contacts, the current epidemic situation, that can be presented in mass communication media, etc. Meanwhile the collective result of vaccination decisions determines the level of population immunity and the strain of the epidemic in current period. When level of vaccination coverage in total population is increased then even agents who are unvaccinated have less risk to become infected. Then we assume that every agent, having this information might decide to decline the vaccination this year and thereby he reduces own vaccination costs. However agents might have incomplete information with the some rumors from the neighbors or friends, or they also may estimate the epidemic situation incorrectly, thus this scenario leads following problem, increasing of unvaccinated individuals provoke the diminution of the herd immunity in the future and thereby collective costs during epidemic period will arise. The reduction of the vaccinated individuals induces the increasing of infected in population the then it leads that the frequency of meeting with infected agents is also increased. Then each unvaccinated agent may transform to the infected and then he should pay treatment costs, that include healthcare expenses, lost productivity and the possibility of pain. Usually treatment costs exceed the vaccination expanses.

Thus in current work we suppose that each agent chooses between two possible alternatives:

– to be vaccinated;
– not to be vaccinated and probably to be infected;

If agent participate in vaccination company then he gets a vaccination costs, in our model these costs are described by functions $h_i(u)$, where u is intensity of vaccination. Vaccination costs contain the immediate monetary cost, the opportunity cost of time spent to get the vaccine and any health effects. We also suppose that vaccination is not absolutely effective and vaccination company should be finished before the epidemic starts.

Infected agents incur treatment costs, which are denoted as functions $f_i(I)$, and when agent convalesce then his treatment costs are reduced to the value $l(R)$, which is benefit function.

Then describe the decision procedure, each season an agent adopts one of the alternative, which determines whether or not he vaccinated. At the end of the season each agent decides whether to change the vaccination decision or not, depending on the current aggregated costs. Then agent i selects at random agent j, and in imitates his role model if opponents payoff is higher. Define probability that agent i adopts behavior of agent j as follows (Fu et al., 2010):

$$\rho_{ij} = \frac{1}{1+\exp(-\beta(p_j - p_i))}, \tag{5}$$

where p_j is agent's payoff on j-th decision, parameter $\beta \in (0, \infty)$.

We incorporate this probability to the basic Susceptible-Infected-Recovered model, which is presented in section 2., thus transmission rate from S to I can be rewritten:

$$\delta = \delta_0 m I \rho_{ij}. \tag{6}$$

3. Structure of optimal control

We use Pontryagin's maximum principle (Pontryagin et al., 1962), to find the optimal control $u = (u_1, u_2)$ to the problem described above in Section 2.. Define the associated Hamiltonian H and adjoint functions λ_S, λ_{I_1}, λ_{I_r}, λ_R as follows:

$$\begin{aligned} H = &-\lambda_0 \sum_{i=1}^{N} (f_i(I(t)) - l_i(R(t)) + h_i(u(t))) + \\ &\lambda_S(-\delta S(t)I(t) - u) + \lambda_I(\delta S(t)I(t) - \sigma I(t)) = \\ &-\lambda_0 \sum_{i=1}^{N} (f_i(I(t)) - l_i(R(t)) + h_i(u(t))) - \\ &\delta S(t)I(t)(\lambda_S - \lambda_I) - \lambda_S u - \lambda_I \sigma. \end{aligned} \tag{7}$$

We construct adjoint system as follows:

$$\begin{aligned}
\dot{\lambda}_I(t) &= -\tfrac{\partial H}{\partial I} = \lambda_0 \sum_{i=1}^{N} f_i'(I(t)) + \delta S(\lambda_S(t) - \lambda_I(t)) + \lambda_I(t)\sigma; \\
\dot{\lambda}_S(t) &= -\tfrac{\partial H}{\partial S} = \lambda_S(t)\delta I(t) - \lambda_2(t)\delta I(t) = \delta I(t)(\lambda_S(t) - \lambda_I(t));
\end{aligned} \tag{8}$$

with the transversality conditions given by

$$\lambda_I(T) = 0,\ \lambda_S(T) = 0,\ \lambda_R(T) = 0. \tag{9}$$

According to Pontryagin's maximum principle, there exist continuous and piecewise continuously differentiable co-state functions λ_i that at every point $t \in [0, T]$ where u_1 and u_2 is continuous, satisfy (8) and (9). In addition, we have $\lambda(t) = (\lambda_0(t), \lambda_S(t), \lambda_I(t), \lambda_R(t))$ òàêàß œòî

$$u \in arg \max_{\underline{u} \in [0,1]} H(\overline{\lambda}, (S, I, R), \underline{u}). \tag{10}$$

To determine an optimal control parameter that maximize Hamiltonian (7) we consider derivative $\dfrac{\partial H}{\partial u}$:

$$\frac{\partial H}{\partial u} = -\lambda_0 \sum_{i=1}^{N} h_i'(u) - \lambda_S = -(\lambda_0 \sum_{i=1}^{N} h_i'(u) + \lambda_S). \tag{11}$$

Now let equal to zero right parts of equations (11), (8):

$$\begin{aligned}
&-(\lambda_0 \sum_{i=1}^{N} h_i'(u) + \lambda_S) = 0; \\
&\lambda_0 \sum_{i=1}^{N} f_i'(I(t)) + \delta S(\lambda_S(t) - \lambda_I(t)) + \lambda_I(t)\sigma = 0; \\
&\delta I(t)(\lambda_S(t) - \lambda_I(t)) = 0.
\end{aligned} \tag{12}$$

From the first equation of system (12), Hamiltonian reaches maximum if and only if next condition is satisfied:

$$(\lambda_0 \sum_{i=1}^{N} h_i'(u) + \lambda_S) < 0. \tag{13}$$

Let be $\lambda_0 = 1$, then expression (13) can be reformulated:

$$\sum_{i=1}^{N} h_i'(u) < -\lambda_S, \tag{14}$$

and we will proof that $\lambda_S < 0$.

From (12) we received that

$$\lambda_S = -\frac{1}{\sigma} \sum_{i=1}^{n} f_i'(I(t)), \tag{15}$$

where $\sigma \geq 0$, $\sum\limits_{i=1}^{n} f_i'(I(t)) \geq 0$ by definition then $\lambda_S < 0$, hence maximum of Hamiltonian is reached on the negative half-space then we should proof that function λ_S is increasing. Consider adjoint system (8) and show that derivative $\dot{\lambda_S} = \delta I(t)(\lambda_S(t) - \lambda_I(t)) \geq 0$.

We will proof this statement base on the next two properties (Khouzani et al., 2011):

Property 1. Let $w(t)$ be a continuous and piecewise differential function of t. Let $w(t_1) = L$ and $w(t) > L$ for all $t \in (t_1, \ldots, t_0]$. Then $\dot{w(t_1^+)} \geq 0$, where $w(t_1^+) = \lim_{x \to x_0} v(x)$.

Property 2. For any convex and differentiable function $y(x)$, which is 0 at $x = 0$, $y'(x)x - y(x) \geq 0$ for all $x \geq 0$.

Step I. Consider instant time moment $t = T$, from transversality conditions (9) we have $\lambda_S(T) - \lambda_I(T) = 0$, and $\dot{\lambda}_S(T) - \dot{\lambda}_I(T) = -\sum_i^n f_i'(I(T)) < 0$, $\dot{\lambda}_I(T) = \sum_i^n f_i'(I(T)) > 0$, therefore function λ_I is increasing on the interval $[0, T]$.

Step 2.(Proof by contradiction).

Let $0 \leq t^* < T$ be the last instant moment at which one of the inequality constraints are performed:

Condition 1. $\lambda_I(t) > 0, \qquad \lambda_S(t) - \lambda_I(t) = 0$ for $t^* < t < T$.
Condition 2. $\lambda_I(t) = 0, \qquad \lambda_S(t) - \lambda_I(t) < 0$ for $t^* < t < T$.

Now consider a difference:

$$\begin{aligned}
\dot{\lambda}_S(t^{*+}) - \dot{\lambda}_I(t^{+*}) = \delta I(T)(\lambda_S - \lambda_I) - (\lambda_0 \sum_{i=1}^{N} f_i'(I(t)) + \delta S(\lambda_S - \lambda_I) + \lambda_I \sigma) = \\
\delta I(T)(\lambda_S - \lambda_I) - \lambda_0 \sum_{i=1}^{N} f_i'(I) + \frac{H}{I} + \frac{\lambda_S}{I} u + \frac{\lambda_I}{I} \sigma I - \lambda_I \sigma \\
+ \frac{\lambda_0}{I} (\sum_{i=1}^{N} (f_i(I(t)) - l_i(R(t)) + h_i(u(t)))) = \\
= \delta I(T)(\lambda_S - \lambda_I) - \frac{\lambda_0}{I} (\sum_{i=1}^{N} f_i'(I(t))I - \sum_{i=1}^{N} f_i(I(t))) + \frac{H}{I} \\
- \frac{\lambda_0}{I} \sum_{i=1}^{N} l_i(R(t)) + \frac{\lambda_0}{I} \sum_{i=1}^{N} h_i(u(t)) + \frac{\lambda_S}{I} u + \frac{\lambda_I}{I} \sigma I - \lambda_I \sigma
\end{aligned} \tag{16}$$

The system ODE is autonomous, i.e., the Hamiltonian and the constraints on the control u do not have an explicit dependency on the independent variable t. Then at time $t = T$ Hamiltonian is:

$$\begin{aligned}
H(T) = -\lambda_0 \sum_{i=1}^{N} (f_i(I(T)) + l_i(R(T)) + h_i(u(T))) - \\
-\delta S(T) I(T)(\lambda_S(T) - \lambda_I(T)) - \lambda_S(T) u(T) - \lambda_I(T) \sigma I(T).
\end{aligned} \tag{17}$$

costs functions follow the next conditions $f_i(I(T)) \geq 0$, $l_i(R(T)) \geq 0$, $h_i(u(T)) \geq 0$ and transversality conditions (9) at time moment T are justified then

$$H(T) \leq 0.$$

Hence as far as functions f, l, h are non-decreasing we have:

$$\begin{aligned} &H(t) - \lambda_0 \sum_{i=1}^{N} (f_i(I(t)) + l_i(R(t)) + h_i(u(t))) = \\ &-\delta S(T)I(T)(\lambda_S(T) - \lambda_I(T)) - \lambda_S(T)u(T) - \lambda_I(T)\sigma I(T) \leq 0. \end{aligned} \tag{18}$$

By property 2. the term is nonnegative $\frac{\lambda_0}{I}(\sum_{i=1}^{N} f_i'(I(t))I - \sum_{i=1}^{N} f_i(I(t))) \geq 0$, from condition 1 $(\lambda_S - \lambda_I) = 0$ and $\lambda_I > 0$ and from (18) we received that $\dot{\lambda}_S(t^{*+}) - \dot{\lambda}_I(t^{+*}) < 0$, then $\frac{d}{dt}(\lambda_S(t^{*+}) - \lambda_I(t^{*+})) < 0$, which contradicts property 1, thus time moment t^{*+} does not exist.

Step II. Consider formula (16) and suppose that condition 2. is satisfied, by property 2. we have $\frac{\lambda_0}{I}(\sum_{i=1}^{N} f_i'(I(t))I - \sum_{i=1}^{N} f_i(I(t))) \geq 0$, for all t it is justified that $H(t) < 0$, therefore $\frac{d}{dt}(\lambda_S(t^{*+}) - \lambda_I(t^{*+})) < 0$. This also contradicts property 1. and then time moment t^{*+} does not exist in case **II**. Hence for all $t \in [0, T]$ condition $\frac{d}{dt}(\lambda_S(t) - \lambda_I(t)) > 0$ is satisfied.

These proofed results can be formulated as lemmas.

Lemma 1. *For all t, $0 < t < T$ following conditions hold $(\lambda_S(t) - \lambda_I(t)) > 0$ and $\lambda_I(t) \leq 0$.*

Lemma 2. *For all t on time interval $0 < t < T$ we have*

$$(\lambda_0 \sum_{i=1}^{N} h_i'(u) + \lambda_S) < 0. \tag{19}$$

Based on previous research (Khouzani et al., 2010; Khouzani et al., 2011; Pontryagin et al., 1962), we show that an optimal control $u(t) = (u_1(t), u_2(t))$ has following form.

Theorem 1. *Optimal control program $u(t)$ has following structure:*

For all t such as $0 < t < t^*$, $u(t)$ satisfies:

$$\frac{\lambda_0}{\sigma} \sum_{i=1}^{N} f_i'(I(t) = \sum_{i=1}^{N} h_i'(u(t)), \tag{20}$$

For all t such as $t^* < t < T$:

$$u(t) = 0.$$

4. Numerical simulations

In this section we present numerical simulation which are used to illustrated the structure of the optimal control and influence of the human decision to the epidemic process. In the example we suppose that population size is $N = 1000$, initial fraction of subpopulations are: $S(0) = 950$, $I(0) = 50$, $R(0) = 0$. Following values of the system parameters are used in the simulation: $h = 0,1$ is model step, $\delta_0 = 0,06$ is transmissibility of decease, $l = 35$ is number of contacts per time unit, $\sigma = 1/15 = 0,06(6)$ is intensity of recovering.

Based on previous research and on the result of the experiments in this work we assume that typical duration of disease is 10-15 days. After simulation we receive that with mentioned initial states and auxiliary parameters the maximum quantity of infected is $I(\bar{t}) = 511$ and epidemic peak is reached at $\bar{t} = 15$ day.

In figure 1 Susceptible-Infected-Recovered model is presented:

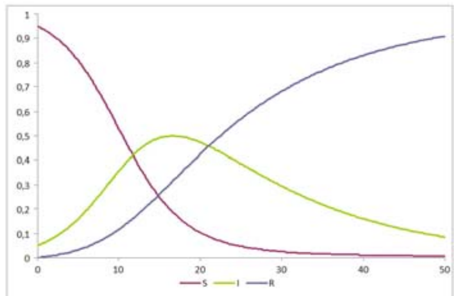

Fig. 1: SIR model without application of the control

Below we present the case, where we apply optimal intensity of vaccination (optimal control strategy), which allow to reduce the number of infected in population. In current model vaccination was used as a control parameters in the system hence agents from subpopulation Susceptible directly transfer to subpopulation Recovered, obtaining immunity. After numerical simulations for the same initial data we get that the maximum quantity of infected is $I^*(\bar{t}) = 426$ at time $\bar{t} = 13$. Therefore we can see that maximum number of Infected is less than in previous case and in comparing with Fig.1 epidemic peak is achieved early.

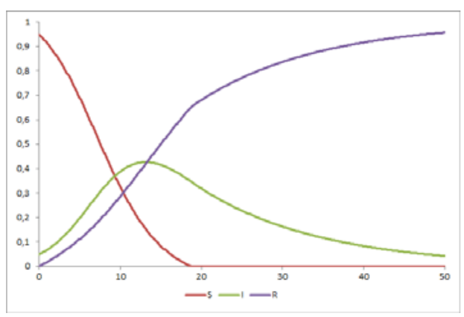

Fig. 2: Application optimal control to the SIR model.

In figure 3 we illustrate the optimal control that minimize aggregated costs on the preventive measures. For the considered initial states optimal control will be switched off at $t^* = 7$ day.

One of the main aim of the work is to show that participation of agents in vaccination company reduces aggregated costs of the entire population, hence in following figures we present aggregated costs for different cases. First individual

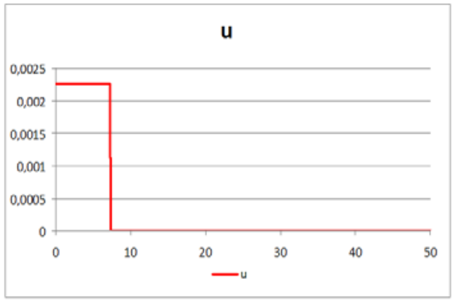

Fig. 3: Optimal control, $u_{opt} = 15$ agents per day.

agents costs are define as follows: treatment costs are $f_i(I(t)) = aI(t) + b$, where $a = 1$, $b = 1.4$, vaccination costs $h_i(u(t)) = k_1u^2 + k_2$, and $k_1 = k_2 = 0.5$, $l_i(R(t)) = cR(t) + d$, $c = 1$, $d = 0.05$.

In figures 4.-5 aggregated costs received for the time interval $[0, T]$ and they are equal to $J(u) = 801.5$ monetary units (m.u.), which can be in US dollars, Rubles or Euros depending on the context.

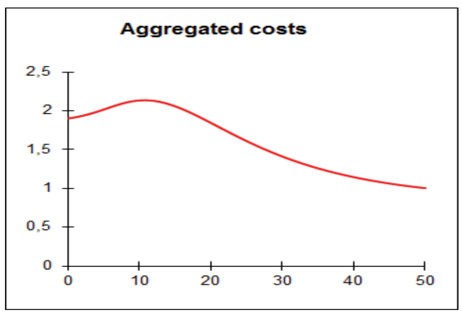

Fig. 4: Aggregated system cost for SIR model without application of the optimal control.

If optimal control is applied to the system then aggregated system costs decrease and value of functional is equal to $J(u) = 707,14$ m.u.

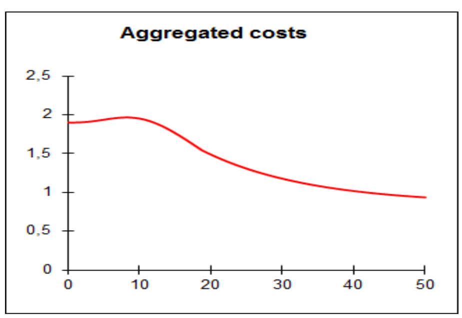

Fig. 5: Aggregated cost for SIR model with application of the optimal control.

To complete the our illustrative example let's consider a modification of the model, where we take into account only agents choices. Let's say that for instance that intensity of vaccination is $u = 5$ agents per day, it means that only five agent accept a decision about the participation in vaccination company. We should add also that $u < u_{opt}$. In such case dynamics in SIR model is changed, the maximum number of infected is achieved at $t^* = 14$ day and equal to $I_{max} = 482$ agents. The result of simulation is presented in figure 6.

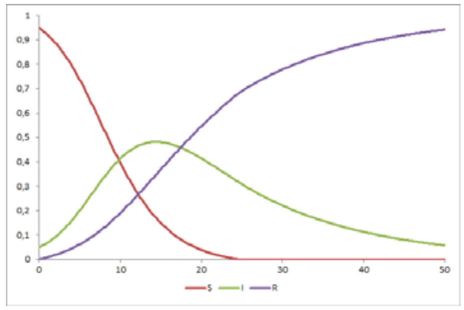

Fig. 6: SIR model with control parameter and decision making procedure.

We can see that the quantity of infected exceeds the number of infected in situation, where we apply optimal control strategy to the population, and the number of agents, choosing vaccination decision is not enough to protect population during the epidemic season. Hence in this case we also can show that aggregates costs increase and value of functional is $J(u) = 761.36$ m.u.

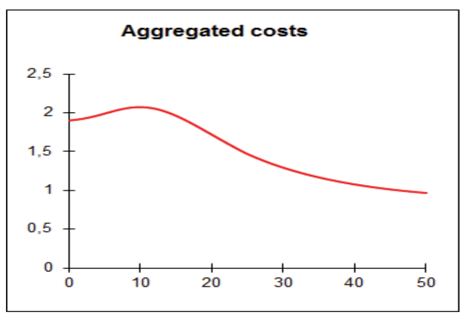

Fig. 7: Aggregated costs with $u = 0.005$ is equal to $J(u) = 761.36$ m.u.

To summarize the results we have to add that since agents in population do not have reliable information about epidemic situation then they may choose incorrect decision, which provoke the degradation of epidemic state in total population.

5. Conclusion

In this paper, we have studied an epidemic model that takes into account the agent motivation to participation in the vaccination company. We incorporate procedure of making decision to the simple Susceptible-Infected-Recovered model and have formulated this model in special case. Using Pontryagin's maximum principle, we have shown the structure of optimal control, which is depending on the agents costs induced by choosing decisions. We supported our results with numerical simulations, observing different cases of epidemic process in entire urban population. In future work we would extend this model including different structure of population, it means that human decision may depend on his social group, not only his costs and to modify the model, using number of contacts as a function of the time.

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A New Characterization of the Pre-Kernel for TU Games Through its Indirect Function and its Application to Determine the Nucleolus for Three Subclasses of TU Games*

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Abstract The main goal is twofold. Thanks to the so-called indirect function known as the dual representation of the characteristic function of a coalitional TU game, we derive a new characterization of the pre-kernel of the coalitional game using the evaluation of its indirect function on the tails of pairwise bargaining ranges arising from a given payoff vector. Secondly, we study three subclasses of coalitional games of which its indirect function has an explicit formula and show the applicability of the determination of the pre-kernel (nucleolus) for such types of games using the indirect function. Three such subclasses of games concern the 1−convex and 2-convex n person games and clan games. A clan game with the clan to be s singleton is known as a big boss game.

Keywords: dual representation, indirect function, pre-kernel, 1- and 2-convex n person games, clan games, big boss games.

1. Introduction and notions

As shown in (Driessen et al., 2010; Driessen et al., 2011), certain practical problems such as co-insurance situations and library situations can be modeled as a cooperative game in characteristic function form. Formally, a cooperative game on player set N is a characteristic function $v : \mathcal{P}(N) \to R$ defined on $\mathcal{P}(N)$ satisfying $v(\emptyset) = 0$. Here $\mathcal{P}(N)$ denotes the power set of the finite player set N, given by $\mathcal{P}(N) = \{S | S \subseteq N\}$, and shortly called a game v on N. In (Martinez-Legaz, 1996), the dual representation of cooperative games based on Fenchel Moreau Conjugation has been introduced, with every game v on N, there is associated the indirect function $\pi^v : R^N \to R$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) \qquad \text{for all } \vec{y} = (y_k)_{k \in N} \in R^N, \tag{1}$$

The excess $e^v(S, \vec{y})$ of a non-empty coalition S at the *salary vector* $\vec{y}$ in the game v represents the net profit the (unique) employer would receive from the selection of

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coalition S, assuming the members of S will produce, using the resources that are available to the employer, a total amount of output the monetary utility of which is measured by $v(S)$, and the (possibly negative) salary required by the player i amounts y_i, $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In the game theory setting, the efficient salary vectors of which all the excess are non-positive, compose the multi-valued solution concept called Core, that is

$$Core(v) = \{\vec{y} \in R^N | e^v(N, \vec{y}) = 0, e^v(S, \vec{y}) \leq 0 \quad \text{for all } S \subseteq N,\, S \neq \emptyset\}, \qquad (2)$$

According to (Martinez-Legaz, 1996), the indirect function $\pi^v : R^N \to R$ of a game v on N is a non-increasing convex function which attains its minimum at level zero, i.e., $\min_{\vec{y} \in R^N} \pi^v(\vec{y}) = 0$.

In this paper, we use indirect function to determine the nucleolus for three subclasses of games concerning $1-$convex and $2-$convex games (Driessen, 1988) (Driessen and Hou, 2010) and clan games. The theory on $1-$convex n person games has been well developed by Theo Driessen. The key feature of this kind of games is the geometrically regular structure of its core. For $2-$convex games, its core coincides with a so-called core catcher associated with appropriately chosen lower and upper Core bounds. For clan games, there is a nonempty coalition called clan, of which each member has veto power; i.e., no coalition can attain any positive reward unless it contains all clan members. With the clan to be a singleton, the clan game reduces to a big boss game.

2. The indirect function of 1-convex and 2-convex n person games and clan games

Given a game (N, v), its corresponding benefits vector $\vec{b^v} = (b_i^v)_{i \in N}$ is defined by $b_i^v = v(N) - v(N \backslash \{i\})$, $i \in N$. Note that the vector $\vec{b^v}$ is an upper bound for core allocations in that $y_i \leq b_i^v$ for all $i \in N$, all $\vec{y} \in Core(v)$. In terms of the characteristic function v, the 1-convexity property requires that, concerning the division problem, the worth $v(N)$ is sufficiently large to meet the coalitional demand amounting its worth $v(S)$, as well as the desirable marginal benefit by any individual not belonging to coalition S. For notation sake, write $\vec{z}(T)$ instead of $\sum_{k \in T} z_k$ for any coalition $T \subseteq N$ and any vector $\vec{z} = (z_k)_{k \in N} \in R^N$, where $\vec{z}(\emptyset) = 0$, and use $\vec{y} \leq \vec{b}^v$ instead of $y_i \leq b_i^v$ for all $i \in N$.

Definition 1. A game v on N is said to be 1-*convex* if it holds

$$\sum_{k \in N} b_k^v \geq v(N) \quad \text{and} \quad v(N) \geq v(S) + \sum_{k \in N \backslash S} b_k^v \qquad \text{for all } S \subseteq N,\, S \neq \emptyset. \qquad (3)$$

Example 1. Let the three-person game v on $N = \{1, 2, 3\}$ be given by $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 1$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 6$, $v(\{2, 3\}) = 7$, $v(N) = 10$. It is left to the reader to check the 1-convexity of this game using the marginal benefit vector $b^v = (3, 4, 6)$. It turns out that core coincides with the triangle with the three vertices $(0, 4, 6)$, $(3, 1, 6)$, $(3, 4, 3)$. In fact, $(y_1, y_2, y_3) \in Core(v)$ is equivalent to $y_1 + y_2 + y_3 = 10$ and $y_1 \leq 3, y_2 \leq 4, y_3 \leq 6$. Under the latter upper core bound assumption $y \leq b^v$, the first part of the following theorem reports that the level equation $\pi^v(y) = c$ for its indirect function π^v is solved by the hyperplane equation

$y_1 + y_2 + \ldots + y_n = v(N) - cm$ provided $c > 0$. Here the larger the strictly positive level c, the smaller $v(N) - c$. In case $c = 0$, then its level equation $\pi^v(y) = 0$ is solved by any hyperplane equation $y_1 + y_2 + \ldots + y_n = d$ where the real number d ranges from $b^v(N)$ to $v(N)$. The lowest hyperplane with $d = v(N)$ represents the core of the 1−convex game.

Theorem 1. *Let v be a 1-convex game on N and we study the indirect function of this game with respect to the following two types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b}^v$.
Type 2: There exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k \in N} y_k\right] \quad \textit{for vectors of type 1.}$$

$$\begin{aligned}(ii)\pi^v(\vec{y}) &= \max\left[0, \quad v(N\backslash\{\ell\}) - \sum_{k \in N\backslash\{\ell\}} y_k\right] \\ &= \max\left[0, \quad v(N) - \sum_{k \in N} y_k + y_\ell - b^v_\ell\right] \quad \textit{for vectors of type 2.}\end{aligned}$$

Proof. (i) Let $S \subseteq N$, $S \neq \emptyset$, and $\vec{y} \in R^N$ with $y_i \leq b^v_i$ for all $i \in N$. From (3), we derive

$$\begin{aligned}v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + \vec{y}(N\backslash S) \\ &\leq v(S) - \vec{y}(N) + \vec{b}^v(N\backslash S) \leq v(N) - \vec{y}(N)\end{aligned} \tag{4}$$

Thus, the restriction of the indirect function π^v to the comprehensive hull of the marginal benefit vector $\vec{b}^v$ attains its maximum either for $S = N$ or $S = \emptyset$.
(ii) For every $\vec{y} \in R^N$ such that there exists a unique $\ell \in N$ with $y_\ell > b^v_\ell$ and $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$, it holds that, on the one hand, $v(S) - \vec{y}(S) \leq v(N) - \vec{y}(N)$ for all $S \subseteq N$ with $\ell \in S$ because the above chain (4) of inequalities still holds due to $\ell \notin N\backslash S$. For all $S \subseteq N$ with $\ell \notin S$, it holds

$$\begin{aligned}v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_\ell + \vec{y}(N\backslash(S \cup \{\ell\})) \\ &\leq v(S) - \vec{y}(N) + y_\ell + \vec{b}^v(N\backslash(S \cup \{\ell\})) \\ &= v(S) - \vec{y}(N) + y_\ell - b^v_\ell + \vec{b}^v(N\backslash S) \\ &\leq v(N) - \vec{y}(N) + y_\ell - b^v_\ell = v(N\backslash\{\ell\}) - \vec{y}(N\backslash\{\ell\}).\end{aligned} \tag{5}$$

In this setting, the indirect function π^v attains its maximum either for $S = N$, $S = N\backslash\{\ell\}$ or $S = \emptyset$, but $S = N$ cancels.

Corollary 1. *For every 1-convex game v on N and the payoff vector $\vec{y} = (y_k)_{k \in N} \in R^N$, it holds:*

$$\vec{y} \in Core(v) \Leftrightarrow \vec{y}(N) = v(N), \pi^v(\vec{y}) = 0 \Leftrightarrow \vec{y}(N) = v(N), \vec{y} \leq \vec{b}^v.$$

The former if and only if implication is trivial, while the latter if and only if implication is shown by the (partial) determination of the indirect function for 1-convex games according to Theorem 1.

In the remainder of this section, we switch from 1-convex to 2-convex games. In this framework, it is useful to introduce the so-called *gap function* $g^v : \mathcal{P}(N) \to R$ of a game v on N, given by $g^v(S) = \vec{b}^v(S) - v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and $g^v(\emptyset) = 0$. In view of (3), a game v on N is 1-convex if and only if the nonnegative gap function attains its minimum at the grand coalition, i.e., $0 \leq g^v(N) \leq g^v(S)$ for all $S \subseteq N$, $S \neq \emptyset$.

Definition 2. (Driessen, 1988) A game v on N is said to be 2-*convex* if the following two conditions hold:

$$g^v(\{i\}) + g^v(\{j\}) \geq g^v(N) \geq g^v(\{i\}) \quad \text{for any players } i, j \in N,\ i \neq j \tag{6}$$

$$v(N) \geq v(S) + \sum_{k \in N \setminus S} b^v_k \qquad \text{for all } S \subseteq N,\ |S| \geq 2 \tag{7}$$

For 2-convexity, the main condition (3) is kept except for singletons, of which the gap is leveled below the gap of the grand coalition, whereas the sum of two such gaps majorizes the gap of the grand coalition.

Theorem 2. *Let v be a 2-convex game on N and we study the indirect function of this game with respect to the following four types of vectors, given $\vec{y} \in R^n$.*
Type 1: $\vec{y} \leq \vec{b}^v$.
Type 2: There exists a unique $\ell \in N$ with $y_\ell > b^v_\ell \geq v(\{\ell\})$ and $v(\{i\}) \leq y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$.
Type 3: There exists a unique $j \in N$ with $y_j < v(\{j\}) \leq b^v_j$ and $v(\{i\}) \leq y_i \leq b^v_i$ for all $i \in N$, $i \neq j$.
Type 4: There exist unique $j, \ell \in N$ with $y_\ell > b^v_\ell \geq v(\{\ell\})$, $y_i \leq b^v_i$ for all $i \in N$, $i \neq \ell$, and $y_j < v(\{j\}) \leq b^v_j$, $y_i \geq v(\{i\})$ for all $i \in N$, $i \neq j$. Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, v(N) - \sum_{k \in N} y_k, \quad (v(\{i\}) - y_i)_{i \in N}\right] \text{ for vectors of type 1.}$$

$$\begin{aligned}(ii)\pi^v(\vec{y}) &= \max\left[0, v(N\setminus\{\ell\}) - \sum_{k \in N\setminus\{\ell\}} y_k\right] \\ &= \max\left[0, v(N) - \sum_{k \in N} y_k + y_\ell - b^v_\ell\right] \text{ for vectors of type 2.}\end{aligned}$$

$$(iii)\pi^v(\vec{y}) = \max\left[v(N) - \sum_{k \in N} y_k, \quad v(\{j\}) - y_j\right] \text{ for vectors of type 3.}$$

$$\begin{aligned}(iv)\pi^v(\vec{y}) &= \max\left[v(N\setminus\{\ell\}) - \sum_{k \in N\setminus\{\ell\}} y_k, \quad v(\{j\}) - y_j\right] \\ &= \max\left[v(N) - \sum_{k \in N} y_k + y_\ell - b^v_\ell, \quad v(\{j\}) - y_j\right] \text{ for vectors of type 4.}\end{aligned}$$

The proof is similar to the previous proof of Theorem(1) and is left to the reader.

Corollary 2. *Let v be a 2-convex game on N and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $v(\{i\}) \leq y_i \leq b_i^v$ for all $i \in N$.*

The former if and only if statement is general and the latter is shown by the structure of the indirect function.

Definition 3. (Potters et al., 1989; Muto et al., 1988; Branzei et al., 2008, page 59) A game v on N is said to be a *clan game* if $b_i^v \geq v(\{i\})$ for all $i \in N$ and there exists a coalition $T \subseteq N$, called the *clan*, such that $v(S) = 0$ whenever $T \not\subseteq S$ and

$$v(N) \geq v(S) + \sum_{k\in N\setminus S} b_k^v \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \text{ with } T \subseteq S \tag{8}$$

A clan game v with an empty clan reduces to an 1-convex game, provided $g^v(N) \geq 0$. A clan game with the clan to be a singleton is known as a big boss game. Although both subclasses are interrelated, the description of its indirect function requires to distinguish two cases (either a singleton or a multi-person clan).

Theorem 3. *Let v be a big boss game on N, say player 1 is the big boss and we study the indirect function of this game with respect to the following four types of vector, given $\vec{y} \in R^n$.*
Type 1: $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1\}$.
Type 2: There exists a unique $\ell \in N\setminus\{1\}$ with $y_\ell > b_\ell^v \geq 0$ and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,\ell\}$.
Type 3: There exists a unique $\ell \in N\setminus\{1\}$ with $y_\ell < 0 \leq b_\ell^v$ and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,\ell\}$.
Type 4: There exist unique $j, \ell \in N\setminus\{1\}$ with $y_\ell > b_\ell^v \geq 0$, $y_j < 0 \leq b_j^v$, and $0 \leq y_i \leq b_i^v$ for all $i \in N\setminus\{1,j,\ell\}$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right] \text{ for vectors of type 1.}$$

$$(ii)\pi^v(\vec{y}) = \max\left[0, \quad v(N\setminus\{\ell\}) - \sum_{k\in N\setminus\{\ell\}} y_k\right]$$

$$= \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 2.}$$

$$(iii)\pi^v(\vec{y}) = \max\left[-y_\ell, \quad v(N) - \sum_{k\in N} y_k\right] \text{ for vectors of type 3.}$$

$$(iv)\pi^v(\vec{y}) = \max\left[-y_j, \quad v(N\setminus\{\ell\}) - \sum_{k\in N\setminus\{\ell\}} y_k\right]$$

$$= \max\left[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 4.}$$

Proof. Let $\vec{y} = (y_k)_{k\in N} \in R^N$.
(i) Suppose that $0 \le y_i \le b_i^v$ for all $i \in N\backslash\{1\}$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \le 0$. In case $1 \in S$, then $v(S) - \vec{y}(S) \le v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \le b_i^v$ for all $i \in N\backslash\{1\}$. This proves part (i).
(ii) Suppose that there exists a unique $\ell \in N\backslash\{1\}$ with $y_\ell > b_\ell^v \ge 0$ and $0 \le y_i \le b_i^v$ for all $i \in N\backslash\{1,\ell\}$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \le 0$. In case $\{1,\ell\} \subseteq S$, then $v(S) - \vec{y}(S) \le v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \le b_i^v$ for all $i \in N\backslash\{1,\ell\}$. In case $1 \in S$, $\ell \notin S$, then (5) applies once again. This proves part (ii).
(iii) Suppose that there exists a unique $\ell \in N\backslash\{1\}$with $y_\ell < 0 \le b_\ell^v$ and $0 \le y_i \le b_i^v$ for all $i \in N\backslash\{1,\ell\}$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \le -y_\ell$. In case $1 \in S$, then $v(S) - \vec{y}(S) \le v(N) - \vec{y}(N)$ as shown in (4), due to (8) together with $y_i \le b_i^v$ for all $i \in N\backslash\{1\}$. This proves part (iii).
(iv) Suppose that there exist unique $j, \ell \in N\backslash\{1\}$ with $y_\ell > b_\ell^v \ge 0$, $y_j < 0 \le b_j^v$, and $0 \le y_i \le b_i^v$ for all $i \in N\backslash\{1,j,\ell\}$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $1 \notin S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \le -y_j$. In case $1 \in S$, the proof proceeds similar to the proof of part (ii).

Corollary 3. *Let v be a big boss game on N and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $0 \le y_i \le b_i^v$ for all $i \in N\backslash\{1\}$.*

Theorem 4. *Let v be a clan game on N, say coalition $T \subseteq N$ with at least two players is the clan. and we study the indirect function of this game with respect to the following four types of vector, given $\vec{y} \in R^n$.*
Type 1: $y_i \ge 0$ for all $i \in N$ and $y_i \le b_i^v$ for all $i \in N\backslash T$.
Type 2: There exists a unique $\ell \in N\backslash T$ with $y_\ell > b_\ell^v \ge 0$, $y_i \le b_i^v$ for all $i \in N\backslash T$, $i \neq \ell$, and $y_i \ge 0$ for all $i \in N$.
Type 3: There exists a unique $\ell \in N$ with $y_\ell < 0$, $y_i \ge 0$ for all $i \in N\backslash\{\ell\}$, and $y_i \le b_i^v$ for all $i \in N\backslash T$.
Type 4: There exist unique $j \in N$, $\ell \in N\backslash T$ with $y_j < 0$, $y_i \ge 0$ for all $i \in N\backslash\{j\}$, and $y_\ell > b_\ell^v \ge 0$, $y_i \le b_i^v$ for all $i \in N\backslash T$, $i \neq \ell$.
Then its indirect function $\pi^v : R^N \to R$ satisfies the following properties:

$$(i)\pi^v(\vec{y}) = \max\left[0, \quad v(N) - \sum_{k\in N} y_k\right] \text{ for vectors of type 1.}$$

$$\begin{aligned}(ii)\pi^v(\vec{y}) &= \max\left[0, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\right] \\ &= \max\left[0, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\right] \text{ for vectors of type 2.}\end{aligned}$$

$$(iii)\pi^v(\vec{y}) = \max\Big[-y_\ell, \quad v(N) - \sum_{k\in N} y_k\Big] \text{ for vectors of type 3.}$$

$$(iv)\pi^v(\vec{y}) = \max\Big[-y_j, \quad v(N\backslash\{\ell\}) - \sum_{k\in N\backslash\{\ell\}} y_k\Big]$$
$$= \max\Big[-y_j, \quad v(N) - \sum_{k\in N} y_k + y_\ell - b_\ell^v\Big] \text{ for vectors of type 4.}$$

The proof of Theorem 4 is similar as the proof of Theorem 3 and is left to the reader.

Corollary 4. *Let v be a clan game with coalition $T \subseteq N$ as the clan and let $\vec{y} = (y_k)_{k\in N} \in R^n$. Then $\vec{y} \in Core(v)$ iff $\vec{y}(N) = v(N)$ and $\pi^v(\vec{y}) = 0$ iff $\vec{y}(N) = v(N)$ and $y_i \geq 0$ for all $i \in N$ and $y_i \leq b_i^v$ for all $i \in N\backslash T$.*

Finally, we remark that a geometrical characterization of a clan game, say with coalition $T \subseteq N$ as the clan, is shown in (Branzei et al., 2008, page 60) requiring that $v(N)\cdot\vec{e}_j \in Core(v)$ for all $j \in T$ and there exists $\vec{x} \in Core(v)$ such that $x_i = b_i^v$ for all $i \in N\backslash T$.

3. Solving the pre-kernel by means of the indirect function

In this section, we characterize the pre-kernel of a game on N by the evaluation of the indirect function of the game at pairwise bargaining ranges arising from the payoff vector involved. Formally, for every pair of players $i, j \in N, i \neq j$, the surplus $s_{ij}^v(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v on N is given by the maximal excess among coalitions containing player i, but not containing player j. That is,

Definition 4. Let v be a game on N and $\vec{y} = (y_k)_{k\in N} \in R^N$.

(i) For every pair of players $i, j \in N$, $i \neq j$, the *surplus* $s_{ij}^v(\vec{y})$ of player i against player j at the (salary) vector $\vec{y}$ in the game v is given by

$$s_{ij}^v(\vec{y}) = \max\Big[e^v(S,\vec{y})| \quad S \subseteq N, \quad i \in S, \quad j \notin S\Big] \tag{9}$$

(ii) The *pre-kernel* $\mathcal{K}^*(v)$ of the game v consist of *efficient* salary vectors of which all the pairwise surpluses are in equilibrium, that is (Maschler et al., 1979)

$$\mathcal{K}^*(v) = \{\vec{y} \in R^N | e^v(N,\vec{y}) = 0, s_{ij}^v(\vec{y}) = s_{ji}^v(\vec{y}) \quad \text{for all } i,j \in N,\, i \neq j.\} \tag{10}$$

For the alternative description of the pre-kernel, with every payoff vector $\vec{x} = (x_k)_{k\in N} \in R^N$, every pair of players $i, j \in N$, $i \neq j$, and every transfer amount $\delta \geq 0$ from player i to player j, there is associated the modified payoff vector $\vec{x}^{ij\delta} = (\vec{x}_k^{ij\delta})_{k\in N} \in R^N$ defined by $x_i^{ij\delta} = x_i - \delta$, $x_j^{ij\delta} = x_j + \delta$, and $x_k^{ij\delta} = x_k$ for all $k \in N\backslash\{i, j\}$.

Theorem 5. *Let v be a game on N and $\vec{x} = (x_k)_{k\in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$.*

(i) For every pair of players $i, j \in N$, $i \neq j$, the indirect function $\pi^v : R^N \to R$ satisfies $\pi^v(\vec{x}^{ij\delta}) = s_{ij}^v(\vec{x}) + \delta$, provided $\delta \geq 0$ is sufficiently large.

(ii) $\vec{x} \in \mathcal{K}^*(v)$ *if and only if the evaluation of the pariwise bargaining ranges arising from* $\vec{x}$ *through the indirect function are in equilibrium, that is, for every pair of players* $i, j \in N$, $i \neq j$, *the indirect function satisfies* $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ *for* δ *sufficiently large.*

Proof. Fix the pair of players $i, j \in N$, $i \neq j$. Firstly, we claim that coalitions not containing player i or containing player j are redundant for maximizing the excesses at the modified payoff vector $\vec{x}^{ij\delta}$, provided the transfer amount $\delta \geq 0$ is sufficiently large. For that purpose, for all coalitions $S \subseteq N\backslash\{i\}$, $T \subseteq N\backslash\{j\}$, note the following two equivalences:

$$v(S \cup \{i\}) - \sum_{k \in S \cup \{i\}} x_k^{ij\delta} \geq v(S) - \sum_{k \in S} x_k^{ij\delta} \quad \text{iff} \quad \delta \geq v(S) - v(S \cup \{i\}) + x_i \tag{11}$$

$$v(T \cup \{j\}) - \sum_{k \in T \cup \{j\}} x_k^{ij\delta} \leq v(T) - \sum_{k \in T} x_k^{ij\delta} \quad \text{iff} \quad \delta \geq v(T \cup \{j\}) - v(T) - x_j \tag{12}$$

From (1) and (11)–(12) respectively, we derive that

$$\pi^v(\vec{x}^{ij\delta}) = \max_{S \subseteq N}\left[v(S) - \sum_{k \in S} x_k^{ij\delta}\right] = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \left[v(S) - \sum_{k \in S} x_k^{ij\delta}\right] \tag{13}$$

where the choice of δ can be improved by

$$\delta \geq \max\left[\max_{S \subseteq N\backslash\{i\}} |v(S \cup \{i\}) - v(S) - x_i|, \quad \max_{T \subseteq N\backslash\{j\}} |v(T \cup \{j\}) - v(T) - x_j|\right]$$

because of $|\alpha| \geq \alpha$ as well as $|\alpha| \geq -\alpha$ for all $\alpha \in R$. Finally, from (13), $x_i^{ij\delta} = x_i - \delta$, and (9) respectively, we conclude that, for $\delta \geq 0$ sufficiently large, the following chain of equalities holds:

$$\pi^v(\vec{x}^{ij\delta}) = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \left[v(S) - \sum_{k \in S} x_k^{ij\delta}\right] = \max_{\substack{S \subseteq N, \\ i \in S, \quad j \notin S}} \left[v(S) - \sum_{k \in S} x_k\right] + \delta = s_{ij}^v(\vec{x}) + \delta$$

This proves part (i). Together with (10), part (ii) follows immediately.

4. Remarks about determination of the nucleolus

The aim of this section is to illustrate the significant role of the indirect function for three classes of games (1-convex, 2-convex and clan games) to determine its nucleolus through a uniform approach replacing its original computation approach. Under these circumstances, the nucleolus belongs always to the pre-kernel, and so it is sufficient to solve the system for its unique solution. Thus we avoid the formal definition of the nucleolus.

Remark 1. Suppose the game v on N is 1-convex. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $\vec{x} \leq \vec{b}^v$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : R^N \to R$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{ij\delta}$ is in accordance with Theorem 1(i)–(ii) dependent on the size of its j-th component $\vec{x}^{ij\delta}_j = x_j + \delta$ in comparison to player j-th marginal benefit b^v_j. From the explicit formula for the indirect function of 1-convex games, we conclude the following:

$$\pi^v(\vec{x}^{ij\delta}) = 0 \qquad \text{if} \quad x^{ij\delta}_j \leq b^v_j, \text{ that is } \delta \leq b^v_j - x_j$$

$$\pi^v(\vec{x}^{ij\delta}) = \max\Big[0, \quad x^{ij\delta}_j - b^v_j\Big] = x_j + \delta - b^v_j > 0 \quad \text{otherwise}$$

For sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ is met if and only if $x_j + \delta - b^v_j = x_i + \delta - b^v_i$, that is $x_j - b^v_j = x_i - b^v_i$ for all $i \neq j$. Together with the efficiency principle $\vec{x}(N) = v(N)$, the unique solution of this system of linear equations is given by

$$x_i = b^v_i - \frac{\alpha}{n} \qquad \text{for all } i \in N, \text{ where} \qquad \alpha = \vec{b}^v(N) - v(N) \geq 0$$

The latter solution is known as the nucleolus and turns out to coincide with the gravity of the core being the convex hull of n extreme points of the form $\vec{b}^v - \alpha \cdot \vec{e}_i$, $i \in N$. Here $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denotes the standard basis of R^n.

We consider once again the 3-person game of the Example 1 in order to illustrate Remark 1 and Theorem 5. Let payoff vector $\vec{x}$ satisfy $\vec{x}(N) = v(N) = 10$ as well as $\vec{x} \leq \vec{b}^v = (3, 4, 6)$. From Remark 1, we obtain that $\pi^v(\vec{x}^{ij\delta}) = x_j + \delta - b^v_j$, $\pi^v(\vec{x}^{ji\delta}) = x_i + \delta - b^v_i$ for sufficiently large δ. By Theorem 5(ii), it holds that $\vec{x} \in \mathcal{K}^*(v)$ iff $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for δ sufficiently large. Thus, $\vec{x} \in \mathcal{K}^*(v)$ iff $x_j + \delta - b^v_j = x_i + \delta - b^v_i$ and due to efficiency, the nucleolus is given by $\vec{x} = (2, 3, 5)$.

Remark 2. Suppose the game v on N is a big boss game, with player 1 as the big boss. For every payoff vector $\vec{x} = (x_k)_{k \in N} \in R^N$ satisfying the efficiency principle $\vec{x}(N) = v(N)$ as well as $0 \leq x_k \leq b^v_k$ for all $k \in N \backslash \{1\}$, and for every pair of players $i, j \in N$, $i \neq j$, the evaluation of the indirect function $\pi^v : R^N \to R$ at the tail of the bargaining range described by the corresponding modified payoff vector $\vec{x}^{j\ell\delta}$ is in accordance with Theorem 3(i)–(iv) dependent on the size of its j-th component $\vec{x}^{j\ell\delta}_j = x_j - \delta$ in comparison to the zero level as well as its ℓ-th component $\vec{x}^{j\ell\delta}_\ell = x_\ell + \delta$ in comparison to player ℓ-th marginal benefit b^v_ℓ. From the explicit formula for the indirect function of big boss games, we conclude the following: for $\{j, \ell\} \subseteq N \backslash \{1\}$, and for $\delta \geq 0$ sufficiently large

$$\pi^v(\vec{x}^{j\ell\delta}) = \max\Big[-(x_j - \delta), \quad (x_\ell + \delta) - b^v_\ell\Big] = \delta - \min\Big[x_j, \quad b^v_\ell - x_\ell\Big]$$

$$\pi^v(\vec{x}^{1\ell\delta}) = \max\Big[0, \quad (x_\ell + \delta) - b^v_\ell\Big] = \delta + x_\ell - b^v_\ell$$

$$\pi^v(\vec{x}^{\ell 1\delta}) = \max\Big[0, \quad -(x_\ell - \delta)\Big] = \delta - x_\ell$$

For all $\ell \in N \backslash \{1\}$ and sufficiently large δ, the equilibrium condition $\pi^v(\vec{x}^{1\ell\delta}) = \pi^v(\vec{x}^{\ell 1\delta})$ is met if and only if $x_\ell - b^v_\ell = -x_\ell$, that is $x_\ell = \frac{b^v_\ell}{2}$ for all $\ell \neq 1$.

Further, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ for any pair $\{j,\ell\} \subseteq N\backslash\{1\}$ is given by

$$\min\Big[x_j, \quad b^v_\ell - x_\ell\Big] = \min\Big[x_\ell, \quad b^v_j - x_j\Big] \quad \text{equalities which are satisfied trivially.}$$

Remark 3. Suppose the game v on N is a clan game, say coalition $T \subseteq N$ with at least two players is the clan. From the explicit formula for the indirect function of clan games, as presented in Theorem 4 (ii)–(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ reduces to the following system of equations: $x_i = x_j$ for all $i,j \in T$, and

$$x_i = \min\Big[b^v_i - x_i, \quad x_j\Big] \quad \text{whenever } i \notin T,\ j \in T$$

$$\min\Big[b^v_j - x_j, \quad x_i\Big] = \min\Big[b^v_i - x_i, \quad x_j\Big] \quad \text{whenever } i,j \notin T$$

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ for all $i \in T$ and $x_i = \min\Big[\lambda, \quad \frac{b^v_i}{2}\Big]$ for all $i \in N\backslash T$, where the parameter $\lambda \in R$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

Remark 4. Suppose the game v on N is 2-convex. From the explicit formula for the indirect function of 2-convex n-person games, as presented in Theorem 2(iv), we conclude that, for $\delta \geq 0$ sufficiently large, the equilibrium condition $\pi^v(\vec{x}^{j\ell\delta}) = \pi^v(\vec{x}^{\ell j\delta})$ reduces to the following system of equations: for every pair of players $j,\ell \in N$, $j \neq \ell$,

$$\min\Big[b^v_\ell - x_\ell, \quad x_j - v(\{j\})\Big] = \min\Big[b^v_j - x_j, \quad x_\ell - v(\{\ell\})\Big]$$

As shown in (Driessen and Hou, 2010), the unique solution is of the parametric form $x_i = v(\{i\}) + \min\Big[\mu, \quad \frac{b^v_i - v(\{i\})}{2}\Big]$ for all $i \in N$, where the parameter $\mu \in R$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

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Game-Theoretic Models of Collaboration among Economic Agents[1]

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Abstract In present article are considered the models explaining the mechanisms of emergence and development of situations, in which it is appropriate for economic agents to collaborate and act together despite of having independent goals. The main attention is concentrated to different approaches to definition of concept of equilibrium for model of collaboration of two agents. The work is devoted to problems in the study of economic instruments, inducing the agents, which initially have independent and uncoordinated systems of goals to commission any beneficial actions. Particularly, we consider an interaction of economic agents when each of them may take the actions, that bring benefit to other. Stimulus to "positive" behavior each agent is a waiting counter actions, that will be useful for him. To identify this class of situations it is proposed to use the term "collaboration". In a model of collaboration between two economic agents is proposed version to express of mixed strategies of players in the form of continuous distribution, which enabled us to formulate two alternative approaches of equilibrium: based on the criterion of minimizing variance of utility of participants and based on the criterion of minimizing of VaR.

Keywords: Game theory, collaboration, Nash equilibrium, value at risk (VaR), quantile.

Models that explain the mechanisms of emergence and situation development, in which it is appropriate for economic agents to collaborate and act together despite of having independent goals have become rather interesting in both theoretical and application way. An interaction of economic agents, where each of them takes actions that bring direct benefit not only to him but to other agents, can serve as the simplest example. An expectation of a beneficial counter action is an incentive for each agent to behave in this way. The most important difference between this behavior model from the "classical" models of rational economic agent's utility optimization is that here the utility of each agent depends directly on decisions made by others, whom he can indirectly influence.

We define such examples of agents' interaction with the term "collaboration". We could as well use the "indication" collaboration for that. On the other hand, such definition could create false associations with models based on cooperative games, also a further framework of the proposed model is solely based on a strategy game with complete information.

The main problem which is considered in this article is to present one possible approach based on the methods of modern game theory, which makes us able to describe and explain the mechanisms of collaborative relations between economic agents.

Obviously collaboration (in the context in which we agreed to consider it) and related issues may arise, for example, between the parties of public and private partnership, alongside with major investment projects or different schemes of financing from various levels of budget sources. Moreover, such models can also be useful in situations that go beyond "pure" economics. For instance, they can be applied to studies of intergovernmental negotiation processes aimed at achievement of agreements, which will complexly take both economic and political interests of the parties into account.

We will consider a simplified situation in order to explain the fundamental ideas of the proposed model. It describes interaction between two parties (agents, participants, players) $i \in I = \{1, 2\}$, who make a decision upon the value of their own contribution to some common project. This contribution (degree) is quantitatively characterized by some arbitrary value from 0 to 1: where "0" stands for lack of affirmative action in the project (non-collaboration, extremely selfish behavior, etc.), and "1" reflects the highest possible level of affirmative action (the maximum propensity to collaborate, ultimately constructive behavior).

If we take into consideration previously set objectives when we define the utility functions of players, we assume that the input (costs) performed by the agents reduce utility they can get, utility can increase due to inputs of his opponents. Linear relations are acceptable in model, because they reflect adequately its fundamental properties. So we define the utility function of the first player, as

$$u_1(x_1, x_2) = b_1 x_2 - a_1 x_1 \tag{1}$$

and the utility function of the second player as

$$u_2(x_1, x_2) = b_2 x_1 - a_2 x_2 \tag{2}$$

Accordingly, a is a value (score, a measure of regret) of a resource unit spent (invested in the project) by the player i and b is utility (effect, measure of satisfaction) for the i–th player, which he gets from a unit invested in the project by another party. Let's imaging such a situation as "classical" finite non-cooperative two-person game. We face the fact that it has an obvious Nash equilibrium in pure strategies

$$x_1^* = 0,\, x_2^* = 0. \tag{3}$$

Obviously our productivity functions are arranged in such a way (see Fig.2) that the best response of the first player to any second player's strategy will be to reduce his share of participation to zero.

$$\max_{x_1 \in [0,1]} \{u_1(x_1, x_2)\} = u_1(0, x_2),\, \forall x_2 \in [0, 1] \tag{4}$$

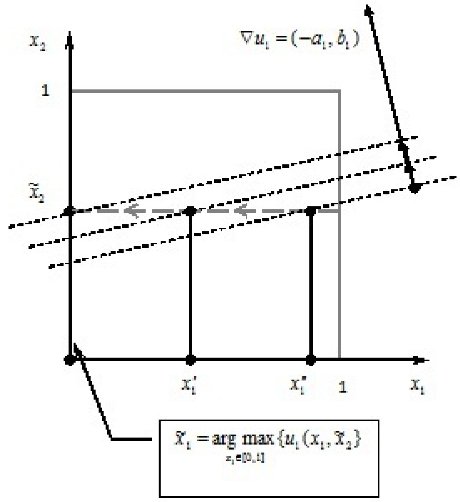

Fig. 1: Productivity functions

$$\max_{x_2 \in [0,1]} \{u_2(x_1, x_2)\} = u_2(x_1, 0), \, \forall x_1 \in [0, 1] \tag{5}$$

Thus, if we follow the concept of Nash equilibrium, we arrive to a pessimistic conclusion that the model described in the framework of collaboration between the players would not happen (the most stable situation is "mutual self–interest"). In this context, this is a particular interest to study modifications of this model in order to explain the mechanisms, which lead to the emergence of a collaborative relationship (collaboration) between economic agents.

First of all we concentrate on the approaches associated with transition from the original game to its mixed extension. Due to the fact that this scenario is based on a continuous set of pure strategies of the players, it seems obvious to set their mixed strategies as probability distributions with densities $p_1(x_1)$ and $p_2(x_2)$ on the interval $[0, 1]$, see Fig.7. According to this suggestion, a particular choice of strategies by players in a particular round of the game can be interpreted as an implementation of independent random variables $\tilde{x_1}$ and $\tilde{x_2}$.

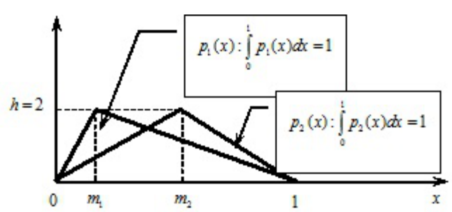

Fig. 2: Mixed strategies densities

This idea of mixed strategies of the players is a generalization of the "traditional" definition of mixed strategies in matrix and bimatrix games, which can be defined as a likelihood $(p_1, \ldots, p_k, \ldots)$ in accordance with every player implement one or another pure strategy. To follow this logic, we would have had to sample the intervals $[0, 1]$ in order to bring in traditional "discontinuous" mixed strategies. This method, however, seems to be not enough justified and reasonable in terms of reflecting the economic realities.

When mixed strategies are defined in the form of continuous distributions, a player's strategic choice is generally reduced to the choice of parameters of these distributions. Due to the fact that the number of parameters in different probability distributions classes is different, we come to a conclusion that definition of the players' strategies within the stated model will vary according to the type of distribution $p_1(x)$ or $p_2(x)$ we've chosen. Actually the value of strategies chosen by the participants in each act of the game can be viewed as a realization of independent random variables $\tilde{x}_1, \tilde{x}_2$, whose densities are known; and utilities $u_1(\tilde{x}_1, \tilde{x}_2), u_2(\tilde{x}_1, \tilde{x}_2)$ are determined as functions of random variables, which characteristics, generally spoken, can be determined with the help of $p_1(x), p_2(x)$.

We should note that specification of participants' strategic choices in the form of continuous probability distributions can be justified by the theory of evolutionary games. Namely, we can assume that we have a community consisting of groups (populations). Different populations of players have different tendencies to collaborate (collaborative behavior). These tendencies are realizations of random variables $\tilde{x}_i$ with densities $p_i(x)$. When members of different populations confront in some acts of the game, their success (or lack of success) can be expressed in terms of utility $\tilde{u}_i$. After that evolution of stochastic characteristics of propensity to cooperate takes place and these indicators reach some "benchmark" stable states, based on the experience accumulated by populations.

Of course if the strategies of participants are determined with continuous probability distributions, we can only compare them correctly if function $p_i(x)$ is restricted by some single parametric class $\mathbf{P}_i$. In this case, parameters of density functions $p_i(x)$ become "obvious" characteristics of strategies. Accordingly, the set of possible situations in a game is defined by the set of all possible combinations $p_i(x)$ of all players.

In terms of the classical Nash approach (Vorobiev, 1984), (Vorobiev, 1985), (Moulin, 1985), (Pecherskiy and Belyaeva, 2001) the equilibrium (solution) in of the described model will be characterized by such joint choice of probability distributions $(p_1^*(x), p_2^*(x))$ from which every participant in the game would not be advantageous to deviate separately from, i.e.:

$$\mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1^*(x), p_2^*(x)\} \geqslant \mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1(x), p_2^*(x)\} \tag{6}$$

$$\mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1^*(x), p_2^*(x)\} \geqslant \mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1^*(x), p_2(x)\} \tag{7}$$

for every $p_1(x) \in \mathbf{P}_1, p_2(x) \in \mathbf{P}_2$, where $\mathbf{E}\{u_i(\tilde{x}_1, \tilde{x}_2) \parallel p_1(x), p_2(x)\}$ is expected value of $u_i(\tilde{x}_1, \tilde{x}_2)$, calculated in the assumption that distribution $\tilde{x}_1$ is determined by the density function $p_1(x)$ and distribution $\tilde{x}_2$ by the density function $p_2(x)$.

Since a randomized model is being described, we cannot deny admissibility and validity of alternative approaches, which determine the equilibrium conditions with respect to other criteria. Particularly, they may be:

– minimization of variances of players' utilities (perhaps with additional restrictions on the lower levels, below which the utility expectation value cannot go);

– minimization of α–quintile values of players' utility function distributions, that is, below which the value of the utility will not fall with a probability $1-\alpha$.

1. Equilibrium based on minimization of utility variance

Let us consider the first mentioned approach in details. To some extent, the ideas of this approach are similar to the ideas in the Markowitz model of portfolio selection that minimizes risk (Binmore, 1987), (Binmore, 1988), (Cheon, 2003). In this case, we may assume that equilibrium in this model will be characterized by such joint selection of probability distributions $p_1^*(x) \in \mathbf{P}_1$ and $p_2^*(x) \in \mathbf{P}_2$, which will provide us with conditions fulfilled:

$$\mathbf{D}\{u_1(\tilde{x_1}, \tilde{x_2}) \parallel p_1^*(x), p_2^*(x)\} \leqslant \mathbf{D}\{u_1(\tilde{x_1}, \tilde{x_2}) \parallel p_1(x), p_2^*(x)\} \tag{8}$$

$$\mathbf{D}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1^*(x), p_2^*(x)\} \leqslant \mathbf{D}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1^*(x), p_2(x)\} \tag{9}$$

$\mathbf{D}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1(x), p_2(x)\}$ — variance of $u_1(\tilde{x}_1, \tilde{x}_2)$, calculated in assumption that distribution of $\tilde{x}_1$ is determined by a density function $p_1(x)$, and distribution of $\tilde{x}_2$ is determined by a density function $p_2(x)$.

In other words, conditions (8)–(9) define the situation, in which participants make an attempts to deviate from, taken by one or another party on an individual basis, lead to an increase in the risk. Variance is used as a measure of risk. A "weak" point of this approach in determination of equilibrium is connected with the fact, that minimal risks can be achieved at an unacceptably low expected utility values. This, in turn, can be "corrected" by introducing a concept of conditional equilibrium, under which one can understand a joint choice of probability distributions $p_1^*(x) \in \mathbf{P}_1$ and $p_2^*(x) \in \mathbf{P}_2$, which provides fulfillment of conditions (8)–(9), as well as conditions

$$\mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2) \parallel p_1(x), p_2(x)\} \geqslant \bar{u}_i,\ i = \{1, 2\} \tag{10}$$

where $\bar{u}_i$ are lower bounds on acceptable levels of expected utility of participants. Subsequent development of the approach (8)–(9) is clearly possible under condition that we specify classes of possible distributions $p_i(x_i)$. We should note that this step is substantial, moreover, it can be critical to the prospects of using this model.

Based on the general properties of solutions, which are made by real economic agents and concern issues of mutual collaboration, we can use an asymmetric triangular distribution for modeling the behavior of variables $\tilde{x}_i$, see Fig. 7[1]. On the interval $[0, 1]$ densities of asymmetric triangular distributions are uniquely determined by the choice of parameter m — the point of mode. It is known that an arbitrary random variable distributed in an asymmetric triangular law on the interval $[0, 1]$ has expected value

$$\mathbf{E}\tilde{x} = \frac{1}{3}(m+1) \tag{11}$$

and variance

$$\mathbf{D}\tilde{x} = \frac{1}{18}(m^2 - m + 1). \tag{12}$$

[1] The distribution which is used here is a generalization of an asymmetric triangular distribution (Simpson's Distribution)

On the basis of (11) and (12) we can obtain an expression for expectation of utility functions of players (1) — (2), considering them after the transition to the mixed extension of the game as functions of random variables $\tilde{x}_1, \tilde{x}_2$:

$$\mathbf{E}\{u_1(\tilde{x}_1, \tilde{x}_2)\} = \mathbf{E}\{b_1\tilde{x}_2 - a_1\tilde{x}_1\} = \frac{1}{3}[b_1(m_2 + 1) - a_1(m_1 + 1)], \tag{13}$$

$$\mathbf{E}\{u_2(\tilde{x}_1, \tilde{x}_2)\} = \mathbf{E}\{b_2\tilde{x}_1 - a_2\tilde{x}_2\} = \frac{1}{3}[b_2(m_1 + 1) - a_2(m_2 + 1)], \tag{14}$$

and their variances as well

$$\begin{aligned}\mathbf{D}\{u_1(\tilde{x}_1, \tilde{x}_2)\} = \mathbf{D}\{b_1\tilde{x}_2 - a_1\tilde{x}_1\} = b_1^2\mathbf{D}\tilde{x}_2 + a_1^2\mathbf{D}\tilde{x}_1 = \\ = \frac{b_1^2}{18}[m_2^2 - m_2 + 1] + \frac{a_1^2}{18}[m_1^2 - m_1 + 1],\end{aligned} \tag{15}$$

$$\begin{aligned}\mathbf{D}\{u_2(\tilde{x}_1, \tilde{x}_2)\} = \mathbf{D}\{b_2\tilde{x}_1 - a_2\tilde{x}_2\} = b_2^2\mathbf{D}\tilde{x}_1 + a_2^2\mathbf{D}\tilde{x}_2 = \\ = \frac{b_2^2}{18}[m_1^2 - m_1 + 1] + \frac{a_2^2}{18}[m_2^2 - m_2 + 1],\end{aligned} \tag{16}$$

Having (15) and (16) we derive that $\mathbf{D}\{u_1(\tilde{x}_1, \tilde{x}_2)\}$ and $\mathbf{D}\{u_2(\tilde{x}_1, \tilde{x}_2)\}$ are convex quadratic functions of parameters m_1 and m_2, and, consequently, they reach a global extremum at the point

$$(m_1^*, m_2^*) = (\frac{1}{2}, \frac{1}{2}), \tag{17}$$

which determines the state of equilibrium in the sense of (8) — (9) for a model of collaboration. Thus, a situation of mutual stability (in terms of minimization of risk criterion) in models constructed on basis of triangular distributions occurs when players choose their strategies relying on symmetric triangular distributions. This reflects the advantage of behavior based on the "golden mean" between extreme selfishness and willingness to maximize collaboration. If guided by the concept of conditional equilibrium (8) — (10), the global minimum point of variances $\mathbf{D}\{u_i(\tilde{x}_1, \tilde{x}_2)\}$ (17) may be outside of set of valid values m_1, m_2, defined by conditions

$$\begin{cases} -a_1m_1 + b_1m_2 \geqslant 3\bar{u}_1 - (b_1 - a_1) \\ \ \ b_2m_1 - a_2m_2 \geqslant 3\bar{u}_2 - (b_2 - a_2) \end{cases}$$

In this case, the procedure of finding conditional equilibrium in the sense of (8) — (10) reduces to solving a series of quadratic programming problems. Of course, the hypothesis, that values $\tilde{x}_i$ are distributed under the triangular law, cannot be regarded as an assumption which has non-alternative benefits. Other interesting and meaningful results for this model can also be obtained for the distributions of other classes. In particular, let us consider the model of collaboration, which is based on the assumption that the distribution of values $\tilde{x}_i$ is exponential[2] with parameters λ_i, i.e.

$$p_i(x) = \lambda_i e^{-\lambda_i x_i}$$

If we compare modifications of densities of triangular and exponential distributions, it is easy to see that the latter reflects the situation of initially low "propensity

[2] Obviously, in this case we assume the possibility of expanding the domain of x_i on the whole positive axle shaft, provided that the probability of $\tilde{x}_i > 1$ is close to zero.

to collaborate" of the economic agents more appropriately, see Fig. 3.One can also note that gamma distribution can be used for more flexible modeling of "propensity of players to collaborate" ratios.

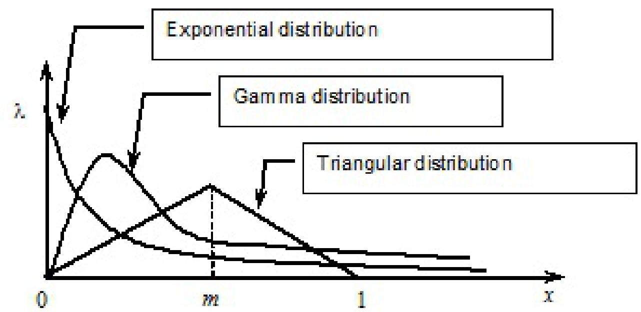

Fig. 3: Exponential, gamma and triangular distributions

If $\tilde{x}_i$ are exponentially distributed then the expected utilities of players will be

$$\mathbf{E}\{u_1(\tilde{x_1},\tilde{x_2})\} = \mathbf{E}\{-a_1\tilde{x_1} + b_1\tilde{x_2}\} = -\frac{a_1}{\lambda_1} + \frac{b_1}{\lambda_2}, \tag{18}$$

$$\mathbf{E}\{u_2(\tilde{x_1},\tilde{x_2})\} = \mathbf{E}\{b_2\tilde{x_1} - a_2\tilde{x_2}\} = \frac{b_2}{\lambda_1} - \frac{a_2}{\lambda_2} \tag{19}$$

In accordance with the formula for adding the variances of independent random variables, the dispersion of the utility can be expressed as

$$\mathbf{D}\{u_1(\tilde{x}_1,\tilde{x}_2)\} = \left(\frac{a_1}{\lambda_1}\right)^2 + \left(\frac{b_1}{\lambda_2}\right)^2 \tag{20}$$

$$\mathbf{D}\{u_2(\tilde{x}_1,\tilde{x}_2)\} = \left(\frac{b_2}{\lambda_1}\right)^2 + \left(\frac{a_2}{\lambda_2}\right)^2 \tag{21}$$

As appears from (21) — (22) functions $\mathbf{D}\{u_i(\tilde{x_1},\tilde{x_2})\}$ have obvious infimums equal to 0, when $\lambda_1 \to \infty, \lambda_2 \to \infty$. This means nothing more than a repetition of "pessimistic outcome", which has been obtained earlier: the variance for exponential distributions will be as smaller, as closer they are concentrated near $x_i = 0$, which corresponds to a situation of lack of collaboration.

The above considerations are valid if $\tilde{x}_i$ are gamma distributed with some parameters κ_i, λ_i .This follows directly from the form of expectation and variance for the corresponding random variables.

$$\mathbf{E}\tilde{x}_i = \frac{\kappa_i}{\lambda_i}, \, \mathbf{D}\tilde{x}_i = \frac{\kappa_i}{\lambda_i^2}$$

2. Equilibrium based on the criterion of minimizing VaR utility

Let us now consider a specific approach, where players make choice about appropriate degrees of collaboration taking distribution of their utility function utility into consideration, i.e.

$$F_{u_1(\tilde{x}_1,\tilde{x}_2)}(u) = \mathbf{P}\{u_1(\tilde{x}_1,\tilde{x}_2) \leqslant u\}$$

and

$$F_{u_2(\tilde{x}_1,\tilde{x}_2)}(u) = \mathbf{P}\{u_2(\tilde{x}_1,\tilde{x}_2) \leqslant u\}$$

A similar approach can be considered as an analogue of the concept of value at risk (VaR), which is widely used in modern risk management. The behavior of the utility distribution function of a player i is a function of independent random variables $\tilde{x}_1, \tilde{x}_2$ in this model. It is presented on Fig. 8. Taking into consideration (1) and (2) we can note that for $x_i \in [0,1]$ $u_i \in [-a_i, b_i]$, and consequently

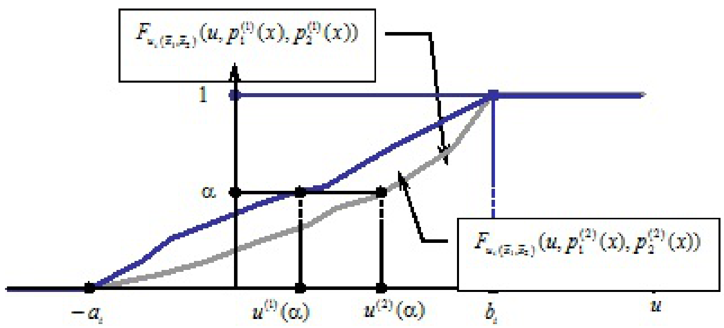

Fig. 4:

At the same time the choice of specific parameter values for $p_1(x)$ and $p_2(x)$ determines how function $F_{u_i(\tilde{x}_1,\tilde{x}_2)}(u)$ will increase on the interval $[-a_i, b_i]$. Thus, for the same level of probability quantiles of the distribution function

$$u(\alpha) = F^{-1}_{u_i(\tilde{x}_1,\tilde{x}_2)}(\alpha, p_1(x), p_2(x))$$

depends on the choice of parameters of probability distributions (densities $p_1(x)$ and $p_2(x)$) of random variables $\tilde{x}_1, \tilde{x}_2$. As we can see from Fig. 8,

$$u^{(1)}(\alpha) < u^{(2)}(\alpha),$$

i.e. α–quantile of the distribution function of the first player's utility , which we get from densities $p_1^{(1)}(x)$ and $p_2^{(1)}(x)$, is lower than the one corresponding densities $p_1^{(2)}(x)$ and $p_2^{(2)}(x)$. Thus, for the first player a strategic option defined by $p_1^{(2)}(x)$ and $p_2^{(2)}(x)$ is preferred, since its threshold below which his utility $u_i(\tilde{x}_1,\tilde{x}_2)$ wouldn't drop will be higher with probability $1-\alpha$.

In this particular approach, **the equilibrium in the model of collaboration can be defined as a set of probability distributions** $(p_1^*(x), p_2^*(x))$ **of some parametric classes** P_1 **and** P_2**, which define strategies of participants which satisfy the following conditions (with a given level of probability** α **and any other probability distributions** $(p_1(x) \in P_1$ **and** $(p_2(x) \in P_2)$

$$F^{-1}_{u_1(\tilde{x}_1,\tilde{x}_2)}(\alpha, p_1^*(x), p_2^*(x)) \geqslant F^{-1}_{u_1(\tilde{x}_1,\tilde{x}_2)}(\alpha, p_1(x), p_2^*(x)) \tag{22}$$

$$F^{-1}_{u_2(\tilde{x}_1,\tilde{x}_2)}(\alpha, p_1^*(x), p_2^*(x)) \geqslant F^{-1}_{u_2(\tilde{x}_1,\tilde{x}_2)}(\alpha, p_1^*(x), p_2(x)) \tag{23}$$

We will now pay a little bit more attention to usage of the following approach in a case when $p_i(x)$ determine random variables $\tilde{x}_i$, which are distributed under the asymmetric triangular law. As it has been already noted, the choice of the actual density $p_i(x)$ is uniquely connected to the choice of parameter m_i, which is a mode, and therefore we can consider players' utility distribution functions $u_i(\tilde{x}_1, \tilde{x}_2)$ as function of m_1 and m_2 using the notation $F_{u_i(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$. We should pay our attention to the fact that even with such a simple functional form of density in the case of an asymmetric triangular distribution, functions $F_{u_i(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$ do not have a "compact" analytic expression. "Method" of finding the value of $F_{u_1(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$ having a particular value for $\bar{u}_1$ is shown on Fig. 2.. There is evident from Fig.5, in order to find the value

$$F_{u_1(\tilde{x}_1,\tilde{x}_2)}(\bar{u}_1, m_1, m_2) = \mathbf{P}\{u_1(\tilde{x}_1, \tilde{x}_2) \leqslant \bar{u}_1 = -a_1x_1 + b_1x_2\}$$

we should calculate the sum $F_I + F_{II} + F_{III} + F_{IV}$, where

$$F_I = \int\limits_0^{m_1} \frac{2x_1}{m_1} \left[\int\limits_0^{\min\left\{\frac{\bar{u}_1 + a_1x_1}{b_1}; m_2\right\}} \frac{2x_2}{m_2} dx_2 \right] dx_1,$$

$$F_{II} = \int\limits_{m_1}^{1} \frac{2x_1 - 2}{m_1 - 1} \left[\int\limits_0^{m_2} \frac{2x_2}{m_2} dx_2 \right] dx_1,$$

$$F_{III} = \int\limits_{\frac{b_1m_2 - \bar{u}_1}{a_1}}^{m_1} \frac{2x_1}{m_1} \left[\int\limits_{m_2}^{\frac{\bar{u}_1 + a_1x_1}{b_1}} \frac{2x_2}{m_2 - 1} dx_2 \right] dx_1,$$

$$F_{IV} = \int\limits_{m_1}^{1} \frac{2x_1 - 1}{m_1 - 1} \left[\int\limits_{m_2}^{\frac{\bar{u}_1 + a_1x_1}{b_1}} \frac{2x_2 - 2}{m_2 - 1} dx_2 \right] dx_1,$$

Thus, in order to find the values of distribution functions $F_{u_i(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$ for arbitrary $u \in [-a_i, b_i]$ we will only have to consider all possible situations of geometry of line $u_i(x_1, x_2)$ and point (m_1, m_2).

In spite of "bad" analytical properties of functions $F_{u_i(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$ we are able to describe their behavior with appropriate accuracy by using numerical methods for specific a_i and b_i. In particular the results of numerical modeling of function $F_{u_1(\tilde{x}_1,\tilde{x}_2)}(u, m_1, m_2)$ with the help of MathCAD software tools for $a_1 = 1$, $b_1 = 2$, $m_2 = 0.8$ are shown on Fig. 2.. In other words, in a situation where the first player gives value to the actions of a second player twice as much as his own costs and the strategy of a second player is determined by the asymmetric triangular distribution with mode equal to 0.8. Fig. 2. depicts graphics of the first player's utility distribution function for cases when his strategy is determined by the asymmetric triangular distribution with mode $m_1^{(1)} = 0.2$ (line **FI_1**) and $m_1^{(2)} = 0.8$ (line **FI_2**).

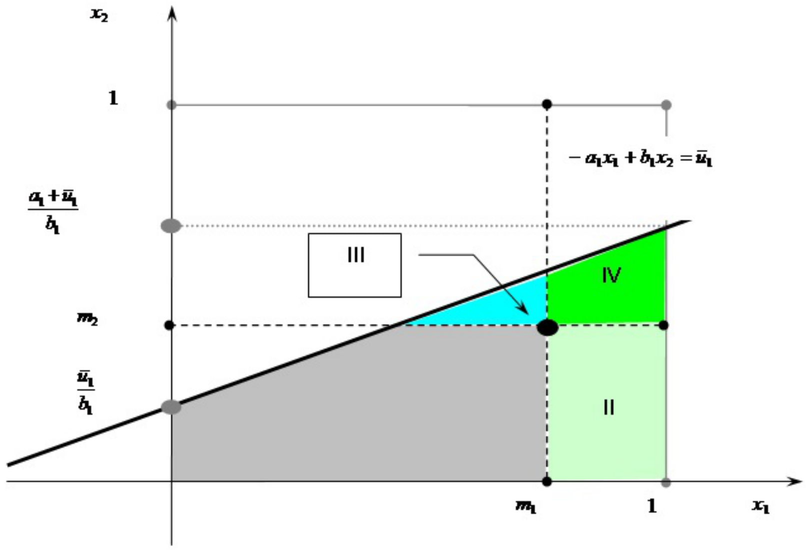

Fig. 5:

As follows from geometry of quantile lines for a level of probability α the quantiles found with respect to the distribution function **FI_2** will be less than quantiles found in respect to the distribution function **FI_1**. Thus in these conditions when the second player chooses the level of collaboration equal to $m_1^{(2)} = 0.8$, it is preferable for the first player to choose the higher level of collaboration $m_1^{(2)} = 0.8$, not $m_1^{(1)} = 0.2$, which would mean more egoistic type of behavior.

In particular - actual values.

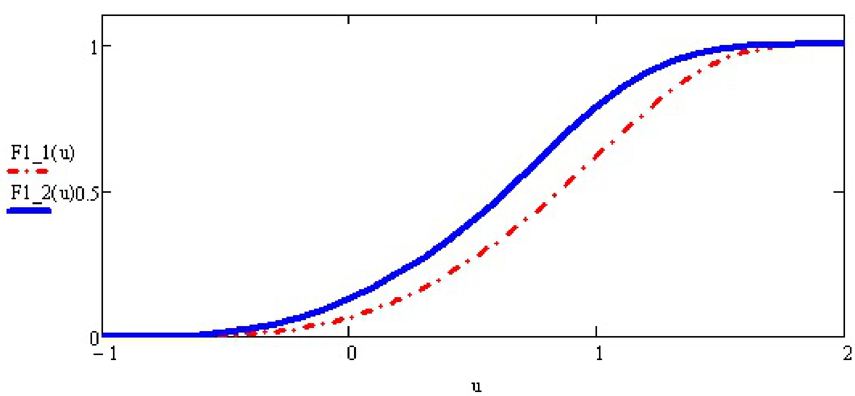

Fig. 6:

It should be admitted that from a mathematical point of view, we should modify the criterion function in respect to which the equilibrium conditions are determined in order to abandon the situation of a non-constructive equilibrium in the proposed model. Roughly speaking, if the players evaluate their results depending on the utility (or expected utility), then situation of non–collaboration becomes stable. At the same time, if they apply different criteria (variance, measure of risk, or VaR utility), the situation of collaboration is preferable.

This is what gives the approaches considered above an added significance in terms of economic meaning. In particular, applying them, we can form the principles of construction and maintenance of mechanisms for collaboration in situations that are initially characterized with selfish behavior of the parties.

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Differential Games with Random Terminal Instants

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Abstract We investigate a noncooperative differential game with two players. Each player has his own random terminal time. After the first player leaves the game, the remaining one continues and gets the final reward for winning. An example is introduced where two firms compete in extracting a unique nonrenewable resource over time. The optimal feedback strategy, i.e. the optimal extraction rate, is calculated in a closed form.

Keywords: Differential game, random terminal time, Hamilton-Jacobi-Bellman equation

1. Introduction

In the last decades many economic models have been investigated with the precious help of the tools provided by differential game theory (see Dockner et al., 2000, Jørgensen and Zaccour, 2007). Both deterministic and stochastic approaches have been widely developed in a wide range of different frameworks.

This paper aims to analyze a class of models of differential games with 2 players. In particular, we consider a framework where the terminal instants of the game are random variables having different cumulative distribution functions. The first player which stops the game is the loser, whereas the remaining player gets a terminal reward and keeps playing. In this case the game collapses into a optimal control problem.

We are going to fully characterize the structure of the game and to determine its dynamic equilibrium structure. Finally, we will feature an example which is a modification of the standard model of extraction (see Rubio, 2006), with linear state dynamics and a logarithmic payoff structure. It will be completely discussed and its optimal feedback solution will be exhibited.

2. Game Formulation

There are two players which participate in differential game $\Gamma(t_0, x_0)$. The game $\Gamma(x_0)$ with dynamics

$$\dot{x} = g(t, x, u_1, u_2), \quad x \in R^n,\, u_i \in U \subseteq \operatorname{comp} R^l, \tag{1}$$
$$x(t_0) = x_0$$

starts from initial state x_0 at the time instant t_0. But here we suppose that each player has a distinct terminal time. The payoff of the game is composed of two components: the integral payoff achieved while playing, and the final reward, assigned to the player which stays alive after the retirement of its rival;

Let T_1 and T_2 be the independent random variables denoting the respective terminal instants of the players, and assume that their c.d.f. $F_1(\cdot)$, $F_2(\cdot)$ and their p.d.f. $f_1(\cdot)$ and $f_2(\cdot)$ are known. Random variables aren't bounded from above, i.e. $T_k \in [t_0; +\infty)$,
$k = 1, 2$.

Suppose, that for all feasible controls of players, participating the game, there exists a continuous at least piecewise differentiable and extensible on $[t_0, \infty)$ solution of a Cauchy problem (1).

Denote the instantaneous payoff of player i at the time τ, $\tau \in [t_0, \infty)$ by $h_i(\tau, x(\tau), u_1, u_2)$, or briefly $h_i(\tau)$. Suppose, that for all feasible controls of players which participate the game, the instantaneous payoff function of each player is bounded, piecewise continuous function of time τ (piecewise continuity is treated as following: function $h_i(\tau)$ could have only finitely many point of discontinuity on each interval $[t_0, t]$ and bounded on this interval).

Thereby, the function $h_i(\tau)$ is Riemann integrable on every interval $[t_0, t]$, in other words for every $t \in [t_0, \infty)$ there exists an integral $\int\limits_{t_0}^{t} h_i(\tau)d\tau$.

So, we have that the expected integral payoff of the player i can be represented as the following mathematical expectation:

$$I_i(t_0, x_0, u_1, u_2) = \mathbb{E}\left[\int_{t_0}^{\min\{T_1, T_2\}} h_i(t)dt\right], \qquad (2)$$

where $\mathbb{E}[\cdot]$ is the mathematical expectation of a function of a random vector (T_1, T_2).

Moreover, we suppose that at the final (random) moment of the game, if player i is the only one remaining in the game, he receives the terminal payoff $\Phi_i(x(T))$, where $\Phi_i(x(T))$ are continuous functions on R^m. Then the expected terminal payoff of the player i can be evaluated as:

$$S_i(t_0, x_0, u_1, u_2) = \mathbb{E}\left[\Phi_i(x(T_j))\mathbb{I}_{[T_i > T_j]}\right], \qquad (3)$$

where $\mathbb{I}_{[\cdot]}$ is the indicator function and $\mathbb{E}[\cdot]$ is the mathematical expectation of a function of a random vector (T_1, T_2).

Then the total expected payoff of the player i is:

$$K_i(t_0, x_0, u_1, u_2) = \mathbb{E}\left[\int_{t_0}^{\min\{T_1, T_2\}} h_i(t)dt + \Phi_i(x(T))\mathbb{I}_{[T_i > T_j]}\right]. \qquad (4)$$

3. Transformation of expected payoff

The total expected payoff (4) is difficult to use in order to find solutions of the game. The standard methods of solution, such as Pontryagin's maximum principle or finding the solution of Hamilton-Jacobi-Bellman equatoin, can not be applied. We need to transform the payoff (4) into the standard integral functional for infinite-horizon differential games.

3.1. Expected integral payoff

At first consider the expected integral payoff (2). We could rewrite it in the following form by the definition of mathematical expectation

$$I_i(t_0,x_0,u_1,u_2) = \int\int_{t_0}^{\min\{\tau_1,\tau_2\}} h_i(t)dt\,dF_{T_1,T_2}(\tau_1,\tau_2), \tag{5}$$

where $F_{T_1,T_2}(\tau_1,\tau_2)$ is the cumulative distribution function of the random vector (T_1,T_2).

Consider the following function of the random vector:

$$T = \min\{T_1,T_2\}.$$

Since the function min{} is a measurable function, then T is a random variable (Borovkov, 1999). Denote by $F(t)$ the cumulative distribution function of the random variable T. Using the cumulative distribution functions of the random variables T_1, T_2, we can write the expression for $F(t)$ in an explicit form (Kostyunin et al., 2011)

$$F(t) = 1-(1-F_1(t))(1-F_2(t))$$

Mathematical expectation (5) could be represented in the equivalent form (Borovkov, 1999)

$$I_i(t_0,x_0,u_1,u_2) = \int\int_{t_0}^{\tau} h_i(t)dt\,dF(\tau).$$

Thus, we could consider the expected integral payoff (5) as the mathematical expectation of a function of a random variable T:

$$I_i(t_0,x_0,u_1,u_2) = \mathbb{E}\left[\int_{t_0}^{T} h_i(t)dt\right], \tag{6}$$

where $\mathbb{E}[\cdot]$ is the mathematical expectation of a function of a random variable T.

If the instantaneous payoff function is nonnegative$h_i(\tau,x,u_1,u_2), \forall\tau,x,u_1,u_2$ then the following equality holds (Kostyunin and Shevkoplyas, 2011)

$$I_i(t_0,x_0,u_1,u_2) = \int\limits_{t_0}^{\infty} h_i(\tau)(1-F(\tau))d\tau. \tag{7}$$

If the instantaneous payoff function does not satisfy the condition of nonnegativity, (7) holds if the following condition is satisfied (Kostyunin and Shevkoplyas, 2011)

$$\lim_{T\to\infty}(F(T)-1)\int\limits_{t_0}^{T} h_i(t)dt = 0. \tag{8}$$

Note, that for a nonnegative instantaneous payoff function $h_i(t)$ the existence of the integral in the right-hand side of (7) implies that (8) holds.

3.2. Expected terminal payoff

Consider the expected terminal payoff (3)

$$S_i(t_0, x_0, u_1, u_2) = \mathbb{E}\left[\Phi_i(x(T_j))\mathbb{I}_{[T_i > T_j]}\right],$$

The expectation in (3) could be expressed as the following Lebesgue-Stieltjes integral:

$$\mathbb{E}\left[\Phi_i(x(T_j))\mathbb{I}_{[T_i > T_j]}\right] = \int \Phi_i(x(t_j))\mathbb{I}_{[t_i > t_j]} dF_{T_1,T_2}(t_1, t_2). \tag{9}$$

Suppose that the function $\Phi_i(x)$ satisfies the condition of nonnegativity. In this case we can use the following theorem on iterated integrals (Borovkov, 1999)

Theorem 1 (Theorem on iterated integrals). *For a Borel function $g(x, y) \geq 0$, and independent random variables ξ_1 è ξ_2:*

$$\int g(x_1, x_2) dF_{\xi_1 \xi_2}(x_1, x_2) = \int \left[\int g(x_1, x_2) dF_{\xi_2}(x_2)\right] dF_{\xi_1}(x_1).$$

Using this theorem, we obtain the following expression for (9)

$$\int_{t_0}^{+\infty} \left[\int_{t_0}^{+\infty} \Phi_i(x(t_j))\mathbb{I}_{[t_i > t_j]} dF_i(t_i)\right] dF_j(t_j).$$

Then we obtain

$$\int_{t_0}^{+\infty} \left[\int_{t_0}^{t_j} \Phi_i(x(t_j))\mathbb{I}_{[t_i > t_j]} dF_i(t_i) + \int_{t_j}^{+\infty} \Phi_i(x(t_j))\mathbb{I}_{[t_i > t_j]} dF_i(t_i)\right] dF_j(t_j).$$

The first term under the integral equals to zero. Further, we find

$$\int_{t_0}^{+\infty} \left[\int_{t_j}^{+\infty} \Phi_i(x(t_j)) dF_i(t_i)\right] dF_j(t_j) =$$
$$\int_{t_0}^{+\infty} \left[\Phi_i(x(t_j)) \int_{t_j}^{+\infty} f_i(t_i) dt_i\right] f_j(t_j) dt_j.$$

Finally, we obtain an expression for the expectation in (3)

$$\mathbb{E}\left[\Phi_i(x(T_j))\mathbb{I}_{[T_i > T_j]}\right] = \int_{t_0}^{+\infty} \Phi_i(x(t_j))(1 - F_i(t_j)) f_j(t_j) dt_j. \tag{10}$$

Then, the sufficient condition for total payoff transformation is given by the following propositions.

Proposition 1. *If the instantaneous payoff function and the terminal payment function are nonnegative*

$$h_i(\tau, x(\tau), u_1, u_2) \geq 0,\ \Phi_i(x(t)) \geq 0,$$

then the total expected payoff of player i (4) could be written as

$$K_i(t_0, x_0, u_1, u_2) = \int_{t_0}^{\infty} \left[h_i(\tau)\left(1 - F(\tau)\right) + \Phi_i(x(\tau)) f_j(\tau)(1 - F_i(\tau))\right] d\tau. \tag{11}$$

Proposition 2. *If the terminal payment function is nonnegative*

$$\Phi_i(x(t)) \geq 0,$$

and the following condition is satisfied

$$\lim_{T\to\infty} (F(T)-1) \int\limits_{t_0}^{T} h_i(t)dt = 0,$$

then the total expected payoff of player i (4) could be written as (11).

4. Hamilton-Jacobi-Bellman equation

Let the game $\Gamma(t_0, x_0)$ develops along the trajectory $x(t)$. Then at the each time instant t, $t \in (t_0; \infty)$ players enter a new game (subgame) $\Gamma(t, x(t))$ with initial state $x(t) = x$.

The expected payoff for player i in this subgame is given by the following equation (Kostyunin et al., 2011)

$$K_i(t, x.u_1, u_2) =$$
$$\frac{1}{(1-F_1(t))(1-F_2(t))} \int\limits_{t}^{+\infty} \left[h_i^*(\tau)\,(1-F(\tau)) + \Phi_i(x^*(\tau)) f_j(\tau)(1-F_i(\tau))\right] d\tau.$$

We denote by $W_i(t, x)$ the i-th optimal value function of the problem starting at $t \in (0, +\infty)$, with initial data $x(t) = x$. The Hamilton-Jacobi-Bellman equation has the same form as in the case where the terminal instants of the players are bounded from above (Kostyunin et al., 2011)

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\frac{f_1(t)}{1-F_1(t)} + \frac{f_2(t)}{1-F_2(t)}\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\frac{f_j(t)}{1-F_j(t)} + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \qquad (12)$$

4.1. Hamilton-Jacobi-Bellman equation and hazard function

Let us remark that the term $\frac{f(\vartheta)}{1-F(\vartheta)}$ in the left-hand side of equation (12) is a well-known function in mathematical reliability theory. It has a name of Hazard function (or failure rate) with typical notation $\lambda(\vartheta)$

$$\lambda(t) = \frac{f(t)}{1-F(t)}. \qquad (13)$$

Using the definition of the Hazard function (13), we get the following form for new Hamilton-Jacobi-Bellman equation (12):

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_1(t) + \lambda_2(t)\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\lambda_j(t) + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \qquad (14)$$

4.2. Exponential distribution case

For exponential distribution of terminal instants $F(t) = 1 - e^{-\lambda t}$, the Hazard function is constant: $\lambda(t) = \lambda$. So, inserting λ_i instead of $\lambda_i(t)$ into (12), we easily get the Hamilton-Jacobi-Bellman equation for player i

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_1 + \lambda_2\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\lambda_j + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \tag{15}$$

5. An example

Consider the following framework, borrowed from (Rubio, 2006) (Example 2.1) and (Dockner et al., 2000) (Example 5.7) and modified with the above discount factor. This example originally describes the joint exploitation of a pesticide, but its structure makes it suitable for our aim. Note that, in contrast to (Rubio, 2006), we confine our attention to the Nash equilibrium under simultaneous play, and we consider the non-stationary feedback case, that is our optimal value function explicitly depends on the initial instant t.

We fix $m = 1$, i.e., a unique state variable $x(t)$, denoting the amount of the resource, whereas the i-th payoff function explicitly depends on the rate of extraction of the i-th player but not on the state variable:

$$h_i(x(t), u_i(t)) = \ln u_i(t),$$

whereas the terminal payoff is given by

$$\Phi_i(x^*(T)) = c_i \ln(x(T_i)).$$

Note that $h_i(\cdot)$ is well-defined and concave for $u_i > 0$.

The transition function is linear and decreasing in the controls, so the dynamic constraint is:

$$\begin{cases} \dot{x} = -u_1 - u_2 \\ x(0) = x_0 > 0 \end{cases}.$$

The kinematic equation ensures that the terminal payoff is well-defined in that the resource cannot equal 0 in finite time.

Using the data of the above model, we obtain:

$$W_i(0,x_0) = \mathbb{E}\left[\int_0^{T_i} \ln u_i^* dt I_{[T_i<T_j]} + \int_0^{T_j} \ln u_i^* dt I_{[T_i>T_j]} + c_i \ln x(T_j)\mathbb{I}_{[T_i>T_j]}\right].$$

The i-th optimal value function of the problem starting at $t \in (0,\omega)$, and with initial condition $x(t) = x$, is given by:

$$W_i(t,x) =$$
$$\frac{1}{(1-F_i(t))(1-F_j(t))}\int_t^{\omega} \left[\ln u_i^*(\tau, x(\tau))\left(1 - F(\tau)\right) + c_i \ln x(\tau) f_j(\tau)(1 - F_i(\tau))\right] d\tau. \tag{16}$$

In compliance with the previous Section, the Hamilton-Jacobi-Bellman equations are given by:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_i(t) + \lambda_j(t)\right] = \\ \max_{u_i}\left[\ln(u_i) + c_i \ln x(t)\lambda_j(t) - \frac{\partial W_i(t,x)}{\partial x}(u_i + u_j^*)\right]. \tag{17}$$

In order to explicitly determine the optimal strategy in the feedback Nash structure, we guess the following ansatz for the solution to (17):

$$W_i(t,x) = A_i(t)\ln x + B_i(t),$$

where $A_i(t)$ and $B_i(t)$ are unknown functions of t, such that the following limits are satisfied:

$$\lim_{t\to\omega} A_i(t) = 0, \qquad \lim_{t\to\omega} B_i(t) = 0. \tag{18}$$

The relevant first order partial derivatives to be employed in (17) are:

$$\frac{\partial W_i(t,x)}{\partial t} = \dot{A}_i(t)\ln x + \dot{B}_i(t), \qquad \frac{\partial W_i(t,x)}{\partial x} = \frac{A_i(t)}{x}.$$

Maximizing the r.h.s. of (17) yields:

$$\frac{1}{u_i^*} - \frac{\partial W_i(t,x)}{\partial x} = 0 \iff u_i^* = \frac{x}{A_i(t)}.$$

Hence, plugging u_i^*, $\dfrac{\partial W_i(t,x)}{\partial t}$ and $\dfrac{\partial W_i(t,x)}{\partial x}$ into (17), we obtain the following equation:

$$-\dot{A}_i(t)\ln x - \dot{B}_i(t) + (A_i(t)\ln x + B_i(t))\left[\lambda_i(t) + \lambda_j(t)\right] = \\ \ln\frac{x}{A_i(t)} + c_i \ln x \lambda_j(t) - \frac{A_i(t)}{x}\left(\frac{x}{A_i(t)} + \frac{x}{A_j(t)}\right). \tag{19}$$

After collecting terms with and without $\ln x$, we determine the following ODEs for the time-dependent coefficients of $W_i(t,x)$:

$$-\dot{A}_i(t) + A_i(t)\left[\lambda_i(t) + \lambda_j(t)\right] - 1 - c_i\lambda_j(t) = 0, \tag{20}$$

$$-\dot{B}_i(t) + B_i(t)\left[\lambda_i(t) + \lambda_j(t)\right] + \ln A_i(t) + 1 + \frac{A_i(t)}{A_j(t)} = 0, \tag{21}$$

composing a Cauchy problem endowed with the transversality conditions:

$$\lim_{t\longrightarrow\omega} A_i(t) = 0, \qquad \lim_{t\longrightarrow\omega} B_i(t) = 0. \tag{22}$$

Proposition 3. *The optimal feedback strategy for the i-th firm is given by:*

$$u_i^*(t,x) = \frac{x}{\int_t^\omega (1 + c_i\lambda_j(\tau))e^{-\int_t^\tau (\lambda_i(\theta)+\lambda_j(\theta))d\theta} d\tau}. \tag{23}$$

Proof. We just consider the Cauchy problem in $A_i(t)$, because the explicit calculation of $B_i(t)$ can be avoided in that $B_i(t)$ does not appear in the expression of u_i^*:

$$\begin{cases} \dot{A}_i(t) = A_i(t)\left[\lambda_i(t) + \lambda_j(t)\right] - 1 - c_i\lambda_j(t) \\ \lim\limits_{t\to\omega} A_i(t) = 0 \end{cases},$$

whose general solution is given by:

$$A_i(t) = e^{\int_0^t(\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left(C - \int_0^t (1 + c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds}d\tau\right), \qquad (24)$$

where the constant C is determined by employing the transversality condition on $A_i(t)$:

$$C = \int_0^\omega (1 + c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds}d\tau,$$

leading to the solution:

$$A_i^*(t) = e^{\int_0^t(\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left[\int_t^\omega (1 + c_i\lambda_j(\tau))e^{-\int_0^\tau(\lambda_i(s)+\lambda_j(s))ds}d\tau\right]. \qquad (25)$$

We can simplify:

$$A_i^*(t) = \int_t^\omega (1 + c_i\lambda_j(\tau))e^{-\int_t^\tau(\lambda_i(s)+\lambda_j(s))ds}d\tau. \qquad (26)$$

Finally, the expression of the optimal feedback strategy for the i-th firm can be achieved from the FOCs of the model:

$$u_i^*(t,x) = \frac{x}{A_i^*(t)} = \frac{x}{\int_t^\omega(1 + c_i\lambda_j(\tau))e^{-\int_t^\tau(\lambda_i(\theta)+\lambda_j(\theta))d\theta}d\tau}. \qquad (27)$$

As a further application, we can consider the circumstance where the two distributions of the firms are the standard exponential distributions, i.e.

$$f_i(t;\ \lambda_i) = \begin{cases} \lambda_i e^{-\lambda_i t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases},$$

whose means are respectively λ_1^{-1}, λ_2^{-1}, both positive, with $\lambda_1 \neq \lambda_2$, ensuring asymmetry.

In this case the hazard functions are constant, i.e. $\lambda_1(t) \equiv \lambda_1$ and $\lambda_2 \equiv \lambda_2$, then substituting in (23) we obtain the two optimal feedback strategies:

$$u_1^*(t,x) = \frac{(\lambda_1 + \lambda_2)x}{(1 + c_1\lambda_2)[1 - e^{-(\lambda_1+\lambda_2)(\omega - t)}]}, \qquad (28)$$

$$u_2^*(t,x) = \frac{(\lambda_1 + \lambda_2)x}{(1 + c_2\lambda_1)[1 - e^{-(\lambda_1+\lambda_2)(\omega - t)}]}. \qquad (29)$$

6. Concluding remarks

This paper intends to be a contribution to the literature of differential games in an area which can be defined as deterministic, but enriched with some stochastic elements. In particular, it is focused on the feature of extraction games that is definitely realistic: the uncertainty about the terminal times of an extracting activity.

The dynamic feedback equilibrium structure has been determined and the specific technicalities of this setting have been pointed out. As an example, a model of nonrenewable resource extraction with a logarithmic utility structure was examined and solved in a closed form.

There exist some possible further extensions, also concerning the example we developed. It would be interesting to check the specific optimal strategies in presence of more complex hazard functions (for example, the Weibull distribution) or endowed with alternative payoff structures. Another interesting development might consist in considering a competition among more than 2 firms, having different terminal times.

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Coalitional Solution in a Game Theoretic Model of Territorial Environmental Production

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Abstract A game-theoretic model of territorial environmental production under Cournot competition is studied. The process is modeled as cooperative differential game with coalitional structure. The Nash equilibrium in the game played by coalitions is computed and then the value of each coalition is allocated according to some given mechanism between its members. The numerical example is given.

Keywords: optimal control, nonlinear system, dynamic programming.

1. Introduction

A game-theoretic model of territorial environmental production is considered. The model is based on the research of Petrosyan and Zaccour, 2003. In the paper of Petrosyan and Zaccour, 2003 the international environmental agreement is modeled, which provides a time-consistent allocation of total costs for all players under which the pollution is reduced.

The model of territorial environmental production is an extension of above mentioned model (Petrosyan and Zaccour, 2003). The region market is considered, where all firms produce homogeneous product under Cournot competition. The production process damages to the environment. Emission of each player is proportional to its output. Any firm has three types of costs: production costs, abatement costs and damage costs.

We consider the voluntary approach to environmental regulation, which became popular in a series of countries. The cooperation of firms leads to increase their profits and decrease of pollution, but the price of product is increased.

The approach of this paper is different. The more general coalitional setting is considered, when not only the grand coalition, but also a coalitional partition of players can be formed. This kind of approach was considered before in . Coalitional values for static games have been studied in a series of papers (Bloch, 1966, Owen, 1997). In a recent contribution, Owen, 1997 proposed a characterization of the Owen value for static games under transferable utility. Owen, 1997 defined the coalitional value for static simultaneous games with transferable payoffs by generalizing the Shapley value to a coalitional framework. In particular, the coalitional value was defined by applying the Shapley value first to the coalition partition and then to cooperative games played inside the resulting coalitions. This approach assumed that coalitions in the first level can cooperate (as players) and form the grand coalition. The game played with coalition partitioning becomes cooperative one with specially defined characteristic function: The Shapley value computed for this characteristic function is then the Shapley-Owen value for the game.

The present paper emerges from idea that it is more natural not to assume that coalitions on the first level can form a grand coalition. At first step the Nash equilibrium in the game played by coalitions is computed. Secondly, the value of each coalition is allocated according to the Shapley value in the form of PMS-vector, that was derived in the paper of Petrosyan and Mamkina, 2006 . The approach was considered earlier in Kozlovskaya et al., 2010. The main result of this paper is the calculation of this solution (PMS-vector). The main result of the paper is construction the dynamic PMS-value in the model of territorial environmental production.

2. Problem Statement

Consider a region market with n firms which produce for simplicity the same product. Let I be the set of firms involved in the game: $I = \{1, 2, \dots, n\}$.

Denote by $q_i = q_i(t)$ the output of firm i at the instant of time t. The price of the product $p = p(t)$ is defined as follows

$$p(t) = a - bQ(t)\ , \tag{1}$$

where $a > 0,\ b > 0$, $Q(t) = \sum\limits_{i=1}^{n} q_i(t)$ – the total output. The price function $p(t)$ is inverse demand function:

$$Q = Q(t) = \frac{a - p(t)}{b}\ .$$

The production cost of any firm equals

$$C_i(q_i(t)) = cq_i(t), \quad c > 0,\ i \in I.$$

The game $\Gamma(s_0, t_0)$ starts at the instant of time t_0 from the initial state s_0, where $s_0 = s(t_0)$ is the stock of pollution at time t_0. Let us denote by $e_i(t)$ the emission of firm i at time t. The emission of firms are linear subject to output:

$$e_i(q_i(t)) = \alpha q_i(t), \quad \alpha > 0. \tag{2}$$

Denote by $\bar{e}_i$ maximum permissible emission for firm i:

$$0 \le e_i(q_i(t)) \le \bar{e}_i. \tag{3}$$

We get from (3) that maximal permissible output of firm i is equal to

$$q_i^{max} = \frac{\bar{e}_i}{\alpha},$$

then maximal permissible total output equals

$$Q^{max} = \frac{\bar{e}}{\alpha},$$

where $\bar{e} = \sum\limits_{i=1}^{n} \bar{e}_i$. Suppose the parameters of model are such that the following inequality is true

$$a - c - \frac{b}{\alpha}\bar{e} \ge 0,$$

which guarantees the nonnegativity of price (1).

Denote by $s = s(t)$ the total stock of accumulated pollution by time t. The dynamics of pollution accumulation is defined by the following differential equation:

$$\dot{s}(t) = \alpha \sum_{k=1}^{n} q_i(t) - \delta s(t),$$
$$s(t_0) = s_0, \tag{4}$$

where δ is the rate of pollution absorption, $\alpha > 0$ is a known parameter. Any firm has two types of costs, which are not directly connected with the production process: abatement costs and damage costs. The abatement costs at moment of time t equals

$$E_i(q_i(t)) = \frac{\gamma}{2} e_i(t)(2\bar{e}_i - e_i(t)) = \frac{\gamma}{2} \alpha q_i(2\bar{e}_i - \alpha q_i),$$
$$\gamma > 0, \qquad 0 \le e_i(t) \le \bar{e}_i.$$

The cost function $E_i(q_i)$ increases and reaches the maximum at $q_i = q_i^{max}$. The function $E_i(q_i)$ is concave. Damage costs depends on the stock of pollution:

$$D_i(s(t)) = \pi_i s(t), \qquad \pi_i > 0, \qquad i \in I.$$

The firm i tries to maximize the profit

$$\Pi_i(s_0, t_0; q) = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\} dt, \tag{5}$$

where $q = q(t) = (q_1(t), q_2(t), \dots, q_n(t)),\ t \ge t_0$ is trajectory of production output, $0 < \rho < 1$ is a discount rate, p is defined by (1).

3. Coalitional Solution

Let $\Delta = (S_1, S_2, \dots, S_m)$ be the partition of the set I, such that $S_i \cap S_j = \emptyset$, $\bigcup_{i=1}^{m} S_i = I,\ |S_i| = n_i,\ \sum_{i=1}^{m} n_i = n.$

Denote by M the set $M = \{1, 2, \dots, , m\}$.

Suppose that each firm i from I is playing in interests of coalition S_k, to which it belongs, trying to maximize the sum of payoffs of its members, i.e.

$$\max_{q_j \in S_k} \sum_{j \in S_k} \Pi_j(s_0; q) =$$
$$= \max_{q_j \in S_k} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \sum_{j \in S_k} \{pq_j - C_j(q_j) - D_j(s) - E_j(q_j)\} dt, \tag{6}$$

where $q = q(t) = (q_1(t), q_2(t), \dots, q_n(t)),\ t \ge t_0$ – trajectory of production output, $0 < \rho < 1$ – discount rate.

Without loss of generality it can be assumed that coalitions S_k are acting as players. Then at first stage the Nash equilibrium is computed. The total cost of coalition S_k is allocated among the players according to Shapley value of corresponding subgame $\Gamma(S_k)$. The game $\Gamma(S_k)$ is defined as follows: let S_k be the set of players involved in the game $\Gamma(S_k)$, $\Gamma(S_k)$ is a cooperative game.

Definition 1. The vector

$$PMS(x,t) = [PMS_1(x,t), PMS_2(x,t), \dots, PMS_n(x,t)],$$

is a PMS-vector, where $PMS_i(x,t) = Sh_i(S_k,x,t)$, if $i \in S_k$, where

$$Sh_i(S_k,x,t) = \sum_{M \supset i, M \subset S_k} \frac{(n_k-m)!(m-1)!}{n_k!}[V(M,x,t) - V(M \backslash \{i\}, x, t)]$$

and $(S_1, S_2, \dots, S_m)$ is the partition of the set I.

3.1. The Construction of Coalitional Solution

Step 1. Computation of the Nash equilibrium in the game of coalitions $S_k,\ k \in M$.
Each firm i from I is playing in interests of coalition S_k, to which it belongs, trying to maximize the sum of payoffs of its members (6).
The Nash equilibrium in the game of coalitions is computed by the solution of the following system:

$$\begin{gathered}\max_{q_j \in S_k} \sum_{j \in S_k} \mathit{\Pi}_j(s_0;q) = \max_{q_j \in S_k} \int\limits_{t_0}^{\infty} e^{-\rho(t-t_0)} \sum_{j \in S_k} \{pq_j - \\ -C_j(q_j) - D_j(s) - E_j(q_j)\}dt \quad k \in M,\end{gathered} \tag{7}$$

subject to equation dynamics (4).

Step 2. Computation of the characteristic function and the Shapley value in the game $\mathit{\Gamma}_V^{S_k}(s_0),\ k = 1,2,\dots,m$. Computation of the characteristic function isn't standard (Petrosyan and Zaccour, 2003): when the characteristic function is calculated for K, the left-out players stick to their Nash strategies

Step 3. Construction of the PMS-vector.
Payoffs of all players $i \in I$ forms a PMS-vector (Petrosyan and Mamkina, 2006).
$PMS(s_0) = (PMS_1(s_0), PMS_2(s_0), \dots, PMS_n(s_0)),\ PMS_i(s_0) = Sh_i^{S_k}(s_0)$, where $Sh^{S_k}(s_0)$ is the Shapley value in the game $\mathit{\Gamma}_V^{S_k}(s_0)$

The Nash equilibrium is calculated with the help of Hamilton-Jacobi-Bellman equation (Dockner et al., 2000). The total cost of coalition S_k is allocated among the players according to Shapley value of corresponding subgame $\mathit{\Gamma}(S_k)$. The game $\mathit{\Gamma}(S_k)$ is defined as follows: let S_k be the set of players involved in the game $\mathit{\Gamma}(S_k)$, $\mathit{\Gamma}(S_k)$ is a cooperative game.

Computation of the characteristic function of this game isn't standard. When the characteristic function is computed for the coalition $K \in S_k$, the left-out players stick to their Nash strategies. Payoffs of all players $i \in I$ forms a PMS-vector (Petrosyan and Mamkina, 2006).

3.2. The Nash Equilibrium in the Game of Coalitions

The solution of the system (7) is equivalent to the solution if the system of Hamilton-Jacobi-Belman equations

$$\begin{gathered}\rho W_{S_k} = \max_{q_j, j \in S_k} \{ \sum_{j \in S_k} (q_j(a - bQ) - cq_j - \pi_j s + \frac{\gamma\alpha}{2} q_j(\alpha q_j - 2\bar{e}_j)) + \\ + \frac{\partial W_{S_k}}{\partial s}(\alpha Q - \delta s)\}, \quad k \in M,\end{gathered} \tag{8}$$

where $Q = \sum\limits_{j \in I} q_j$, W_{s_k} is the Bellman function subject to equation od dynamics (4). By the first Step to find the Nash equilibrium, consider the system (8).

Differentiating with respect to $q_i,\ i \in S_k$ the right hand side of the equation (8) leads to

$$a - bQ - b\sum_{j \in S_k} q_j - c + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_{S_k}}{\partial s} = 0, \quad i \in I,\ k \in M. \tag{9}$$

Let us denote $Q_{S_k} = \sum\limits_{j \in S_k} q_j$.Then $Q = \sum\limits_{j=1}^{m} Q_{S_j}$, the system (9) is obtained in the following form

$$a - b\sum_{j=1}^{m} Q_{S_j} - bQ_{S_k} - c + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_{S_k}}{\partial s} = 0, \quad i \in I,\ k \in M. \tag{10}$$

Summing equations (10) with respect to S_k gives

$$n_k(a - c - b\sum_{j=1}^{m} Q_{S_j}) - n_k bQ_{S_k} + \gamma\alpha^2 Q_{S_k} - \gamma\alpha\bar{e}^{S_k} + \alpha n_k\frac{\partial W_{S_k}}{\partial s} = 0, \quad k \in M, \tag{11}$$

where $\bar{e}^{S_k} = \sum\limits_{j \in S_k} \bar{e}_j$. Solving (11) subject to Q_{S_k}, find

$$Q_{S_k} = \frac{n_k(a - c - bQ) - \gamma\alpha\bar{e}^{S_k} + \alpha n_k\frac{\partial W_{S_k}}{\partial s}}{bn_k - \alpha^2\gamma}, \quad k \in M. \tag{12}$$

Summing (12) with respect to the set M leads to

$$Q = \sum_{j=1}^{m} \frac{n_j(a - c - bQ) - \gamma\alpha\bar{e}^{S_j} + \alpha n_j\frac{\partial W_{S_j}}{\partial s}}{bn_j - \alpha^2\gamma} =$$
$$= \sum_{j=1}^{m} \frac{n_j(a - c + \alpha\frac{\partial W_{S_j}}{\partial s}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma} - Q\sum_{j=1}^{m}\frac{bn_j}{bn_j - \alpha^2\gamma},$$

then one can find:

$$Q = \frac{\sum\limits_{j=1}^{m} \dfrac{n_j(a - c + \alpha\frac{\partial W_{S_j}}{\partial s}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma}}{1 + \sum\limits_{j=1}^{m}\dfrac{bn_j}{bn_j - \alpha^2\gamma}}. \tag{13}$$

Substituting (13) in (12) gives the formula for Q_{S_k}. Then solving (9) leads to:

$$q_i = \frac{\bar{e}_i}{\alpha} - \frac{1}{\alpha^2\gamma}(a - c - bQ - bQ_{S_k} + \alpha\frac{\partial W_{S_k}}{\partial s}), \quad i \in S_k \tag{14}$$

It can be shown by the usual way that the Bellman function

$$W_{S_k} = A_{S_k} x + B_{S_k}, \quad k = 1, 2, \dots, m, \tag{15}$$

satisfies the Hamilton-Jacobi-Bellman equation (8) [13]. One can notice that

$$\frac{\partial W_{S_k}}{\partial x} = A_{S_k}. \tag{16}$$

Substituting (15) and (16) in formula (14) gives:

$$\hat{q}_i = \frac{\bar{e}_i}{\alpha} - \frac{1}{\alpha^2 \gamma}(a - c - b\hat{Q} - b\hat{Q}_{S_k} + \alpha A_{S_k}), \quad i \in S_k,$$

where

$$\hat{Q}_{S_k} = \frac{n_k(a - c - bQ) - \gamma\alpha\bar{e}^{S_k} + \alpha n_k A_{S_k}}{bn_k - \alpha^2\gamma}, \quad k \in M,$$

and

$$\hat{Q} = \frac{\sum\limits_{j=1}^{m} \dfrac{n_j(a - c + \alpha A_{S_j}) - \gamma\alpha\bar{e}^{S_j}}{bn_j - \alpha^2\gamma}}{1 + \sum\limits_{j=1}^{m} \dfrac{bn_j}{bn_j - \alpha^2\gamma}},$$

it means that

$$q_i^n = \begin{cases} \hat{q}_i, & \hat{q}_i \in [0, \frac{\bar{e}_i}{\alpha}] \\ \frac{\bar{e}_i}{\alpha}, & \hat{q}_i > \frac{\bar{e}_i}{\alpha} \\ 0, & \hat{q}_i < 0 \end{cases} \qquad i \in I. \tag{17}$$

Substituting (15) and (16) into the formula (8) leads to:

$$\begin{gathered} \rho A_{S_k} s + \rho B_{S_k} = Q_{S_k}^n (a - c - Q^n) - \sum_{j \in S_k} \pi_j s + \\ + \frac{\gamma\alpha}{2} \sum_{j \in S_k} q_j^n (\alpha q_j^n - 2\bar{e}_j) + A_{S_k}(\alpha Q^n - \delta s), \quad k \in M. \end{gathered} \tag{18}$$

From (18), we get the coefficients A_{S_k} and B_{S_k}:

$$\begin{gathered} A_{S_k} = -\frac{\sum\limits_{j \in S_k} \pi_j}{\rho + \delta} \\ B_{S_k} = \frac{1}{\rho}\Big(Q_{S_k}^n (a - c - bQ^n) + A_{S_k} Q^n + \frac{\gamma\alpha}{2} \sum_{j \in S_k} q_j^n (\alpha q_j^n - 2\bar{e}_j)\Big), \end{gathered} \tag{19}$$

where q_i^n is defined by the formula (17), and

$$Q_{S_k}^n = \sum_{j \in S_k} q_j^n, \quad k \in M, \tag{20}$$

$$Q^n = \sum_{j \in M} Q_{S_j}^n, \tag{21}$$

3.3. Computation of the Characteristic function

Computation of the characteristic function of this game is not standard. When the characteristic function is computed for the coalition $K \subset I$, we suppose that the left-out players have used their Nash equilibrium strategies. The advantage of this approach is the following: such characteristic function is easier to compute. This approach requires to solve only one equilibrium problem, all others being standard dynamic optimization problems, while standard approach requires to solve $2^n - 2$ equilibrium problems, which are harder then a dynamic optimization one. But this approach has a limitation, because in general the characteristic function is not superadditive. The superadditivity of the characteristic function was considered in Kozlovskaya et al., 2010, Zenkevich and Kozlovskaya, 2010.

Suppose that for parameters of the model the following conditions hold:

$$\begin{aligned} \frac{1}{b(n+1)-\alpha^2\gamma}\Big(a-c-\frac{b\alpha(A-\gamma\bar{e})}{b-\alpha^2\gamma}\Big) &\le \frac{1}{b-\alpha^2\gamma}\Big(\frac{b}{\alpha}\bar{e}_i-\alpha A_i\Big), \\ \frac{\bar{e}_i}{\alpha}+\frac{1}{2bn-\alpha^2\gamma}(a-c+\alpha A-\frac{2b}{\alpha}\bar{e}) &\ge 0, \quad i \in I, \end{aligned} \tag{22}$$

where

$$\bar{e} = \sum_{j\in I} \bar{e}_j,$$
$$A = -\frac{\sum\limits_{j\in I} \pi_j}{\rho+\delta}.$$

Conditions (22) are the sufficient conditions of superadditivity of the characteristic function.

Computation of th Nash equilibrium in the game $\boldsymbol{\Gamma_V^{S_k}(s_0)}$ To find the Nash equilibrium the system of Hamilton-Jacobi-Bellman equations must be solved:

$$\max_{e_i} \Pi_i(s;q) = \max_{e_i} \int\limits_t^\infty e^{-\rho(\tau-t)}\{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\}d\tau, \quad i \in S_k. \tag{23}$$

The solution of the system (23) is equivalent to the solution of the system of Hamilton-Jacobi-Bellman equations.

$$\begin{aligned} \rho W_i = \max_{q_i}\{q_i(a-bQ) - cq_i - \pi_i s + \frac{\gamma\alpha^2}{2}q_i^2 - \gamma\alpha\bar{e}_i q_i + \\ + \frac{\partial W_i}{\partial s}(\alpha Q - \delta s)\}, \quad i \in S_k. \end{aligned} \tag{24}$$

Differentiaiting the right hand side (24) with respect to q_i and equating to 0 leads to

$$a - bQ - bq_i - c + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_i}{\partial s} = 0, \quad i \in S_k$$

Recall that players from $I \backslash S_k$ stick to the strategies (17), where $Q_{S_j}^n$ is defined by the formula (20)

$$a-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ_{s_k}-bq_i-c+\gamma\alpha^2q_i-\gamma\alpha\bar{e}_i+\alpha\frac{\partial W_i}{\partial s}=0,\quad i\in S_k \tag{25}$$

Summing (25) by S_k gets

$$n_k(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j})-n_kbQ_{s_k}-bQ_{S_k}+\gamma\alpha^2Q_{S_k}-\gamma\alpha\bar{e}^{S_k}+\alpha\sum_{j\in S_k}\frac{\partial W_j}{\partial s}=0,$$

We obtain

$$Q^N_{S_k}=\frac{n_k(a-c-b\sum\limits_{j\in M\setminus\{k\}}Q^n_{S_j})-\gamma\alpha\bar{e}^{S_k}+\alpha\sum\limits_{j\in S_k}\frac{\partial W_j}{\partial s}}{b(n_k+1)-\alpha^2\gamma}. \tag{26}$$

One can find from (25), that

$$q^N_i=\frac{\bar{e}_i}{\alpha}+\frac{1}{b-\alpha^2\gamma}(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ_{S_k}+\alpha\frac{\partial W_i}{\partial s}-\frac{b}{\alpha}\bar{e}_i). \tag{27}$$

On account of (22), $0\le q^N_i\le\frac{\bar{e}_i}{\alpha}$. The Bellman functions have the linear form:

$$W_i=A_is+B_i,\quad i\in S_k. \tag{28}$$

Substituting (28) into (24), we obtain

$$\begin{aligned}\rho A_is+\rho B_i=q^N_i(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k})-\pi_is+\\+\frac{\gamma\alpha}{2}q^N_i(\alpha^N_i-2\bar{e}_i)+A_i(\alpha(\sum_{j\in M\setminus\{k\}}Q^n_{S_j}+Q^N_{S_k})-\delta s)\end{aligned} \tag{29}$$

from (29) one can find

$$\begin{gathered}A_i=-\frac{\pi_i}{\rho+\delta}\\ B_i=\frac{1}{\rho}(q^N_i(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k})+\frac{\gamma\alpha}{2}q^N_i(\alpha^N_i-2\bar{e}_i)+\\ +\alpha A_i(\sum_{j\in M\setminus\{k\}}Q^n_{S_j}+Q^N_{S_k})),\end{gathered} \tag{30}$$

where $Q^n_{S_j}$ is defined by (3.2.), q^N_i is defined by (3.3.) and

$$Q^N_{S_k}=\frac{n_k(a-c-b\sum\limits_{j\in M\setminus\{k\}}Q^n_{S_j})-\gamma\alpha\bar{e}^{S_k}+\alpha A_{S_k}}{b(n_k+1)-\alpha^2\gamma},$$

$$q^N_i=\frac{\bar{e}_i}{\alpha}+\frac{1}{b-\alpha^2\gamma}(a-c-b\sum_{j\in M\setminus\{k\}}Q^n_{S_j}-bQ^N_{S_k}+\alpha A_i-\frac{b}{\alpha}\bar{e}_i).$$

Computation of the characteristic function for the intermidiate coalition L in the game $\Gamma_V^{S_k}(s_0)$ Let $L \in S_k,\ |L| = l,\ |S_k| = n_k$. Players from L maximize

$$\max_{q_i, q_i \in L} \Pi_i(s;q) = \max_{q_i, q_i \in L} \int_t^\infty e^{-\rho(t-\tau)}\{pq_i - C_i(q_i) - D_i(s) - E_i(q_i)\}d\tau, \tag{31}$$

on the assumption of the left-out players stick to their Nash equilibrium strategies q_i^N. The solution of (31) is equivalent to the solution of the following Hamilton-Jacobi-Bellman equation.

$$\begin{aligned} \rho W_L = \max_{q_j \in L}\{\sum_{j\in L} q_j(a-bQ) - c\sum_{j\in L} q_j - \sum_{j \in L}\pi_j s + \\ +\frac{\gamma\alpha^2}{2}\sum_{j\in L} q_j^2 - \gamma\alpha\sum_{j\in L}\bar{e}_j q_j + \frac{\partial W_L}{\partial s}(\alpha Q - \delta s)\}. \end{aligned} \tag{32}$$

Differentitating the right hand side of (32) with respect to q_i and equating to 0 gives:

$$a - c - bQ - b\sum_{j\in L} q_j + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_L}{\partial s} = 0. \tag{33}$$

Suppose the players from $I\backslash S_k$ stick to q_i^n (17) and t he players from $S_k\backslash L$ stick to q_i^N (3.3.), so from (33) it can be obtained

$$a - c - b\sum_{j\in M\backslash\{k\}} Q_{S_j}^n - b\sum_{j\in S_k\backslash\{L\}} q_j^N - 2b\sum_{j\in L} q_j + \gamma\alpha^2 q_i - \gamma\alpha\bar{e}_i + \alpha\frac{\partial W_L}{\partial s} = 0. \tag{34}$$

By the same way it can be found:

$$q^L = \sum_{j\in L} q_j^L = \frac{l(a - c - b(\sum\limits_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum\limits_{j\in S_k\backslash\{L\}} q_j^N)) - \gamma\alpha\bar{e}^L + \alpha A_L}{2bl - \alpha^2\gamma}, \tag{35}$$

and then

$$q_i^L = \frac{\bar{e}_i}{\alpha} + \frac{1}{b - \alpha^2\gamma}(a - c - b(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L) + \alpha A_L - \frac{b}{\alpha}\bar{e}^L). \tag{36}$$

Because of the condition (22), $0 \le q_i^L \le \frac{\bar{e}_i}{\alpha}$. The characteristic function is defined by the following formula:

$$W_L = A_L s + B_L, \tag{37}$$

where

$$A_L = -\frac{\sum\limits_{j\in L}\pi_j}{\rho + \delta}$$

$$\begin{aligned} B_L = \frac{1}{\rho}(q^L(a - c - b(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L)))+ \\ +\frac{\gamma\alpha}{2}\sum_{j\in L} q_j^L(\alpha q_j^L - 2\bar{e}_j) + \alpha A_L(\sum_{j\in M\backslash\{k\}} Q_{S_j}^n + \sum_{j\in S_k\backslash\{L\}} q_j^N + q^L). \end{aligned}$$

3.4. Characteristic function

We have proved that characteristic function of the game $\Gamma_V^{S_k}(s_0)$ is given by the following formula:

$$V(K,s)=\begin{cases}0, \quad K=\emptyset,\\ W_i(s), \quad K=\{i\},\\ W_{S_k}(s), \quad K=S_k,\\ W_L(s), \quad K=L,\end{cases}$$

where $W_i(s)$, $W_L(s)$, $W_{S_k}(s)$ is defined by (15), (37), (28).

3.5. The PMS-vector in the game $\Gamma_V^{S_k}(s_0)$

Let $s^n(t),\ t\geq t_0$ be the coaltiotnal trajectory, and players from coalition S_k players are agreed to divide the total payoff $V(S_k,s_0)$ according to Shapley value:

$$Sh(s)=(Sh_1(s),Sh_2(s),\dots,Sh_n(s)),$$

where $SH_i(s)$ is defined by (**??**). The structure of the Shapley value is the following

$$Sh_i(s^n(t))=A_is^n(t)+Bsh_i,$$

4. The Numerical Example of the Coalitional Solution

All computations were executed in MAPLE 10.

4.1. Parameters of the Model

Consider the game of territorial environmental production of 7 players: $I=\{1,2,3,4,5,6,7\}$. Let the parameters of the model be the following:
$t_0=0$ – the initial instant of time ,
$s_0=0$ – the initial stock of pollution,
$p(t)=8000-10\sum\limits_{i=1}^{7}q_i(t)$ - the price function,
$c=3$ – specific production costs,
$\rho=0.07$ – discount rate,
$\alpha=4$ – coefficient that characterizes the specific emission volume,
$\delta=0.2$ – natural rate of pollution absorption,
$\gamma=0.055$ – abatement costs coefficient
$\bar{e}=(600,450,510,480,550,410,430)$ - maximum permissible emissions,
$\pi=(4.7,5.3,5,5.1,4.8,5.2,5.05)$ - damage costs coefficients.
It follows from (2) and (3) that maximum permissible outputs of players are equal to

$$q^{max}=(150,112.5,127.5,120,137.5,102.5,107.5)$$

4.2. Results

Consider the following cases:

1. the Nash equilibrium
2. full cooperation
3. coalitional partition $\Delta_1=(\{1,2,3\},\{4,5\},\{6,7\})$

4. coalitional partition $\Delta_2 = (\{1,2\},\{3,4\},\{5,6,7\})$
5. coalitional partition $\Delta_3 = (\{1,2,3,4\},\{5,6,7\})$
6. coalitional partition $\Delta_4 = (\{1,2\},\{3,4\},\{5\},\{6\},\{7\})$
7. coalitional partition $\Delta_5 = (\{1,2,3,4\},\{5\},\{6\},\{7\})$

Table 1: Results

	max	NE	COO	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
p	-575	1085.99	4292.47	2155.02	2156.65	2869.6	2098.54	3173.57
q_1	150	96.64	80.46	82.34	117.6	85.3	79.7	56.29
q_2	112.5	99.28	42.96	44.84	80.1	47.8	42.2	18.79
q_3	127.5	98.32	57.96	59.84	102.9	62.8	65	33.79
q_4	120	98.89	50.46	90.21	95.4	55.3	57.5	26.29
q_5	137.5	97.68	67.96	107.7	84.5	109	135.7	137.5*
q_6	102.5	100.41	32.96	97.28	49.5	74	102.5*	102.5*
q_7	107.5	100.17	37.96	102.3	54.5	79	107.5*	107.5*

The first string of the table contains the prices of product in all 7 cases. The price of product is the highest in the case off full cooperation, the price is the lowest, when the players compete. The dynamics of pollution in any of 7 cases are the following:

$$
\begin{aligned}
s^N(t) &= 13828.02 - 13828.02e^{-02t},\\
s^I(t) &= 7415.07 - 7415.07e^{-02t},\\
s^{\Delta_1}(t) &= 11689.95 - 11689.95e^{-02t},\\
s^{\Delta_2}(t) &= 11686.7 - 11686.7e^{-02t},\\
s^{\Delta_3}(t) &= 10260.8 - 10260.8e^{-02t},\\
s^{\Delta_4}(t) &= 11802.9 - 11802.9e^{-02t},\\
s^{\Delta_5}(t) &= 9652.9 - 9652.9e^{-02t}.
\end{aligned}
$$

Functions $s^{\Delta_1}(t)$, $s^{\Delta_2}(t)$ $s^{\Delta_4}(t)$ are almost coincides, so let us denote it by $s^{\Delta_1}(t)$ (Pic. 1). The emissions are maximin in the case of competition at any t and minimum in the case of cooperation. On Fig. 2-8 profits of any player are represented. The profit is lowest in the Nash equilibrium (competitive case) for any player. On Fig. 9 and 10 the profit functions of players in the case of cooperation and competition are represented. On Fig. 11-15 the the profit functions of players in the case of coalitional partitions are represented.

Appendix

$$V(\{1\}, s^N(t)) = 443161.9 + 240710e^{-0.2t}$$
$$V(\{2\}, s^N(t)) = 410614.1 + 271439e^{-0.2t}$$
$$V(\{3\}, s^N(t)) = 436648.3 + 256074.5e^{-0.2t}$$
$$V(\{4\}, s^N(t)) = 434696.7 + 261196e^{-0.2t}$$
$$V(\{5\}, s^N(t)) = 454212 + 245831.5e^{-0.2t}$$
$$V(\{6\}, s^N(t)) = 460253.9 + 266317.5e^{-0.2t}$$
$$V(\{7\}, s^N(t)) = 479901 + 258635.2e^{-0.2t}$$
$$Sh_1(s^I(t)) = 2596830.2 + 129041.3e^{-0.2t}$$
$$Sh_2(s^I(t)) = 2534222.5 + 145514.7e^{-0.2t}$$
$$Sh_3(s^I(t)) = 2633017.4 + 137278e^{-0.2t}$$
$$Sh_4(s^I(t)) = 2643934.6 + 140023.6e^{-0.2t}$$
$$Sh_5(s^I(t)) = 2630935.5 + 131786.9e^{-0.2t}$$
$$Sh_6(s^I(t)) = 2693927.9 + 142769.1e^{-0.2t}$$
$$Sh_7(s^I(t)) = 2704317.1 + 138650.8e^{-0.2t}$$
$$PMS_1^1(s^1(t)) = 1004458.7 + 203491.8e^{-0.2t}$$
$$PMS_2^1(s^1(t)) = 9836653.3 + 229469.5e^{-0.2t}$$
$$PMS_3^1(s^1(t)) = 1019834.5 + 216480.6e^{-0.2t}$$
$$PMS_4^1(s^1(t)) = 2116292.1 + 220810.2e^{-0.2t}$$
$$PMS_5^1(s^1(t)) = 2116722 + 207821.4e^{-0.2t}$$
$$PMS_6^1(s^1(t)) = 2135539.1 + 225139.9e^{-0.2t}$$
$$PMS_7^1(s^1(t)) = 2149420.1 + 218645.4e^{-0.2t}$$
$$PMS_1^2(s^2(t)) = 2098449.6 + 203435.1e^{-0.2t}$$
$$PMS_2^2(s^2(t)) = 2106725.7 + 229405.6e^{-0.2t}$$
$$PMS_3^2(s^2(t)) = 2110817.6 + 216420.4e^{-0.2t}$$
$$PMS_4^2(s^2(t)) = 2116851.4 + 220748.8e^{-0.2t}$$
$$PMS_5^2(s^2(t)) = 1004927.8 + 207763.6e^{-0.2t}$$
$$PMS_6^2(s^2(t)) = 1021950.9 + 225077.2e^{-0.2t}$$
$$PMS_7^2(s^2(t)) = 1052447.7 + 218584.6e^{-0.2t}$$
$$PMS_1^3(s^3(t)) = 1618274.1 + 1786139e^{-0.2t}$$
$$PMS_2^3(s^3(t)) = 1620465.4 + 2014156.7e^{-0.2t}$$
$$PMS_3^3(s^3(t)) = 1634972.8 + 1900147.8e^{-0.2t}$$
$$PMS_4^3(s^3(t)) = 1638652.2 + 1938150.8e^{-0.2t}$$
$$PMS_5^2(s^3(t)) = 2698271.6 + 1824141.9e^{-0.2t}$$
$$PMS_6^3(s^3(t)) = 2740960.6 + 1976153.7e^{-0.2t}$$
$$PMS_7^3(s^3(t)) = 2289477 + 1919149.3e^{-0.2t}$$

$$PMS_1^4(s^4(t)) = 912508.3 + 205458.4e^{-0.2t}$$
$$PMS_2^4(s^4(t)) = 893504 + 231687e^{-0.2t}$$
$$PMS_3^4(s^4(t)) = 941322.5 + 218572.7e^{-0.2t}$$
$$PMS_4^4(s^4(t)) = 879177.1 + 222944.2e^{-0.2t}$$
$$PMS_5^4(s^4(t)) = 3133855.8 + 209829.8e^{-0.2t}$$
$$PMS_6^4(s^4(t)) = 2125635.4 + 2273156.4e^{-0.2t}$$
$$PMS_7^4(s^4(t)) = 2294008.6 + 2207584.7e^{-0.2t}$$
$$PMS_1^5(s^5(t)) = 718845.5 + 168031.3e^{-0.2t}$$
$$PMS_2^5(s^5(t)) = 702952 + 189482.1e^{-0.2t}$$
$$PMS_3^5(s^5(t)) = 724518 + 178756.7e^{-0.2t}$$
$$PMS_4^5(s^5(t)) = 724522.1 + 182331.8e^{-0.2t}$$
$$PMS_5^5(s^5(t)) = 5447153.4 + 171606.4e^{-0.2t}$$
$$PMS_6^5(s^5(t)) = 3859509.6 + 185907e^{-0.2t}$$
$$PMS_7^5(s^5(t)) = 4100063.7 + 180544.2e^{-0.2t}$$

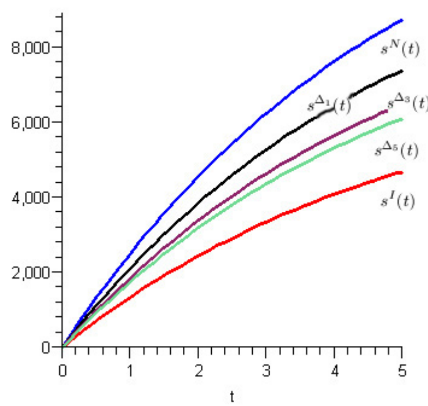


Fig. 1: Dynamics of pollution

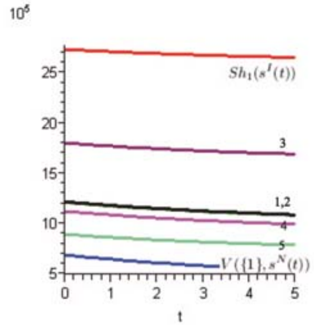


Fig. 2: Profit functions of 1st player

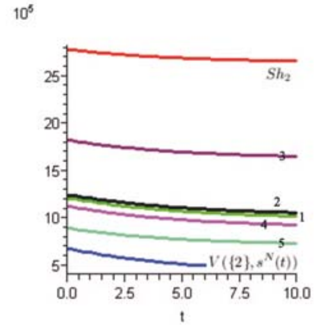


Fig. 3: Profit functions of 2nd player

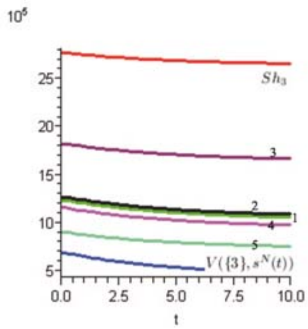


Fig. 4: Profit functions of 3st player

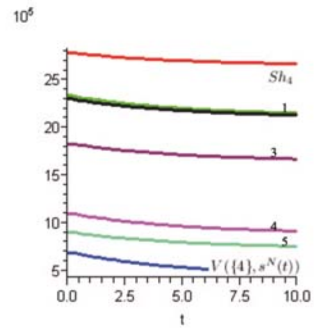


Fig. 5: Profit functions of 4nd player

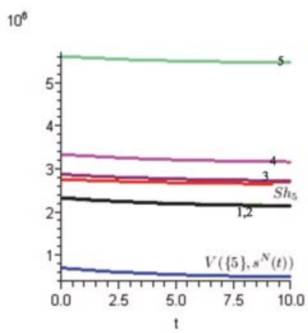


Fig. 6: Profit functions of 5st player

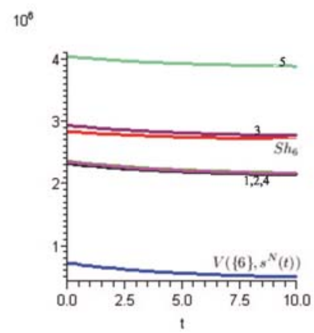


Fig. 7: Profit functions of 6nd player

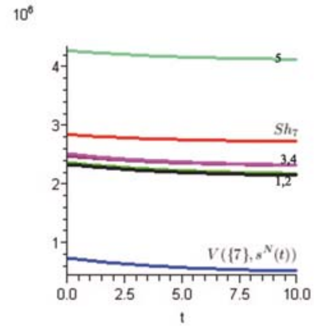


Fig. 8: Profit functions of 7st player

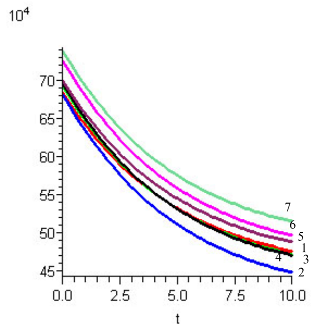


Fig. 9: Profit functions of player in the Nash equilibrium

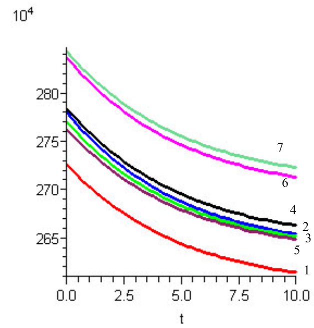


Fig. 10: Profit functions of player in the cooperation

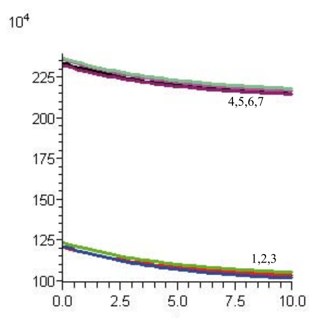


Fig. 11: Profit functions of player in the case Δ_1

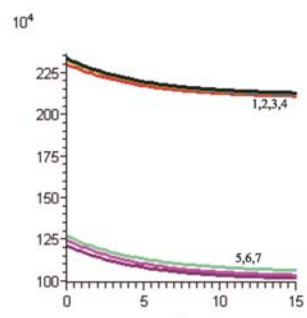
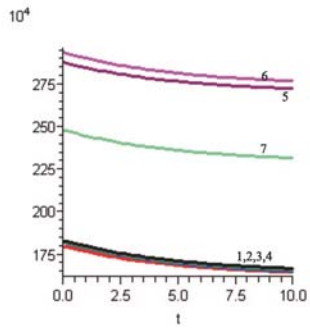


Fig. 12: Profit functions of player in the case Δ_2



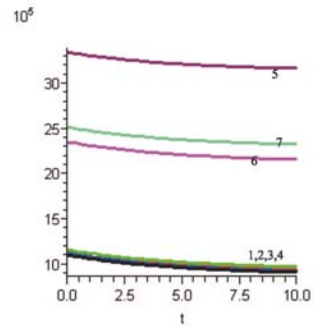


Fig. 13: Profit functions of player in the case Δ_3

Fig. 14: Profit functions of player in the case Δ_4

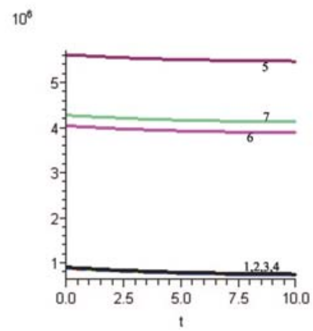


Fig. 15: Profit functions of player in the case Δ_5

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On a Mutual Tracking Block for the Real Object and its Virtual Model-Leader

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Abstract The research is devoted to a feedback control problem of stochastic stable mutual tracking for motions of a real dynamical object, and some virtual computer simulated model-leader, under dynamical and informational disturbances. The control and disturbance actions in the model are determined by proposed random tests. To obtain solution to the considered problem we apply the so-called extremal minimax and maximin shift conditions. Theoretical results are illustrated by numerical simulations.

Keywords: feedback control, nonlinear system, extremal shift.

1. Introduction

The investigations in this work are based on the approaches, methods and constructions from the theory of stochastic processes, theory of stability, theory of optimal control and differential games, tracing and observation of the processes and so on, which proposed and are developed in the works of Bellman, 1957, Isaacs, 1965, Krasovskii and Subbotin, 1974, Kurzhanski, 1977, Mishchenko, 1972, Osipov and Kryazhimskii, 1995, Pontryagin et al., 1962 and many other authors. This work uses the ideas of the books (Krasovskii and Subbotin, 1974; Krasovskii and Krasovskii, 1994). The stochastic process for the solution of considered problem is based on the appropriate constructions of the so-called extremal shift (Krasovskii, 1980) of given controlled x-object to its virtual w-model-leader.

2. Mutual tracking block. Extremal shift.

The dynamics of x-object is described by the vector ODE – nonlinear in controls u and disturbances v:

$$\dot{x} = A(t)x + f(t,u,v) + h_{din}(t), \quad t_0 \leq t \leq \theta, \tag{1}$$

subject to restrictions:

$$u \in P = \{u^{[1]}, \dots, u^{[M]}\}, \quad v \in Q = \{v^{[1]}, \dots, v^{[N]}\}. \tag{2}$$

Here symbols M and N are given numbers. Symbol $h_{din}(t)$ denotes a random vector-function restricted by the following constrains:

$$|h_{din}(t)| \leq H, \quad E\{h_{din}(t)\} \leq \delta_{din},\ t \in [t_0, \theta], \tag{3}$$

where H stands for a sufficiently large constant, δ_{din} is a small constant, where $E\{\cdots\}$ is the mathematical expectation (Liptser and Shiryaev, 1974).

Let us consider the case, when the *saddle point condition for the small game* (McKinsey, 1952), i.e.:

$$min_{u\in P}\, max_{v\in Q}\, \langle l\cdot f(t,u,v)\rangle = max_{v\in Q}\, min_{u\in P}\, \langle l\cdot f(t,u,v)\rangle, \qquad (4)$$

where l is any n-dimensional vector and the symbol $\langle l\cdot f(t,u,v)\rangle$ denotes the inner product in R^n,*is not satisfied* for the function $f(t,u,v)$.

Let us choose a partition $t_k \in \Delta\{t_k\} = \{t_0, t_1, \dots, t_k < t_{k+1}, \dots, t_K = \theta\}$, where K is a large number, and consider the finite-difference equation for x-object:

$$x[t_{k+1}] = x[t_k] + (A(t_k)x[t_k] + f(t_k,u,v) + h_{din}(t_k))(t_{k+1}-t_k). \qquad (5)$$

Together with a real x-object we consider the motion of an abstract w-model:

$$w[t_{k+1}] = w[t_k] + (A(t_k)w[t_k] + \sum_{i=1}^{M}\sum_{j=1}^{N} f(t_k,u^{[i]},v^{[j]})p_i q_j + h_{din}(t_k))(t_{k+1}-t_k). \qquad (6)$$

Here numbers $p_i, i=1,\dots,M$ and $q_j, j=1,\dots,N$ satisfy conditions:

$$p_i \geq 0, i=1,\dots,M, \sum_{i=1}^{M} p_i = 1, \quad q_j \geq 0, j=1,\dots,N, \sum_{j=1}^{N} q_j = 1. \qquad (7)$$

We assume that the motion of w-model is simulated by a computer, implemented in a regulator, and considered as the "leader" (or "pilot") for the motion of x-object.

Further, we consider the case, when position $\{t_k, x[t_k]\}, k=0,\dots,K$, of x-object is estimated with some informational error $\Delta_{inf}[t_k]$, such that at each time moment $t_k \in \Delta t_k$ only the distorted position $\{t_k, x^*[t_k]\}$ is known, where:

$$x^*[t_k] = x[t_k] + \Delta_{inf}[t_k]. \qquad (8)$$

Here $\Delta_{inf}[t_k]$ is a random vector.

Control actions for x-object and w-model, which provide mutual tracking in the combined process x-object, x-model-leader, are constructed as follows.

At the moment $t_k, k=0,\dots,K-1$, a vector of actions $u^0[t] = u^0[t_k] \in P,\ t \in [t_k, t_{k+1})$, for the real x-object is chosen by probability test:

$$P(u^0[t_k] = u^{[i]} \in P) = p_i^o, \quad i=1,\dots,M. \qquad (9)$$

Here symbol P denotes probability (Liptser and Shiryaev, 1974) and probabilities $p_i^o : p_i^o \geq 0,\ i=1,\dots,M,\ \sum_{i=1}^{M} p_i^0 = 1$, are chosen from the so-called *Extremal Minimax Shift Condition*:

$$min_p\, max_q\, \langle l^*[t_k], \sum_{i=1}^{M}\sum_{j=1}^{N}\ f(t_k,u^{[i]},v^{[j]})p_i q_j\rangle = \langle l^*[t_k], \sum_{i=1}^{M}\sum_{j=1}^{N}\ f(t_k,u^{[i]},v^{[j]})p_i^0 q_j^*\rangle, \qquad (10)$$

under restrictions (7). Here $l^*[t_k] = x^*[t_k] - w[t_k]$.

Let the "control action" $q^0[t_k]$ for the virtual w-model be chosen from the *Extremal Maxmin Shift Condition*:

$$max_p\ min_q\ \langle l^*[t_k], \sum_{i=1}^{M}\sum_{j=1}^{N}\ f(t_k,u^{[i]},v^{[j]})p_iq_j\rangle = \langle l^*[t_k], \sum_{i=1}^{M}\sum_{j=1}^{N}\ f(t_k,u^{[i]},v^{[j]})p_i^*q_j^0\rangle, \tag{11}$$

Probabilities $\{q_j\}$ that define the stochastic disturbances $v[t_k] \in Q$ on x-object, and "actions" $\{p_i\}$ for w-model may take arbitrary values subject to conditions (7).

Theorem 1. *Under described above choices (10) and (11) of the random actions $u^0[t_k]$ for x-object and "actions" $q^0[t_k]$ for w-model, for any chosen beforehand numbers V^* and $0 < \beta < 1$, there exist sufficiently small numbers $\delta_0 > 0,\ \delta_{inf} > 0,\ \delta_{din} > 0,\ \delta > 0$, such that the following inequality holds:*

$$P(V(t,l[t]) \le v^*, \quad \forall t \in [9,\theta]) \ge 1-\beta, \tag{12}$$

if $l[t_0] \le \delta_0,\ E\{|l(t)-l^*(t)|\ l(t)\} \le \delta_{inf}$, for any admissible $l[t] = x[t]-w[t],\ t \in [0,\theta],\ E\{h_{din}\{t\}\}$, and $\Delta t = t_{k+1}-t_k \le \delta$. Here:

$$V(t,l[t]) = V(t,x[t],w[t]) = |x[t]-w[t]|^2e^{\lambda t}. \tag{13}$$

Presented results are illustrated by a model example and its numerical simulation.

Example 1. In this section we apply the elaborated algorithms to computer for tracing of a motions x-object and w-model for the concrete 2-dimensional system. Let us consider the model problem such that the control x-object (1) is described by the finite-differential equation (5), where in our concrete case we assume:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A(t) = \begin{pmatrix} 0\ 1 \\ 1\ 0 \end{pmatrix}, \tag{14}$$

$$u \in P = u^{[1]} = -1, u^{[2]} = 1, \quad v \in Q = v^{[1]} = -1, v^{[2]} = 1, \tag{15}$$

and the function $f(t,u,v)$ has the form:

$$f(t,u,v) = \begin{Bmatrix} 0,5u+(u+v)^2+v & for & t \in [0,\frac{\vartheta}{4}) \cup [\frac{\theta}{2},\frac{3\vartheta}{4}), \\ u+(u+v)^2+0,5v & for & t \in [\frac{\vartheta}{4},\frac{\theta}{2}) \cup [\frac{3\vartheta}{4},\vartheta]. \end{Bmatrix} \tag{16}$$

As it was described above, $h_{din}(t_k)$ in (1) is a dynamical error (1) that has a random character. And we used the positional stochastic feedback control scheme in which the informational image $x^*[t_k]$ at the current moment $t_k \in \Delta t_k$ satisfies the condition (8). Here the w-model (6), (7) that corresponds to x-object (5),(14),(16) has the form:

$$w[t_{k+1}] = w[t_k] + (A(t_k)w[t_k] + \widetilde{f_{pq}}(t_k) + h_{din}(t_k))(t_{k+1}-t_k), \tag{17}$$

where:

$$\widetilde{f_{pq}}(t_k) = \sum_{i=1}^{2}\sum_{j=1}^{2} f(t_k,u^{[i]},v^{[j]})p_iq_j. \tag{18}$$

Here $f(t_k, u^{[i]}, v^j)$ is a function (16), and: $p_i \geq 0, i = 1, 2,\ p_1 + p_2 = 1, \quad q_j \geq 0, j = 1, 2,\ q_1 + q_2 = 1$.

Under the values of parameters of the $\{x, w\}$ system (5),(14)–(18): $x_1[0] = -1.0$, $x_2[0] = 1.0$, $w_1[0] = -0.95$, $w_2[0] = 1.05$, $\vartheta = 4.0$, $\Delta t = t_{k+1} - t_k = \vartheta = 0.01$, $E|h_{inf}| \leq \delta_{inf} = 0.01$, $E|h_{din}(t)| \leq \delta_{din} = 0.01$, we obtain the results of the computer simulation for the motions of the x-object (solid line) and w-model (dashed line) presented at the figure 1.

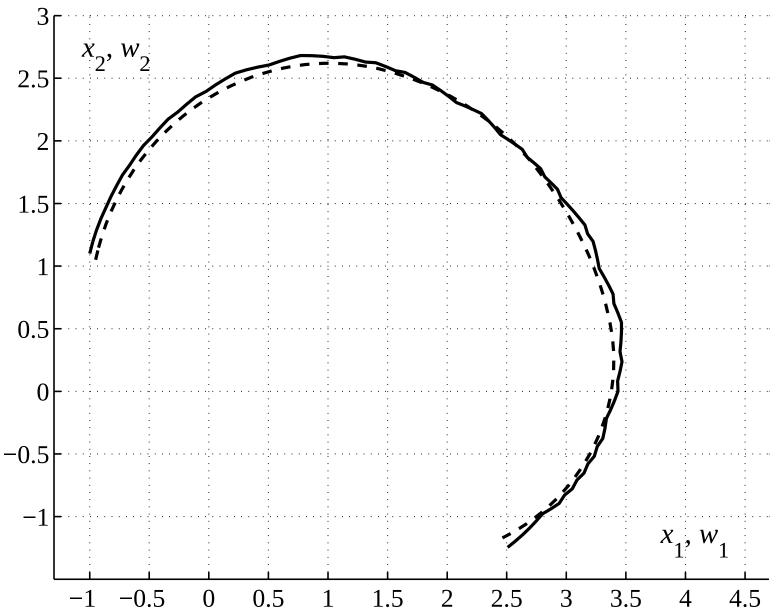

Fig. 1

At this Fig. 1 we have the phase portrait of the motion of x-object and w-model. In this case we chose the control actions $u^0[t] = u^0[t_k] \in P, \quad t_k \leq t < t_{k+1}$ for x-object and "actions" $q^0[t_k]$ for w-model under the algorithms (Extremal Minimax and Maximin Shifts Conditions) from section 2 . The control actions $v[t] = v[t_k] \in Q, \quad t_k \leq t < t_{k+1}$ for x-object and "actions" $p[t_k]$ for w-model we constructed by some random mechanism.

3. Conclusion

By the Theorem 1 the solution of the considered problem of the mutual tracing of the motions of the real controlled object and its virtual model-leader is established. The illustrative example and its computer simulation is given.

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Game-Theoretic Model on a Cognitive Map and its Tolerance to Errors in Input Data to Analyze a Conflict of Interests Between Russia and Norway in Barents Sea*

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Abstract Some problems of complex control in the fields connected with public life (i.e., social-economic, political and other fields) include ill-structured control object. Situation appears ill-structured if the basic parameters have qualitative (not quantitative) nature, and their values are subjective expert evaluations. Cognitive maps serve to solve control problems for ill-structured situations. Cognitive map is a model representing knowledge of the expert (or a group of experts) regarding situation; this model is described in the form of weighted directed graph. The nodes of cognitive map correspond to those concepts being employed to describe the situation. The concept may be treated as a variable (for instance, "national defence capacity") which may have different values, such as "high", "low" and so on. Weighted arc is interpreted as direct cause-effect relationship between two concepts. Suppose several decision-makers (agents) take part in the process of decision making in an ill-structured situation given that the utility of each of them depends both on his self actions and the actions of the others, than interactions of the agents can be seen as a game on the cognitive map. In the game cognitive map represents a model of ill-structured control object and clearly describe the dynamics of the situation. The use of cognitive maps in the game gives more detailed and visual simulation of the environment of the conflict in the form of simple causal links, so as to describe the goals and strategies of the agents in terms of the environment which makes it more convenient to simulate the real conflicts adequately. Since the input data for the model are expert evaluation prone to subjectiveness, it is necessary to estimate the tolerance of model results to errors in input data. Experts evaluate the "importance percentage" of a target concept compared the others target concepts and the weight of edges in cognitive map as the type of the causal links and its strength. In this paper we consider the problem of model tolerance to errors in input data and illustrate it on the material of conflict of interests between Russia and Norway in the Barents Sea.

Keywords: game, cognitive map, conflict of interests, dominant strategy, tolerance to errors.

1. Introduction

Cognitive maps were previously introduced by Axelrod (1976) to clarify and improve decision making process. A cognitive map is a weighted digraph-based mathemat-

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ical model of a decision maker belief system about some limited domain, such as a policy problem. Cognitive map nodes correspond to situation concepts. Concepts are interpreted as variables whose values may vary. Weighted edges are interpreted as direct causal links from one concept to another. Analysis of possible situation developments depending on the control (in terms of an influence on some concepts) is one of the possible applications of cognitive maps. Both direct (situation development prediction with the fixed control) and inverse (search of the appropriate control) cognitive analysis problems are considered for this purpose.
The game-theoretic model of interactions between several agents at a dynamic system in the form of a situation cognitive map was generally considered by Novikov (2008). Since the input data for the model are expert evaluation prone to subjectiveness, it is necessary to estimate the tolerance of model results to errors in input data. Experts evaluate the "importance percentage" of a target concept compared the others target concepts and the weight of edges in cognitive map as the type of the causal links and its strength. In this paper we consider the problem of model tolerance to errors in input data and illustrate it on the material of conflict of interests between Russia and Norway in the Barents Sea.

2. Description of model

2.1. General model

A linear cognitive map C is called a weighted digraph, if its nodes (concepts) and edges (causal links) meet the conditions stated below, and the undermentioned rule regarding node value dynamics is given. By $M = \{1, \ldots, m\}$ denote the set of all concepts. A causal concept is a concept where an edge starts; an effect concept is a concept where an edge ends. Thereafter let an adjacency matrix of the digraph W be a matrix with elements $w_{ji} \in R$, if elements of the matrix correspond to weights of graph edges, which define types and strengths of causal links. Strength of the causal link from the j-th causal concept to the i-th effect concept is equal to the absolute value of the edge weight $|w_{ji}|$. The sign of the edge weight corresponds to the link type: if $w_{ji} > 0$, then the causal link from the j-th concept to the i-th one is positive, if $w_{ji} < 0$, then the causal link is negative (Roberts, 1976).
All results were obtained for discrete time and the zero-time initial state. An pulse process of a cognitive map is defined by the rule (1) with the initial concept vector $x(0) = (x_1(0), x_2(0), \ldots, x_m(0))$, $x(0) \in R^m$, and the vector $p = (p_1, p_2, \ldots, p_m)$, $p \in R^m$ of an external pulse to each node at the zero time point (Roberts, 1976).

$$x_i(t+1) = x_i(t) + p_i(t), p_i(t) = \begin{cases} p_i, & \text{if } t = 0; \\ \sum_{j \in M} w_{ji} \cdot p_j(t-1), & \text{if } t = 1, 2, 3, \ldots \end{cases} \quad (1)$$

Let us fix the discrete time point T $(T > 0)$. Then the concept vector $x(T)$ is defined by the expression:
$x(T) = x(0) + p(0) + p(1) + \cdots + p(T-1) = x(0) + p + p \cdot W + \cdots + p \cdot W^{T-1} = x(0) + p \cdot (E + W + \cdots + W^{T-1}) = x(0) + p \cdot {}_TQ$.
Where E is an identity matrix. Let a matrix ${}_TQ = E + W + \cdots + W^{T-1}$ be a matrix of an influence reachability by the time T for the adjacency matrix W. Then the sum of the consequent increments for the concept x_j is as follows:

$$\sum_{t=0}^{T} p_j(t) = \sum_{k \in M} {}_T q_{kj} \cdot p_k. \tag{2}$$

Where ${}_T q_{kj}$ are elements of the matrix ${}_T Q$. Let us consider the problem of semistructured situation control at a linear cognitive map-based model. Let control actions be external pulses to each node at the zero-time point p; where $p_j = 0$, if there is no control to the node j. A control effect is a set of all concept values at the time point :

$$x_j(T) = x_j(0) + \sum_{t=0}^{T} p_j(t), j \in M. \tag{3}$$

A control target is defined by desirable values for all or some concepts $x(T) = (\ x_1(T),\ x_2(T),\ \ldots,\ x_m(T)),\ x(T) \in R^m$ (Roberts, 1976).
The agent with the number $i \in N$ has a nonempty subset of concepts $M_i \in M$ he can control. Let M_i be a set of controlled concepts of the i-th agent. For any two agents $i, j \in N : M_i \cap M_j = \emptyset$ and $\cup_{k \in N} M_k \in M$. By m_i denote the number of concepts at the set M_i.

A control action of each agent is contained in a vector of mutual control actions $p = (p_1, p_2, \ldots, p_m)$. Let the strategy s_i of the i-th agent be a vector of ordered components of the vector p with indices from the set $\{k_1, k_2, \ldots, k_{m_i}\} = M_i$: $s_i = (p_{k_1}, p_{k_2}, \ldots, p_{k_{m_i}})$. Each agent defines only "his" components of the vector p during the influence on the situation. If there are no agent who influences on the concept, then the corresponding component is null: $(\forall j \in M - \cup_{k \in N} M_k)$, $p_j = 0$. Control actions require some expense of limited resources. Let us impose the basic restrictions to control actions for each concept in the form of the interval of acceptable values: $(\forall j \in \cup_{k \in N} M_k)$ $p_j \in [-1, 1]$. Then *the set of i-th agent strategies* S_i can be represented as the Cartesian product m_i of intervals $[-1, 1]^{m_i}$. Let the hypercube $(s_1, s_2, \ldots, s_n) \in S_1 \times \cdots \times S_n$ be the set of all agent strategies $S_1 \times \cdots \times S_n$.
Let us define the utility function $f_i(x_1(T), x_2(T), \ldots, x_m(T))$ on the result set for each agent. The control target of i-th agent is the maximization the function f_i. If the i-th agent want to increase (alternatively, decrease) in the value of the concept x_j , then it is desirable for him to maximize the expression $(x_j(T) - x_j(0))$ (similarly $-(x_j(T) - x_j(0))$). If the agent i can define desirable values for several concepts, then the weighted sum should be maximized according to the above stated expressions for such concepts. Each coefficient is interpreted as an "importance percentage" of restrictions on the corresponding concept. The utility function of the i-th agent is as follows:

$$f_i(x_1(T), x_2(T), \ldots, x_m(T)) = \sum_{j \in M} \gamma_{ij} \cdot (x_j(T) - x_j(0)). \tag{4}$$

Where $|\gamma_{ij}|$ is the "importance percentage" of the j-th concept value for the i-th agent, $\gamma_{ij} \in [-1, 1]$, the sum of all $|\gamma_{ij}|$ at the right hand side of the expression (4) is equal to 1. The sign of the coefficient γ_{ij} indicates the direction of variation of the concept value (being beneficial to the agent). In particular, provided $\gamma_{ij} > 0$ the i-th agent strives for infinite increasing the j-th concept value. If $\gamma_{ij} < 0$, then the i-th agent seeks to infinitely decrease the value of the j-th concept. Finally, $\gamma_{ij} = 0$ means the i-th agent does not care about the value of the j-th concept.

Let the target concept of the i-th agent be a concept with $\gamma_{ij} \neq 0$ at the utility function (4). After the definition of all game parameters, let us represent the game at the normal form:

$$\Gamma_C = \{N, \{S_i\}_{i \in N}, \{f_i\}_{i \in N}, C, T\}. \tag{5}$$

Let substitute $x_j(T)$ with the right-hand side of the expression (2) and (3) in formula (4). Proceeding in this manner, one derives

$$f_i = \sum_{j \in M} \gamma_{ij} \cdot (x_j(T) - x_j(0)) = \sum_{k \in M} \left(\sum_{j \in M} \gamma_{ij} \cdot {}_T q_{kj} \right) \cdot p_k = \sum_{k \in M} {}_T \alpha_{ik} \cdot p_k. \tag{6}$$

Dominant strategies of the agent i are defined by:

$$p_k = sign({}_T \alpha_{ik}), k \in M_i. \tag{7}$$

The model (5) is based on the model considered in Novikov (2008) but has some difference. It has a point of control effect T (target time) that let to make analysis more detail.

2.2. A model of conflict of interests between Russia and Norway in Barents Sea

Norway and Russia have sovereign rights over shelf space in the Barents Sea, which includes: 1) the Russian continental shelf (the right of Russia), 2) the Norwegian continental shelf (the right of Norway), 3) the offshore area of Svalbard (the right is governed by the Svalbard Treaty in Paris, 1920) and 4) continental shelves space disputed zone. Disputed territory is about 175 thousand sq. km. Disputed area after 40 years of negotiations was divided into two approximately equal parts in Russian-Norwegian treaty on maritime delimitation in the Barents Sea on September 15, 2010 (hereinafter the Treaty).

There was constructed cognitive map representations of the situation surrounding the signing of the Treaty (see Fig. 1) based on the materials from open source with expert evaluations of the situation in the Barents Sea and the Treaty. During constructing the model we should took into account the proportionality of the propagation time from concept to concept along the arc. The estimated time of impact along the arcs model in about 4-5 years. Time effect of impact from concept ♯12 to ♯7 is about 10 years, so between them added a dummy concept which is not marked in Fig. 1.

Control concept for Russia – concept ♯1, for Norway – concept ♯2. The initial impact $+1$ for each of these concepts is interpreted as a desire to conclude the Treaty. The impact -1 as the absence of such aspirations, and on the contrary, his rejection. The impact value equal to zero, can be interpreted as indifference of the gamer on this issue. The target concepts for Russia will consider two: ♯3 and ♯8 (with "importance percentage" $\gamma_{1,3} = 0.5$ and $\gamma_{1,8} = 0.5$) for Norway ♯4 and ♯11 (with $\gamma_{2,4} = 0.5$ and $\gamma_{2,11} = 0.5$). The solution of the game is the equilibrium with dominant strategies. A set of solutions were found for different target times T (see Fig. 2).

We shall explain two broken lines represented on Fig. 2 in greater detail. In the model (5) the desirable variations of values of target concepts are established for the fixed point in time T in the future (target time T). Thus if the target time T

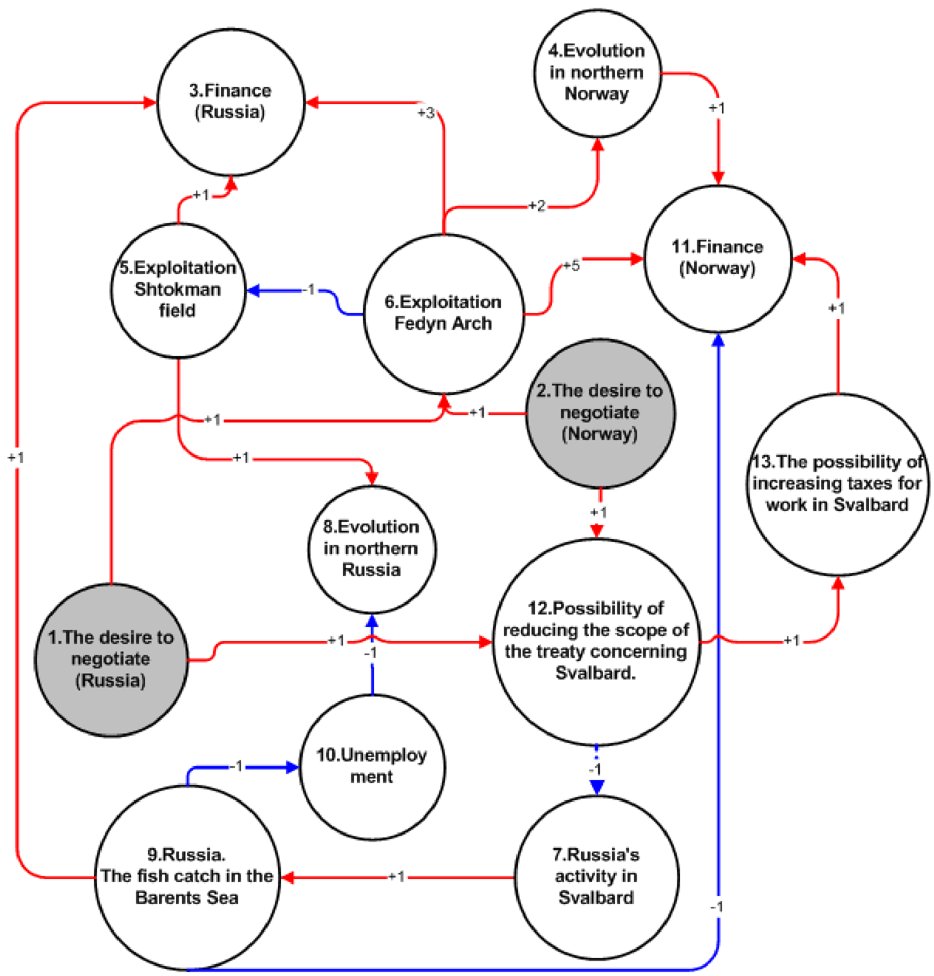

Fig. 1: Cognitive map that reflects the causal links between concepts in the problem of the disputed territory in the Barents Sea. (The target concepts Russia is ♯3 and ♯8, Norway is ♯4 and ♯11)

is small, it means, that agents are inclined to statement of short-term targets and wait for fast results from the action. Than more value so more "far-sightedness" of agent targets. There is a solution of game (5) in the form of equilibrium of dominant strategies according (7) at fixed . Different values are corresponds with different games, accordingly and different dominant strategies of agents. On axis "X" on Fig. 2 different values of target time , that is the different games in which targets of agents are changing from short term (for values 1, 2, 3, 4, 5, 6 on axis) to long run (9, 10, 11, 12, etc.). The lines in Fig. 2 shows how the "far-sightedness" agents in terms of targets, affects the optimal strategy to choose, in accordance with the targets. As can be seen from Fig. 2 division of the disputed territory in the Barents Sea in two is profitable for Norway. It is profitable without depends on target time T. In the case of Russia the situation is quite different. According to Fig. 2 the signing of the Treaty will beprofitable for Russia in the short term, but not favourable in the long run.

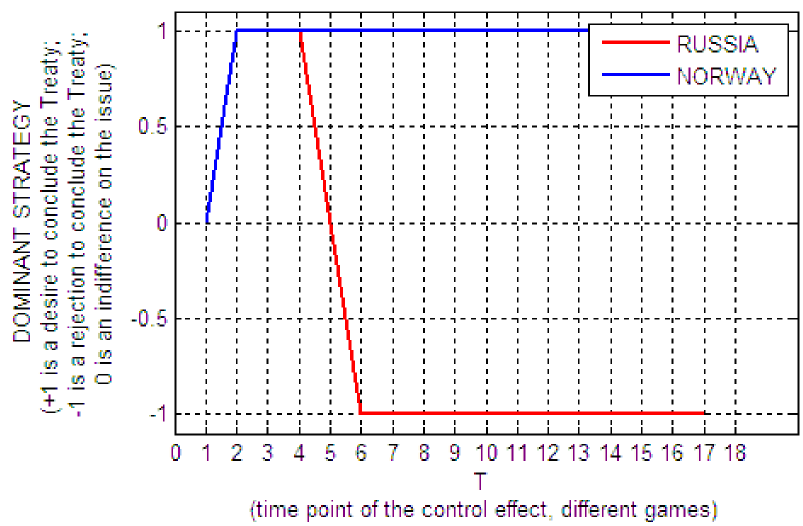

Fig. 2: The set of equilibriums in dominant strategies for games depending on the target time T. The axis "X" is the different values of the target time T, the vertical axis "Y" corresponding to the equilibrium strategies of agents: Russia (red) and Norway (blue).

The results of the work model (Fig. 2) were obtained with using expert evaluation of the importance of target concepts ($\gamma_{1,3}$= 0.5, $\gamma_{1,8}$ = 0.5, $\gamma_{2,4}$ = 0.5 and $\gamma_{2,11}$ = 0.5) and expert evaluation of the weights of the arcs in the digraph of cognitive maps (Fig. 1). Let estimate the tolerance of the model results to errors in the expert evaluations.

3. Estimate of tolerance to errors in input data

3.1. Estimate of tolerance to errors in a target coefficient γ_{ij}

Let consider the situation where the expert make an error in one of the weights coefficients γ_{is} in (6). If expert did not make an error, the value of ${}_T\alpha_{ik}$ would be (8). Because of the error the value of ${}_T\alpha_{ik}$ changes to (9). It is enough to satisfy condition (10) to keep strategy (7) unchanged after making the error. From this condition we obtain an estimate of error in one coefficient in (6).

$$ {}_T\alpha_{ik} = \gamma_{i1\,T}q_{k1} + \gamma_{i2\,T}q_{k2} + \cdots + \gamma_{is\,T}q_{ks} + \cdots + \gamma_{i,m\,T}q_{k,m}. \tag{8} $$

$$ {}_T\alpha^{\varepsilon}_{ik} = \gamma_{i1\,T}q_{k1} + \gamma_{i2\,T}q_{k2} + \cdots + (\gamma_{is} \pm \varepsilon^{i}_{ks})_T q_{ks} + \cdots + \gamma_{i,m\,T}q_{k,m}. \tag{9} $$

$$ {}_T\alpha_{ik} \cdot {}_T\alpha^{\varepsilon}_{ik} = {}_T\alpha_{ik}({}_T\alpha_{ik} \pm \varepsilon^{i}_{ks} \cdot {}_Tq_{ks}) > 0 \Rightarrow \varepsilon^{i}_{ks} < \left| \frac{{}_T\alpha_{ik}}{{}_Tq_{ks}} \right|. \tag{10} $$

We claim that:

Proposition 1. *Suppose* there is an error of expert estimate in only one target coefficient γ_{is} $(s \in M_i)$ of the agent utility function f_i and it doesn't exceed the value (11) *then* the dominant strategy of the agent i is invariable.

$$ \varepsilon^{i}_{s} = \min_{k \in M_i} \left| \frac{{}_T\alpha_{ik}}{{}_Tq_{ks}} \right| \tag{11} $$

Indeed, this follows from the necessary of working the condition (10) for all control concepts of agent i. The value (11) is called the *estimate of tolerance to errors in value of* γ_{is} or *allowable error.*

Fig. 3 shows the results of calculations of (11) for the target concepts of both agents (Russia ♯ 3, ♯ 8 and Norway ♯ 4, ♯ 11). Fig. 3 shows that the least sensitive to changes (due to errors) is concept ♯ 4. The value of $\gamma_{2,4}$ is not critical to the invariance of the result when $T > 2$. The reason is probably in the causal link (♯4 $\rightarrow$ ♯11). We shall return to this fact further in analysis of allowable errors for the weights of the arcs of the digraph.

The estimates of tolerance to errors in value of γ_{ij} (11) are within the range of allowed values [-1, 1] for the other concepts. However, allowable error in γ_{ij} is quite large for the concepts ♯3 and ♯11. The allowable error in γ_{ij} is much smaller for the concept ♯8. It is clear that the correct evaluation of γ_{ij} for the concept ♯8 is the most important for choosing the optimal strategy. It requires a high confidence in the correct evaluation $\gamma_{1,8}$.

Note that the values of the allowable errors for the concepts ♯ 3 and ♯ 8 at $T = 5$ are equal to zero. It is happened because the optimal strategy for Russia at $T = 5$ is an omission (see Fig. 2). This strategy is not stable to errors in the coefficient values γ_{ij}, because (7).

A similar analysis of allowable errors for the target concepts can be used for their selecting. For example the allowable error in concept ♯4 is bigger than the high limit of the value range $\gamma_{2,4} \in [-1, 1]$. The concept ♯ 4 should not be selected as the target concept in a game with cognitive map (Fig. 1) because it is not critical for the invariance result.

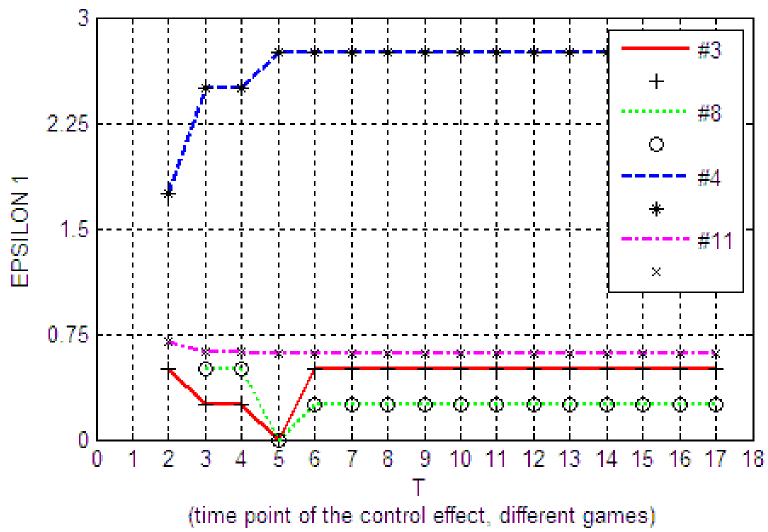

Fig. 3.

3.2. Estimate (lower bound) of tolerance to error in all target coefficients

Let consider the situation where the expert make an error in all weight coefficients in (6). If expert did not make an error, the value of ${}_T\alpha_{ik}$ would be (8). Because of the error the value of ${}_T\alpha_{ik}$ changes to (12). It is enough to satisfy condition (13) to keep strategy (7) unchanged after making the error. From this condition we obtain an estimate of error (lower bound) in all coefficients in (6).

$$ {}_T\alpha^{\varepsilon}_{ik} = (\gamma_{i1} \pm \varepsilon^i_k)_T q_{k1} + (\gamma_{i2} \pm \varepsilon^i_k)_T q_{k2} + \cdots + (\gamma_{is} \pm \varepsilon^i_k)_T q_{ks} + \cdots + (\gamma_{i,m} \pm \varepsilon^i_k)_T q_{k,m}. \tag{12} $$

We claim that:

$$ {}_T\alpha_{ik} \cdot {}_T\alpha^{\varepsilon}_{ik} = {}_T\alpha_{ik}({}_T\alpha_{ik} \pm \varepsilon^i_k \cdot \sum_{s=1}^{m} {}_T q_{ks}) > 0 \Rightarrow \varepsilon^i_k < \left| \frac{{}_T\alpha_{ik}}{\sum_{s=1}^{m} {}_T q_{ks}} \right|. \tag{13} $$

Proposition 2. *Suppose* there are some errors of expert estimates in target coefficient γ_{is} $(s \in M_i)$ of the agent utility function f_i and each of them doesn't exceed the value (14) *then* the dominant strategy of the agent i is invariable.

$$ \varepsilon^i = \min_{k \in M_i} \left| \frac{{}_T\alpha_{ik}}{\sum_{s=1}^{m} {}_T q_{ks}} \right|. \tag{14} $$

Indeed, this follows from the necessary of working the condition (13) for all control concepts of agent i. The value (14) is called the *estimate of tolerance to errors in all values of* γ_{is} or *allowable error in all values* γ_{is} $(s \in M_i)$.
Fig. 4 shows the results of calculations of (14) for both agents (Russia and Norway). Fig. 4 shows that the allowable error in all values γ_{is} is very small. It is an illustration of the unstable situation of Russia in this game (5). The optimal solution for Russia in the model is not stable for a fixed system of priorities $\gamma_{1,3} = 0.5$ and $\gamma_{1,8} = 0.5$.

The values of the target coefficients $\gamma_{2,4} = 0.5$ and $\gamma_{2,11} = 0.5$ for Norway are stable enough. The optimal solution for Norway stays the same for all γ_{ij} in range ± 0.5. Note that the values of the allowable errors for the target concepts for Russia at $T = 5$ are equal to zero. It is happened because the optimal strategy for Russia at $T = 5$ is an omission (see Fig. 2). This strategy is not stable to errors in the coefficient values γ_{ij}, because (7).

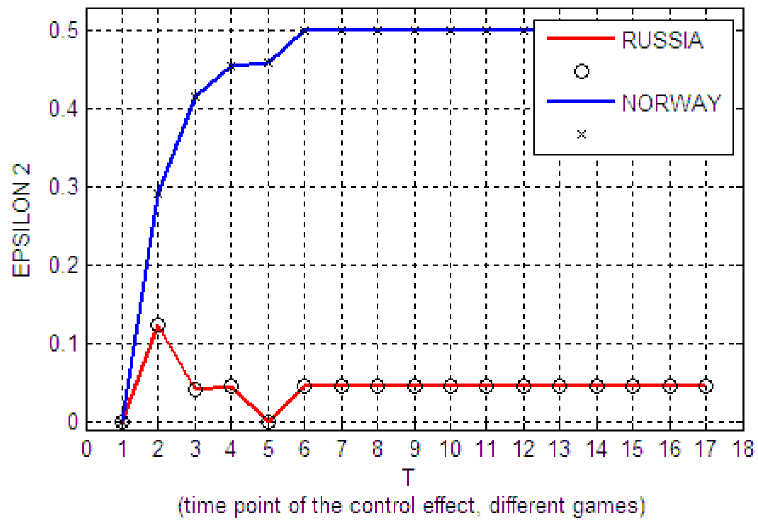

Fig. 4.

3.3. Estimate (lower bound) of tolerance to error in a weight of arc in digraph of cognitive map

Let consider the situation where the expert make an error in δ in a weight of arc w_{rs} of cognitive map. In this case, the adjacency matrix of a digraph to be the next:

$$W_\delta = \begin{pmatrix} w_{11} & \cdots & w_{1s} & \cdots & w_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{r1} & \cdots & w_{rs} \pm \delta & \cdots & w_{rm} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{m1} & \cdots & w_{ms} & \cdots & w_{mm} \end{pmatrix}$$

In this situation the error δ is the reason for errors in elements of *matrix of an influence reachability by the time* $T\,{}_TQ$:
${}_TQ_\varepsilon = (E + W_\delta + W_\delta^2 + \cdots + W_\delta^{T-1})$.
The elements of the matrix $({}_TQ_\varepsilon - {}_TQ)$ correspond to the changes in every element of matrix of an influence reachability by the time T caused by an error δ in adjacency matrix W. The change in one element can be represented as (15), where $P_k(w_{ks})$ the algebraic sum of products of elements of matrix W. It is necessary to estimate the error ε in the value of elements of matrix ${}_TQ$ for getting the estimation the

tolerable error δ in the matrix element w_{rs}.

$$\left|{}_T q_{ij}^{\varepsilon} - {}_T q_{ij}\right| = \left|\delta \cdot P_1(\{w_{ks}\}) + \delta^2 \cdot P_2(\{w_{ks}\}) + \cdots + \delta^{T-1} \cdot P_{T-1}(\{w_{ks}\})\right| < \varepsilon. \tag{15}$$

Similar to the arguments (12), (13) we obtain an estimate of error ε_k^i for the elements of ${}_T Q$ (16)-(17).

$${}_T\alpha_{ik}^{\varepsilon} = \gamma_{i1}({}_T q_{k1} \pm \varepsilon_k^i) + \gamma_{i2}({}_T q_{k2} \pm \varepsilon_k^i) + \cdots + \gamma_{is}({}_T q_{ks} \pm \varepsilon_k^i) + \cdots + \gamma_{i,m}({}_T q_{k,m} \pm \varepsilon_k^i). \tag{16}$$

$${}_T\alpha_{ik} \cdot {}_T\alpha_{ik}^{\varepsilon} = {}_T\alpha_{ik}({}_T\alpha_{ik} \pm \varepsilon_k^i \cdot \sum_{s=1}^{m} {}_T\gamma_{is}) > 0 \Rightarrow \varepsilon_k^i < \left|\frac{{}_T\alpha_{ik}}{\sum_{s=1}^{m} {}_T\gamma_{is}}\right|. \tag{17}$$

The value ε_k^i is so error in every element of matrix ${}_T Q$ that the value of the control concept k in the dominant strategy of the agent i is invariable. It is clear that if error in every element of matrix ${}_T Q$ less than (18) then the dominant strategies of every agent is invariable.

$$\varepsilon = \min_{i \in N} \min_{k \in M_i} \left|\frac{{}_T\alpha_{ik}}{\sum_{s=1}^{m} \gamma_{is}}\right| \tag{18}$$

We claim that:
Proposition 3. *Suppose* there is an error of expert estimate in a weight of arc w_{rs} of cognitive map and it doesn't exceed the value δ from (15), where ε is calculated as (18) *then* the dominant strategies of every agent is invariable.
The value δ is called the *estimate of tolerance to error in a weight of arc in digraph of cognitive map or allowable error in* $w_{rs}(r, s \in M)$.
We can calculate the allowable error δ in the weight of w_{rs} using the expression (15) with ε from (18). Note that the expression standing on the left of the inequality sign in (15) is a continuous function from δ, which always intersects the Ox-axis at 0. Consequently, if $\varepsilon > 0$ then there are values in a neighborhood of 0 witch satisfy (15).
On the basis of Symbolic Math Toolbox MATLAB were calculated the allowable error δ in all weights of arcs in digraph of cognitive map on Fig. 1 (Fig. 5).

Fig. 5 shows the dependence of the allowable error δ from target time T in the game (5). The weights of the links ($\sharp 1 \to \sharp 6$) and ($\sharp 2 \to \sharp 6$) have the most low values of allowable errors δ. This is illustration of the importance of the concept $\sharp$ 6 for agents target. The weight of the arc ($\sharp 4 \to \sharp 11$) is less critical, but important. The reason is probably in the fact that both concepts $\sharp$ 4 and $\sharp$ 11 are the target concepts for Norway. This connection provides an additional agreement between two sub-targets. The values of links ($\sharp 1 \to \sharp 12$) and ($\sharp 2 \to \sharp 12$) are also important. The arcs ($\sharp 1 \to \sharp 6$), ($\sharp 2 \to \sharp 6$), ($\sharp 1 \to \sharp 12$) and ($\sharp 2 \to \sharp 12$) are the main causal links witch providing connectivity the control concepts with all other in cognitive map.

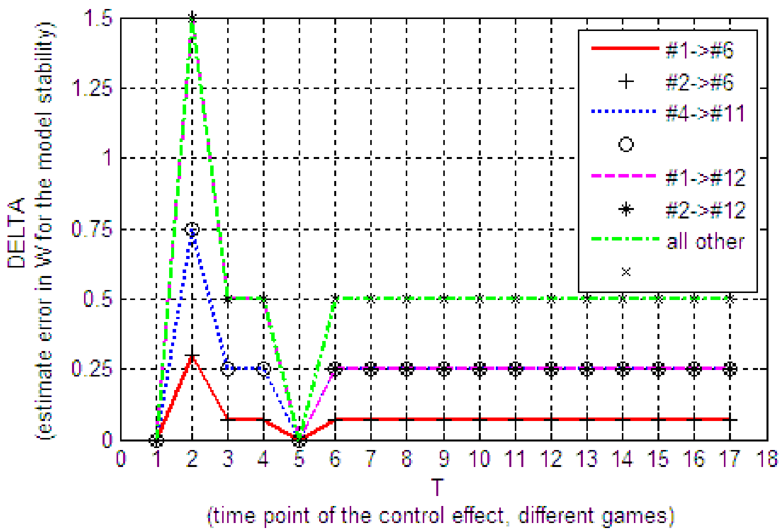

Fig. 5.

4. Conclusion

This paper deals with the issue of input data error tolerance of the game on cognitive map according to expert evaluations. The model is constructed on the basis of conflict of interests between Russia and Norway on maritime delimitation in the Barents Sea. As a result was evaluated the error in the coefficients of the utility functions of agents as well as in the weights of the arcs of the digraph of cognitive maps in this model. The estimation of errors tolerance made it possible to understand the structural properties of the model and to assess the degree of selecting target concepts feasibility.

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The Strategy of Tax Control in Conditions of Possible Mistakes and Corruption of Inspectors

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Abstract A generalization of the game-theoretical model of tax control adjusted for possible corruption and inspectors mistakes is considered. The hierarchical model has a three-level structure: at the highest level of a hierarchy is an administration of tax authority, in the middle is an inspector, subordinated to tax administration, and at the lowest level are n taxpayers. It is supposed, that an interaction between risk-neutral players of different levels of a hierarchy corresponds to scheme "principal-to-agent".

The model is studied for the case when the penalty is proportional to the level of evasion. It is supposed that a tax inspector may turn out a bribetaker or make ineffective tax audit, i.e. make a mistake and don't reveal an existing tax evasion.

In the case of corruption a tax control supposed to be effective, i.e. reveals existing tax evasions always. As in previous models, it is supposed that fact of corruption is very difficult to reveal and an inspector is punished only for negligent audit.

In the case of ineffective auditing it is assumed that the tax inspector can mistake and miss an existing evasion with the probability, which can be considered as a part of negligent inspectors of their total number.

For every possible situation the players profit functions and optimal strategies are found.

Keywords: tax auditing, tax evasion, corruption, ineffective auditing.

1. Introduction

One of the most important aspects of modeling of taxation is the tax control. Mathematical models of tax inspection considering a corruption earlier were studied in (Chander and Wilde, 1992), (Hindriks and Keen and Muthoo, 1999) and (Vasin and Panova, 1999). Due to the mathematical tradition, founded in these works the game-theoretical model of tax audit adjusted for possible corruption and inspectors' mistakes is considered.

In the basis of this model there is a hierarchical game, described in (Kumacheva and Petrosyan, 2009). In the mentioned game the tax authority (high level of the hierarchy) and the finite number of taxpayers (low level of the hierarchy) are players, is considered.

To investigate the case of corruption let's consider an improved hierarchical model, which has a three-level structure: at the highest level of a hierarchy is an administration of tax authority, in the middle is an inspector, subordinated to tax administration, and at the lowest level are n taxpayers. As in the previous models, such as (Chander and Wilde, 1998) and (Vasin and Morozov, 2005), it is supposed that the interaction between the tax authority and each taxpayer corresponds to the scheme “principal-to-agent”. The players’ behaviour is supposed to be risk neutral.

For studying the case of inspectors’ mistake let’s suppose that the auditing is not 100%-effective and consider a probability of an inspector’s mistake as a parameter of the model.

2. The Base Model

In the studied model a set of n taxpayers is considered; each of them has income level equal to i_k, where $k = \overline{1,n}$. The income of the taxpayer r_k is declared at the end of a tax period, where $r_k \leq i_k$ for each $k = \overline{1,n}$. Let t be the tax rate, π be the penalty rate. These rates are assumed to be constant.

As in (Kumacheva and Petrosyan, 2009) and (Boure and Kumacheva, 2010), it is considered here that the audit of the k-th taxpayer is made by the tax authority with the probability p_k ($0 \leq p_k \leq 1$). Model is constructed following the assumption, that the taxpayers are aware of these probabilities.

If the evasion is revealed as the result of the tax audit, then the evaded taxpayer should pay the penalty, which depends on the evasion’s level. In (Boure and Kumacheva, 2010) the model was studied in four cases of penalties, which are known from (Vasin and Morozov, 2005):

1. the net penalty is proportional to evasion;
2. the penalty is proportional to difference between true and payed tax;
3. the penalty is restricted by the given level of the agent’s minimal income in the case of his nonoptimal behaviour;
4. the post-audit payment is proportional to the revealed evaded income.

Let’s consider the first case, when the penalty is proportional to evasion. In other words, if the evasion is revealed, the taxpayer should pay the underpaid tax and the penalty, both of which depend on the evasion’s level.

Without consideration of possible corruption the expected tax payment of the k-th taxpayer in this case of penalty is defined from the equation

$$u_k = tr_k + p_k(t + \pi)(i_k - r_k), \tag{1}$$

where the first summand is always paid by the taxpayer (pre-audit payment), and the second – as the result of the tax auditing, made with probability p_k (post-audit payment).

Let’s make the model more sophisticated by upgrading to a three-level game. The tax authority is divided on the administration and a subordinated inspector, who may be a corruptionist. As earlier, it is supposed, that the interaction between risk-neutral players of the different levels of the hierarchy corresponds to the scheme “principal-to-agent”.

The tax authority sends an inspector for the tax audit with the probability p_k, which costs c_k, $k = \overline{1,n}$. For the bribe b_k audit inspector can agree not to inform his administration about the evasion revealed. With the probability $\tilde{p_k}$ the tax administration makes corruption-free re-auditing of this taxpayer, which costs $\tilde{c_k}$. Both of the audits are supposed to be effective, i. e. they reveal the existing evasion.

If a result of re-auditing is the revelation of the fact that the evasion was concealed by the inspector, the taxpayer must pay $(t + \pi)(i_k - r_k)$ (as earlier) and the inspector must pay a fine $f \cdot (i_k - r_k)$, where f is an inspector's penalty coefficient. Following (Hindriks and Keen and Muthoo, 1999), it is supposed, that the fact of corruption is very difficult to reveal and an inspector is punished only for negligent audit.

2.1. The Condition of an Evasion

The k-th taxpayer evades, if his expected payments in the case of evasion are less than the tax, which he pays, declaring his true income, i. e. the inequality

$$tr_k + p_k(t + \pi)(i_k - r_k) < ti_k,$$

which holds or, that is equivalent to:

$$p_k(t + \pi)(i_k - r_k) < t(i_k - r_k). \tag{2}$$

The condition (2) is violated, if the probability of audit $p_k = p^*$ for each $k = \overline{1, n}$, where

$$p^* = \frac{t}{t + \pi}. \tag{3}$$

2.2. The Condition of a Bribe Existence

Let's suppose, that there was an audit and an inspector identified an evasion of the k-th taxpayer. Let's define the condition, in which it is more profitable for a taxpayer to pay a bribe to an inspector, then to pay a post-audit payment. Expected payments of a taxpayer in a case, when a bribe was given, but the tax evasion was revealed whatever as a result of re-audit, is $\tilde{p_k}(t + \pi)(i_k - r_k) + b_k$. The bribe is profitable for a taxpayer, when

$$\tilde{p_k}(t + \pi)(i_k - r_k) + b_k < (t + \pi)(i_k - r_k). \tag{4}$$

It follows from this inequality, that the value of profitable bribe for the k-th taxpayer should be less than his tax payments when there was no re-audit:

$$b_k < (1 - \tilde{p_k})(t + \pi)(i_k - r_k). \tag{5}$$

It is profitable for an inspector to take a bribe, if it is more than an expected penalty, which an inspector should pay in the case of re-auditing. That is

$$b_k > \tilde{p_k} f(i_k - r_k). \tag{6}$$

Thus, we can obtain a mutually an inspector and a taxpayer beneficial bribe condition:

$$\tilde{p_k} f(i_k - r_k) < b_k < (1 - \tilde{p_k})(t + \pi)(i_k - r_k). \tag{7}$$

It means, that a bribe is possible only, when an interval

$$(\tilde{p_k} f(i_k - r_k); (1 - \tilde{p_k})(t + \pi)(i_k - r_k))$$

exists. It doesn't exist, if the probability of re-auditing takes value

$$\tilde{p}^* = \frac{t + \pi}{t + \pi + f}, \tag{8}$$

which is defined as the solution of the equation, which is got as a marginal case of the inequality (7).

3. The Stages of the Game

The studied hierarchical game can be divided on the next stages.

On the first stage a tax inspector is not considered as a separate level of hierarchy and an interaction between the tax authority and a taxpayer is studied. Being the high-level player, the tax authority makes the first move, choosing a pair of vectors: $p = (p_1, \ldots, p_n)$ and $\tilde{p} = (\tilde{p_1}, \ldots, \tilde{p_n})$. Components of vector p are values of probabilities of audits of each taxpayer, and components of vector $\tilde{p}$ are values of probabilities of the re-auditings of activities of tax inspectors. The second move is made by taxpayers: they make decisions to evade or to pay their taxes honestly, that is to declare $r_k < i_k$ or $r_k = i_k$, $k = \overline{1,n}$. If there was no tax audit, the game can be considered as finished on this stage.

If the tax authority sends an inspector for auditing of the k-th taxpayer, the second stage of the game begins. There is an interaction between an inspector and a taxpayer on this stage. Let the first taxpayer's move is a choice of the strategy of evasion. Then the strategy of his second step is a decision whether to give a bribe to an inspector or not. The strategy of the inspector, who revealed evasion, is the choice whether to take a bribe or not.

The third stage is an interaction between the administration of the tax authority and both of the subordinated levels of hierarchy. The realization of this stage does not depend on the results of the previous stage and happens, if the tax authority makes a re-audit of an inspector's activity.

Thus, players' strategies are the following. For each $k = \overline{1,n}$ the administration of the tax authority chooses probabilities p_k and $\tilde{p_k}$ of auditing of a taxpayer and re-auditing of an inspector's activity correspondingly. On the first move the taxpayer makes a decision to evade or not, on the second move – to give a bribe to an inspector or not. The strategy of the inspector, who revealed evasion, is the choice whether to take a bribe or not.

4. Possible Situations

In the model considered there are three possible situations:

1. there were an evasion and a given bribe;
2. a taxpayer evaded, but there was no corruption;
3. an honest payment due to a declaration.

Furtheron, let's consider each of them separately.

4.1. An Evasion with a Corruption

The situation of an evasion with a bribe is possible in the following cases:

1. Let conditions (2) and (7) be fulfilled, i. e. an evasion is profitable for a taxpayer and a bribe is profitable for both sides (the taxpayer and the inspector).
2. Let the condition (2) isn't fulfilled, i. e. there is a big risk of revelation of an evasion of a taxpayer. But the interval, defined in (7), exists, therefore, it is possible to reach an agreement about a bribe.

As in the previous models (Kumacheva and Petrosyan, 2009) and (Boure and Kumacheva, 2010), let's consider expected tax payments of the k-th taxpayer, k =

$= \overline{1,n}$. Let's change (1), assuming a possibility of a bribe. Then we obtain that in both cases expected tax payments are

$$u_k = tr_k + p_k[\tilde{p_k}(t+\pi)(i_k - r_k) + b_k]. \tag{9}$$

The expected payoff w_k of the k-th taxpayer is:

$$w_k = i_k - tr_k - p_k[\tilde{p_k}(t+\pi)(i_k - r_k) + b_k].$$

The inspector takes a bribe, but can be audited and fined, therefore, his expected payoff, got from auditing of the k-th taxpayer, (over his wages), is

$$J_k = p_k(b_k - \tilde{p_k} f(i_k - r_k)).$$

The tax authority's profit function in this case has the form

$$R_k = tr_k + p_k\left[\tilde{p_k}\left((t+\pi+f)(i_k - r_k) - \tilde{c_k}\right) - c_k\right]. \tag{10}$$

It should be noted that the tax authority's net income, a taxpayer's declared income and, therefore, his expected payoff in general depend on the strategy of the tax authority. Thus, the functions $R_k(p_k, \tilde{p_k})$, $r_k(p_k, \tilde{p_k})$ and $w_k(p_k, \tilde{p_k})$ will be considered further.

Choosing a strategy $r_k(p_k, \tilde{p_k})$ at the first stage of the game, the taxpayer analyzes the possibility of both an evasion and a fact of corruption. Some combination of the mentioned distortions can be realized when the expected post-audit payments of the taxpayer (the second summand of (9)) is less then his underpaid taxes:

$$p_k[\tilde{p_k}(t+\pi)(i_k - r_k) + b_k] < t(i_k - r_k).$$

Taking into account, that a bribe, which the taxpayer means to give, should satisfy the inspector ((6) is fulfilled), the last inequality takes a form:

$$p_k\tilde{p_k}(t+\pi+f) < t.$$

If this condition isn't fulfilled, the probabilities p_k and $\tilde{p_k}$ relate as follows:

$$p_k\tilde{p_k} = \frac{t}{t+\pi+f}. \tag{11}$$

A fact of choosing strategies, that satisfied (11), by the tax authority, does not let the simultaneous implementation of the evasion and bribe. Let's consider the next situations.

4.2. An Evasion without Corruption

Let p_k satisfies the condition (2). The k-th taxpayer evaded, made his declared income lower than his true level. However, he risked vainly, and the tax authority send an inspector, who revealed the tax evasion. Negotiations about a bribe are doomed to failure, because the probability of re-audit is chosen by the tax administration correspondingly to (11).

In this situation the k-th taxpayer's profit function is

$$w_k = i_k - tr_k - p_k(t+\pi)(i_k - r_k);$$

the tax authority's payoff is defined as

$$R_k = tr_k + p_k\left[((t+\pi)(i_k - r_k) - c_k) - \tilde{p_k}\tilde{c_k}\right] =$$
$$= tr_k + p_k\left((t+\pi)(i_k - r_k) - c_k\right) - \frac{t}{t+\pi+f}\,\tilde{c_k}.$$

The inspector gets nothing over his usual wages.

4.3. The Honest Payment Corresponding to the Declaration

Let's suppose, that the condition (2) is violated. If p_k and $\tilde{p_k}$ relate as in (11), a taxpayer will not risk to evade, understanding that if an evasion is revealed it will be impossible to reach an agreement with an inspector about mutually beneficial bribe.

In the considered situation the expected tax authority's net income R_k (profit function), got from the taxation of the k-th taxpayer is

$$R_k = t\,i_k - p_k(c_k + \tilde{p_k}\tilde{c_k}),$$

the taxpayer's declared income $r_k^* = i_k$, the bribe $b_k = 0$. I. e., the taxpayer's payoff is $w_k = i_k - ti_k$ (his true income level less honestly paid tax), an inspector's benefit over his usual wages is $J_k = 0$ (he does not get a bribe).

Proposition 1. *The maximum tax authority's income, got from the taxation of the k-th taxpayer, is reached when audit probability $p_k = p^*$ and re-audit probability $\tilde{p_k} = \tilde{p}^*$:*

$$\max_{p_k,\tilde{p_k}} R_k(p_k,\tilde{p_k}) = R_k(p^*,\tilde{p}^*) = ti_k - \frac{t}{t+\pi}\,c_k - \frac{t}{t+\pi+f}\,\tilde{c_k}. \tag{12}$$

Herewith the taxpayer's maximum payoff $w_k(p^*,\tilde{p}^*) = i_k - ti_k$ is reached when his declared income $r_k^* = r_k(p^*,\tilde{p}^*) = i_k$; when the taxpayer and the tax authority have such strategies, the inspector's benefit over his usual wages is $J_k = 0$.

Proof. At the first stage the tax authority's optimal strategy is a choice of the minimum value of the probability of audit, that guarantees violation of the condition (2), that is, $p_k = p^*$, where p^* is defined from the equality (3). From the results, given in (Boure and Kumacheva, 2010) it follows that this strategy is the tax authority's optimal strategy in order to maximize its income.

The k-th taxpayer will declare $r_k^* = i_k$, if on the second stage of his interaction with the tax authority the possibility of corruption is excluded, i. e., $\tilde{p_k}$ relates with $p_k = p^*$ by the condition (11). If p^* from (3) is put in (11), the minimum value of the probability of re-audit, which guarantees violation of the condition (7), will be obtained. It means that $\tilde{p_k} = \tilde{p}^*$, where $\tilde{p}^*$ is defined from (8), for each $k = \overline{1,n}$.□

4.4. Cases, Allowing Unprofitable Activities of the Tax Authority

The case of unprofitable activities of the tax authority, when the parameters t, π and c_k relate so as inequality

$$(t+\pi)i_k < c_k \tag{13}$$

holds was considered in (Boure and Kumacheva, 2010).

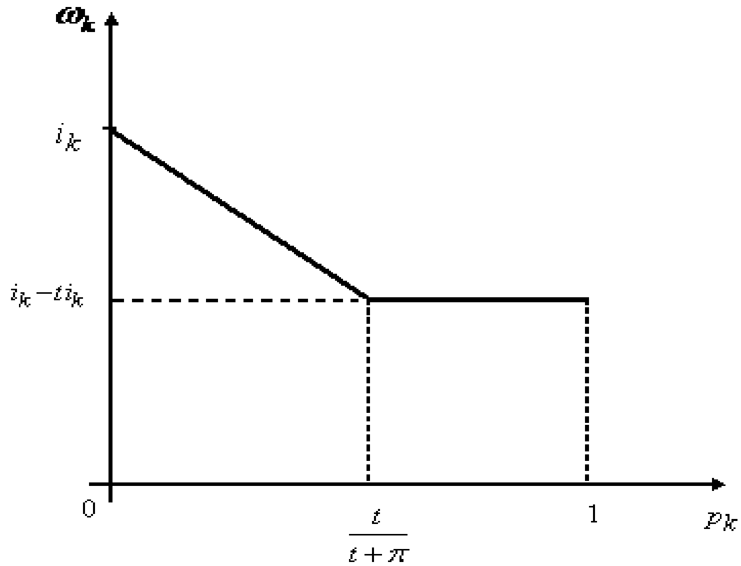

Fig. 1: Dependence the k-th taxpayer's expected profit w_k on the probability of audit p_k

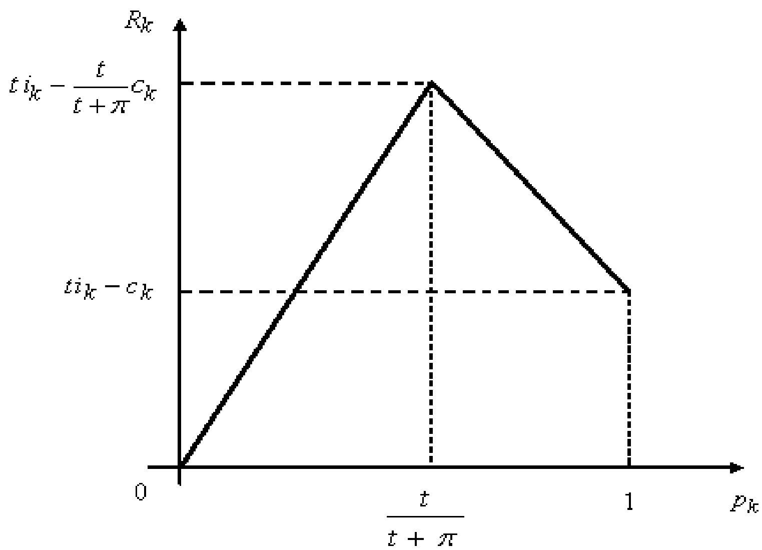

Fig. 2: Dependence the tax authority's expected profit R_k on the probability of audit p_k

The proposition was formulated for this case. This proposition implies that for the tax authority it is optimal (in order to maximize its profit function) not to audit the k-th taxpayer, because each value of the probability of the audit $p_k > 0$ gives only dead losses, i. e. $R_k \leq 0$.

If in the case of unprofitable activities of the tax authority the players act optimally (corresponding to the mentioned proposition), the game is finished on the first stage, as an inspector is not sent for auditing. Thus, there is no sense in the further interaction between players of different levels of the hierarchy.

If the inequality

$$(t+\pi)i_k \geq c_k \tag{14}$$

is fulfilled for the parameters t, π and c_k, the second stage of the game is implemented. Re-auditing with any value of the probability will be unprofitable for the tax authority, if the relations of the parameters t, π, f, c_k and $\tilde{c_k}$ such, as following inequality

$$\tilde{c_k} > (t+\pi+f)\left(i_k - \frac{c_k}{t+\pi}\right) \tag{15}$$

holds.

In this case the optimal (in order to maximize its net tax income) strategy is $\tilde{p_k} = 0$. But then for any value of p_k the taxpayer and the inspector have an opportunity to reach an agreement about bribe b_k, and, thus, the taxpayer can evade with impunity. In this case the optimal audit strategy is $p_k = 0$.

It is obvious that if (6) holds, the right side of inequality (15) becomes a negative and, therefore, it is fulfilled for any relation of $\tilde{c_k}$, π and f. Thus, inequality, opposite to (15), implies (4).

5. The Optimal Player's Strategies

Following (Chander and Wilde, 1998) and (Vasin and Morozov, 2005), let's notice, that the tax authority's strategy in general is some optimal contract (Vasin and Morozov, 2005) or optimal scheme (Chander and Wilde, 1998) $(t, \pi, p, \tilde{p}, f)$, where t, π and f are the parameters of long-term tax control, and $p = (p_1, \ldots, p_n)$ and $\tilde{p} = (\tilde{p_1}, \ldots, \tilde{p_n})$ are the strategies, chosen by the tax authority in each tax period for the k-th taxpayer, $k = \overline{1,n}$.

As it was in (Boure and Kumacheva, 2010), the tax authority's net income is defined as a sum of the payoffs R_k, $k = \overline{1,n}$. It's obvious, that

$$\max_{p,\tilde{p}} R = \sum_{k=1}^{n} R_k(p^*, \tilde{p}^*).$$

Correspondingly to the previous proposition, the maximum value of the net tax income from taxation of the k-th taxpayer is reached on a restricted class of strategies of the tax authority, which fulfills (11). The taxpayer's best reply on the tax authority's activity (due to the mentioned optimal strategies) is defined in the same proposition.

The generalization of the considered reasonings is formulated in the next theorem.

Theorem 1. *1. If a relation of parameters t, π, f, c_k and $\tilde{c_k}$ allows to make a profitable audit of the k-th taxpayer (the inequality, opposite to (15), holds), the* *maximum of the tax authority's income, got from taxation of the k-th taxpayer, is reached when the strategy of auditing (3)*

$$p_k = p^* = \frac{t}{t+\pi}$$

and the strategy of re-auditing (8)

$$\tilde{p_k} = \tilde{p}^* = \frac{t+\pi}{t+\pi+f}$$

and has a form (12). In conditions of such strategy of the tax authority the k-th taxpayer's optimal strategy (in order to maximize his payoff) is $r_k^*(p^*, \tilde{p}^*) = i_k$*; his payoff is* $w_k(p^*, \tilde{p}^*) = i_k - ti_k$.

2. *In the case, when for parameters* t, π, f, c_k *and* $\tilde{c_k}$ *holds (15), the maximum of the tax authority's income is reached when the strategy of auditing* $p^* = 0$ *and the strategy of re-auditing* $\tilde{p}^* = 0$*; its value is* $R_k = 0$*. In this case the k-th taxpayer optimal strategy is* $r_k^*(0,0) = 0$*; his payoff is* $w_k(0,0) = i_k$.
The inspector's payoff is $J_k = 0$ *in both cases.*

Thus, taxpayers' and the tax authority's optimal strategies are found in conditions of possible corruption.

6. Possible Mistakes of Inspectors

Let's suppose that the auditing is not 100%-effective. It means that tax inspectors can make unintentional mistakes and miss an existing evasion.

Let's consider a parameter μ, which has two different meanings.

On the one hand μ is the probability of an inspector's mistake. Then, from the probabilistic point of view we obtain that the value $(1-\mu)$ is the effectiveness of auditing. Therefore it can be included as an additional specifying multiply of the probability of auditing p_k in every equality.

On the other hand μ can be considered as a part of negligent inspectors of their total number. Then, the probability of re-auditing $\tilde{p_k}$ depends on μ. As in (Hindriks and Keen and Muthoo, 1999), it is considered that there is no way to identify if the auditing was negligent or the inspector was corrupted. So, as in the case of corruption, the negligent inspector pays a fine $f \cdot (i_k - r_k)$ and the tax evader pays penalty $(t+\pi)(i_k - r_k)$. To construct the optimal strategy the tax administration needs to obtain an estimation $\widehat{\mu}$ of the probability μ.

7. Conclusion

In this paper the game-theoretical model of tax control, based on the hierarchical game with a three-level structure and adjusted for possible corruption and inspectorsŠ mistake, is considered. The playersŠ profit functions and optimal strategies are found considering two mentioned features.

In the previous papers with the familiar problems (Chander and Wilde, 1992) and (Hindriks and Keen and Muthoo, 1999) and (Vasin and Panova, 1999) a binary distribution of taxpayers' income was considered. The game-theoretical model, presented in this paper, differs from the mentioned models by the assumption about a nonuniformity of taxpayers not only on the income level, but on the costs of auditing for the tax authority. Another specific feature, on which this hierarchical model was constructed, is the assumption that a strategy of the tax authority doesn't depend on the taxpayer's income, declared in given tax period.

However, it should be noted, that results, obtained for this model, highly correlate with previous conclusions, published in the papers (Chander and Wilde, 1992) and (Vasin and Panova, 1999), which are devoted to problems of taxation.

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Asymmetric Equilibria in Stahl Search Model *

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Abstract The paper explores the classic consumer search model introduced by Stahl in (Stahl, 1989). Literature uses the unique symmetric Nash Equilibrium, but does little to discuss asymmetric Equilibria. This paper describes all possible asymmetric Nash Equilibria of the original model, under the common literature assumption of consumer reserve price. Those include strategies of three types: pure, continuous mixing and a mixture of the previous two types. The findings suggest that on some level, lower than the symmetric Equilibrium, price dispersion will still exist, together with some level of price stickiness, both observed in reality.

Keywords: Sequential Consumer Search, Oligopoly, Asymmetric NE
JEL Classification Numbers: D43, D83, L13.

1. Introduction

Empirical studies, such as (Bazucs and Imre, 2009) or (Martin-Oliver et al., 2005), have established that significant price dispersion exists even for homogeneous goods. As the literature suggests, this effect is observed in many market structures and is persistent. One of the explanations for this phenomenon is that consumers search for the cheapest price. Since searching is costly, consumers may settle down for a slightly higher price. In the literature many papers deal with search models, for example (Burdett and Judd, 1983), (Burdett and Smith, 2009), (Carlson and McAffee, 1983), (Stahl, 1989), (Varian, 1980) and (Watanabe, 2010). Search models were developed originally in order to provide a solution to the Diamond Paradox (Diamond, 1971), which predicted a complete market failure. The search models vary in the scope, the length, the stopping condition or the information revealed during the consumer search.

Additional Empiric studies, for example (Janssen et al., 2004), reveal that the model introduced by Stahl in (Stahl, 1989) perform very well and predicts correctly the pricing model of 86 out of 87 tested products. Moreover, (Baye et al., 2009) empirically shows the existence of the two consumer types predicted by this model. Therefore, this paper will concentrate on the Stahl search model.

The Stahl model is dealt extensively in the literature, and is a a very popular model. Numerous extension to the Stahl Model were introduced, and the various extensions are dealing with nearly every aspect of the model. Among those are introducing heterogeneous searchers. Example for such extensions are (Chen and Yhang, 2011) and (Stahl, 1996), where the searchers have different cost for each additional store they visit. They can differ by the search scope, as discussed in (Astone-Figari and Yankelevich, 2010), where some stores are near, and thus will be searched first. Another extension introduced advertisement costs, as discussed,

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for example by (Chioveanu and Zhou, 2011). There are also models where already the first price is costly, such as (Janssen et al., 2005), or no possibility to freely return to previously visited store, such as (Janssen and Parakhonyak, 2008). The literature has discussion regarding the sequential search in the model and looks also at non-sequential search, for example in (Janssen and Moraga Gonzales, 2004), or the unknown production cost as shown in (Janssen et al., 2009). Most assumptions of the model introduced by Stahl in (Stahl, 1989) are discussed extensively, except one main assumption, used extensively in the literature. This is the focus on symmetric equilibria, where all sellers select an identical strategy. One of reasons is the mathematical complexity: (Carlson and McAffee, 1983) and (Rotschild, 1973) showed that in symmetric equilibria consumer reserve price must exist, and in asymmetric ones it may not. Reserve price assumption is common in the literature, and therefore, the paper considers only NE with reserve price, yet justifies the rationality behind it. Nevertheless, one should note that additional Equilibria without reserve price may exist, and fall beyond the scope of this paper.

For comparison, in the Varian search model introduced in (Varian, 1980), it is shown in (Baye et al., 1992) that there are asymmetric equilibria, but those can be ignored. In the Stahl model there might be additional equilibria when different settings are considered. Additionally, it is shown in (Baye and Morgan, 1999) that one can receive additional equilibria in commonly known games, when the scope is broadened. This paper finds a family of asymmetric equilibria to the original model, where strategies are of (at most) three types - some sellers (at least two) mix over the entire available price interval with a seller invariant distribution, whereas the second group (might be empty) selects the reserve price as a pure strategy. The third group (might be empty) has a pricing distribution which consists of a mass point at the reserve price, and use the same distribution as the first group up to a seller specific cutoff price.

An additional outcome of this model can explain price stickiness, as described for example in (Davis and Hamilton, 2003). Many equilibria found here have mass points on certain prices. This implies that with some probability the price in the previous round can be the same also in the next round, even though the seller is mixing. In reality it is known that that prices do not change too often and are sticky. The results of this model can provide an insight on why it is so, as prices selected with mass points can remain unchanged during several periods.

The structure of the paper is as follows: first the Stahl model is formally introduced. Then knowledge and structure of the game are discussed. Afterward the structure of the asymmetric NE of the model is discussed, followed by an example of such Equilibrium. Lastly the implications of the results are discussed, and suggestions on how those results can be empirically tested.

2. Model

The Stahl model, as introduced in (Stahl, 1989) is formally described below. Notation was adjusted to the recent literature on the Stahl model.

There are N sellers, selling an identical good. Each seller owns a single store. The production cost is normalized to 0, and assume that the seller can meet the demand. Additionally, there are buyers, each of whom wishes to buy a unit of the good. The mass of buyers is normalized to 1. This implies that there are many small buyers, each of which is strategically insignificant.

The sellers are identical, and set their price once at the first stage of the game. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place.

The buyers are of two types. A fraction μ of buyers are shoppers, who know where the cheapest price is, and they buy at the cheapest store. In case of a draw they randomize uniformly over all cheapest stores, spreading equally among the cheapest stores. The rest are searchers, who sample prices. Sampling price in the first, randomly and uniformly selected, store is free. It is shown in (Janssen et al., 2005) that if it is not the case then some searchers would avoid purchase, and in all other aspects the results would be the same. If the price at the store is satisfactory - the searcher will buy there. However, if the price is not satisfactory - the searcher will go on to search, sequentially, in additional stores, where each additional search has a cost c. The second (or any later) store is randomly and uniformly selected from the previously unvisited stores, and the searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing uniformly in case of a draw.

The buyers need to be at both types (namely, $0 < \mu < 1$). If there are only shoppers - it is the Bertrand competition setting (Baye and Morgan, 1999), and if there are only searchers the Diamond Paradox (Diamond, 1971) is encountered, both well studied.

Before going on, make a technical assumption on the model. In order to avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals. This limitation allows all of the commonly used distributions and mixtures between such.

Additionally, note two very basic observations:

- Sellers cannot offer a price above some finite bound M. This has the interpretation of being the maximal valuation of a buyer for the good.
- Searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.

2.1. Reserve Price and Knowledge

In the symmetric Stahl model the consumers have a reserve price in NE. The reserve price determines the behavior of consumers - the searcher is satisfied and searches no further if and only if the price is (weakly) below her reserve price, unless all stores are visited. If the price is below the reserve price - the search stops and the consumer purchases the good, if not - the search will continue. If all prices are above the reserve price - the cheapest store will be selected, after searching in all stores. In order to maintain in one line with the vast literature of the model, and being able to compare the results reserve price existence is assumed. However, one needs to specify when and how the reserve price is determined. The reserve price is determined simultaneously to the price strategy choice of the sellers. The reserve price is identical to all searchers, as was also in the original model. It will be denoted throughout the paper as P_M. How the reserved price is determined is dealt with below.

Below is the setting that allows searching, as difference in prices can provide incentives to it. Moreover, it extends the symmetric Stahl model knowledge available to the searchers, as the reserve price is c above the expected price of a seller. There, they knew the mixed strategy chosen by the sellers, and their behavior (whether to search further or not) was adjusted accordingly. Here, as the strategies of the sellers do not have to be identical, a price observed implies something on prices not observed yet. After observing price p in a store, the searcher can estimate the probability that the strategy of the seller is a specific one, and from that induce the expected price in other stores. Therefore, it is important to introduce beliefs and explain how exactly these are adjusted while searching.

The searchers have beliefs regarding the prices set. For each possible (pure and mixed) strategy s of the model is attached a belief, stating how many sellers are actually using this strategy denoted as $n(s)$ (clearly the sum of $n(s)$ is n, the number of stores). Each strategy has an expected price, denoted $e(s)$. Now, it is easy to explain how the searcher will determine whether she searches on or not.

Suppose the searcher observed the price p. Let the probability that this price p came from strategy s be denoted as $prob(p, s)$. For this the searcher calculates chance that s is selected by some seller and the probability that p is the realization of strategy s (relevant for mixed strategies). One needs to note that if some strategies (with positive $n(s)$) have a mass point on p only those will be considered, and if there are no mass points on p the densities will play a role. Formally:

$$prob(p, s) = \frac{n(s)f(s)}{\sum_{p \in s'} n(s')p(s')} \tag{1}$$

Now, if the searcher thinks that strategy s was selected, searching further will yield the expected price in all the other stores. Therefore, it is the expected price, only that $n(s)$ is now one lower (as s was observed in one of the stores). If $n(s) \leq 1$ s will be simply omitted from further calculations:

$$\frac{\sum_{s':n(s')>0, s' \neq s} n(s')e(s') + [n(s) - 1]^{+} e(s)}{\sum_{s}' n(s')} \tag{2}$$

Searchers search further only when the expected price in a search is at least c lower than the lowest observed price. Below is an example of how to calculate an expected search price, and additionally illustrates that no reserve price may exist:

Example 1 (Expected Search Price Calculation). Suppose the search cost c is 0.9 and pricing strategies, equally probable from the beliefs of a searcher, are as follows:

1. Uniform in [1, 9], Exp. value of 5
2. Uniform in [5, 9], Exp. value of 7
3. Pure strategy of 7.

After observing the price of 7 one is certain with prob. 1 that she had encountered the third strategy seller. An additional search will yield the average between the expected values of the two strategies - namely - 6, making a search worthy.

After observing the price of $7+\varepsilon$ One knows that she had encountered one of the mixed strategies, and due to a likelihood ratio - twice more probable that it is the second strategy. Therefore, with probability $1/3$ it is the first str. and probability $2/3$ the second str.

If the first strategy was encountered, then an additional search will end up in ether second or third strategy - both with expected price of 7.

If it is the second strategy, then an additional search will end up with expected price of 5 or of 7, as both can occur with equal probability (due to the beliefs) expected price in an additional search in this case is 6.

Combining the two possibilities, when taking into account that the second case is twice more probable, the expected price in an additional search is $(2{\cdot}6{+}7)/3 = 6.333$, making another search not profitable.

Here one sees the problematic assumption of the reserve price - it might be the case that it does not exist. However, in order to maintain in one line with the literature I concentrate on NE with a reserve price. Therefore, when one has a suspected a profile to be a NE one still needs to check whether the searchers there behave rationally, when adopting a reserve price. Therefore, the set of all possible NE may be wider, as some NE without a reserve price may exist, and fall beyond the scope of this paper.

2.2. Game Structure

The game is played between the sellers, searchers and the shoppers. The time line of the game is as follows:

1. Sellers select pricing strategies and consumers set reserve price.
2. Realizations of prices occur for sellers with mixed strategies.
3. Shoppers go and purchase the item at the cheapest store
4. Searchers select a store and observe the price in the store
5. If the price observed is weakly below P_M the searcher is satisfied and purchases the item, if not the search continues
6. All unsatisfied searches select one additional store, pay c and sample the price there.
7. If the price observed is below P_M the searcher is satisfied and purchases the item, if not the search continues
8. ...
9. When the searcher observed all stores and observed only prices above P_M she would buy at the cheapest store encountered.

When the reserve price and pricing strategies are being determined the knowledge of the various agents of the game is as follows:

– Sellers are aware of the reserve price set by the searchers
– Ssearchers have beliefs about which strategies were actually played by the sellers (see subsection 2.1.).
– Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to the shoppers when offering price p is denoted $\alpha_i(p)$. Let q be defined as the expected quantity that seller i sells when offering price p. The expected quantity sold by the seller consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store multiplied by the fraction of shoppers. This is also the market share of the seller.

Note that the reserve price ensures that the searcher will purchase at the last visited store, unless all stores were searched.

The utilities of the game are as follows:

– The seller utility is the price charged multiplied by the expected quantity sold.

– The consumer utility is a large constant M, from which item price and search costs are subtracted.

The NE of the game has a Bayesian structure, and is as follows:

– Searchers have a reserve price.
– The searchers beliefs coincide with the actual strategies played.
– The reserve price is rational for the searchers
– No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.

Remark 1. As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.

3. Equilibrium Structure

Before stating out the main results of the original model, a number of definitions is required. The reserve price is denoted as P_M. Additionally, a specific price denoted as P_L, and it is the price solving the following equation:

$$P_L(\mu + \frac{1-\mu}{n}) = P_M \frac{1-\mu}{n}$$
$$P_L = P_M \frac{1-\mu}{(n-1)\mu + 1}$$

If the support of seller i strategy is a positive measure interval from P_L to some price $p_i < P_M$, and in addition mass point at P_M, it will be said that seller i has a cutoff price of p_i.

Now it is possible to describe the NE of the Stahl model:

Theorem 1. *In any NE of the Stahl model with a reserve price there are at most three groups of strategies, as follows:*

1. *At least two sellers who have the full support of $[P_L, P_M]$ with some NE dependent continuous full support distr. function F.*
2. *A group of sellers (possibly empty) that select P_M as a pure strategy*
3. *A group of sellers (possibly empty) with an individual cutoff price, such that below the cutoff price the distribution used is the same F as from the first group. Above the cutoff price there is only a mass point at P_M.*

Additionally, all sellers have the same profit of $P_M(1-\mu)/n$.

Proof Shifted to the appendix.

Remark 2. For any combination where the third group is empty and the first group has at least two sellers exists a corresponding NE. Moreover, the sellers have the same expected profit of $P_M \frac{1-\mu}{n}$ and the searchers buy at the first store they visit. To see this simply adjust the shoppers share to reflect the game when only searchers visiting the mixing sellers exist.

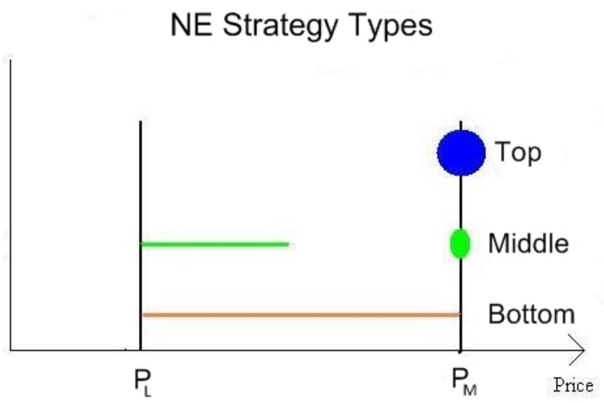

Fig. 1: The three types of strategies available in a NE of the extended model

Illustration of the three types of strategies can be seen on figure 1.

Theorem proof will be provided in the appendix. However, the first step is required to understand certain results on the extended model. Therefore, it is provided below with a short proof. Several examples will be provided in a later section.

Before continuing I wish to provide some very basic, yet important insights, valid also for the extended model:

Remark 3. As noted already in (Stahl, 1989), due to undercutting no pure NE exist. This is true for the extended model too for the same reasoning.

Lemma 1. *In both models, no seller offers a price above P_M in NE.*

Let p be the highest (or supremum) price offered in NE, and $p > P_m$. Such supremum exists as it is assumed that there is a finite bound on the prices. Let me distinguish between several cases:

- A unique mass point at p implies profit 0 to the seller offering it. Searchers would go on searching and find something cheaper, whereas shoppers would buy at a cheaper price w.p. 1. A deviation to offer the price c would be a profitable one.
- No mass points at price p implies profit 0 to all offering it. In case of a supremum price - profit is arbitrarily close to 0. In such case deviation to c is profitable.
- Some (but not all) offer price p with a mass point. The same case as with a single mass point: the searcher would go on searching until she finds a price cheaper than p.
- All sellers offer p with a mass point - undercutting is profitable. With some positive probability (that all offer price p) you would get all the market instead of just $1/n$ of it.

To sum it up - for a seller offering a price $p > P_M$ there is a profitable deviation in all cases. □

Corollary 1. *Any NE is socially optimal. This is since the total utilities of the sellers and consumers sums up to a constant, as long as the searchers buy at the first store they visit.*

I now show a lemma which will assist in determining the reserve price condition for the searchers:

Lemma 2. *Suppose that in a NE every seller has the expected price of at least $P_M - c$. Then setting P_M as a reserve price is rational for the searchers.*

It is not possible to observe a price above P_M, therefore, the searcher always stops searching after the first store visited. It is still required to show that after the first price observed it is not rational for the searcher to continue searching.

Suppose a price q was observed. As $q \leq P_M$ it is required to show that an expected price in a search is at least $q - c$. As the expected price in a search is a convex combination of some of the expected prices of sellers it is larger than a lower bound on such expected values. The lower bound on these expected values is $P_M - c$. Therefore, the expected price obtained in an additional store is at least $P_M - c > q - c$, making an additional search unprofitable. □

Note that the condition here is only a sufficient one, and it might be the case that additional reserve prices may be rational for searchers. Therefore, the asymmetric NE found here may do not cover all the possible NE of the model.

3.1. Equilibrium Distribution

Here I elaborate on the structure of the F function which is used in equilibrium by sellers, and what reserve price can be used. Suppose that in equilibrium we have $B = \{1, 2, \ldots b\}$ sellers with 'bottom' strategy (mixing over entire support), T sellers with 'top' strategy (pure reserve price) and $M = \{1, 2, \ldots g\}$ sellers with 'middle' strategy (cutoff price strategy), with the cutoff prices of $cp_1, cp_2, \ldots cp_g$ and mass points at the reserve price are with mass of $a_1, a_2, \ldots a_g$.

Let the set of sellers with cutoff point below some price p be denoted as $L(p)$.

From the structure of the equilibrium all sellers have equal profit. Additionally, all sellers have P_M in support and the reserve price attracts no shoppers. Therefore, the profit for all sellers is:

$$\pi = P_M(1 - \mu)/n \tag{3}$$

For any price p the expected profit needs to be equal to the expression above. At price p seller i has a certain probability $\alpha_i(p)$ to attract shoppers, if she is the cheapest. This can be calculated as follows:

- For each seller $j \neq i$, calculate the probability that j offers a price above p
- Multiply these probabilities

Let p be a price in (P_L, P_M). For group O this probability is clear and equal to $1 - F(p)$. For group T - it is zero. For group G we need to distinguish between two cases: ether $p \in L(p)$ and the probability is $1 - F(p)$, or $p \notin L(p)$ and then it is equal $a(p)$. Combining the cases we get that the expression for the expected profit is as follows:

$$\pi = P_M(1 - \mu)/N = p[(1 - \mu)/n + \mu(\prod_{j \in B \cup M(p)} (1 - F(p)) \prod_{j \in M \setminus M(p)} (a_j))] \tag{4}$$

As the F function is the same we can simplify and get:

$$p[(1 - \mu)/n + \mu((1 - F(p))^{b + |M(p)|} \prod_{j \in M \setminus M(p)} (1 - a_j))] = P_M(1 - \mu)/n \tag{5}$$

Extracting $F(p)$ form this equation will yield:

$$F(p) = \sqrt[b+|M(p)|]{1 - (\frac{P_M}{p} - 1)\frac{1-\mu}{n\prod_{j\in M\setminus M(p)}(a_j)}} \tag{6}$$

Note that at the point of the cutoff price $a_j(p) = 1 - F(p)$, and therefore F will be continuous, and as a certain expression instead of decreasing remains constant will also be differentiable. Therefore, it is still possible to calculate the density and expected value regularly. The last step, based on lemma 2, require finding the expected value $E(F)$, and setting the reserve price at $E(F) + c$. As this step is technical and the expressions involved are in many cases cannot be explicitly calculated. This step is not done here for the general case, and at the section with examples specific cases are provided.

4. NE Example

Consider the Stahl model with 3 sellers and a shoppers fraction of $\mu = 1/4$.

The following asymmetric NE exists:

- The searchers have a reserve price of $P_M = c/(1 - ln2) > c$ and $P_L = P_M/2$.
- One of the sellers offers the reserve price as a pure strategy.
- The other two sellers use a continuous distribution function $F(p) = 2 - P_M/p$ on $[P_M/2, P_M]$

Note that 1/4 is the mass of searchers visiting each of the stores initially.

The pure str. agent receives the profit of $P_M/4$.

Suppose the mixed str. agent selects a price $p \in [P_L, P_M)$. Then, her expected profit would be:

$$p(\frac{1-F(p)}{4} + \frac{1}{4}) = \frac{p}{4}(2 - F(p)) = \frac{p}{4}\frac{P_M}{p} = \frac{P_M}{4} \tag{7}$$

Clearly, if the pure str. agent selects a price in (P_L, P_M) her prob. to sell to shoppers is $(1-F(p))^2 < (1-F(p)$, and therefore, such deviation is not profitable. Similarly, selecting P_L would lead to the same profit as selecting P_M.

Any agent selecting prices above P_M would not sell to anyone, and selecting a price below P_L yields less profit.

One last thing to check is the searcher condition. Sufficient for this would be to check that the expected price of the mixed str. seller is at least $P_M - c$.

The density function, which is the derivative of the distribution function, is P_M/p^2. Therefore, the expected value is:

$$E(F) = \int_{P_M/2}^{P_M} pf(p) = \int_{P_M/2}^{P_M} P_M/p = P_M(ln(P_M/P_L)) = P_M(ln2) \tag{8}$$

Thus, the expected price of a mixing seller, $E(F) = P_M ln2$. Since $P_M(1-ln2) = c$, it is easy to see that $P_M - E(F) = c$, or $P_M - c = E(F)$ as required.

If a searcher did not observe the price of P_M but a lower one, she know that she had encountered a mixed price agent. Additional search will yield with prob. 0.5 another mixed agent with expected price of $P_M - c$, or prob. 0.5 of a pure agent and price P_M. Combined - expected price in an additional search is $P_M - c/2$, making the additional search not profitable after observing a price below P_M, due to the search price c.

If a searcher observed a price of P_M she know that she encountered a pure str. seller, and if she searches further she will get the expected price of $P_M - c$. Here the searcher is indifferent whether to search on or not. Therefore, it is an equilibrium.

5. Discussion and Summary

The three types of strategies in the NE have some economic motivation. The mixing seller wishes to compete over the shoppers when the pure reserve price seller does not to bother with the shoppers. Those kind of behavior are common in the economic world, and not in all cases all will compete as predicted by the symmetric NE. If only a single seller decides to compete, she will have monopolistic profits, which would attract additional competitors, and therefore, in NE at least two sellers will compete for the shoppers.

The cutoff price is for sellers that do not wish to be bothered with small probabilities. There are several effects that may cause a seller to refrain from sufficiently low probability events, for example see (Barron and Yechiam, 2009). Then, such seller will compete for shoppers, but only at prices that yield the benefit of getting the shoppers from high enough probability. When the probability to attract shoppers is lower than this individual threshold, the seller prefers to refrain from the shoppers market and select the reserve price with mass point instead.

The structure of NE allow to run several empiric tests on a database containing pricing and chain size data. Sellers may play an asymmetric NE, and the results here suggest some differences from the classical Stahl Model. There will be a higher probability for reserve price. In any asymmetric NE some sellers select the reserve price with a mass point. This implies that the reserve price will be more commonly selected. Similarly, larger discounts will be more rare, as the reserve price will be more common.

The results here open several important questions, which leave place for a fruitful future research. Firstly, the assumption here is that a reserve price exists. There may be additional NE without a reserve price, and an interesting question is whether such exist and how do these look like. This will allow to fully characterize all NE of the model and fully explain behavior of sellers. An additional question is combined with the determination of the reserve price. What is the full set of reserve prices under a certain setting, as here only a lemma provides a sufficient condition for the rationality of it. Moreover, which reserve price will the consumers set in order to minimize their price. On the other hand - with which NE should the sellers respond. What is the best NE for sellers and what is the best NE for consumer, will sellers prefer to mix, or to have a specific cutoff price? This question of consumer welfare and seller welfare will provide an important insight on behavior of these groups, and can provide a policy decision for a regulator in order to set the price lower or higher.

The Stahl model is a very important tool and the model is being used and applied in numerous papers. I hope that this paper provides an additional important insight which will make the Stahl model more applicable and more realistic. Additionally, any of the further research topics suggested here will provide yet another important block to the model, and to explaining behavior of consumers and sellers.

5.1. Acknowlegments.

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A Omitted Proofs

Here I show the proof to theorem 1. This is shown in a sequence of lemmas, first dealing with the regular Stahl model and then dealing with the extended model.

A1. Mass Points and Highest offered Price

Lemma 3. *There are no mass points at any price that can attract shoppers with positive prob.*

If at price q there is a mass point by a single seller i, price just above it is strictly less profitable for all others, and therefore would not be selected, as there the chance to attract shoppers drops discontinuously. Thus, seller i can set the mass point higher and gain more profit. In the case of mass points by several sellers at price p undercutting is possible, which probability to attract shoppers discontinuously. Therefore, there are no mass points at prices that can attract shoppers. $\square$

Lemma 4. *All sellers select P_M as the supremum point of their strategy support.*

From lemma 1 it cannot be higher than P_M.

Suppose that the supremum price of seller i is $p < P_M$. For any price above p and below P_M the probability to sell to shoppers is 0. Therefore, in equilibrium no seller would select a price in (p, P_M). Additionally, suppose that seller i has the lowest support supremum.

All sellers cannot have a mass point at p, as in such case undercutting would be profitable. From previous lemma seller i has no mass point at price p. Thus, probability to sell to shoppers at price p is 0, for all other shoppers, and no other seller would have a mass point at this price. Therefore, a deviation exists to seller i, where i selects prices arbitrarily close to P_M instead of prices arbitrarily close to p is profitable. $\square$

Remark 4. Note that this implies equal profit to all sellers in any equilibrium, or all but one have equal and one higher.

If at least two sellers do not have a mass point at P_M the probability that shoppers buy at P_M is 0. Moreover, if only one seller has no mass point at P_M she has weakly higher profit than all other sellers.

The two lemmas combined imply that there can be no mass points at any price except for P_M.

A2. Single Interval and Profit Equivalence

Definition 1. Let $\alpha_i(p)$ be denoted as the probability that p is the cheapest price, if seller i selects it. Explicitly: what is the probability of seller i to sell to shoppers given she selects price p. As the distribution is with no mass points except (maybe)

P_M, one can define $\alpha_i(p)$ as the product of 'Probability that seller j sets price above p', which is denoted as $\beta_j(p)$. Formally:

$$\beta_j(p) = 1 - F_j(p) \tag{9}$$

$$\alpha_j(p) = \prod_{j \neq i} \beta_j(p) \tag{10}$$

Lemma 5. *Exists an interval I such that the union of the seller strategies is contained in I and dense in it.*

Suppose exists an interval $[a, b]$ ($a < b < P_M$) such that sellers select prices only below a and above b, and exist prices both below a and above b. Let $p-$ be the highest price below a that is in the support union of the sellers. A seller can deviate from $p-$ and prices just below it to b, and sellers arbitrarily close to all of her previous quantity:

The searchers behavior does not change, as the prices are below P_M. Since the probability for someone to select a price just below p is arbitrarily small, the decrease in probability to sell to shoppers is arbitrarily small.

The profit form raising the price is much higher than such arbitrarily small loss, as it is at least $(b-p)(1-\mu)/n$, as the searchers pay strictly more after the deviation. Therefore, if the support is not continuous there is a profitable deviation. □

Corollary 2. *Exists an interval $I = [P_L, P_M]$, such that any NE strategy profile the sellers randomize continuously over I, and possibly some sellers set mass points at P_M.*

Lemma 6. *The previous lemma holds also for two sellers. Meaning - any interval has a non empty intersection with the support of at least two sellers.*

Suppose that all points in an interval $[p, p']$ ($p <' p < P_M$) are selected at most by one seller. Additionally, from previous lemma this seller needs to have in support the entire interval. Than exists a profitable deviation for her would be to set a mass point at p' instead of selecting the original distribution over the interval. □

Corollary 3. *Any interval between P_L and P_M has points in the support of at least two sellers.*

Lemma 7. *All sellers have the same profit.*

The only case that needed to be shown is as follows: If $n - 1$ sellers have the same profit, the other seller cannot have a profit above them. It was shown before that if at least two sellers do not have mass points at P_M all sellers have equal profit. If only one seller has no mass point at P_M then she must have a higher profit. This is since she can always deviate to a pure strategy offering P_M.

Suppose seller i is the only seller who does not offer a mass point at P_M. Let p_i be the lowest (infimum if needed) price in the support of i. As it has a higher profit than all other players this price cannot be the lowest price in the support union. Note that due to previous lemmas seller i sets no mass point at p_i, $F_i(p_i) = 0$. If no seller selects a price below p_i then it is not possible for seller i to have a higher profit than other sellers, as other sellers could get the same profit as i gets with p_i.

Denote a seller $j \neq i$, and examine the profits of seller i and j. As noted before, $\pi_i > \pi(j)$.

The profit of seller i offering p_i is (remember all searchers visit exactly one store):

$$\pi_i(p_i) = p_i((1 - F_j(p) \prod_{k \neq i,j} (1 - F_k(p_i))\mu + (1 - \mu)/n) \tag{11}$$

The profit of seller j:

$$\pi_j(p_i) = p_i((1 - F_i(p) \prod_{k \neq i,j} (1 - F_k(p_i))\mu + (1 - \mu)/n) \tag{12}$$

Since $0 = F_i(p_i) \leq F_j(p_i)$ the profit of j when offering p_i is weakly higher than the profit of i when offering p_i. This contradicts the fact that seller i must have a higher profit than seller j. □

I have shown that the support union is equal to some interval I. Let P_L be the lowest price in this interval. As the mixing sellers need to be indifferent between all the strategies they mix one can say that:

$$P_L = \frac{(1 - \mu)/n}{\mu + (1 - \mu)/n} P_M \tag{13}$$

Clearly, the searchers do not search at this price, as P_L is the cheapest price that can exist in EQ. Additionally, when a seller selects this price she is certain to sell to shoppers.

A3. Symmetry

In this subsection I will discuss the symmetries in the NE distribution functions, and see where they can differ.

Lemma 8. *The following inequality needs to be satisfied for any $p \in (P_L, P_M)$:*

$$p(\alpha_i(p)\mu + \frac{1 - \mu}{n}) \leq P_M(\frac{1 - \mu}{n}) \tag{14}$$

If it is strictly larger than price p will be more profitable than P_M, which due to previous lemmas cannot occur in NE. Moreover, if seller i selects price p there must be an equality, as the profit i gets from any price she selects has to be equal to $P_M(\frac{1-\mu}{n})$. □

The following observation will be crucial in understanding the asymmetric NE:

Corollary 4. *Only the seller(s) with the maximal α among the sellers may select the corresponding price.*

Note that since there are no mass points α and β change continuously, except possibly at P_M. Note that at P_M the α of each seller approaches 0 continuously as P_M is approached, as the probability to sell to shoppers with price P_M is 0. Adding the fact that there are no mass points below P_M, it is clear that α a continuous function.

Lemma 9. *Let I be an open interval in $[P_L, P_M]$. Suppose that seller i selects a price in I with some positive probability. Let j be a different seller. Then, also seller j must select a price from I with same positive probability, or not to select any prices in or above the interval I, except P_M.*

From previous lemma it is known that each interval is selected by at least two sellers, and done so without mass points. Note that the only way to select elements continuously is to select a dense subset of an interval.

Assume that seller i sets a positive probability to a dense subset of $I = (p', p*)$, whereas seller j does not select any prices in this interval. Let $p \in I$.

Note that β_i is strictly decreasing in the neighborhood of p, and β_j remains constant there. This is since i select prices in the neighborhood of p and j not.

Note that from the definition it is known that $\alpha_i/\beta_j = \alpha_j/\beta_i$. Since β_i is decreasing and β_j, and conclude that α_j is decreasing more rapidly then α_i.

Therefore, for any price $p \in I$, the ratio of the parameters is $\alpha_i(p) > \alpha_j(p)$, except maybe the infimum of the interval.

Similarly, if both i and j do not select prices in an interval then α_i and α_j decrease in such interval at the same rate.

Let $\hat{p}$ be the infimum of an interval that is to the right of I, and is selected by j. For $\hat{p}$, the α parameters need to satisfy $\alpha_j(p) \geq \alpha_i(p)$. If both select this price - equality, if only j does so - weak inequality.

Note that at p the opposite inequality holds, and in all points between p and $\hat{p}$, the parameter β_j is decreasing less than β_i. Since all the α's and β's change continuously everywhere except P_M, it is the case that j cannot offer such prices.

Concluding, if a seller does not select an interval within $[P_L, P_M)$ she would not select any price above it, except possibly P_M, where the equation holds due to zero probability to sell to shoppers. □

As shown in Lemma 2, for the reserve price to make sense, the following condition is sufficient: The expected price of a seller is at least $P_L - c$.

Combining the lemmas the theorem 1 is obtained.

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Differential Bargaining Games as Microfoundations for Production Function *

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Abstract In the present paper the game theory is applied to an important open question in economics: providing microfoundations for often-used types of production function. Simple differential games of bargaining are proposed to model a behavior of workers and capital-owners in processes of formation of possible factor prices and participants' weights (moral-ethical assessments). These games result, correspondingly, in a factor price curve and a weight curve – structures dual to a production function. Ultimately, under constant bargaining powers of the participants, the Cobb-Douglas form of the production function is received.

Keywords: bargaining, differential games, production factors, choice of technology, duality, production function.

1. Introduction

It is well known that the acceptance of concrete types of production functions in economics, such as the Cobb-Douglas and the CES forms, was rather occasional and till now not enough attempts have been made to explain and justify the wide used types of production function – e.g. (Matveenko, 1997; Acemoglu, 2003; Jones, 2005; Lagos, 2006; Nakamura, 2009; Matveenko, 2010; Dupuy, 2012). In the paper models resulting in the Cobb-Douglas production function are constructed on base of differential games of bargaining and by use of dual relations in production and distribution. A simple differential game of price bargaining is introduced as a benchmark and then is modified to a differential game of bargaining for prices of capital and labor and to a differential game of weights (moral-ethical assessments of the factor owners. Each of these three differential games exploits one or another of duality relations existing in the economy (cf. (Cornes, 1992)).

One of the duality relations used in the paper is usually represented as the duality between the production function $Y = F(K, L)$ and the cost function $C(p_K, p_L, Y)$. The first of these functions shows the maximal output in dependence on production factors: capital and labor, while the second one shows the minimal cost in dependence on prices of the production factors and the output. We study a similar duality by use of the well-known representation of the production function by use of the Euler theorem:

$$F(K.L) = \frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L = p(x)x,$$

where $x = (K, L)'$ is the vector of production factors (capital and labor) and $p(x) = (\partial F/\partial K, \partial F/\partial L)$ is the corresponding price vector (the vector of marginal

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products). There exists a set Π of the price vectors corresponding the production function, such that the Euler theorem can be written in the "extremal" version:

$$F(K,L) = \min_{p\in\Pi} px, \tag{1}$$

which means that the production function represents a result of a choice of the price vector from the set Π. Let $M = \{x : F(x) = 1\}$ be the unit level line of the production function F. A conjugate problem for (1) is the problem of a choice of the bundle of production factors $x = (K,L)$ from the set M to provide a unit output with minimal cost:

$$F^*(p) = \min_{x\in M} px.$$

Rubinov (Rubinov and Glover, 1998; Rubinov, 2000) introduced some other types of duality using instead of the usual inner product its analogues, such as Leontief function $\min_{i=1,\ldots,n} l_i x_i$. Notably, the latter is similar to the inner product but uses the idempotent operation of summation: $\bigoplus = \min$. Matveenko (1997; 2010) and Jones (2005) found a representation for neoclassical production functions which reminds (1) but uses the Leontief function as an inner product; in the two factor case:

$$F(K,L) = \max_{l\in\Psi} \min\{l_K K, l_L L\}.$$

In Section 2 we introduce the benchmark differential game of price bargaining. In Section 3 a differential game of factor price curve formation is considered. In Section 4 a differential game of weight curve formation is studied which, together with the model in Section 3, provides a foundation for the Cobb-Douglas production function. Section 5 concludes.

2. Benchmark differential game of price bargaining

The term *bargain* relates both to a process of bargaining and to a result of this process. Both sides of bargaining are being studied in the bargaining theory – a special chapter of the game theory, However, traditionally, the bargaining theory deals more with results of bargains rather than with processes of bargaining. Nash (1950) proposed a system of axioms leading to a so called symmetric Nash bargaining solution; later an asymmetric solution was found and axiomatized . For the reviews of the axiomatic approach in the bargaining see (Roth, 1979; Thomson, 1994; Serrano, 2008). The models of processes of bargaining are usually based on assumptions concerning economic benefits gained by participants under one or other running of the process of bargaining (see (Muthoo, 1999)). For example, a participant can bear some costs connected with the duration of the bargaining process. In practice, however, in many cases the course of a bargaining process depends in much not on expectations of economic benefits by participants but on their skills to bargain (see (Schelling, 1956; Blainey, 1988)). These skills can be associated with bargaining powers of the participants. The notion of bargaining power is often used in game theory, though, different authors put different sense into this notion . In this Section we propose a simple differential game as a model of a bargaining process. In different versions of the game the bargaining powers of the players are either given exogenously or are defined endogenously in the game itself.

In the benchmark example of bargaining (Muthoo, 1999) an object is on sale (e.g. a house). A seller (player S) wishes to sell the house for a price exceeding

$\bar{p}_S^0$(the latter is the minimal price acceptable for player S). A buyer (player B) is ready to purchase the house for a price not exceeding $\bar{p}_B^0$ (the maximal acceptable price for player B). Here $\bar{p}_B^0 > \bar{p}_S^0$, what ensures the possibility of the bargain. The seller starts from a start price, $p_S(0) > \bar{p}_S^0$, and then decreases her price, while the buyer simultaneously starts from a price $p_B(0) < \bar{p}_B^0$ and then increases her price. It is assumed, naturally, that $p_B(0) < p_S(0)$. A price trajectory $p_B(t), p_S(t)$ formed in continuous time stops at a moment T when $p_B(T) = p_S(T)$. It follows that $p_B(t) < p_S(t)$ for $t \in [0, T)$. The selling price will be referred as p^* . A surplus of the selling price over (under) the minimal (maximal) admissible price of a player can be considered as the player's utility:

$$u_S = p^* - \bar{p}_S^0, u_B = \bar{p}_B^0 - p^*. \tag{2}$$

A set Ω of all possible pairs of utilities on plane (u_B, u_S) is

$$\Omega = \{(u_B, u_S) : u_B + u_S = \bar{p}_B^0 - \bar{p}_S^0, u_B, u_S \geq 0\}.$$

A simplest model of price bargaining appears under an assumption that each player $i = B, S$ changes her price with a constant velocity equal to the bargaining power of her opponent. A strong opponent forces the player to change her price faster:

$$p_S(t) = p_S(0) - b_B t, p_B(t) = p_B(0) + b_S t.$$

The game stops at the moment T which is found from equation:

$$p_S(0) - b_B T = p_B(0) + b_S T,$$

i.e. at the moment

$$T = \frac{p_S(0) - p_B(0)}{b_S + b_B}$$

when the selling price is:

$$p^* = p_S(T) = p_B(T) = \frac{b_S}{b_S + b_B} p_S(0) + \frac{b_B}{b_S + b_B} p_B(0). \tag{3}$$

So, the selling price is the convex combination of the start prices proposed by the players summed with weights equal to their relative bargaining powers. If each player i knows the minimal (maximal) price accessible for the opponent and establishes it as her start price, then the play stops at the moment:

$$T = \frac{\bar{p}_B^0 - \bar{p}_S^0}{b_S + b_B}$$

with the selling price:

$$p^* = \frac{b_S}{b_S + b_B} \bar{p}_B^0 + \frac{b_B}{b_S + b_B} \bar{p}_S^0$$

and with the utilities of the players equal to

$$u_i = \frac{b_i}{b_S + b_B} (\bar{p}_B^0 - \bar{p}_S^0), i = B, S. \tag{4}$$

Theorem 1. *Price p^* corresponds the asymmetric Nash bargaining solution of the bargaining problem under utilities (2) and bargaining powers b_S, b_B .*

Proof. The asymmetric Nash bargaining solution is here a solution of the problem of maximization of the function $u_B^{b_B} u_S^{b_S}$ on the set Ω . The first order optimality condition, $b_B u_S + b_S u_B$, and the constraint, $u_B + u_S = \bar{p}_B^0 - \bar{p}_S^0$, define the asymmetric Nash bargaining solution is found, which coincides with (4)).

The case when the players change prices under constant growth rates (rather than constant velocities) is similar. Since the growth rate of price is the velocity of changing the logarithm of the price, an equation similar to (3) is fulfilled: the bargaining stops under a price the logarithm of which is equal to the convex combination of the logarithms of the start prices with weights equal to the relative bargaining powers of the players.

In a more complex case the velocity of changing price by a player depends on the actions of her opponent. If the seller decreases her price slowly then the buyer also increases her price slowly because she does not want the game to stop on a too high price. Similarly, if the buyer increases her price slowly then the seller decreases her price slowly. Let the growth rates of price change, $g_i = \frac{\dot{p}_i}{p_i}, i = B, S$, be constant. The bargaining power of player i can be defined as the value inverse to $|g_i|$:

$$b_B = \frac{1}{g_B}, b_S = -\frac{1}{g_S}.$$

Then

$$\frac{g_B}{g_S} = -\frac{b_S}{b_B},$$

i.e.

$$\frac{dp_B}{dp_S}\frac{p_S}{p_B} = -\frac{b_B}{b_S} = const,$$

The game interpretation of this differential equation is the following. Each player i chooses a control g_i , and the controls are connected by the relation:

$$g_B \geq |g_S|\frac{b_S}{b_B},$$

which means that in the bargaining process the faster the seller decreases her price the faster the buyer increases hers. Moreover, a higher bargaining power of the buyer relaxes this constraint (this means a lower degree of reaction to the opponent's actions), and a higher bargaining power of the seller reinforces the constraint. At the same time the seller is limited by the opposite constraint:

$$g_S \geq |g_B|\frac{b_B}{b_S},$$

which means that the faster the buyer increases her price the faster the seller decreases hers. An increased bargaining power of the buyer forces the seller to diminish her price faster, and an increased own bargaining power allows the seller to diminish her price slower. Simultaneous fulfillment of inequalities (7) and (8) implies the Equation (6).

3. Bargaining for production factor prices and corresponding choice of technologies

In the just described benchmark differential game the players change their proposals concerning the same price. Now we turn to differential games in which the interests of the players relate to different prices. At each moment of time one of the players *attacks*, another one *defends*. Only the attacker is satisfied by the direction of her price change while the defender hinders changes in her price.

In the present Section the following pair of dual objects will be under consideration:

(i) a neoclassical production function $F(K, L)$ which is characterized by its factor curve: $M = \{(K, L) : F(K, L) = 1\}$, i.e. the set of bundles of resources allowing the unit output, and

(ii) the factor price curve $\Pi = \{(p_K, p_L)\}$ i.e. the set of such bundles of prices under which the unit output under unit costs is possible.

3.1. Usual causality

Given production function $F(K, L)$, the price curve Π can be found from the following system of equations:

$$F(K, L) = 1, \tag{5}$$

$$p_K K + p_L L = 1, \tag{6}$$

$$\frac{\partial F/\partial K}{\partial F/\partial L} = \frac{p_K}{p_L}. \tag{7}$$

Equations (5) and (6) are conditions of the unit output under unit costs. Equation (7) is a condition of efficiency of production; it can be interpreted as a condition of output maximization under given costs.

The system (5)-(6) establishes a one-to-one correspondence between points of the factor curve, M, and points of the factor price curve, Π. Indeed, by the Euler theorem, the Equation (9) can be written as

$$\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L = 1, \tag{8}$$

then Equations (6)-(7) imply:

$$\frac{\partial F}{\partial K} = p_K, \frac{\partial F}{\partial L} = p_L. \tag{9}$$

In particular, for the Cobb-Douglas production function, $F(K, L) = AK^\alpha L^{1-\alpha}$, the system (9) takes the form:

$$\alpha K^{\alpha-1} L^{1-\alpha} = p_K,$$

$$(1-\alpha)AK^\alpha L^{1-\alpha} = p_L.$$

Excluding the ratio K/L from these two equations we find the factor price curve:

$$Bp_K^\alpha p_L^{1-\alpha} = 1$$

where $B = A^{-1}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$.

For the CES function $F(K, L) = (\alpha(A_K K)^p + (1-\alpha)(A_L L)^p)^{\frac{1}{p}}$ where $p \in (-\infty, 0) \cup (0, 1)$ the system (13) takes the form:

$$\alpha A_K^p(\alpha A_K^p + (1-\alpha)K^{-p}(A_L L)^p)^{\frac{1}{p}-1} = p_K,$$

$$(1-\alpha)A_L^p(\alpha(A_K K)^p L^{-p} + (1-\alpha)A_L^p)^{\frac{1}{p}-1} = p_L.$$

Excluding $K^{-p}L^{p}$ from these equations we receive, after some transformations, the following equation of the factor price curve in form of CES function:

$$B\left[\beta\left(\frac{p_K}{A_K}\right)^q+(1-\beta)\left(\frac{p_L}{A_L}\right)^q\right]^{1/q}=1,$$

where

$$B=(\alpha^{\frac{1}{1-p}}+(1-\alpha)^{\frac{1}{1-p}})^{\frac{p-1}{p}},\beta=\frac{\alpha^{\frac{1}{1-p}}}{\alpha^{\frac{1}{1-p}}+(1-\alpha)^{\frac{1}{1-p}}},q=\frac{p}{1-p}.$$

3.2. Reversed causality

Usual causality, considered in the previous Subsection, presupposes that the prices are primarily determined by the physical side of production - physical technologies and available bundles of production resources. However, another direction of causality is possible: institutions, reflected by the prices, can define which products have to be produced and by use of which technologies. We propose now a model in which the factor price curve, Π , is defined in a pure institutional way. This model belongs to a class of island models - such where partially independent segments of a market are considered.

There are two types of agents: workers and entrepreneurs. A single product is produced in a continuum of segments – islands, some of them are "inhabited" by the agents of both types. On each of the inhabited islands in each moment of time there are definite prices of labor and capital in terms of the product. In random moments of time from randomly chosen islands either a part of workers or a part of entrepreneurs moves to an uninhabited island. Since this moment the prices in the inhabited island are fixed. After that a part of the other social group also moves to the "new" island and there the groups start bargaining about the factor prices. Those who have come first possess an advantage and try to increase their factor price - they attack. Those who have come later try not to allow their factor price to fall too much - they defend. As start prices in the bargaining process the groups use the prices in the "old" island at the moment when the first group left. It is assumed that the social groups always have constant bargaining powers, b_K, b_L . Weakening this assumption is left for a future research. Opposed to the case of the selling/purchasing bargaining game considered in Section 2, now the prices relate to different goods (labor and capital). The attacker, a, is interested in maximizing the growth rate of her factor price while the defender, d, is interested in minimizing (the module of) the growth rate of her factor price. In the simplest case, similarly to the case considered in Section 2, it can be assumed that players have constant growth rates of their factor prices, $g_i=\dot{p}_i/p_i$, where $g_a>0$ for the attacker; $g_d<0$ for the defender; and the price growth rates are linked with the bargaining powers by the equation:

$$|g_d|=\frac{b_a}{b_d}g_a.$$

According to this equation, a higher relative bargaining power b_d/b_a of the defender allows her to reach a slower decline in her factor price, i.e. a smaller $|g_d|$. Vice versa, an increase in the bargaining power of the attacker forces the defender to agree to a larger decline in her factor price. Equation (14) describing the price change process turns into:

$$\frac{dp_a}{dp_d} = -\frac{b_d}{b_a} = const,$$

which can be written as

$$\frac{dp_K}{p_K} b_K = -\frac{dp_L}{p_L} b_{L.}$$

Solving this differential equation we receive the price curve Π :

$$p_K^{b_K} p_L^{b_L} = C. \tag{10}$$

If initially the price vector belongs the curve Π given a constant C then the vector stays in the same curve further.

To describe the strategic behavior of the players in more details, let the attacker's problem be to maximize her price growth rate, g_a , under the following constraint:

$$|g_d| \geq g_a \frac{b_a}{b_d},$$

and, correspondingly, let the defender's problem be to minimize the module of her price growth rate, g_d, under (17). The inequality (17) means that the attacker forces the defender to increase her price reduction rate. An increased bargaining power of the attacker reinforces this constraint, while an increase in the bargaining power of the defender relaxes it. There exists a continuum of Nash equilibria, (g_a, g_d) , and all of them satisfy the equation

$$\frac{g_a}{|g_d|} = \frac{b_d}{b_a}$$

This equation, independently on which player (K or L) is the attacker, leads to the price curve (10). Now let us show in what way the price curve (16) leads to the Cobb-Douglas type of production function. We will use the representation of neoclassical production function by use of a menu of Leontief technologies (Matveenko, 1997; Matveenko, 2010; Jones, 2005). Matveenko (2010) has shown that to each neoclassical production function $F(K, L)$ a unique technological menu Ψ corresponds which consists of effectiveness coefficients of the Leontief function and is such that

$$F(K, L) = \max_{l \in \Psi} \min\{l_K K, l_L L\}.$$

Moreover, there exists a simple one-to-one correspondence between the points $(K, L) \in M$ of the factor curve and the points $l \in \Psi$ of the technological menu:

$$(l_K, l_L) \in \Psi \Leftrightarrow \left(\frac{1}{l_K}, \frac{1}{l_L}\right) = (\tilde{K}, \tilde{L}) \in M.$$

The function

$$F^{\circ}(l_K, l_L) = \frac{1}{F\left(\frac{1}{l_K}, \frac{1}{l_L}\right)}$$

is referred to as a *conjugate (polar) function*. Representation (18) follows from the following Lemma.

Lemma 1. *Let $F(x_1, x_2, ..., x_n)$ be an increasing positively homogeneous of 1st power function of n positive variables, M - its unit level set, and Ψ - the unit level set of the conjugate function:*

$$M = \{x : F(x_1, x_2, ..., x_n) = 1\},$$

$$\Psi = \{l : F\left(\frac{1}{l_1}, \frac{1}{l_2}, ..., \frac{1}{l_n}\right) = 1\}.$$

Then

$$F(x_1, x_2, ..., x_n) = \max_{l \in \Psi} \min\{l_1 x_1, l_2 x_2, ..., l_n x_n\}.$$

See proof in (Matveenko, 2010).

When a pair of prices is defined on an island, the island chooses a suitable technology on base of one or another pure economic criterion (efficiency) or an institutional criterion (fairness). We assume that the whole set ("cloud") of available Leontief technologies is extensive enough to include all those technologies which any islands would choose to use. The technological menu Ψ is narrower and consists of those technologies which would be chosen. Below three mechanisms of choice are identified resulting in the same technological menu Ψ and the factor curve .

Mechanism A. Given factor prices p_K^0, p_L^0 , an island chooses such Leontief technology (l_K, l_L) which guarantees receiving factor shares equal to the relative bargaining powers of the social groups : $\alpha = \frac{b_K}{b_K + b_L}$ for the capital and $1 - \alpha = \frac{b_L}{b_K + b_L}$ for the labor. For this technology, such volumes of factors $\tilde{K}, \tilde{L}$ exist for which:

$$l_K \tilde{K} = l_L \tilde{L} = 1 - \alpha = 1, p_K^0 \tilde{K} = \alpha, p_L^0 \tilde{L} = 1 - \alpha.$$

Such kind of choice of the Leontief technologies by all islands results in the following factor curve:

$$M = \{(K, L) : p_K K = \alpha, p_L L = 1 - \alpha, (p_K, p_L) \in \Pi\} = \{(K, L) : AK^\alpha L^{1-\alpha}\},$$

where $A = \frac{C}{\alpha^\alpha (1-\alpha}^{1-\alpha}$. Thus, the Leontief technologies chosen by all the islands define the Cobb-Douglas production function: $F(K, L) = AK^\alpha L^{1-\alpha}$.

Mechanism B. Given factor prices p_K^0, p_L^0 , an island chooses such Leontief technology $(l_K, l_L) = (\frac{1}{K^0}, \frac{1}{L^0})$ which is *competitive* in the sense that, under this technology, the cost of the unit production on the island is equal to 1, while the cost on any other island is greater than 1. So, the usage of this technology is profitable only on the present island. In other words,

$$p_K^0 K^0 + p_L^0 L^0 = 1 < p_K K^0 + p_L L^0$$

for any bundle of prices $p_K, p_L \in \Pi, (p_K, p_L) \neq (p_K^0, p_L^0)$. It follows that p_K^0, p_L^0 is a solution for the problem:

$$\min_{(p_K, p_L) \in \Pi} (p_K K^0 + p_L L^0).$$

The first order optimality condition for this problem is:

$$\frac{p_L^0 L^0}{p_K^0 K^0} = \frac{1 - \alpha}{\alpha},$$

and we come to the Mechanism A.

Mechanism C. Given factor prices p_K^0, p_L^0 , an island chooses a Leontief technology (l_K, l_L) (or, what is equivalent, $(K, L) \in M$) ensuring fulfillment of a fairness principle:

$$\max_{(K,L)\in M} \min\{\frac{p_K^0 K}{b_K}, \frac{p_L^0}{b_L}\},$$

which is analogous to the Rawlsian maximin principle: a gain of the most hurt agent has to be maximized. Here the gain of an agent is her revenue but with account of her bargaining power: a participant's gain increases if her relative bargaining power increases. The solution is characterized by the equation:

$$\frac{p_K^0 \tilde{K}}{b_K} = \frac{p_L^0 \tilde{L}}{b_L},$$

hence,

$$\frac{p_L^0 \tilde{L}}{p_K^0 \tilde{K}} = \frac{1-\alpha}{\alpha},$$

and again we come to the Mechanism A.

4. Differential game of weights formation

In this Section we provide a microfoundation for the Mechanism A decribed in Section 3. We propose a differential game in which the players (workers and capital-owners) form a weight curve - a set of possible assessments (weights); the curve is used by an arbiter to choose a vector of weights in a concrete bargain.

Three common features present in many real bargains and negotiations. Firstly, it is a presence of an arbiter in which role often a community acts, in a framework of which the bargainers interact. Examples are so called 'international community', including governments and elites of countries, and different international organizations; a 'collective' or a union in a firm; a local community; a 'scientific community', etc. The community acts as an arbiter realizing a control for bargains in such way that unfair, from the point of view of the arbiter, bargains are less possible, at least as routine ones. An outcome of an unfair bargain can be, with a help of the arbiter, revised, if not formally than through a conflict. Such conflicts rather often arise, both on a local and on a national levels, as well as in international relations. Secondly, bargains inside a fixed set of participants are often not 'one-shot' but represent a routine repeated process in which a 'public opinion' of the community is important; and the latter is being formed along with the bargains. Usually it is unknown in advance what concrete bargains will take place and in what time, and the process of formation of the public opinion processes uninterruptedly to prepare it for future bargains. The public opinion can be modeled as a set of the vectors of weights - the moral-ethical assessments which can be used by the arbiter as coefficients for the participants' utilities . Possibilities of formation of public opinion are limited both by possibilities of access to media and by image-making abilities of the participants. Thirdly, the moral-ethical assessments formed by participants are usually not univalent, but allow a variance: the public opinion practically always can stress both positive and negative features of a participant; concrete weights differ in different concrete bargains depending on circumstances. Thus, it can be useful to speak not about a single vector of weights but rather about a curve (in case of two participants) or a surface of admissible assessments.

Thus, the public opinion can be modeled as a weight curve (or a weight surface). In its approval or disapproval of a possible result of a concrete bargain the arbiter acts in accordance with a Rawlsian-type maximin principle, paying attention to the most infringed participant, but taking into account admissible vectors of weights for utilities, the set of which is formed in advance by the participants.

We consider a two stage game. On the first stage, two players (workers and capital-owners) form a curve $\Lambda = (\lambda_K, \lambda_L$ consisting of vectors of admissible reputational assessments (*weights*). On the second stage, for a concrete bargain, an arbiter (community) chooses an admissible pair of weights from the weight curve and divides the product Y among the players ($Y = Y_K + Y_L$) to achieve the maximin

$$\max_{y \in \Omega} \max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\}, \tag{11}$$

where

$$\Omega = \{y = (Y_K, Y_L) : Y = Y_K + Y_L\}$$

is the set of outputs.

Let us describe the first stage of the game in detail. A player's gain depends negatively on her weight and depends positively on the opponent's weight . Hence, each player is interested in decreasing her weight and in increasing the opponent's weight. However, in the process of the weight curve formation, the player i would agree to a decrease in the opponent's weight in some part of Λ at the expense of an increase in her own weight, as far as the opponent similarly temporizes in another part of Λ. Since the system of weights is essential only to within a multiplicative constant, the players can start the formation of the weight curve Λ from an arbitrary pair of weights and then construct parts of the curve to the left and to the right of the initial point. The player who attacks maximizes, at each moment of time, the module of her weight's growth rate while the defender minimizes her weight's growth rate. This takes place under the following constraint:

$$|g_a| \leq g_d \frac{b_a}{b_d}, \tag{12}$$

which means that a higher bargaining power of the attacker helps her to enlarge the constraint, while an increase in the bargaining power of the defender makes the constraint stricter.

In equilibrium (12) is fulfilled as an equality. Thus, the constancy of the bargaining powers of the participants implies :

$$\frac{d\lambda_L}{d\lambda_K} \frac{\lambda_K}{\lambda_L} = -\frac{b_K}{b_L} = const. \tag{13}$$

The more the bargaining power of a player is the better reputational assessment she gains for herself. Solving the differential equation (13) we receive the weight curve λ :

$$\lambda_K^{b_K} \lambda_L^{b_L} = C = const.$$

Now we turn to the second stage of the game.

Lemma 2. *For each outcome, the following equality is valid:*

$$max_{\lambda \in \Lambda} min\{\lambda_K Y_K, \lambda_L Y_L\} = A Y_K^{\frac{b_K}{b_K+b_L}} Y_L^{\frac{b_L}{b_K+b_L}},$$

where $A = const.$

Proof. It follows from Lemma 1 when it is applied to the set Λ.

According to (14), the solution of the arbiter's problem (11) is none other than the asymmetric Nash bargaining solution.

It is easily seen that the players receive shares proportional to their bargaining powers. This provides a support to the Mechanism A described in Section 3. This mechanism, as we have seen there, generates the Cobb-Douglas production function. Notice, that a constancy of bargaining powers can explain a constancy of factor shares in some countries on a definite stage of their development.

5. Conclusion

In this paper a new approach is proposed for understanding a relation between a physical side of economy (resources and technologies) and its institutional side (distributional relations between social groups). The idea of the models presented here is that the distributional behavior can be described by a differential game of bargaining.

Three differential games are proposed to describe a behavior of economic agents in processes of prices and weights formation. In the benchmark model of price bargaining players are interested in changing the same price in opposite directions. It is shown that under some conditions this game leads to the Nash bargaining solution. This benchmark game is modified to games in which players change (different) prices of their owned resources or change weights (moral-ethical assessments). One of these games describes bargaining of workers and capital-owners for their factor prices. In another game the same players bargain for weights (moral-ethical assessments); these weights enter a Rawlsian-type criterion which is used by an arbiter (community) in concrete bargains. These games result in construction of structures – a price curve in one case and a weight curve in another – which are dual to the production function. Ultimately, under constant bargaining powers of the participants, these games lead to the Cobb-Douglas form of production function.

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Pricing and Transportation Costs in Queueing System

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Abstract A non-cooperative four-person game which is related to the queueing system $M/M/2$ is considered. There are two competing stores and two competing transport companies which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The stream forms the Poisson process with intensity λ. The problem of pricing and determining the optimal intensity for each player in the competition is solved.

Keywords: Duopoly, equilibrium prices, queueing system.

1. Introduction

A non-cooperative four-person game which is related to the queueing system $M/M/2$ is considered. There are two competing stores P_1 and P_2 and two competing transport companies C_1 and C_2 which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The stream forms the Poisson process with intensity λ.. Suppose that $\lambda < \mu_1 + \mu_2$. Let shops declare the price for the produced product. After that transport companies declare the price of the service and carry passengers to the store, and the company C_1 carries passengers to P_1, when the company C_2 carries passengers to P_2. Customers choose the service with minimal costs. This approach was used in the Hotelling's duopoly (Hotelling, 1929; D'Aspremont et al., 1979; Mazalova, 2012) to determine the equilibrium price in the market. But the costs of each customer are calculated as the price of the product and transport charges. In this model, costs are calculated as the sum of prices for services and product plus losses of staying in the queue. Thus, the incoming stream is divided into two Poisson flows with intensities λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \lambda$. So the problem is following, what price for the service, the price for the product and the intensity of services is better to announce for the companies and shops. Such articles as (Altman and Shimkin, 1998; Levhari and Luski, 1978; Hassin and Haviv, 2003; Mazalova, 2013; Koryagin, 2008; Luski, 1976) are devoted to the similar game-theoretic problems of queuing processes.

Game-theoretic model of pricing. Consider the following game. Players P_1 and P_2 declare the price for the produced product p_1 and p_2 respectively. The customers have to use a transport to get to the shop. There are two competing transport companies C_1 and C_2 which serve the stream of customers with exponential distribution with parameters μ_1 and μ_2 respectively. The transport companies declare the price of the service c_1 and c_2 respectively and carry passengers to the store, and the company C_1 carries passengers to P_1, when the company C_2 carries passengers to P_2. So the customers choose the service with minimal costs, and the incoming stream is divided into two Poisson flows with intensities λ_1 and λ_2, where $\lambda_1 + \lambda_2 = \lambda$. In this case the costs of each customer will be

$$c_i + p_i + \frac{1}{\mu_i - \lambda_i}, \quad i = 1, 2,$$

where $1/(\mu_i - \lambda_i)$ is the expected time of staying in a queueing system (Saati, 1961). Then the intensities of the flows λ_1 and $\lambda_2 = \lambda - \lambda_1$ for the corresponding services can be found from

$$c_1 + p_1 + \frac{1}{\mu_1 - \lambda_1} = c_2 + p_2 + \frac{1}{\mu_2 - \lambda_2}. \tag{1}$$

So, the payoff functions for each player are

$$H_1(c_1, c_2, p_1, p_2) = \lambda_1 c_1, \quad H_2(c_1, c_2, p_1, p_2) = \lambda_2 c_2,$$

$$K_1(c_1, c_2, p_1, p_2) = \lambda_1 p_1, \quad K_2(c_1, c_2, p_1, p_2) = \lambda_2 p_2.$$

We are interested in the equilibrium in this game.

Symmetric model. Let start from the symmetric case, when the services are the same, i. e. $\mu_1 = \mu_2 = \mu$. Assuming that the stores fixed their prices p_1 and p_2, let us find the the equilibrium behavior for the transport companies. The equation (1) for the intensity λ_1 is

$$c_1 + p_1 + \frac{1}{\mu - \lambda_1} = c_2 + p_2 + \frac{1}{\mu - \lambda + \lambda_1}. \tag{2}$$

Differentiating (2) by c_1 we can find

$$1 + \frac{1}{(\mu - \lambda_1)^2} \frac{d\lambda_1}{dc_1} = -\frac{1}{(\mu - \lambda + \lambda_1)^2} \frac{d\lambda_1}{dc_1},$$

from which

$$\frac{d\lambda_1}{dc_1} = -\left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right)^{-1}. \tag{3}$$

Now we can find Nash equilibrium strategies c_1^* and c_2^* for fixed p_1, p_2 and c_2, i. e. we can find the maximum of $H_1(c_1, c_2, p_1, p_2)$ by c_1. The first order condition for the maximum of payoff function is

$$\frac{dH_1(c_1, c_2, p_1, p_2)}{dc_1} = \lambda_1 + c_1 \frac{d\lambda_1}{dc_1} = 0,$$

wherefrom

$$c_1^* = \frac{\lambda_1}{\frac{d\lambda_1}{dc_1}}. \tag{4}$$

substituting (3) to (4), we will get

$$c_1^* = \lambda_1 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right). \tag{5}$$

For another transport company it is

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda + \lambda_1)^2} \right). \tag{6}$$

Now we can find the Nash equilibrium for players P_1 and P_2. Let us find the maximum of $K_1(c_1, c_2, p_1, p_2)$ by p_1 when p_2 is fixed, assuming that transport companies use the equilibrium strategies. The first order condition for the maximum of payoff function is

$$\frac{dK_1(c_1, c_2, p_1, p_2)}{dp_1} = \lambda_1 + p_1 \frac{d\lambda_1}{dp_1} = 0,$$

from where

$$p_1^* = \frac{\lambda_1}{\frac{d\lambda_1}{dp_1}}.$$

substituting the equilibrium prices of the transport companies (5)-(6) to (2) and differentiating it by p_1, we will get

$$\frac{d\lambda_1}{dp_1} = -\left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda + \lambda_1)^2} + (2\lambda_1 - \lambda)\big(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda + \lambda_1)^3}\big) \right)^{-1}. \tag{7}$$

So,

$$p_1^* = \lambda_1 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_1 - \lambda)\big(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda_2)^3}\big) \right).$$

For another store it is

$$p_2^* = \lambda_2 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_2 - \lambda)\big(\frac{2}{(\mu - \lambda_2)^3} - \frac{2}{(\mu - \lambda_1)^3}\big) \right).$$

Thus we get the system of equations that defines the equilibrium prices as transport companies and stores.

$$\begin{aligned}
c_1 + p_1 + \frac{1}{\mu - \lambda_1} &= c_2 + p_2 + \frac{1}{\mu - \lambda_2} \\
c_1^* &= \lambda_1 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda_2)^2} \right) \\
c_2^* &= \lambda_2 \left(\frac{1}{(\mu - \lambda_1)^2} + \frac{1}{(\mu - \lambda_2)^2} \right) \\
p_1^* = \lambda_1 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} \right. &\left. + (2\lambda_1 - \lambda)\big(\frac{2}{(\mu - \lambda_1)^3} - \frac{2}{(\mu - \lambda_2)^3}\big) \right) \\
p_2^* = \lambda_2 \left(\frac{3}{(\mu - \lambda_1)^2} + \frac{3}{(\mu - \lambda_2)^2} + (2\lambda_2 - \lambda)\big(\frac{2}{(\mu - \lambda_2)^3} \right. &\left. - \frac{2}{(\mu - \lambda_1)^3}\big) \right) \lambda_1 + \lambda_2 = \lambda.
\end{aligned}$$

Using the symmetry of the problem, the solution of this system is

$$\begin{aligned} \lambda_1 = \lambda_2 &= \frac{\lambda}{2} \\ c_1^* = c_2^* &= \frac{\lambda}{(\mu - \frac{\lambda}{2})^2} \\ p_1^* = p_2^* &= \frac{3\lambda}{(\mu - \frac{\lambda}{2})^2} \end{aligned} \tag{8}$$

It is easy to check, that the second order condition for the maximum of payoff function is also satisfied.

$$\frac{d^2 H_1}{dc_1^2} = 2\frac{d\lambda_1}{dc_1} + c_1 \frac{d^2\lambda_1}{dc_1^2}.$$

$$\frac{d^2 K_1}{dp_1^2} = 2\frac{d\lambda_1}{dp_1} + p_1 \frac{d^2\lambda_1}{dp_1^2}.$$

Differentiating (3) by c_1 and (7) by p_1 we find

$$\frac{d^2\lambda_1}{dc_1^2} = \left(\frac{d\lambda_1}{dc_1}\right)\left[\frac{2}{(\mu-\lambda_1)^3} - \frac{2}{(\mu-\lambda+\lambda_1)^3}\right].$$

$$\frac{d^2\lambda_1}{dp_1^2} = \left(\frac{d\lambda_1}{dp_1}\right)\left[\frac{10}{(\mu-\lambda_1)^3} - \frac{10}{(\mu-\lambda+\lambda_1)^3} + (2\lambda_1-\lambda)\big(\frac{6}{(\mu-\lambda_1)^4} + \frac{6}{(\mu-\lambda+\lambda_1)^4}\right].$$

In the equilibrium $\lambda_1 = \lambda/2$, from which $\frac{d^2\lambda_1}{dc_1^2} = 0$ è $\frac{d^2\lambda_1}{dp_1^2} = 0$. So,

$$\frac{d^2 H_1(c_1^*, c_2^*, p_1^*, p_2^*)}{dc_1^2} = 2\frac{d\lambda_1}{dc_1} = -\left(\mu - \frac{\lambda}{2}\right)^2 < 0.$$

$$\frac{d^2 K_1(c_1^*, c_2^*, p_1^*, p_2^*)}{dp_1^2} = 2\frac{d\lambda_1}{dp_1} = -\frac{\left(\mu - \frac{\lambda}{2}\right)^2}{3} < 0.$$

So, if one of the players uses the strategy (8), the maximum of payoff of another player is reached at the same strategy. That means that this set of strategies is equilibrium.

Asymmetric model. Let us assume now, that transport services are not equal, i. e. $\mu_1 \neq \mu_2$, suppose that $\mu_1 > \mu_2$. Let us find the equilibrium in the pricing problem in this case. Let us fix p_1, p_2 and c_2 and find the best reply of the player C_1. As well as in the symmetric case we get

$$\frac{dH_1(c_1, c_2, p_1, p_2)}{dc_1} = \lambda_1 + c_1\frac{d\lambda_1}{dc_1} = 0,$$

wherefrom

$$c_1^* = \frac{\lambda_1}{d\lambda_1/dc_1}.$$

Differentiating (1),we find

$$c_1^* = \lambda_1\left(\frac{1}{(\mu_1-\lambda_1)^2} + \frac{1}{(\mu_2-\lambda_2)^2}\right).$$

For another transport company it is

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right).$$

Table 1: The value of (c_1^*, c_2^*), (p_1^*, p_2^*) and (λ_1, λ_2) at $\lambda = 10$

		μ_2				
μ_1		6	7	8	9	10
6	$(c_1^*;c_2^*)$	(10;10)				
	$(p_1^*;p_2^*)$	(30;30)				
	$(\lambda_1;\lambda_2)$	(5;5)				
7	$(c_1^*;c_2^*)$	(5,918;5,804)	(2,5;2,5)			
	$(p_1^*;p_2^*)$	(17,035;16,707)	(7,5;7,5)			
	$(\lambda_1;\lambda_2)$	(5,049;4,951)	(5;5)			
8	$(c_1^*;c_2^*)$	(4,953;4,797)	(1,781;1,743)	(1,11;1,11)		
	$(p_1^*;p_2^*)$	(13,636;13,208)	(5,26;5,15)	(3,33;3,33)		
	$(\lambda_1;\lambda_2)$	(5,08;4,92)	(5,053;4,947)	(5;5)		
9	$(c_1^*;c_2^*)$	(4,553;4,375)	(1,494;1,437)	(0,866;0,848)	(0,625;0,625)	
	$(p_1^*;p_2^*)$	(12,165;11,689)	(4,3;4,136)	(2,597;2,533)	(1,875;1,875)	
	$(\lambda_1;\lambda_2)$	(5,1;4,9)	(5,097;4,903)	(5,054;4,946)	(5,5)	
10	$(c_1^*;c_2^*)$	(4,342;4,15)	(1,346;1,276)	(0,743;0,713)	(0,514;0,503)	(0,4;0,4)
	$(p_1^*;p_2^*)$	(11,371;10,869)	(3,781;3,586)	(2,176;2,088)	(1,535;1,502)	(1,2;1,2)
	$(\lambda_1;\lambda_2)$	(5,113;4,887)	(5,132;4,868)	(5,103;4,897)	(5,055;4,945)	(5;5)

Now we can find the best replies for the P_1 and P_2.

$$\frac{dK_i(c_1, c_2, p_1, p_2)}{dp_i} = \lambda_i + p_i \frac{d\lambda_i}{dp_i} = 0, \quad i = 1, 2,$$

from which

$$p_i^* = \frac{\lambda_i}{d\lambda_i / dp_i} i = 1, 2.$$

Using the same arguments as in the symmetric model, we obtain the system of equations that determine the equilibrium prices as transport companies and stores.

$$c_1 + p_1 + \frac{1}{\mu_1 - \lambda_1} = c_2 + p_2 + \frac{1}{\mu_2 - \lambda_2}$$

$$c_1^* = \lambda_1 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right)$$

$$c_2^* = \lambda_2 \left(\frac{1}{(\mu_1 - \lambda_1)^2} + \frac{1}{(\mu_2 - \lambda_2)^2} \right)$$

$$p_1^* = \lambda_1 \left(\frac{3}{(\mu_1 - \lambda_1)^2} + \frac{3}{(\mu_2 - \lambda_2)^2} + (2\lambda_1 - \lambda)\left(\frac{2}{(\mu_1 - \lambda_1)^3} - \frac{2}{(\mu_2 - \lambda_2)^3}\right) \right)$$

$$p_2^* = \lambda_2 \left(\frac{3}{(\mu_1 - \lambda_1)^2} + \frac{3}{(\mu_2 - \lambda_2)^2} + (2\lambda_2 - \lambda)\left(\frac{2}{(\mu_2 - \lambda_2)^3} - \frac{2}{(\mu_1 - \lambda_1)^3}\right) \right)$$

$$\lambda_1 + \lambda_2 = \lambda.$$

In Table 1 the values of the equilibrium prices with different μ_1, μ_2 at $\lambda = 10$ and are given.

2. Conclusion

It is seen from the table, that the higher the intensity of service of one transport company is, the higher payoff this transport company and the store, which is connected to this company, get. So, they can increase the price of the service and the price for the product.

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Optimal Strategies in the Game with Arbitrator*

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Abstract The organization of negotiations by using arbitration procedures is an actual problem in game theory. We consider a non-cooperative zero-sum game, related with an arbitration scheme, generalized well known final-offer procedure. The nash equilibrium in this game in mixed strategies is found.

Keywords: game, arbitration procedure, equilibrium, mixed strategies.

1. Introduction

The problem of some resource allocation among several participants take one of the central places in the modern theory of economical regulation. This situations occur in business (a Labour and a Manager consider the question on an improvement in the wage rate), in the market models (a Buyer, who wants to purchase some merchandise at a lower price, and the Seller, whose purpose is to sell this merchandise at a more beneficial price), insurance models, etc. This is a multicriterial problem, for which there are several solving approaches. We use game-theoretical methods of negotiation theory. In order to run the negotiations the participants call in the third independent party of one or several arbitrators participates. By the solution, we mean the Nash equilibrium in this game. The procedures with arbitrator's participation are called arbitration procedures. The problems of negotiation organization by using arbitration procedures are topical presently and in connection with of virtual enterprises appearing in the global Internet network.

There are various models of arbitration procedures. One of them is the final-offer arbitration procedure. This procedure was described in the papers (Farber,1980; Chatterjee, 1981; Kilgour, 1994).

We will find an equilibrium in the arbitration games in the terms of salary problem; however this approach may be also applied for other problems of resources allocation with arbitrator's participation.

So, we consider a non-cooperative zero-sum game in which two players L and M, called respectively the Labour and the Manager, have a dispute on an improvement in the wage rate. The player L makes an offer x, and the player M - an offer y; x and y are arbitrary real numbers. If $x \leq y$ there is no conflict, and the players agree on a payoff equal to $(x+y)/2$. If, otherwise, $x > y$, the parties call in the arbitrator A. Assume that the arbitrator's solution is a discrete random variable and denote it by z. In the final-offer arbitration scheme the arbitrator chooses the offer, which

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is closer to its solution z, i.e., the payoff function in this scheme has a form

$$H_z(x,y) = \begin{cases} \frac{x+y}{2}, & if \quad x \le y, \\ x, & if \;\; x > y, |x-z| < |y-z|, \\ y, & if \;\; x > y, |x-z| > |y-z|, \\ z, & if \;\; x > y, |x-z| = |y-z|. \end{cases} \tag{1.1}$$

Since in the function (1.1) the arbitrator's solution z is a random variable, we take for the payoff function the mathematical expectation of this function: $H(x,y) = EH_z(x,y)$.

Further, let $x \in [0,+\infty)$, $y \in (-\infty, 0]$. If $z = 0$ almost everywhere, it is evidently that the point of equilibrium in this game is the pair of pure strategies: $(0,0)$. In the papers (Mazalov et al., 2005; Mazalov et al., 2006; Mentcher, 2009) for the cases in which z is distributed in the final set of integer points the Nash equilibria in this game in mixed strategies were found.

Now we consider a generalization of the final-offer arbitration procedure. Namely, let $x \in [0,+\infty)$, $y \in (-\infty, 0]$, $\alpha > 0$ and

$$H_z(x,y) = \begin{cases} x^\alpha, & if \; |x-z| < |y-z|, \\ -(-y)^\alpha, & if \; |x-z| > |y-z|, \\ z, & if \; |x-z| = |y-z|. \end{cases} \tag{1.2}$$

Let the arbitrator chooses one of the $2n+1$ numbers: $-n$, $-(n-1)$,Ě,-1, 0, 1, Ě, $n-1$, n - with equal probabilities $p = \frac{1}{2n+1}$. This game does not have a solution in pure strategies, and we will be looking for the equilibrium in mixed strategies. Denote by $f(x)$ and $g(y)$ the mixed strategies of the players L and M, respectively. We have

$$f(x) \ge 0, \; \int\limits_0^{+\infty} f(x)dx = 1; \;\; g(y) \ge 0, \; \int\limits_{-\infty}^{0} g(y)dy = 1.$$

Due to the symmetry, the game value is equal to zero, and the optimal strategies are symmetric in respect to the y-axis, i.e. $g(y) = f(-y)$. Hence, it suffices to construct the optimal strategy only for one player, for example L.

We find the optimal strategy for the player L in the following form:

$$f(x) = \begin{cases} 0, & if \; 0 \le x < c, \\ \varphi(x), & if \; c < x < c+2, \\ 0, & if \; c+2 < x < +\infty, \end{cases} \tag{1.3}$$

where the function $\varphi(x)$ is positive and continuously differentiable in the interval $(-(c+2), -c)$.

Denote by $H(f(x), y)$ the payoff function of the player M for the strategy $f(x)$ choosen by the player L. The function $H(f(x), y)$ is continuous on the entire semi-axis $(-\infty, 0]$ and twice continuously differentiable in the interval $(c, c+2)$. The strategy (1.3) will be optimal, if $H(f(x), y) = 0$ for $y \in [-(c+2), -c]$ and $H(f(x), y) \ge 0$ for $y \in (-\infty, -(c+2)) \cup (-c, 0]$.

Assume that $y \in [-(c+2), -c]$, then $-y \in [c, c+2]$, and

$$H(f(x), y) = \frac{1}{2n+1}\left[n\int_c^{c+2}(-(-y)^\alpha)f(x)dx + \int_c^{-y} x^\alpha f(x)dx + \right.$$

$$+\int_{-y}^{c+2}(-(-y)^{\alpha})f(x)dx+n\int_{c}^{c+2}x^{\alpha}f(x)dx\Bigg]. \tag{1.4}$$

If $f(x)$ is an optimal strategy, then

$$0=H(f(x),-c-0)=\frac{1}{2n+1}\left[-(n+1)c^{\alpha}+n\int_{c}^{c+2}x^{\alpha}f(x)dx\right],$$

$$0=H(f(x),-(c+2)+0)=\frac{1}{2n+1}\left[-n(c+2)^{\alpha}+(n+1)\int_{c}^{c+2}x^{\alpha}f(x)dx\right]. \tag{1.5}$$

From (1.5) we obtain the equation

$$\left(\frac{n+1}{n}\right)c^{\alpha}=\left(\frac{n}{n+1}\right)(c+2)^{\alpha}.$$

Whence we conclude that

$$c=\frac{2}{(1+\frac{1}{n})^{\frac{2}{\alpha}}-1} \tag{1.6}$$

and

$$\int_{c}^{c+2}x^{\alpha}f(x)dx=\sqrt{c^{\alpha}(c+2)^{\alpha}}. \tag{1.7}$$

For the strategy (1.3) in order to be optimal, it is necessary that $0<c\leq 2n$, whence we obtain $0<\alpha\leq 2$.

Furthere, it is necessary that $H'(f(x),y)=H''(f(x),y)=0$ in the interval $(-(c+2),-c)$. We have

$$H'(f(x,y))=\frac{1}{2n+1}\left[n\alpha(-y)^{\alpha-1}-2(-y)^{\alpha}f(-y)+\alpha(-y)^{\alpha-1}\int_{-y}^{c+2}f(x)dx\right], \tag{1.8}$$

$$H''(f(x,y))=\frac{1}{2n+1}\Big[-n\alpha(\alpha-1)(-y)^{\alpha-2}+3\alpha(-y)^{\alpha-1}f(-y)+$$

$$+2(-y)^{\alpha}f'(-y)-\alpha(\alpha-1)(-y)^{\alpha-2}\int_{-y}^{c+2}f(x)dx\Bigg]. \tag{1.9}$$

If now $H'(f(x),y)=H''(f(x),y)=0$ in the interval $(-(c+2),-c)$, then from (1.8)-(1.9) we obtain

$$(\alpha-1)(-y)^{-1}H'(f(x),y)+H''(f(x),y)=0,$$

whence

$$(\alpha+2)f(-y)-2yf'(-y)=0. \tag{1.10}$$

Assume that $x=-y$, then $x\in(c,c+2)$, $f(x)=\varphi(x)$ and

$$2x\varphi'(x)+(\alpha+2)\varphi(x)=0. \tag{1.11}$$

The solution of this equation is the function

$$\varphi(x)=\beta x^{-(\frac{\alpha}{2}+1)}. \tag{1.12}$$

Determine the constant β. From (1.8) we obtain

$$0 = H'(f(x), -c-0) = \frac{1}{2n+1}\left[\alpha(n+1)c^{\alpha-1} - 2c^{\alpha}\frac{\beta}{c^{\frac{\alpha}{2}+1}}\right],$$

whence

$$\beta = \frac{\alpha(n+1)}{2}c^{\frac{\alpha}{2}}.$$

Therefore, the function $f(x)$ from (1.3) has a form

$$f(x) = \begin{cases} 0, & \textit{if } \ 0 \le x < c, \\ \frac{\alpha(n+1)}{2}\cdot\frac{c^{\frac{\alpha}{2}}}{x^{\frac{\alpha}{2}+1}}, & \textit{if } c < x < c+2, \\ 0, & \textit{if } c+2 < x < +\infty, \end{cases} \tag{1.13}$$

where

$$c = \frac{2}{\left(1+\frac{1}{n}\right)^{\frac{2}{\alpha}} - 1}.$$

2. Optimal strategies

Theorem 1. *If $\alpha \in (0, 2]$ and $n = 1$, then for the player L the strategy*

$$f(x) = \begin{cases} 0, & \textit{if } \ 0 \le x < c, \\ \frac{\alpha\cdot c^{\frac{\alpha}{2}}}{x^{\frac{\alpha}{2}+1}}, & \textit{if } c < x < c+2, \\ 0, & \textit{if } c+2 < x < +\infty, \end{cases} \tag{2.1}$$

where $c = \frac{2}{4^{\frac{1}{\alpha}}-1}$ is optimal.

Proof. Assuming in (1.13) $n = 1$, we come to the formula (2.1) with corresponding constant c. Check the fulfilment of optimal conditions.

Assume that $y \in [-(c+2), -c]$, then

$$H(f(x), y) = \frac{1}{3}\left[-(-y)^{\alpha} + \int_c^{-y} \alpha c^{\frac{\alpha}{2}} x^{\frac{\alpha}{2}-1} dx - (-y)^{\alpha}\int_{-y}^{c+2} \alpha c^{\frac{\alpha}{2}} x^{-\frac{\alpha}{2}-1} dx + 2c^{\alpha}\right] =$$

$$= \frac{1}{3}\left[-(-y)^{\alpha} + 2c^{\frac{\alpha}{2}}(-y)^{\frac{\alpha}{2}} - 2c^{\alpha} + (-y)^{\alpha} - 2c^{\frac{\alpha}{2}}(-y)^{\frac{\alpha}{2}} + 2c^{\alpha}\right] = 0. \tag{2.2}$$

Assume that $y \in (-\infty, -(c+4)]$, then

$$H(f(x), y) = \int_c^{c+2} x^{\alpha} f(x) dx = 2c^{\alpha}. \tag{2.3}$$

Assume that $y \in [-(c+4), -(c+2)]$, then $-y \in [c+2, c+4]$, $-2-y \in [c, c+2]$ and

$$H(f(x), y) = \frac{1}{3}\left[\int_c^{-2-y} x^{\alpha} f(x) dx - (-y)^{\alpha}\int_{-2-y}^{c+2} f(x) dx + 2\int_c^{c+2} x^{\alpha} f(x) dx\right] =$$

$$= \frac{2}{3}c^{\frac{\alpha}{2}}\left[(-2-y)^{\frac{\alpha}{2}} + c^{\frac{\alpha}{2}} + \frac{(-y)^{\alpha}}{(c+2)^{\frac{\alpha}{2}}} - \frac{(-y)^{\alpha}}{(-2-y)^{\frac{\alpha}{2}}}\right]. \tag{2.4}$$

We have

$$H(f(x), -(c+2)-0) = \frac{2}{3}c^{\frac{\alpha}{2}}\left[c^{\frac{\alpha}{2}} + c^{\frac{\alpha}{2}} + (c+2)^{\alpha} - 2(c+2)^{\alpha}\right] = 0. \tag{2.5}$$

Further, assume in (2.4) $-2-y=t$, $t \in [c, c+2]$ and consider the function

$$\widetilde{H}(t) = \frac{2}{3}c^{\frac{\alpha}{2}}\left[t^{\frac{\alpha}{2}} + c^{\frac{\alpha}{2}} + \frac{(t+2)^{\alpha}}{(c+2)^{\frac{\alpha}{2}}} - \frac{(t+2)^{\alpha}}{t^{\frac{\alpha}{2}}}\right].$$

The functions $g(t) = \frac{(t+2)^{\alpha}}{(c+2)^{\frac{\alpha}{2}}}$ and

$$h(t) = t^{\frac{\alpha}{2}} - \frac{(t+2)^{\alpha}}{t^{\frac{\alpha}{2}}} = t^{\frac{\alpha}{2}}\left[1 - \left(1+\frac{2}{t}\right)^{\alpha}\right]$$

are monotonouse increasing in the interval $[c, c+2]$. Finally, we conclude that the function $H(f(x), y)$ is monotonouse decreasing in the interval $[-(c+4), -(c+2)]$ from $2c^{\alpha}$ to 0 and therefore is positive in the interval $[-(c+4), -(c+2))$.

Assume that $y \in [-c, 0]$, then $-y \in [0, c]$, $2-y \in [2, c+2]$ and

$$H(f(x), y) = \frac{1}{3}\left[-2(-y)^{\alpha} + \int_{c}^{2-y} x^{\alpha} f(x)dx - \int_{2-y}^{c+2} (-y)^{\alpha} f(x)dx\right] =$$

$$= \frac{1}{3}\left[-(-y)^{\alpha} + 2c^{\frac{\alpha}{2}}(2-y)^{\frac{\alpha}{2}} - 2c^{\alpha} - 2c^{\frac{\alpha}{2}} - 2c^{\frac{\alpha}{2}}\frac{(-y)^{\alpha}}{(2-y)^{\frac{\alpha}{2}}}\right]. \tag{2.6}$$

We have

$$H(f(x), -c+0) = 0, H(f(x), -0) = \frac{2}{3}c^{\frac{\alpha}{2}}(2^{\frac{\alpha}{2}} - c^{\frac{\alpha}{2}}) \geq 0. \tag{2.7}$$

Furthere,

$$H'(f(x), y) = \frac{\alpha}{3}\left[(-y)^{\alpha-1} + c^{\frac{\alpha}{2}}\frac{-(2-y)^{\alpha} + 4(-y)^{\alpha-1} + (-y)^{\alpha}}{(2-y)^{\frac{\alpha}{2}+1}}\right]. \tag{2.8}$$

Assume that $\alpha \in (0, 1]$, then $c \in (0, \frac{2}{3}] \subset (0, 1]$.

We have

$$-(2-y)^{\alpha} + 4(-y)^{\alpha-1} + (-y)^{\alpha} = -(2-y)^{\alpha} + (-y)^{\alpha-1}(4-y) \geq (4-y) - (2-y) = 2.$$

Therefore, $H'(f(x), y) > 0$ in the interval $(-c, 0)$ and take into consideration (2.7), we conclude that $H(f(x), y) > 0$ in this interval.

Assume that $\alpha \in (1, 2]$, then $c \in (\frac{2}{3}, 2]$. We have

$$H'(f(x), -c+0) = \frac{\alpha}{6}\frac{c^{\alpha-1}(8-c)}{c+2} > 0,$$

$$H'(f(x), -0) = -\frac{\alpha c^{\frac{\alpha}{2}}}{3 \cdot 2^{1-\frac{\alpha}{2}}} < 0. \tag{2.9}$$

Therefore, in the interval $(-c, 0)$ exists if only one point y_0, for which $H'(f(x), y_0) = 0$. If y_0 is the unique point, then y_0 is the point of maximum for the function

$H(f(x),y)$ and take into consideration (2.7) we conclude that $H(f(x),y)>0$ in the interval $(-c,0)$.

Assume in (2.8) $-y=t$, $t\in[0,c]$, $y_0=-t_0$.

Then

$$\widetilde{H}'(t)=\frac{\alpha}{3}\left[t^{\alpha-1}+c^{\frac{\alpha}{2}}\frac{t^{\alpha-1}(t+4)-(t+2)^{\alpha}}{(t+2)^{\frac{\alpha}{2}+1}}\right].$$

If now $\widetilde{H}'(t)=0$, then we obtain the equation

$$t^{\alpha-1}(t+2)^{\frac{\alpha}{2}+1}+c^{\frac{\alpha}{2}}t^{\alpha}\left(1-\left(1+\frac{2}{t}\right)^{\alpha}\right)=-4c^{\frac{\alpha}{2}}t^{\alpha-1}. \tag{2.10}$$

The function from the left part of (2.10) is monotonouse increasing in the interval $(-c,0)$, but the function from the right part is monotonouse decreasing in the same interval. Consequently, it exists an unique point t_0, for which $\widetilde{H}'(t_0)=0$.

□

In particular, if $\alpha=1$, we have

$$f(x)=\begin{cases}0, & \textit{if}\, 0\le x<\frac{2}{3},\\ \sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{x^3}}, & \textit{if}\, \frac{2}{3}<x<\frac{8}{3},\\ 0, & \textit{if}\, \frac{8}{3}<x<+\infty.\end{cases} \tag{2.11}$$

This result was published in the paper (Mazalov et al., 2005). The graf $H(f(x),y)$ has the form, presented in Fig. 1.

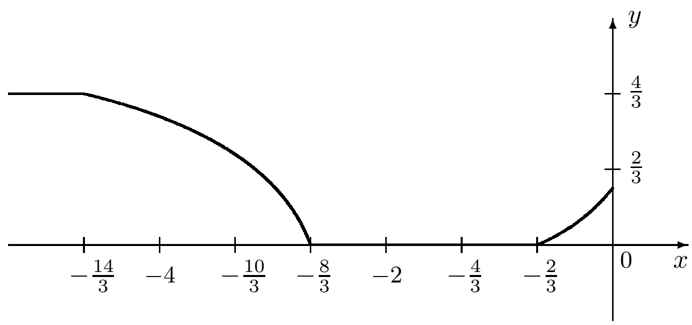

Fig. 1.

For $\alpha=2$ we have

$$f(x)=\begin{cases}0, & \textit{if}\;\; 0\le x<2,\\ \frac{4}{x^2}, & \textit{if}\;\; 2<x<4,\\ 0, & \textit{if}\;\; 4<x<+\infty.\end{cases} \tag{2.12}$$

The graf $H(f(x),y)$ has the form, presented in Fig. 2.

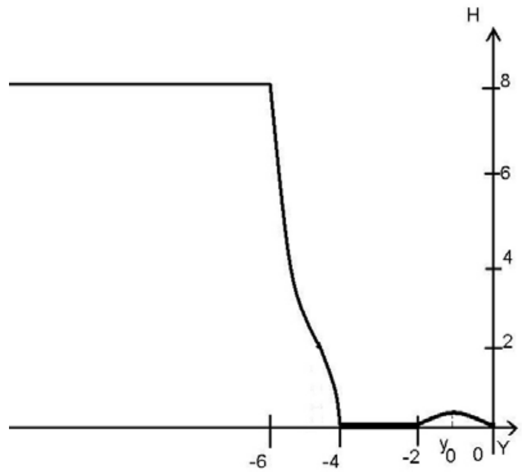

Fig. 2.

Theorem 2. *If $\alpha = 1$, then for the player L the strategy*

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < c, \\ \frac{(n+1)\sqrt{c}}{2\sqrt{x^3}}, & \text{if } c < x < c+2, \\ 0, & \text{if } c+2 < x < +\infty, \end{cases} \tag{2.13}$$

where $c = \frac{2n^2}{2n+1}$ is optimal.

Proof. Assuming in (1.13) $\alpha = 1$, we come to the formula (2.13) with corresponding constant c. Check the fulfilment of optimal conditions.

Assume that $y \in (-\infty, -(c+2) - 2n]$, then

$$H(f(x), y) = \int_c^{c+2} xf(x)dx = \sqrt{c(c+2)} = \frac{2n(n+1)}{2n+1}. \tag{2.14}$$

Furthere, let $k = 3\left[\frac{n}{2}\right] + 2$, if n is odd and $k = 3\frac{n}{2}$, if n is even. For $y \in [-(c+2)-2n+2r, -c-2n+2r]$, where $r = 0, 1, ..., n, ..., k-1$ and $y \in [-(c+2)-2n+2r, 0]$, if $r = k$, we find

$$H(f(x), y) = \frac{1}{2n+1}\left[ry + \left(\int_c^{-2n+2r-y} xf(x)dx + \int_{-2n+2r-y}^{c+2} yf(x)dx\right) + \right.$$

$$\left. +(2n-r)\int_c^{c+2} xf(x)dx\right] =$$

$$= \int_c^{c+2} xf(x)dx - \frac{1}{2n+1}\left[r\int_c^{c+2}(x-y)f(x)dx + \int_{-2n+2r-y}^{c+2}(x-y)f(x)dx\right]. \tag{2.15}$$

Differentiating (2.15), we obtain

$$H'(f(x),y) = \frac{1}{2n+1}\left[r + \int_{-2n+2r-y}^{c+2} f(x)dx + (2n-2r+2y)f(-2n+2r-y)\right] =$$

$$= \frac{r-n}{2n+1}\left(1+\frac{2n(n+1)}{\sqrt{2(2n+1)(-2n+2r-y)}}\right) \tag{2.16}$$

It follows from (2.16) that in the interval $[-(c+2), -c]$, where $r = n$, the expected payoff $H(f(x), y)$ is constant and because we used the equality $H(f(x), -c-0) = 0$ it yields $H(f(x), y) = 0$ in the interval $[-(c+2), -c]$.

For $r < n$ (2.16) gives $H'(f(x), y) < 0$ and for $r > n$ — $H'(f(x), y) > 0$ in the interval $[-2n+2r-(c+2), -2n+2r-c]$.

Consequently, the function $H(f(x), y)$ is positive outside the interval $[-(c+2), -c]$. That proves the optimality of the strategy (2.13). □

Theorem 3. *If $\alpha = 2$, then for the player L the strategy*

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 2n, \\ \frac{2n(n+1)}{x^2}, & \text{if } 2n < x < 2n+2, \\ 0, & \text{if } 2n+2 < x < +\infty \end{cases} \tag{2.17}$$

is optimal.

Proof. Assuming in (1.13) $\alpha = 2$, we come to the formula (2.17). Check the fulfilment of optimal conditions.

Assume then $y \in (-\infty, -(4n+2)]$, then

$$H(f(x), y) = \int_{2n}^{2n+2} x^2 f(x)dx = 4n(n+1). \tag{2.18}$$

Furthere, let $y \in [-(2n+2k+2), -(2n+2k)]$, where $k = -n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n$. Then

$$H(f(x),y) = \frac{1}{2n+1}\left[(n-k)\int_{2n}^{2n+2}(-y^2)f(x)dx + \int_{2n}^{-2k-y} x^2 f(x)dx + \right.$$

$$\left. + \int_{-2k-y}^{2n+2}(-y^2)f(x)dx + (n+k)\int_{2n}^{2n+2} x^2 f(x)dx\right] =$$

$$= \frac{1}{2n+1}\left[-(n-k)y^2 + 2n(n+1)(-2k-y-2n) + \right.$$

$$\left. + 2n(n+1)y^2\left(\frac{1}{2n+2}+\frac{1}{y+2k}\right) + 4n(n+1)(n+k)\right]. \tag{2.19}$$

For $k = 0$ we have $y \in [-(2n+2), -2n]$ and $H(f(x), y) = 0$ in this interval.

Furthere, assume that $y = -2n-2k$. We have

$$H(f(x), -2n-2k) = \frac{4n(n+k)(k-1)}{2n+1}. \tag{2.20}$$

From (2.20) we obtain that $H(f(x), -2n-2k) = 0$ for $k = -n$, $k = 0$ and $k = 1$; and $H(f(x), -2n-2k) > 0$ for all other considered values of k.

Besides

$$H'(f(x), y) = \frac{2k}{2n+1}\left[y - \frac{4n(n+1)k}{(y+2k)^2}\right], \tag{2.21}$$

$$H''(f(x), y) = \frac{2k}{2n+1}\left[1 + \frac{8kn(n+1)}{(y+2k)^3}\right]. \tag{2.22}$$

If now $k \geq 1$, then $H'(f(x), y) < 0$ and the function $H(f(x), y)$ is monotonous decreasing in the interval $[-(4n+2), -(2n+2)]$ from $4n(n+1)$ to 0. If $k \leq -1$, then $H''(f(x), y) < 0$ and the function $H(f(x), y)$ is concave in the interval $[-(2n+2k+2), -(2n+2k)]$.

Take into cosideration preceding arguments, we conclude that $H(f(x), y) > 0$ in the interval $(-2n, 0)$ and $H(f(x), 0) = 0$.

$\square$

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Solidary Solutions to Games with Restricted Cooperation

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Abstract In TU–cooperative game with restricted cooperation the values of characteristic function $v(S) > 0$ are defined only for $S \in \mathcal{A}$, where $\mathcal{A}$ is a collection of some nonempty coalitions of players.
We examine generalizations of both the proportional solutions of claim problem (Proportional and Weakly Proportional solutions, the Proportional Nucleolus, and the Weighted Entropy solution) and the uniform losses solution of claim problem (Uniform Losses and Weakly Uniform Losses solutions, the Nucleolus, and the Least Square solution). These generalizations are U–equal sacrifice solution, the U–nucleolus and qU–solutions, where U and q are strictly increasing continuous functions.
We introduce Solidary (Weakly Solidary) solutions, where if a total share of some coalition in $\mathcal{A}$ is less than its claim, then the total shares of all coalitions in $\mathcal{A}$ (that don't intersect this coalition) are less than their claims. The existence conditions on $\mathcal{A}$ for two versions of solidary solution are described. In spite of the fact that the versions of the solidary solution are larger than the corresponding versions of the proportional solution, the necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the U–nucleolus in two versions of the solidary solution coincide with conditions on $\mathcal{A}$ for inclusion of the proportional nucleolus in the corresponding versions of the proportional solution. The necessary and sufficient conditions on $\mathcal{A}$ for inclusion qU–solutions in two versions of the solidary solution coincide with conditions on $\mathcal{A}$ for inclusion of the Weighted Entropy solution in the corresponding versions of the proportional solution.
Moreover, necessary and sufficient conditions on $\mathcal{A}$ for coincidence the U–nucleolus with the U–equal sacrifice solution and conditions on $\mathcal{A}$ for coincidence qU–solutions with the U–equal sacrifice solution are obtained.

Keywords: claim problem; cooperative games; proportional solution; weighted entropy; nucleolus.

1. Introduction

A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of nonempty coalitions of agents, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T\in\mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T. We assume that $\mathcal{A}$ covers N and $N \notin \mathcal{A}$.

A *set of imputations of* $(N, \mathcal{A}, c, v)$ is the set

$$\{\{y_i\}_{i\in N} : \; y_i \geq 0, \; \sum_{i\in N} y_i = c\}.$$

A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of its set of imputations. Then $F(N, \mathcal{A}, c, v)$ is a *solution of* $(N, \mathcal{A}, c, v)$. We denote $y(S) = \sum_{i\in S} y_i$.

If $\mathcal{A} = \{\{i\} : i \in N\}$ then a *claim problem* arises, therefore, a cooperative game with restricted cooperation can be considered as a claim problem with coalition demands.

Solutions of claim problem and their axiomatic justifications are described in surveys (Moulin, 2002) and (Thomson, 2003). For claim problems, the Proportional solution, the Uniform Losses solution and their generalization Equal Sacrifice solution are well known. The papers (Naumova, 2011, 2012) and this paper consider generalizations of these solutions to games with restricted cooperation.

For claim problems, the Proportional solution, the Proportional Nucleolus, and the Weighted Entropy solution give the same results. In the case of generalized claim problems, the Proportional solution is the most natural generalization, but this set can be empty for some games. The larger set is the Weakly Proportional solution, where the ratios of total shares of coalitions to their claims are equal for disjoint coalitions in $\mathcal{A}$. This set can also be empty. The Proportional Nucleolus and the Weighted Entropy solution are always nonempty and define uniquely total shares of coalitions in $\mathcal{A}$. These solutions can give different results.

For claim problems, the Uniform Losses solution, the nucleolus, and the Least Square solution give the same results. For generalized claim problems, the Uniform Losses solution and the Weak Uniform Losses solution are the most natural generalizations but they can be empty. The Nucleolus and the Least Square solution can give different results, but each of them is always nonempty and define uniquely total shares of coalitions in $\mathcal{A}$.

Necessary and sufficient conditions on $\mathcal{A}$ that provide the existence of the Proportional solution (Weakly Proportional solution) are obtained in (Naumova, 2011) and these conditions coincide with conditions that provide the existence of the Uniform Losses solution (Weakly Uniform Losses solution).

Necessary and sufficient conditions on $\mathcal{A}$ that provide inclusion of the Weighted Entropy solution in the Proportional solution are the same as conditions on $\mathcal{A}$ for inclusion of the Least Square solution in the Uniform Losses solution. The same are conditions for inclusion of the Proportional Nucleolus in the Proportional solution and conditions for inclusion of the Nucleolus in the Uniform Losses solution. These conditions were obtained in (Naumova, 2011). That paper also contains necessary and sufficient conditions on $\mathcal{A}$ for coincidence the Weighted Entropy solution and the Weakly Proportional solution.

The paper (Naumova, 2012) considers only generalizations of the Proportional solution of claim problems. Generalizations of the Weighted Entropy solution that are called g–solutions are introduced. Necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the g–solution in the Weakly Proportional solution are the same for all g. These conditions permit to obtain for each g the necessary and sufficient conditions on $\mathcal{A}$ for coincidence the g–solution and the Weakly Proportional solution. The obtained conditions are the same as conditions for coincidence the Weighted Entropy solution and the Weakly Proportional solution. The paper (Naumova, 2012) also contains necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the Proportional Nucleolus in the Weakly Proportional solution. The proofs of that paper are not suitable for obtaining conditions on $\mathcal{A}$ for inclusion of the Nucleolus and the Least Square solution in the Weakly Uniform Losses solution.

In this paper we consider two topics. First, for strictly increasing continuous functions U, we introduce U-equal sacrifice solutions that generalize both the Proportional solution and the Uniform Losses solution, U–nucleolus that generalize both the Proportional Nucleolus and the Nucleolus, and qU–solutions that generalize both q–solutions and the Least Square solution. All results of the paper (Naumova, 2012) concerning the proportional case are generalized. In particular, we obtain conditions on $\mathcal{A}$ that provide inclusion of the Nucleolus in the Weakly Uniform Losses solution and conditions on $\mathcal{A}$ that provide inclusion of the Least Square solution in the Weakly Uniform Losses solution.

Moreover, we obtain the necessary and sufficient conditions on $\mathcal{A}$ that provide coincidence of the U–nucleolus and the Weakly U–equal sacrifice solution.

Second, we introduce new solution concepts of Solidary solution (Weakly Solidary solution) that contain U–equal sacrifice solutions (Weakly U–equal sacrifice solutions).

For almost all solutions of claim problems, if one agent gets less than its claim then each agent gets less than its claim, i.e., the *solidarity property* takes place. Two versions of Solidary solutions are obtained by generalizations of the solidarity property to games with restricted cooperation.

In spite of the fact that the versions of the Solidary solution are larger than the corresponding versions of the Proportional solution, the conditions on $\mathcal{A}$ that ensure existence results for the versions of the Solidary solutions are the same as for the corresponding versions in the proportional case. Moreover, the conditions on $\mathcal{A}$ that provide inclusions of the U–nucleolus in the Solidary solution (Weakly Solidary solution) are the same as conditions on $\mathcal{A}$ that provide inclusions of the Proportional Nucleolus in the Proportional (Weakly Proportional) solution. The conditions on $\mathcal{A}$ that provide inclusion of the qU–solution in the Solidary (Weakly Solidary) solution are the same as conditions on $\mathcal{A}$ for inclusion of the Weighted Entropy solution in the Proportional (Weakly Proportional) solution.

The paper is organized as follows. The definitions of U–equal sacrifice solutions, U–nucleolus, qU–solutions, the Solidary solutions and the relations between U–equal sacrifice solutions and the Solidary solutions are given in Section 2. Some properties of qU–solutions that will be used in next sections are obtained in Section 3. Conditions on $\mathcal{A}$ for existence the U–equal sacrifice and the Weakly U–equal sacrifice solutions are described in Section 4. Necessary and sufficient conditions on $\mathcal{A}$ for inclusion of the qU–solution in the U–equal sacrifice solution and in the Solidary solution and for inclusion of the U–nucleolus in the U–equal sacrifice solution and in the Solidary solution are obtained in Section 5. In Section 6 we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the U–nucleolus in the Weakly U—equal sacrifice solution and in the Weakly Solidary solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion of qU–solution in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution. In Section 7 we describe necessary and sufficient conditions on $\mathcal{A}$ for coincidence the qU–solution with the Weakly U–equal sacrifice solution and conditions on $\mathcal{A}$ for coincidence the U–nucleolus with the Weakly U–equal sacrifice solution.

2. Definitions

Definition 1. A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of coalitions of agents, $N \notin \mathcal{A}$, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T \in \mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T.

We assume that $\mathcal{A}$ covers N.

Definition 2. A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of *its set of imputations* $\{\{y_i\}_{i\in N} : \; y_i \geq 0, \; \sum_{i\in N} y_i = c\}$. We denote $y(S) = \sum_{i\in S} y_i$.

Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Denote $U(0) = \lim_{t\to 0} U(t)$.

Definition 3. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *U–equal sacrifice solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$, $y(T) > 0$ implies $U(y(T)) - U(v(T)) \leq U(y(S)) - U(v(S))$.

Definition 4. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Proportional solution* of $(N, \mathcal{A}, c, v)$ iff $y(T)/v(T) = y(S)/v(S)$ for all $S, T \in \mathcal{A}$.

Definition 5. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Uniform Losses solution of* $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$, $y(T) > 0$ implies $y(T) - v(T) \leq y(S) - v(S)$, i.e., y belongs to the U–equal sacrifice solution for $U(t) = t$.

Definition 6. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly U–equal sacrifice solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$,
$y(T) > 0$ implies $U(y(T)) - U(v(T)) \leq U(y(S)) - U(v(S))$.

Definition 7. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Proportional solution* of $(N, \mathcal{A}, c, v)$ iff for $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$, $y(T)/v(T) = y(S)/v(S)$.

Definition 8. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Uniform Losses solution* of $(N, \mathcal{A}, c, v)$ iff for all $S, T \in \mathcal{A}$ with $S \cap T = \emptyset$,
$y(T) > 0$ implies $y(T) - v(T) \leq y(S) - v(S)$.

Remark 1. Let $U(0) = -\infty$. Then for each x in the U–equal sacrifice solution, $U(x(Q)) - U(v(Q)) = U(x(S)) - U(v(S))$ for all $S, Q \in \mathcal{A}$. If x belongs to the Weakly U–equal sacrifice solution, then for each S, Q in the same for each $Q, S \in \mathcal{A}$ with $Q \cap P = \emptyset$, either $x(Q) = x(S) = 0$ or $U(x(Q)) - U(v(Q)) = U(x(S)) - U(v(S))$.

Proof. Let x belong to the U–equal sacrifice solution. Since $\mathcal{A}$ covers N and $x(N) > 0$, there exists $T \in \mathcal{A}$ such that $x(T) > 0$. For each $S \in \mathcal{A}$, we have and $U(x(T)) - U(v(T)) \leq U(x(S)) - U(v(S))$, hence $U(x(S)) > -\infty$ and $x(S) > 0$, then we get the equality.

The case of Weakly U–equal sacrifice solution is considered similarly. □

Therefore, the Proportional solution coincides with the ln–equal sacrifice solution and the Weakly Proportional solution coincides with the Weakly ln–equal sacrifice solution.

Definition 9. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Solidary solution* of $(N, \mathcal{A}, c, v)$ iff $x(Q) < v(Q)$ for some $Q \in \mathcal{A}$ implies $x(T) < v(T)$ for all $T \in \mathcal{A}$.

Definition 10. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Solidary solution* of $(N, \mathcal{A}, c, v)$ iff $x(Q) < v(Q)$ for some $Q \in \mathcal{A}$ implies $x(T) < v(T)$ for all $T \in \mathcal{A}$ with $Q \cap T = \emptyset$.

Proposition 1. *Each U–equal sacrifice solution is contained in the Solidary solution. Each Weakly U–equal sacrifice solution is contained in the Weakly Solidary solution.*

Proof. Let y belong to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and $y(Q) < v(Q)$. Then $U(y(Q)) - U(v(Q)) < 0$. Let $T \in \mathcal{A}$. If $y(T) = 0$, then $y(T) < v(T)$, and if $y(T) > 0$, then $U(y(T)) - U(v(T)) \leq U(y(Q)) - U(v(Q)) < 0$, hence $y(T) < v(T)$.

The case of the Weakly Solidary solution is considered similarly. □

Let U be a strictly increasing continuous function defined on $(0, +\infty)$.

Definition 11. Let $X \subset R^n$, $f_1, \ldots, f_k$ be functions defined on X. For $z \in X$, let π be a permutation of $\{1, \ldots, k\}$ such that $f_{\pi(i)}(z) \leq f_{\pi(i+1)}(z)$, $\theta(z) = \{f_{\pi(i)}(z)\}_{i=1}^{k}$. Then $y \in X$ belongs to the *nucleolus with respect to* $f_1, \ldots, f_k$ *on* X iff $\theta(y) \geq_{lex} \theta(z)$ for all $z \in X$.

Definition 12. A vector $y = \{y_i\}_{i \in N}$ belongs to the U– *nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ on X, where $f_T(z) = U(z(T)) - U(v(T))$ and X is defined as follows. If $U(0) > -\infty$ then X is the set of imputations of $(N, \mathcal{A}, c, v)$ and if $U(0) = -\infty$ then X is the set of imputations z of $(N, \mathcal{A}, c, v)$ such that $z(T) > 0$.

For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the U–nucleolus of $(N, \mathcal{A}, c, v)$ is nonempty and defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Definition 13. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Proportional nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T)/v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

The Proportional nucleolus coincides with the ln–nucleolus.

Definition 14. An imputation $y = \{y_i\}_{i \in N}$ belongs to the *Nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{f_T\}_{T \in \mathcal{A}}$ with $f_T(z) = z(T) - v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

Note that even in the case when $\mathcal{A} = 2^N \setminus \{N, \emptyset\}$, the Nucleolus of $(N, \mathcal{A}, c, v)$ does not coincide with the nucleolus of the corresponding TU game because the set of imputations in our definition does not depend on the values of singletons.

$q - U$–**solutions**

Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $\mathcal{Q}(U)$ be a class of strictly increasing continuous functions q defined on $(-\infty, +\infty)$ such that $q(0) = 0$ and $\lim\limits_{x \to 0} \int\limits_a^x q(U(t))dt < +\infty$ for each $a > 0$.

Definition 15. A vector $y = \{y_i\}_{i \in N}$ belongs to the *qU–solution* of $(N, \mathcal{A}, c, v)$ iff y minimizes

$\sum\limits_{S \in \mathcal{A}} \int\limits_{v(S)}^{z(S)} q(U(t) - U(v(S)))dt$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

Examples of qU-solutions

1. $U(t) = \ln t$, $q(t) = t$, then

$$\int_{v(S)}^{z(S)} q(U(t) - U(v(S)))dt = z(S)[\ln(z(S)/v(S)) - 1] + v(S)$$

and the qU–solution is the *Weighted Entropy solution* (Naumova, 2000, 2008, 2010).

2. $U(t) = \ln t$, $q(t) = (exp(t))^p - 1$, where $p > 0$, then we obtain the minimization problem for $\sum\limits_{S \in \mathcal{A}} z(S)[\frac{z(S)^p}{(p+1)v(S)^p} - 1]$ that was considered in (Yanovskaya, 2002).

3. $U(t) = t = q(t)$, then we obtain the *Least Square solution* that solves the minimization problem for $\sum\limits_{T \in \mathcal{A}} (z(T) - v(T))^2$ on the set of imputations.

3. Existence rezults

The U–nucleolus and the qU–solution are always nonempty sets. Now we describe conditions on $\mathcal{A}$ which ensure that U–equal sacrifice solutions, Weakly U–equal sacrifice solutions, Solidary solutions, Weakly Solidary solutions are nonempty sets. We found that these conditions are the same for all U and coincide with the corresponding versions for Solidarity solutions.

Theorem 1. *Let U be a strictly increasing continuous function defined on* $(0, +\infty)$. *Then the following 3 statements are equivalent.*
1. The U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ is nonempty for all $c > 0$, all v with $v(T) > 0$.
2. The Solidary solution of $(N, \mathcal{A}, c, v)$ is nonempty for all $c > 0$, all v with $v(T) > 0$.
3. $\mathcal{A}$ is a minimal covering of N.

Proof. Let $\mathcal{A}$ be a minimal covering of N. Then for each $S \in \mathcal{A}$ there exists $j(S) \in S \setminus \cup_{Q \in \mathcal{A} \setminus \{S\}} Q$. Denote $J = \{j(S) : S \in \mathcal{A}\}$. For $(N, \mathcal{A}, c, v)$, take $y = \{y_i\}_{i \in N}$ such that $y_i = 0$ for all $i \in N \setminus J$, $\sum_{i \in N} y_i = c$, and $\{y_{j(S)}\}_{S \in \mathcal{A}}$ is the U–equal sacrifice solution of the claim problem $(J, c, \{v(S)\}_{S \in \mathcal{A}})$. Then y belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and by Proposition 1, y belongs to the Solidary solution of $(N, \mathcal{A}, c, v)$.

Let the Solidary solution of $(N, \mathcal{A}, c, v)$ be nonempty for all $c > 0$, all v with $v(T) > 0$. Suppose that $\mathcal{A}$ is not a minimal covering of N, then there exists $S \in \mathcal{A}$ such that $\mathcal{A} \setminus \{S\}$ covers N. Take $c > 0$, $v(S) > c$, $v(Q) = \epsilon$, where $0 < \epsilon < c/|\mathcal{A}|$ for all $Q \in \mathcal{A} \setminus \{S\}$. Let y belong to the Solidary solution of $(N, \mathcal{A}, c, v)$. Then $y(S) \le c < v(S)$ and for each $Q \in \mathcal{A} \setminus \{S\}$, $y(Q) < \epsilon$, hence $\sum_{i \in N} y_i \le |\mathcal{A}|\epsilon < c$, but this contradicts to $\sum_{i \in N} y_i = c$. $\square$

Now we describe conditions on $\mathcal{A}$ that ensure existence of Weakly U–equal sacrifice solutions and Weakly Solidary solutions. The following result of the author will be used.

Theorem 2 (Naumova, 1978, Theorem 2 or 2008 Corollary 1). *Let $c > 0$, $I(c) = \{x \in R^{|N|} : x_i \geq 0,\ x(N) = c\}$, Gr be an undirected graph with the set of nodes $\mathcal{A}$, $\{\succ_x\}_{x \in I(c)}$ be a family of relations on $\mathcal{A}$, and for each $K \in \mathcal{A}$*

$$F^K = \{x \in I(c) : L \not\succ_x K \text{ for all } L \in \mathcal{A}\}.$$

Let $\{\succ_x\}_{x \in I(c)}$ satisfy the following 5 conditions.
1. $\succ_x$ is acyclic on $\mathcal{A}$.
2. If $K \in \mathcal{A}$ and $x_i = 0$ for all $i \in K$, then $x \in F^K$.
3. The set F^K is closed for each $K \in \mathcal{A}$.
4. If $K \succ_x L$, then K and L are adjacent in the graph Gr.
5. If a single node is taken out from each component of Gr, then the remaining elements of $\mathcal{A}$ do not cover N.

Then there exists $x^0 \in I(c)$ such that $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$.

Theorem 3. *Let $G(\mathcal{A})$ be the undirected graph, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$. Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Then the following 3 statements are equivalent.*
1. The Weakly solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$.
2. The Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$.
3. $\mathcal{A}$ satisfies the following condition.
C0. If a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

Proof. Suppose that $\mathcal{A}$ satisfies C0. Fix $(N, \mathcal{A}, c, v)$. For each imputation x, consider the following relation on $\mathcal{A}$: $P \succ_x Q$ iff $P \cap Q = \emptyset$, $x(Q) > 0$, and $U(x(P)) - U(v(P)) < U(x(Q)) - U(v(Q))$. Then x^0 belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ iff $K \not\succ_{x^0} L$ for all $K, L \in \mathcal{A}$. This family of relations and the graph $G(\mathcal{A})$ satisfy all conditions of Theorem 2, hence the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ is a nonempty set. In view of Proposition 1, this implies that the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set.

Now suppose that the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$ and let us prove that C0 is satisfied. Suppose that $\mathcal{A}$ does not satisfy the condition C0. Let m be the number of components of $G(\mathcal{A})$, $S_1, \ldots, S_m$ be the nodes taken out from each component of $G(\mathcal{A})$ such that $\mathcal{A} \setminus \{S_1, \ldots, S_m\}$ cover N.

Let us take $c > 0$, $v(S_i) = c$ for all $i = 1, \ldots, m$, $v(Q) = \epsilon$ for remaining $Q \in \mathcal{A}$, where $\epsilon|\mathcal{A}| < c$. Let y belong to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$. If $Q \cap S_i = \emptyset$, then $y(S_i) > 0$ implies $y(S_i) < v(S_i)$, therefore $y(Q) < \epsilon$ for $Q \neq S_i$, and as such Q cover N, we get $y(N) \leq |\mathcal{A}|\epsilon < c = y(N)$. This contradiction completes the proof. □

4. Properties of qU–solutions

Property 1. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $U(t) \to -\infty$ as $t \to 0$, $q \in \mathcal{Q}(U)$, $q \to \infty$ as $t \to \infty$, and x belong to the qU–solution of $(N, \mathcal{A}, c, v)$. Then $x(S) > 0$ for all $S \in \mathcal{A}$.

Proof. Suppose that there exist $(N, \mathcal{A}, c, v)$, $S \in \mathcal{A}$, and x in qU–solution of $(N, \mathcal{A}, c, v)$ such that $x(S) = 0$. Let $0 < \epsilon < \min\{x_k : x_k > 0\}$. Let

$$M = \max_{T:T\in\mathcal{A}, x(T)>0} \max_{t\in[x(T)-\epsilon, x(T)+\epsilon]} |q(U(t) - U(v(T)))|.$$

Fix $\delta > 0$ such that $\delta < \min\{\epsilon, \min_{T\in\mathcal{A}} v(T)\}$ and $|q(U(\delta) - U(v(S)))| > 2^{|N|}M$. Let $i \in S$, $j \in N$, $x_j > 0$.

Take $z \in R^{|N|}$ such that $z_i = x_i + \delta$, $z_j = x_j - \delta$, $z_k = x_k$ for $k \neq i, j$. Then $\sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{z(T)} q(U(t) - U(v(T)))dt - \sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{x(T)} q(U(t) - U(v(T)))dt =$
$\sum\limits_{T\in\mathcal{A}:i\in T, j\notin T} \int\limits_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt - \sum\limits_{T\in\mathcal{A}:i\notin T, j\in T} \int\limits_{x(T)-\delta}^{x(T)} q(U(t) - U(v(T)))dt.$

If $i \notin T$, $j \in T$ then $|\int\limits_{x(T)-\delta}^{x(T)} q(U(t) - U(v(T)))dt| \leq \delta M$.

If $T = S$ then $\int\limits_{x(S)}^{x(S)+\delta} q(U(t) - U(v(S)))dt = \int\limits_{0}^{\delta} q(U(t) - U(v(S)))dt < -2^{|N|}M\delta$.

If $i \in T$, $j \notin T$, $x(T) = 0$, then $\int\limits_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt < 0$ since $\delta < v(T)$.

If $i \in T$, $j \notin T$, $x(T) > 0$, then $|q(U(t) - U(v(T)))| \leq M$ as $t \in [x(T), x(T) + \delta]$, hence $|\int\limits_{x(T)}^{x(T)+\delta} q(U(t) - U(v(T)))dt| \leq \delta M$.

Thus,
$\sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{z(T)} q(U(t) - U(v(T)))dt - \sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{x(T)} q(U(t) - U(v(T)))dt <$
$(|\mathcal{A}| - 1)\delta M - 2^{|N|}M\delta < 0$ and x is not in the qU–solution of $(N, \mathcal{A}, c, v)$. □

Property 2. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$, then $f(z) = \sum\limits_{Q\in\mathcal{A}} \int\limits_{v(Q)}^{z(Q)} q(U(t)-U(v(Q)))dt$ is a continuous convex function of z defined on the set of imputations of $(N, \mathcal{A}, c, v)$ and for all $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the qU–solution of $(N, \mathcal{A}, c, v)$ defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Proof. Let $a > 0$, $\psi(r) = \int\limits_{a}^{r} q(U(t))dt$ for $r \geq 0$. If $\lim\limits_{t\to 0} q(U(t)) > -\infty$, then $\psi(r)$ is a strictly convex function on $[0, +\infty)$. If $\lim\limits_{t\to 0} q(U(t)) = -\infty$, then $\psi(r)$ is a convex function on $[0, +\infty)$ and a strictly convex function on $(0, +\infty)$. Therefore $f(z)$ is a convex function of z and in view of Property 1, if y and z belong to qU–solution of $(N, \mathcal{A}, c, v)$, then $y(T) = z(T)$ for all $T \in \mathcal{A}$. □

Property 3. Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$. Then for each x in the qU–solution of $(N, \mathcal{A}, c, v)$, $x_i > 0$ implies

$$\sum_{T\in\mathcal{A}:i\in T} q(U(x(T)) - q(U(v(T))) \leq \sum_{T\in\mathcal{A}:j\in T} q(U(x(T)) - U(v(T)))$$

for all $j \in N$.

Proof. Let x belong to the qU–solution of $(N,\mathcal{A},c,v)$. Note that in view of Property 1, $q(U(x(Q))-U(v(Q)))$ are well defined for all $Q\in\mathcal{A}$. Let $x_i>0$. Suppose that there exists $j\in N$ such that

$$\sum_{T\in\mathcal{A}:j\in T} q(U(x(T))-U(v(T)))<\sum_{T\in\mathcal{A}:i\in T} q(U(x(T))-U(v(T))).$$

Consider $\epsilon\geq 0$ and $y(\epsilon)\in R^{|N|}$ such that $\epsilon<x_i$, $y(\epsilon)_i=x_i-\epsilon$, $y(\epsilon)_j=x_j+\epsilon$, $y(\epsilon)_k=x_k$ for $k\neq i,j$. Let

$$F(\epsilon)=\sum_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{y(\epsilon)(Q)} q(U(t)-U(v(Q)))dt-\sum_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{x(Q)} q(U(t)-U(v(Q)))dt,$$

then

$F(\epsilon)=\sum\limits_{Q\in\mathcal{A}:i\in Q,j\notin Q}\int\limits_{x(Q)}^{x(Q)-\epsilon} q(U(t)-U(v(Q)))dt+$

$\sum\limits_{Q\in\mathcal{A}:i\notin Q,j\in Q}\int\limits_{x(Q)}^{x(Q)+\epsilon} q(U(t)-U(v(Q)))dt,$

$F'(0)=-\sum\limits_{Q\in\mathcal{A}:i\in Q,j\notin Q} q(U(x(Q))-U(v(Q)))+$

$\sum\limits_{Q\in\mathcal{A}:i\notin Q,j\in Q} q(U(x(Q))-U(v(Q)))<0.$

Hence, $F(\epsilon)<0$ for some $\epsilon>0$ and x does not belong to the qU–solution of $(N,\mathcal{A},c,v)$. □

Property 4. Let U be a strictly increasing continuous function defined on $(0,+\infty)$, $q\in\mathcal{Q}(U)$, and x be an imputation of $(N,\mathcal{A},c,v)$ such that $x_i>0$ implies

$$\sum_{T\in\mathcal{A}:i\in T} q(U(x(T))-q(U(v(T)))\leq\sum_{T\in\mathcal{A}:j\in T} q(U(x(T))-U(v(T)))$$

for all $j\in N$.

Then x belongs to the qU–solution of $(N,\mathcal{A},c,v)$.

Proof. For each imputation z of $(N,\mathcal{A},c,v)$, let $f(z)=\sum\limits_{Q\in\mathcal{A}}\int\limits_{v(Q)}^{z(Q)} q(U(t)-U(v(Q)))dt$.

If $z_j>0$ for all $j\in N$ then f is differentiable at z and

$$\frac{\partial}{\partial z_j}f(z)=\sum_{T\in\mathcal{A}:\,T\ni j} q(U(z(T))-U(v(T))). \tag{1}$$

If z and w are imputations of $(N,\mathcal{A},c,v)$ such that $z_j,w_j>0$ for all $j\in N$, then, in view of Property 2,

$$f(w)-f(z)\geq\sum_{j\in N}\frac{\partial f(z)}{\partial z_j}(w_j-z_j). \tag{2}$$

Note that if $x_i>0$ then for all $Q\ni i$, $x(Q)>0$ and $q(U(x(Q))-U(v(Q)))$ are well defined. Hence for all $j\in N$, $\sum\limits_{T\in\mathcal{A}:\,T\ni j} q(U(x(T))-U(v(T)))$ are well defined.

Let y be an imputation of $(N, \mathcal{A}, c, v)$. There exist imputations z^k and w^k with positive coordinates such that $\lim_{k\to+\infty} z^k = x$, $\lim_{k\to+\infty} w^k = y$, then it follows from (2) and (1) that

$$f(y) - f(x) \geq \sum_{j\in N}(y_j - x_j) \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))). \tag{3}$$

Let $x_i > 0$, then (1) implies

$$\sum_{j\in N} x_j \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))) = c \sum_{T\in\mathcal{A}:\, T\ni i} q(U(x(T)) - U(v(T))), \tag{4}$$

$$\sum_{j\in N} y_j \sum_{T\in\mathcal{A}:\, T\ni j} q(U(x(T)) - U(v(T))) \geq c \sum_{T\in\mathcal{A}:\, T\ni i} q(U(x(T)) - U(v(T))). \tag{5}$$

It follows from (3),(4), (5) that $f(y) - f(x) \geq 0$, i.e., x belongs to the qU–solution of $(N, \mathcal{A}, c, v)$. □

5. When generalized solutions satisfy solidarity properties?

We describe conditions on the collection of coalitions $\mathcal{A}$ that ensure the inclusion of the U–nucleolus (qU–solution) in the U–equal sacrifice solution and in the Solidary solution. We prove that these conditions depend neither on U nor on q and are the same.

Theorem 4. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$. Then the following 3 statements are equivalent.*
1. $\mathcal{A}$ is a partition of N.
2. The U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.
3. The U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.

Proof. Let $\mathcal{A}$ be a partition of N, then the U–nucleolus always coincides with the U-equal sacrifice solution, and by Proposition 1, it is contained in the Solidary solution.

Let the U–nucleolus be always contained in the Solidary solution. Suppose that there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) > 1$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/(4|N|)$.

Let x belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$, then $x(P) < v(P)$ and due to the solidarity property this implies $x(T) < \epsilon$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i < \epsilon$ for all $i \in N \setminus P$. As long as $\mathcal{A}$ covers N, $x(P) > 3/4$. Since x belongs to the U–nucleolus and $\mathcal{A}_P = \{T \in \mathcal{A} \setminus \{P\} : T \cap P \neq \emptyset\} \neq \emptyset$, we have $x_i = 0$ for all $i \in P \setminus \cup_{T\in\mathcal{A}_P} T$. Then $x(S) \geq x(P)/|P|$ for some $S \in \mathcal{A}_P$. Therefore,

$$x(S) \geq 3/(4|N|) > \epsilon,$$

but this contradicts to $x(S) < \epsilon$. □

Theorem 5. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$. Then the following 3 statements are equivalent.*
1. $\mathcal{A}$ is a partition of N.
2. The qU–solution of $(N, \mathcal{A}, c, v)$ is contained in the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.
3. The qU–solution of $(N, \mathcal{A}, c, v)$ is contained in the Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$.

Proof. Let $\mathcal{A}$ be a partition of N. Then for each imputation x of $(N, \mathcal{A}, c, v)$,

$$\sum_{T \in \mathcal{A}:\, T \ni i} q(U(x(T)) - U(v(T))) = q(U(x(S)) - U(v(S))) \quad \text{for all} \quad S \in \mathcal{A},\ i \in S.$$

Let x belong to the qU–solution of $(N, \mathcal{A}, c, v)$, then by Property 3, $x(S) > 0$ for some $S \in \mathcal{A}$ implies $q(U(x(S)) - U(v(S))) \leq q(U(x(T)) - U(v(T)))$ for all $T \in \mathcal{A}$. As q is a strictly increasing function, this implies $U(x(S)) - U(v(S)) \leq U(x(T)) - U(v(T))$. Thus, x belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Then, by Proposition 1, x belongs to the Solidary solution of $(N, \mathcal{A}, c, v)$.

Let the qU–solution be always contained in the Solidary solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 2$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/|N|$.

Let x belong to the qU–solution of $(N, \mathcal{A}, 1, v)$. Then $x(P) < v(P)$ and it follows from the solidarity property that $x(T) < \epsilon$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i < \epsilon$ for all $i \in N \setminus P$. If $x_i \leq \epsilon$ for all $i \in P$, then $x(N) \leq \epsilon|N| < 1$, hence there exists $j_0 \in P \setminus \cup_{T \in \mathcal{A} \setminus \{P\}} T$ such that $x_{j_0} > \epsilon$. Let $i_0 \in P \cap Q$. By Property 3,
$q(U(x(P)) - U(v(P))) = \sum_{T \in \mathcal{A}:\, T \ni j_0} q(U(x(T)) - U(v(T))) \leq$
$\sum_{T \in \mathcal{A}:\, T \ni i_0} q(U(x(T)) - U(v(T)))$.

Since $x(T) < v(T)$ for all $T \in \mathcal{A}$ and $q(0) = 0$, this implies

$$0 \leq \sum_{T \in \mathcal{A}:\, T \ni i_0\, T \neq P} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q)) < 0.$$

In view of this contradiction, $\mathcal{A}$ is a partition of N. □

6. When generalized solutions satisfy weak solidarity properties?

In this section we obtain conditions on the collection of coalitions $\mathcal{A}$ that ensure the inclusion of the U–nucleolus in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution. We prove that these conditions coincide. We also obtain necessary and sufficient conditions on $\mathcal{A}$ that ensure the inclusion of qU–solutions in the Weakly U–equal sacrifice solution. These conditions depend neither on U nor on q and coincide with the conditions that ensure the inclusion of the qU–solutions in the Weakly Solidary solution.

For $i \in N$, denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

Definition 16. A collection of coalitions $\mathcal{A}$ is *weakly mixed at N* if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;

C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C3) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Remark 2. If $k \leq 2$ then C3 follows from C1 and C2.

Remark 3. If $\mathcal{A}$ is a weakly mixed collection of coalitions, then it satisfies the condition C0 of Theorem 3.

Proof. Let $\mathcal{A}$ be weakly mixed at N. Take $j_0 \in N$ such that $|\mathcal{A}_{j_0}| \geq |\mathcal{A}_i|$ for all $i \in N$. Let $\mathcal{A}_{j_0} = \{Q_t\}_{t \in M}$, where $Q_t \in \mathcal{B}^t$, $M \subset \{1, \ldots, k\}$.

Let $S_t \in \mathcal{B}^t$ for all $t \leq k$. Since $\mathcal{A}$ is weakly mixed, there exists $i_o \in \bigcap_{t \in M} S_t$. In view of definition of j_0, $\mathcal{A}_{i_0} = \{S_t : t \in M\}$. Therefore, if for each $t \in \{1, \ldots, k\}$, S_t is taken out from $\mathcal{A}$, then the remaining elements of $\mathcal{A}$ do not cover i_0. □

Example 1. Let $N = \{1, 2, \ldots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N.

Example 2. $N = \{1, 2, \ldots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12, 13\}\}$.
Then $\mathcal{A}$ is weakly mixed at N.

Example 3. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
$\mathcal{B}^3 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$,
then $\mathcal{C}$ satisfies C0, C1, and C2, but does not satisfy C3 (for $i = 1$ and $Q = \{1, 2\}$), hence $\mathcal{C}$ is not weaky mixed at N.

Proposition 2. *Let U be a strictly increasing continuous function defined on* $(0, +\infty)$ *and the U–nucleolus of $(N, \mathcal{A}, t, v)$ be contained in the Weakly Solidary solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$.*

Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let us take the following v: $v(S) = v(P) = 1$, $v(T) = \epsilon$ for all $T \in \mathcal{A} \setminus \{S, P\}$, where $0 < \epsilon < 1/2|N|$. Let $j \in P \cap Q$.

Let x belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$. First, we prove that $x(Q) \geq x(P)/|P|$. Assume the contrary, then $x(P \cap Q) < x(P)/|P|$, hence there exists $i_0 \in P \setminus Q$ such that $x_{i_0} > x(P)/|P|$.

Let $i_0 \notin T$ for all $T \in \mathcal{A} \setminus \{P\}$ then we take $y \in R^{|N|}$: $y_{i_0} = 0$, $y_j = x_j + x_{i_0}$, $y_i = x_i$ otherwise. Then $y(P) = x(P)$, $y(Q) > x(Q)$, $y(T) \geq x(T)$ for all $T \in \mathcal{A}$, hence x does not belong to the U–nucleolus.

Let $i_0 \in T$ for some $T \in \mathcal{A} \setminus \{P\}$, then $T \neq S$ and $x(T) > x(P)/|P| > x(Q)$. This implies

$$U(x(T)) - U(v(T)) = U(x(T)) - U(\epsilon) > U(x(Q)) - U(v(Q)).$$

Let $z = z(\delta) \in R^{|N|}$, $z_{i_0} = x_{i_0} - \delta$, $z_j = x_j + \delta$, $z_i = x_i$ otherwise. If $\delta > 0$ and δ is sufficiently small, then for $T \in \mathcal{A}_{i_0} \setminus \{P\}$,

$$U(z(T)) - U(v(T)) > U(z(Q)) - U(v(Q)) > U(x(Q)) - U(v(Q)),$$

otherwise $z(T) \geq x(T)$, hence $\theta(z(\delta)) >_{lex} \theta(x)$. Thus

$$x(Q) \geq x(P)/|P|.$$

Weak solidarity condition for Q and S implies $x(Q) < \epsilon$, hence $x(P) < \epsilon|P|$. We consider 4 cases.

Case 1. There exists $j_0 \notin P \cup Q \cup S$ such that $x_{j_0} > \epsilon$. Then for all $T \ni j_0$, $x(T) > v(T)$. Let $w = w(\delta) \in R^{|N|}$, $w_{j_0} = x_{j_0} - \delta$, $w_j = x_j + \delta$, $w_i = x_i$ otherwise. Then for $\delta > 0$, $w(Q) > x(Q)$ and for sufficiently small δ, $w(Q) < v(Q)$, and $w(T) > v(T)$ for all $T \ni j_0$, hence we get $\theta(w(\delta)) >_{lex} \theta(x)$, and the Case 1 is impossible.

Case 2. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$ and $x(S) \leq x(P)$. Then

$$x(N) \leq x(Q) + 2\epsilon|P| + \epsilon|N \setminus (P \cup S \cup Q)| \leq 2\epsilon|N| < 1$$

and this contradicts $x(N) = 1$.

Case 3. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$ and $x_i \leq \epsilon$ for all $i \in S$. This implies $1 = x(N) \leq \epsilon|N| < 1$, hence this case is impossible.

Case 4. $x_i \leq \epsilon$ for all $i \notin P \cup Q \cup S$, $x(S) > x(P)$ and $x_{i_0} > \epsilon$ for some $i_0 \in S$. Then $x(T) > v(T)$ for $T \neq S, T \ni i_0$. Let $y = y(\delta) \in R^{|N|}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_i = x_i$ otherwise. Then for $\delta > 0$, $y(Q) > x(Q)$, $y(P) > x(P)$ and for sufficiently small $\delta > 0$, we get $\theta(y(\delta)) >_{lex} \theta(x)$. This contradiction completes the proof. □

Theorem 6. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$.*

If $\mathcal{A}$ is a weakly mixed collection of coalitions at N then for all $c > 0$, all v with $v(T) > 0$, the U–nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let, moreover, either U be a convex function or $U(t) \to +\infty$ as $t \to +\infty$. Let the U–nucleolus of $(N, \mathcal{A}, c, v)$ be contained in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. Then $\mathcal{A}$ is a weakly mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be weakly mixed at N and x belong to the U–nucleolus of $(N, \mathcal{A}, c, v)$. We prove that x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Suppose the contrary, i.e., there exist $S, Q \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $U(x(Q)) - U(v(Q)) < U(x(S)) - U(v(S))$ and $x(S) > 0$. Take $i_0 \in S$ such that $x_{i_0} > 0$. Since $\mathcal{A}$ is weakly mixed, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Take $\delta > 0$ such that

$$U(x(Q) + \delta)U(v(Q)) < U(x(S) - \delta) - U(v(S))$$

and $\delta < x_{i_0}$. Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Then $y(P) < x(P)$ only for $P = S$ and $y(Q) > x(Q)$. Since $U(y(Q)) - U(v(Q)) < U(y(S)) - U(v(S))$, this contradicts the definition of the U-nucleolus. Therefore, x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and by Proposition 1, x belongs to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let either U be is a convex function or $U(t) \to +\infty$ as $t \to +\infty$ and the U–nucleolus be always contained in the Weakly Solidary solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 3. Then $\mathcal{A}$ satisfies C2 by the definition of $G(\mathcal{A})$ and satisfies C1 in view of Proposition 2. Suppose that $\mathcal{A}$ is not weakly mixed. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ for all $j \in N$. Let $0 < \epsilon < 1/|N|$. We can take v with the following properties:
$v(S) = 1$,
$U(v(P)) > 2U(1) - U(1/|N|)$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.
Let x belong to the U–nucleolus and to the Weakly Solidary solution of $(N, \mathcal{A}, 1, v)$. Since $S \cap Q = \emptyset$, $x(N) = 1$, and $v(S) + v(Q) > 1$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Take $\delta > 0$ such that $\delta < 1/|N|$ and for each $T, P \in \mathcal{A}$,

$$U(x(T)) - U(v(T)) < U(x(P)) - U(v(P))$$

implies

$$U(x(T) + \delta) - U(v(T)) < U(x(P) - \delta) - U(v(P)).$$

Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} + \delta$, $y_{j_0} = x_{j_0} - \delta$, $y_i = x_i$ otherwise.

We prove that there exists $P \in \mathcal{A}$ such that $y(P) > x(P)$ and $U(x(P)) - U(v(P)) < U(x(T)) - U(v(T))$ for all $T \in \mathcal{A}$ with $y(T) < x(T)$ and this would imply that x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$. Consider 2 cases.

Case 1. $j_0 \notin S$. Let $y(T) < x(T)$, then $T \ni j_0$ and $v(T) = \epsilon$, hence

$$U(x(T)) - U(v(T)) \geq U(x_{j_0}) - U\epsilon > 0.$$

Since $U(x(Q)) - U(v(Q)) < 0$ and $y(Q) > x(Q)$, we can take $P = Q$, hence x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$ in this case.

Case 2. $j_0 \in S$. Then there exists $P \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, where $y(P) > x(P)$. Let us check that

$$y(T) < x(T) \quad \text{implies} \quad U(x(P)) - U(v(P)) < U(x(T)) - U(v(T)).$$

If $T = S$ then

$$U(x(S)) - U(v(S)) \geq U(1/|N|) - U(1) > U(1) - U(v(P)) \geq U(x(P)) - U(v(P)).$$

If $T \neq S$ then $v(T) = \epsilon$ and $U(x(T)) - U(v(T)) \geq U(x_{j_0}) - U\epsilon > 0$. Since U is strictly increasing, $U(v(P)) > U(1)$, hence $U(x(P)) - U(v(P)) < 0$ and $U(x(P)) - U(v(P)) < U(x(T)) - U(v(T))$. Thus, x does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, v)$ in this case. □

Corollary 1. *The Proportional Nucleolus of* $(N, \mathcal{A}, c, v)$ *is contained in the Weakly Proportional solution and in the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$, v *with* $v(T) > 0$ *if and only if* $\mathcal{A}$ *is a weakly mixed collection of coalitions at* N.

Corollary 2. *The Nucleolus of* $(N, \mathcal{A}, c, v)$ *is contained in the Weakly Uniform Losses solution and in the the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$, v *with* $v(T) > 0$ *if and only if* $\mathcal{A}$ *is a weakly mixed collection of coalitions at* N.

Definition 17. A collection of coalitions $\mathcal{A}$ is *mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C4) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Note that if $\mathcal{A}$ is mixed at N then $\mathcal{A}$ is weakly mixed at N.

Example 4. If $\mathcal{A}$ is weakly mixed at N and all $i \in N$ belong to the same number of coalitions, then $\mathcal{A}$ is mixed at N.

Example 5. Let $N = \{1, 2, \dots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N.

Example 6. Let $N = \{1, 2, \dots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N but not mixed at N. (For $i = 3$, the condition C4 is not realized.)

Proposition 3. *Let the qU–solution of* $(N, \mathcal{A}, c, v)$ *be contained in the Weakly Solidary solution of* $(N, \mathcal{A}, c, v)$ *for all* $c > 0$*, all* v *with* $v(T) > 0$*. Then the case* $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ *is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to the qU–solution of $(N, \mathcal{A}, 1, v)$. Since x satisfies the weakly solidarity property, $v(P) + v(S) > 1$, and $S \cap P = \emptyset$, we have $x(S) < v(S)$. Since $Q \cap S = \emptyset$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P\}$, hence $v(T) = \epsilon$ and $x(T)/v(T) > 1$. Thus,

$$\sum_{T \in \mathcal{A}\colon T \ni j_0, T \not\ni i_0} q(U(x(T)) - U(v(T))) \geq 0. \tag{6}$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and it follows from the weak solidarity property that $x(T) < v(T)$. If $v(T) = 1$, then $v(T) \geq x(T)$. Therefore

$$\sum_{T \in \mathcal{A}\colon T \not\ni j_0, T \ni i_0} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q))) < 0. \tag{7}$$

It follows from (6) and (7) that

$$\sum_{T \in \mathcal{A}\colon T \ni j_0} q(U(x(T)) - U(v(T))) > \sum_{T \in \mathcal{A}\colon T \ni i_0} q(U(x(T)) - U(v(T))),$$

but this contradicts Property 3. $\square$

Theorem 7. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$.*

The qU–solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly U–equal sacrifice solution and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be a mixed collection of coalitions. Let x belong to the qU–solution of $(N, \mathcal{A}, c, v)$. We prove that x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Suppose that there exist $Q, S \in \mathcal{A}$ such that $Q \cap S = \emptyset$, $x(Q) > 0$, and $U(x(Q)) - U(v(Q)) > U(x(S)) - U(v(S))$. There exists $i_0 \in Q$ with $x_{i_0} > 0$. Since $\mathcal{A}$ is mixed, there exists $j_0 \in N$ such that $\mathcal{A}_{j_0} = \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Then

$$\sum_{T \in \mathcal{A}\colon T \not\ni j_0, T \ni i_0} q(U(x(T)) - U(v(T))) = q(U(x(Q)) - U(v(Q))),$$

$$\sum_{T \in \mathcal{A}\colon T \ni j_0, T \not\ni i_0} q(U(x(T)) - U(v(T))) = q(U(x(S)) - U(v(S))),$$

hence

$$\sum_{T \in \mathcal{A}\colon T \ni i_0} q(U(x(T)) - U(v(T))) > \sum_{T \in \mathcal{A}\colon T \ni j_0} q(U(x(T)) - U(v(T))),$$

but this contradicts Property 3. Thus, x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, and due to Proposition 1, x belongs to the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$.

Let the qU–solution be always contained in the Weakly Solidary solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 3. By the definition of $G(\mathcal{A})$, $\mathcal{A}$ satisfies C2. In view of Proposition 3, $\mathcal{A}$ satisfies C1. Suppose that $\mathcal{A}$ is not mixed at N. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ with $S \cap Q = \emptyset$ such that for each $j \in N$, $\mathcal{A}_j \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$.

Let $L > 1$. Since q and U are strictly increasing continuous functions, there exists $\epsilon > 0$ such that
$\epsilon < 1/|N|$,
$U(1) - U(1 - \epsilon|N|) \leq U(L) - U(1)$,
$q(U(1 - \epsilon|N|) - U(1)) + q(U(1/|N|) - U(\epsilon)) \geq 0$.

We take the following v:
$v(S) = 1$,
$U(v(P)) = L$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to the qU–solution and to the Weakly Solidary solution of $(N, \mathcal{A}, 1, v)$. Since $v(S)+v(Q) > 1$ and $x(N) = 1$, we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$. We shall prove that there exists $j_1 \in N$ such that $x_{j_1} > 0$ and

$$\sum_{T\in\mathcal{A}:T\ni i_0} q(U(x(T)) - U(v(T))) < \sum_{T\in\mathcal{A}:T\ni j_1} q(U(x(T)) - U(v(T))), \tag{8}$$

and this would contradict Property 3 of qU–solutions.

The following 3 cases are possible.
1. There exists $j_1 \notin S$ such that $x_{j_1} \geq \epsilon$.
2. $x_j < \epsilon$ for all $j \notin S$ and $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \neq \{Q\}$.
3. $x_j < \epsilon$ for all $j \notin S$ and $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} = \{Q\}$.

Case 1. Since $v(P) > 1$ for all $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,

$$\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_1}} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q))) < 0.$$

Since $j_1 \notin S$, $x(T) \geq v(T) = \epsilon$ for all $T \in \mathcal{A}_{j_1} \setminus \mathcal{A}_{i_0}$, therefore,

$$\sum_{T\in\mathcal{A}_{j_1}\setminus\mathcal{A}_{i_0}} q(U(x(T)) - U(v(T))) \geq 0,$$

this implies (8).

Case 2. We have $x(S) \geq 1 - \epsilon|N|$ and $j_0 \in S$. There exists $P^0 \in \mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\}$, then $\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0}} q(U(x(T)) - U(v(T))) \leq q(U(x(Q)) - U(v(Q)))+$
$q(U(x(P^0)) - (v(P^0))) < q(U(x(P^0)) - U(v(P^0)))$.

If $T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}$ then either $T = S$ or $x(T) > v(T) = \epsilon$, therefore

$$\sum_{T\in\mathcal{A}_{j_0}\setminus\mathcal{A}_{i_0}} q(U(x(T)) - U(v(T))) \geq q(U(x(S)) - U(v(S))).$$

Since $U(1 - \epsilon|N|) - U(1) \geq U(1) - U(L)$, we get

$$q(U(x(S))-U(S)) \geq q(U(1-\epsilon|N|)-U(1)) \geq q(U(1)-U(L)) \geq q(U(x(P^0))-U(v(P^0)))$$

and this implies (8) for $j_1 = j_0$.

Case 3. Since $\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0} = \{Q\}$ and $\mathcal{A}_{j_0} \neq \mathcal{A}_{i_0}\cup\{S\}\setminus\{Q\}$, there exists $T_0 \in \mathcal{A}\setminus\mathcal{A}_{i_0}$ such that $j_0 \in T_0$ and $T_0 \neq S$. Then

$$\sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}_{j_0}} q(U(x(T)) - U(v(T))) = q(U(x(Q)) - U(v(Q))) < 0,$$

In view of $x(S) \geq 1 - \epsilon|N|$, $x(T_0) \geq 1/|N|$, $v(T_0) = \epsilon$, and restrictions on ϵ, we have
$\sum_{T \in \mathcal{A}_{j_0} \setminus \mathcal{A}_{i_0}} q(U(x(T)) - U(v(T))) \geq q(U(x(S)) - U(v(S)))+$
$q(U(x(T_0)) - U(v(T_0))) \geq q(U(1 - \epsilon|N|) - U(1)) + q(U(1/|N|) - U(\epsilon)) \geq 0.$
Thus, we obtain (8) for $j_1 = j_0$. □

Corollary 3. *The Weighted Entropy solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Proportional solution and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a mixed collection of coalitions at N.*

Corollary 4. *The Least Square solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Uniform Losses solution and in the Weakly Solidary solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a mixed collection of coalitions at N.*

7. When different U–generalizations give the same result?

In this section necessary and sufficient conditions on $\mathcal{A}$ that provide the coincidence of the U–nucleolus with the Weakly U–equal sacrifice solution and conditions on $\mathcal{A}$ that provide the coincidence of the qU–solution with the Weakly U–equal sacrifice solution. These conditions are the same for all U and $q \in \mathcal{Q}(U)$. The result concerning qU–solutions is a generalization of the corresponding results concerning the Weighted Entropy solution (Naumova, 2011, Theorem 4) and g–solutions (Naumova, 2012, Theorem 4), but the proof of this paper also permits to solve the problem of coincidence the Least Square solution with the Uniform Losses solution. The result concerning the U–nucleolus is completely new.

Definition 18. A collection of coalitions $\mathcal{A}$ is *totally mixed at N* if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N and for each collection $\{S_i\}_{i=1}^{k}$ ($S_i \in \mathcal{P}^i$), we have $\cap_{i=1}^{k} S_i \neq \emptyset$.

Example 7. Let $N = \{1, 2, 3, 4\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
then $\mathcal{C}$ is a totally mixed collection of coalitions at N.

Theorem 8. *Let U be a continuous strictly increasing function defined on $(0, +\infty)$ and either U is a convex function or $U(t) \to +\infty$ as $t \to +\infty$. Then the U–nucleolus of $(N, \mathcal{A}, c, v)$ coincides with the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.*

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is weakly mixed at N and it follows from Theorem 4 that the U–nucleolus of $(N, \mathcal{A}, c, v)$ is always contained in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Since in this case for all x in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, $x(S)$ are uniquely defined, this implies coincidence of the U–nucleolus and the Weakly U–equal sacrifice solution of $(N,\mathcal{A},c,v)$.

Now suppose that the Weakly U–equal sacrifice solution of $(N,\mathcal{A},c,v)$ coincides with the U–nucleolus of $(N,\mathcal{A},c,v)$ for all $c>0$, all v with $v(T)>0$. Note that for each x in the U–nucleolus of $(N,\mathcal{A},c,v)$,

$$x_i>0 \quad \text{and} \quad \mathcal{A}_j\supset\mathcal{A}_i \quad \text{imply} \quad \mathcal{A}_j=\mathcal{A}_i. \tag{9}$$

By Proposition 2, $\mathcal{A}=\bigcup_{i=1}^k\mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 2, for each collection $\{S_i\}_{i=1}^k$ with $S_i\in\mathcal{P}^i$, we have $\cap_{i=1}^kS_i\neq\emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $q<k$ such that $\cup_{i=1}^q\mathcal{B}^i$ does not cover N and $\cup_{i=1}^q\mathcal{B}^i\cup\mathcal{B}^j$ covers N for each $j>q$. Denote $N^0=\bigcup_{S\in\cup_{i=1}^q\mathcal{B}^i}S$. We consider 2 cases.

Case 1. For each $j=q+1,\dots,k$, there exists $S_j\in\mathcal{B}^j$, such that if S_j is taken out from $\mathcal{B}^j$ for each $j=q+1,\dots,k$, then the remaining elements of $\cup_{j=q+1}^k\mathcal{B}^j$ cover $(N\setminus N^0)$.

We prove that for each y in the U–nucleolus of $(N,\mathcal{A},t,v)$, $y(N\setminus N^0)=0$. Suppose that there exist x in the U–nucleolus of $(N,\mathcal{A},t,v)$ and $j_0\in N\setminus N^0$ such that $x_{j_0}>0$. Let $\mathcal{A}_{j_0}=\{Q_i\}_{i\in M}$, then $Q_i\in\mathcal{B}^i$, $i\in\{q+1,\dots,k\}$. Since $\mathcal{A}$ is weakly mixed by Theorem 4, there exists $j_1\in N$ such that $\mathcal{A}_{j_1}\supset\{S_i\}_{i\in M}$.

If $\mathcal{A}_{j_1}=\{S_i\}_{i\in M}$, then $j_1\in N\setminus N^0$ by the definition of N^0, hence the Case 1 is impossible.

Let $\mathcal{A}_{j_1}\neq\{S_i\}_{i\in M}$. Since $\mathcal{A}$ is weakly mixed, there exists $j_2\in N$ such that $\mathcal{A}_{j_2}\supset\mathcal{A}_{j_1}\cup\{Q_i\}_{i\in M}\setminus\{S_i\}_{i\in M}$. Then $\mathcal{A}_{j_2}\supset\neq\mathcal{A}_{j_0}$, but this contradicts (9).

Take $\tilde v(S)=|S|/|N|$ for all $S\in\mathcal{A}$, $\tilde x_i=1/|N|$ for all $i\in N$, then $\tilde x$ belongs to the Weakly U–equal sacrifice solution of $(N,\mathcal{A},1,\tilde v)$ Proportional solution of $(N,\mathcal{A},1,\tilde v)$ and $\tilde x(N\setminus N^0)>0$. By the proved above, $\tilde x$ does not belong to the U–nucleolus of $(N,\mathcal{A},1,\tilde v)$, hence Case 1 is impossible.

Case 2. If $S_j\in\mathcal{B}^j$ is taken out from $\mathcal{B}^j$, for all $j=q+1,\dots,k$, then the remaining elements of $\cup_{j=q+1}^k\mathcal{B}^j$ do not cover $N\setminus N^0$.

For each $j=q+1,\dots,k$, $S_j\in\mathcal{B}^j$, we have $S_j\cap(N\setminus N^0)\neq\emptyset$. Indeed, suppose that $S_{j_0}\subset N^0$ for some $j_0>q$. Then if we take S_{j_0} and arbitrary $S_j\in\mathcal{B}^j$ for $j>q$, $j\neq j_0$ out from $\cup_{j=q+1}^k\mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k\mathcal{B}^j$ cover $N\setminus N^0$ as if $\{N^0\}\cup\mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C}=\{(N\setminus N^0)\cap S:\, S\in\mathcal{B}^j,\, |\mathcal{B}^j|>1,\, j>q\}.$$

Note that $P,S\in\cup_{j=q+1}^k\mathcal{B}^j$, $P\neq S$, $P\cap(N\setminus N^0)\in\mathcal{C}$ imply $P\cap(N\setminus N^0)\neq S\cap(N\setminus N^0)$.

Indeed, suppose that $P\cap(N\setminus N^0)=S\cap(N\setminus N^0)$. There exists $P^1\in\mathcal{A}$ such that $P\cap P^1=\emptyset$. If we take S,P^1 and arbitrary $S_j\in\mathcal{B}^j$ for $j>q$ with $P\notin\mathcal{B}^j$ out from $\cup_{j=q+1}^k\mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^k\mathcal{B}^j$ cover $N\setminus N^0$ because $\{N^0\}\cup\mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0}\ni S$, but this is impossible in the considered case.

For arbitrary problem $(N,\mathcal{A},c,v)$, where $\mathcal{A}$ is under the Case 2, consider the problem $(N\setminus N^0,\mathcal{C},c,w)$, where $w(T)=v(S)$ for $T=S\cap(N\setminus N^0)\in\mathcal{C}$. As was proved above, w is well defined. Under the Case 2, due to Theorem 2, there exists y in the Weakly U–equal sacrifice solution of $(N\setminus N^0,\mathcal{C},c,w)$. Let $x\in R^{|N|}$, $x_i=0$

for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then x belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ and $x(T) = 0$ for all $T \in \bigcup_{i=1}^{q} \mathcal{B}^i$.

Take $c = 1$ and $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$. As was proved above, there exists z in the Weakly U–equal sacrifice solution of of $(N, \mathcal{A}, 1, \tilde{v})$ with $x(T) = 0$ for some $T \in \mathcal{A}$. If $U(0) = -\infty$, z does not belong to the U–nucleolus by the definition of the U–nucleolus. Let $U(0) > -\infty$. We have $U(z(T)) - U(\tilde{v}(T)) < 0$ and $U(\tilde{x}(S)) - U(\tilde{v}(S)) = 0$ for all $S \in \mathcal{A}$, hence z does not belong to the U–nucleolus of $(N, \mathcal{A}, 1, \tilde{v})$. Thus, Case 2 is impossible. □

Corollary 5. *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v with $v(T) > 0$ if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.*

Corollary 6. *The Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with the Weakly Uniform Losses solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v with $v(T) > 0$ if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.*

Theorem 9. *Let U be a strictly increasing continuous function defined on $(0, +\infty)$, $q \in \mathcal{Q}(U)$.*

The qU–solution of $(N, \mathcal{A}, c, v)$ coincides with the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is a totally mixed collection of coalitions at N.

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is mixed at N and it follows from Theorem 7 that the qU–solution of $(N, \mathcal{A}, c, v)$ is always contained in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$. Since $x(S)$ are uniquely defined for all x in the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, this implies coincidence of the qU–solution and the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$.

Now suppose that the Wealky U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$ coincides with the qU–solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. By Proposition 3, $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 3, for each collection $\{S_i\}_{i=1}^{k}$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^{k} S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $p < k$ such that $\cup_{i=1}^{p} \mathcal{B}^i$ does not cover N and $\cup_{i=1}^{p} \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > p$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^{p} \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = p+1, \ldots, k$, there exists $S_j \in \mathcal{B}^j$, such that if S_j is taken out from $\mathcal{B}^j$ for all $j > p$, then the remaining elements of $\cup_{j=p+1}^{k} \mathcal{B}^j$ cover $(N \setminus N^0)$.

Let $j_0 \in N \setminus N^0$, $\mathcal{A}_{j_0} = \{Q_i\}_{i \in M}$, then $Q_i \in \mathcal{B}^i$, $i \in \{p+1, \ldots, k\}$. Since $\mathcal{A}$ is mixed by Theorem 7, there exists $j_1 \in N$ such that $\mathcal{A}_{j_1} = \{S_i\}_{i \in M}$, then $j_1 \in N \setminus N^0$, hence Case 1 is impossible.

Case 2. If $S_j \in \mathcal{B}^j$ is taken out from $\mathcal{B}^j$, $j = p+1, \ldots, k$, then the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ do not cover $N \setminus N^0$.

For each $j = p+1, \ldots, k$, $S_j \in \mathcal{B}^j$, we have $S_j \cap (N \setminus N^0) \neq \emptyset$. Indeed, suppose that $S_{j_0} \subset N^0$ for some $j_0 > p$. Then if we take S_{j_0} and arbitrary $S_j \in \mathcal{B}^j$ for $j > p$,

$j \neq j_0$ out from $\cup_{j=p+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=p+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ as if $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C} = \{(N \setminus N^0) \cap S : \, S \in \mathcal{B}^j, \, |\mathcal{B}^j| > 1, \, j > p\}.$$

Note that $P, S \in \cup_{j=p+1}^k \mathcal{B}^j$, $P \neq S$, $P \cap (N \setminus N^0) \in \mathcal{C}$ imply $P \cap (N \setminus N^0) \neq S \cap (N \setminus N^0)$.

Indeed, suppose that $P \cap (N \setminus N^0) = S \cap (N \setminus N^0)$. There exists $P^1 \in \mathcal{A}$ such that $P \cap P^1 = \emptyset$. If we take S, P^1 and arbitrary $S_j \in \mathcal{B}^j$ for $j > p$ with $P \notin \mathcal{B}^j$ out from $\cup_{j=p+1}^k \mathcal{B}^j$, the remaining elements of $\cup_{j=p+1}^k \mathcal{B}^j$ cover $N \setminus N^0$ because $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0} \ni S$, but this is impossible in the considered case.

For arbitrary problem $(N, \mathcal{A}, c, v)$, where $\mathcal{A}$ is under the Case 2, consider the problem $(N \setminus N^0, \mathcal{C}, c, w)$, where $w(T) = v(S)$ for $T = S \cap (N \setminus N^0) \in \mathcal{C}$. As was proved above, w is well defined. Under the Case 2, due to Theorem 3, there exists y in the Weakly U–equal sacrifice solution of $(N \setminus N^0, \mathcal{C}, c, w)$. Let $x \in R^{|N|}$, $x_i = 0$ for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then x belongs to the Weakly U–equal sacrifice solution of $(N, \mathcal{A}, c, v)$, $x(N^0) = 0$.

Let $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$, then $\tilde{x}$ belongs to the U–equal sacrifice solution of $(N, \mathcal{A}, 1, \tilde{v})$ as if $U(\tilde{x}(S)) - U(\tilde{v}(S)) = 0$ for all $S \in \mathcal{A}$ and $\tilde{x}(N^0) > 0$. By Property 2 of qU–solutions, for z in the qU–solution, $z(S)$ are uniquely defined at each $S \in \mathcal{A}$, but $0 = x(T) \neq \tilde{x}(T) > 0$ for $T \in \mathcal{B}^p$. Thus, in Case 2, qU–solution does not coincide with the Weakly U–equal sacrifice solution for some problem. $\square$

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On an Algorithm for Nash Equilibria Determination in the Informational Extended Bimatrix Games

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Abstract The informational aspect for the non-cooperative games becomes an important element for the most of the make decision problems. In this article the informational extended games ${}_1\Gamma$ and ${}_2\Gamma$ are defined. For these informational extended bimatrix games we present two modes for construction of the extended matrices and an algorithm for determination of Nash equilibria. For this algorithm we make some modifications and present an algorithm for determination of Nash equilibria in the informational extended bimatrix games in the case, in which the dimensions of the matrices are too big. Using this algorithm we can also determine the number of Nash equilibria in the informational extended game, without using the extended matrices.

Keywords: Informational extended bimatrix games, Nash equilibria, solution in pure strategies, extended matrices, algorithm of Nash equilibria determination

1. Introduction

Usually the information to make decision problems is the most import "element". Especially this is important if we consider the case of non-cooperative games. Thus the informational aspect represents a real fillip for the elaboration of new study methods for the non-cooperative theory. The informational aspect in the game theory may be manifested by: the devise of possession information about strategy's choice, the payoff functions, the order of moves, and optimal principles of players; using methods of possessed information in the strategy's choice by players (Hâncu, Novac, 2005). The inclusion of information as an important element of the game have imposed a new structure to the game theory: the games in complete information (the games in extended form), the games with not complete information, the games in imperfect information (the Bayes games). The player's possession of supplementary information about unfolding of the game can influence appreciably the player's payoffs.

An important element for the players represent the possession of information about the behaviour of his opponents. Thus for the same sets of strategies and same payoff functions it is possible to obtain different results, if the players have supplementary information. So the information for the players about the strategy's choice by the others players have a significant role for the unfolding of the game.

2. Basic definitions

Consider bimatrix game in the normal form $\Gamma = \langle N, X_1, X_2, A, B\rangle$, where, $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ (A and B are the payoff matrices for the first and the second player respectively. Each player can choose one of his strategies and his purpose is to maximize his payoff. The player can choose his strategy independently of his opponent and the player does not know the chosen strategy of his opponent.

In this article we will determine the Nash equilibria for the informational extended bimatrix games using the well known definition.

Definition 1. The pair (i^*, j^*), $i^* \in X_1, j^* \in X_2$ is called Nash equilibrium (NE) for the game Γ, if the next relations hold

$$\begin{cases} a_{i^*j^*} \geqslant a_{ij^*}, \forall i \in X_1, \\ b_{i^*j^*} \geqslant a_{i^*j}, \forall j \in X_2. \end{cases}$$

Notation: $(i^*, j^*) \in NE(\Gamma)$.

There are bimatrix games for which the set of Nash equilibria is empty: $NE(\Gamma) = \emptyset$ (i.e. solutions do not exist in pure strategies).

However for every bimatrix game we can construct some informational extended games. If one of the players knows the strategy chosen by the other, we consider that it is one form of the informational extended bimatrix game for the initial game. Even if the initial bimatrix game has no solutions in pure strategies, for the informational extended games at least one solution in pure strategies (Nash equilibria) always exists. Proof of this assertion we will present below. In the case of informational extended games the player which knows the chosen strategy of his opponent has one advantage and he will obtain one of his greater payoff.

According to (Kukushkin and Morozov, 1984), let us define two forms of informational extended games ${}_1\Gamma$ and ${}_2\Gamma$. We consider that for the game ${}_1\Gamma$ the first player knows the chosen strategy of the second player, and for the game ${}_2\Gamma$ the second player knows the chosen strategy of the first player.

If one of players knows the chosen strategy of the other, then the set of strategies for this player can be represented by a set of mappings defined on the set of his opponent's strategies.

Definition 2 (The game ${}_1\Gamma$ according to Kukushkin and Morozov, 1984). The informational extended bimatrix game ${}_1\Gamma$ can be defined in the normal form by: ${}_1\Gamma = \langle N, \overline{X_1}, X_2, \overline{A}, \overline{B}\rangle$, *where* $N = \{1,2\}$, $\overline{X_1} = \{\varphi_1 : X_2 \longrightarrow X_1\}$, $\overline{A} = \{\overline{a}_{ij}\}$, $\overline{B} = \{\overline{b}_{ij}\}$, $i = \overline{1, m^n}$, $j = \overline{1,n}$.

For the game ${}_1\Gamma$ we have $\overline{X_1} = \{1, 2, \ldots, m^n\}$, $X_2 = \{1, 2, \ldots, n\}$, $|\overline{X_1}| = m^n$, and the matrices $\overline{A}$ and $\overline{B}$ have dimension $[m^n \times n]$ and are formed from elements of initial matrices A and B respectively.

The matrices $\overline{A}$ and $\overline{B}$ will be constructed in the next mode:

Let us denote by $A_{i\cdot} = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$, $B_{i\cdot} = \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $i = \overline{1,m}$ the rows i in the matrices A and B, respectively.

Choosing one element from each of these rows $A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot}$, we will build one column in the matrix $\overline{A}$. The columns from the matrix $\overline{B}$ are built in the same mode, choosing one element from each of the rows $B_{1\cdot}, B_{2\cdot}, \ldots, B_{m\cdot}$.

Thus, the matrices $\overline{A}$ and $\overline{B}$ have the dimension $[m^n \times n]$.

Example 1 (For the game ${}_1\Gamma$).
We consider the game Γ defined by the matrices:

$$A = \begin{pmatrix} 5 & \underline{\mathbf{3}} & 5 \\ 2 & 1 & 6 \end{pmatrix}, \; B = \begin{pmatrix} 4 & \underline{\mathbf{7}} & 5 \\ 3 & 9 & 2 \end{pmatrix}.$$

For this game we build the matrices for the informational extended bimatrix game ${}_1\Gamma$.

$$\overline{A} = \begin{pmatrix} 5 & \underline{\mathbf{3}} & 5 \\ 5 & \underline{\mathbf{3}} & 6 \\ 5 & 1 & 5 \\ 5 & 1 & 6 \\ 2 & \underline{\mathbf{3}} & 5 \\ 2 & \underline{\mathbf{3}} & 6 \\ 2 & 1 & 5 \\ 2 & 1 & 6 \end{pmatrix}, \; \overline{B} = \begin{pmatrix} 4 & \underline{\mathbf{7}} & 5 \\ 4 & \underline{\mathbf{7}} & 2 \\ 4 & 9 & 5 \\ 4 & 9 & 2 \\ 3 & \underline{\mathbf{7}} & 5 \\ 3 & \underline{\mathbf{7}} & 2 \\ 3 & 9 & 5 \\ 3 & 9 & 2 \end{pmatrix}.$$

The underlined elements (in the matrices above) represent the player's payoff corresponding to the Nash equilibria of the game.

Definition 3 (The game ${}_2\Gamma$ according to Kukushkin and Morozov, 1984). The informational extended bimatrix game ${}_2\Gamma$ can be defined in the normal form by: ${}_2\Gamma = \left\langle N, X_1, \overline{X_2}, \widetilde{A}, \widetilde{B} \right\rangle$, where $\overline{X_2} = \{\varphi_2 : X_1 \longrightarrow X_2\}$, $\left|\overline{X_2}\right| = n^m$, $\widetilde{A} = \{\widetilde{a}_{ij}\}$, $\widetilde{B} = \left\{\widetilde{b}_{ij}\right\}$, $i = \overline{1,m}$, $j = \overline{1,n^m}$.

For the game ${}_2\Gamma$ we have $X_1 = \{1, 2, \ldots, m\}, \overline{X_2} = \{1, 2, \ldots, n^m\}$ and the matrices $\widetilde{A}$ and $\widetilde{B}$ have dimension $[m \times n^m]$ and are formed from elements of initial matrices A and B respectively.

The extended matrices $\widetilde{A}$ and $\widetilde{B}$ will be built in analogical mode as in the case of the game ${}_2\Gamma$.

Let us denote by $A_{\cdot j} = \{a_{1j}, a_{2j}, \ldots, a_{mj}\}$, $B_{\cdot j} = \{b_{1j}, b_{2j}, \ldots, b_{mj}\}$, $j = \overline{1,n}$ the columns j in the initial matrices A and B, respectively). Each of rows in the matrix $\widetilde{A}$ (or in the matrix $\widetilde{B}$, respectively) will be built choosing one element from each of the columns $A_{\cdot j}$ (or from the columns $B_{\cdot j}$, respectively).

Example 2 (For the game ${}_2\Gamma$).
We consider the game Γ defined by the matrices:

$$A = \begin{pmatrix} 3 & 5 \\ \underline{\mathbf{4}} & 6 \\ 1 & \underline{\mathbf{9}} \end{pmatrix}, \; B = \begin{pmatrix} 5 & 0 \\ \underline{\mathbf{9}} & 7 \\ 1 & \underline{\mathbf{5}} \end{pmatrix}.$$

For this game we build the matrices for the informational extended bimatrix game ${}_2\Gamma$.

$$\widetilde{A} = \begin{pmatrix} 3 & 3 & 3 & 3 & 5 & 5 & 5 & 5 \\ \underline{\mathbf{4}} & 4 & 6 & 6 & 4 & 4 & 6 & 6 \\ 1 & \underline{\mathbf{9}} & 1 & \underline{\mathbf{9}} & 1 & \underline{\mathbf{9}} & 1 & \underline{\mathbf{9}} \end{pmatrix}, \; \widetilde{B} = \begin{pmatrix} 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 \\ \underline{\mathbf{9}} & 9 & 7 & 7 & 9 & 9 & 7 & 7 \\ 1 & \underline{\mathbf{5}} & 1 & \underline{\mathbf{5}} & 1 & \underline{\mathbf{5}} & 1 & \underline{\mathbf{5}} \end{pmatrix}.$$

As in the first example, here the underlined elements (in the matrices) represent the player's payoff corresponding to the Nash equilibria of the game.

3. Properties of the informational extended bimatrix games

The next theorem represents the condition of the Nash equilibria existence for the informational extended bimatrix games ${}_1\Gamma$ and ${}_2\Gamma$ (according to Novac, 2009).

Theorem 1. *For every bimatrix game Γ we have the following:*

$$NE\left({}_1\Gamma\right) \neq \emptyset,\ NE\left({}_2\Gamma\right) \neq \emptyset;\ \textit{and}\ NE\left(\Gamma\right) \subset\ NE\left({}_1\Gamma\right),\ NE\left(\Gamma\right) \subset\ NE\left({}_2\Gamma\right).$$

Proof (According to Novac, 2009).
We will prove the theorem for the game ${}_2\Gamma$, i.e. we will build the Nash equilibrium for this informational extended game, using the initial matrices for the game Γ. Let us to denote by $A_{i\cdot} = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$, and $B_{i\cdot} = \{b_{i1}, b_{i2}, \ldots, b_{in}\}$ the row i from the matrices A and B, respectively, $i = \overline{1,m}$. Next we determine the maximum element from each row $B_{i\cdot}$, $\left(i = \overline{1,m}\right)$. We denote the maximum elements from each row by $b_{ij_i} = \max\limits_j \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $\left(i = \overline{1,m}\right)$. Using this maximum elements $b_{1j_1}, b_{2j_2}, \ldots, b_{mj_m}$, we build a column in the extended matrix $\widetilde{B}$, (this column will be one of columns from the extended matrix $\widetilde{B}$ according to the building of extended matrices, described above). Next we build a sequence of elements, using the corresponding elements from the matrix A (i.e. from the same rows and columns as the maximum elements determined above from matrix B); these elements are: $a_{1j_1}, a_{2j_2}, \ldots, a_{mj_m}$, which will represent a column from the extended matrix $\widetilde{A}$ corresponding to the same column from the extended matrix $\widetilde{B}$. We denote by $a_{i^*j^*} = \max\limits_i \{a_{1j_1}, a_{2j_2}, \ldots, a_{mj_m}\}$ the maximum element from this new obtained column. The index of the row i^* will represent the strategy of the first player and this index will correspond to the same row from the extended matrices which will contain these maximum elements; then $j^* = j_{i^*}$ will represent the column from the matrix A which contains the element $a_{i^*j^*}$ determined above. The pair of determined elements $a_{i^*j^*}$ and $b_{i^*j^*}$ from the initial matrices will represent the payoff for the first and the second player, respectively (for the informational extended game ${}_2\Gamma$).

Remark, that because the extended matrices will contain n^m columns, thus the index of the column (from the extended matrices) which will contain the determined elements $a_{i^*j^*}$ or $b_{i^*j^*}$ will not correspond to the same index of the column j^* from the initial matrices, i.e. $j' \neq j^*$, where $j' \in \overline{X}_2, j^* \in X_2$, and $b_{i^*j^*} = b_{i^*j'}$ (because this element will be from the column built above using the maximum elements from each row of the matrix B).

So, it follows that (i^*, j') is a Nash equilibrium for the game ${}_2\Gamma$ and the elements $a_{i^*j^*} = \widetilde{a}_{i^*j'}$ and $b_{i^*j^*} = \widetilde{b}_{i^*j'}$ will be the payoff for the first and the second player, respectively. The optimal strategy j' for the second player in this case can be determined using the indexis $j_1, j_2, ..., j_m$ (according to the algorithm which is presented below in the next section). Thus, there exists (at least one) Nash equilibrium for the informational extended game ${}_2\Gamma$ and this Nash equilibrium is $(i^*, j') \in NE({}_2\Gamma)$.

The proof for the game ${}_1\Gamma$, can be done in analogical mode, building a row from the extended matrices, using the maximum elements from each column of the matrix A. □

For the informational extended games ${}_1\Gamma$ and ${}_2\Gamma$ we can prove the following statements (according to Novac, 2004, 2009).

Assertion 1. *If* $\exists i^* \in X_1$, $\exists j^* \in X_2$ *for which* $a_{i^*j^*} = \max\limits_i \max\limits_j a_{ij}$, $b_{i^*j^*} = \min\limits_i \min\limits_j b_{ij}$ *and* $\forall i \in X_1$, $\forall j \in X_2 : (i,j) \neq (i^*, j^*)$ *so that* $a_{ij} < a_{i^*j^*}$, $b_{ij} > b_{i^*j^*}$; *then:*

1) in the game ${}_2\Gamma$ all columns k (from $\widetilde{A}$ which contain the element $a_{i^*j^*}$, and from $\widetilde{B}$ which contain the element $b_{i^*j^*}$) do not contain NE equilibria;

2) in the game ${}_1\Gamma$ the column j^* (in the matrices $\overline{A}$ and $\overline{B}$) do not contains NE equilibria.

Assertion 2. *If $\exists i^* \in X_1$, $\exists j^* \in X_2$ so that $a_{i^*j^*} = \min_i \min_j a_{ij}$ and $b_{i^*j^*} = \max_i \max_j b_{ij}$, and $\forall i \in X_1, \forall j \in X_2 : (i,j) \neq (i^*, j^*)$ so that $a_{ij} > a_{i^*j^*}$, $b_{ij} < b_{i^*j^*}$; then:*

1) in the game ${}_2\Gamma$ the row i^* (the $\widetilde{A}_{i^*\cdot}$ and the $\widetilde{B}_{i^*\cdot}$) does not contain NE equilibria;

2) in the game ${}_1\Gamma$ all rows k (the $\overline{A}_{k\cdot}$, and the $\overline{B}_{k\cdot}$ which contain the elements $a_{i^*j^*}$ and $b_{i^*j^*}$, respectively) do not contain NE equilibria.

From the assertions 1 and 2 the next two statements result.

Assertion 3. *Consider that $\exists i^* \in X_1$, $\exists j^* \in X_2$ so that $a_{i^*j^*} = \max_i \max_j a_{ij}$ and $b_{i^*j^*} = \min_i \min_j b_{ij}$.*

1) If $\forall i \in X_1 \setminus \{i^*\}$, $\forall j \in X_2 : a_{ij} < a_{i^*j^*}$, and $\forall j \in X_2 \setminus \{j^*\} : b_{i^*j} > b_{i^*j^*}$, then in the game ${}_2\Gamma$ each of columns k ($\widetilde{A}_{\cdot k}$, $\widetilde{B}_{\cdot k}$ which contains the elements $a_{i^*j^*}$ and $b_{i^*j^*}$, respectively) does not contain NE equilibria.

2) If $\forall i \in X_1$, $\forall j \in X_2 \setminus \{j^*\}$: $b_{ij} > b_{i^*j^*}$ and $\forall i \in X_1 \setminus \{i^*\} : a_{ij^*} < a_{i^*j^*}$, then in the game ${}_1\Gamma$ the column j^* ($\overline{A}_{\cdot j^*}$ and $\overline{B}_{\cdot j^*}$) does not contain NE equilibria.

Assertion 4. *Consider that $\exists i^* \in X_1$, $\exists j^* \in X_2$ so that $a_{i^*j^*} = \min_i \min_j a_{ij}$ and $b_{i^*j^*} = \max_i \max_j b_{ij}$.*

1) If $\forall i \in X_1 \setminus \{i^*\}$, $\forall j \in X_2 : a_{ij} > a_{i^*j^*}$, and $\forall j \in X_2 \setminus \{j^*\} : b_{i^*j} < b_{i^*j^*}$, then in the game ${}_2\Gamma$ the row i^* ($\widetilde{A}_{i^*\cdot}$, $\widetilde{B}_{i^*\cdot}$) does not contain NE equilibria.

2) If $\forall i \in X_1$, $\forall j \in X_2 \setminus \{j^*\} : b_{ij} < b_{i^*j^*}$ and $\forall i \in X_1 \setminus \{i^*\} : a_{ij^*} > a_{i^*j^*}$, then in the game ${}_1\Gamma$ each of rows k ($\overline{A}_{k\cdot}$, $\overline{B}_{k\cdot}$ which contains the elements $a_{i^*j^*}$ and $b_{i^*j^*}$, respectively) does not contain NE equilibria.

For proof of the Assertions 1-4 see (Novac, 2009).

Example 3 (For Assertions 2 and 4). Consider the game Γ defined by:

$A = \begin{pmatrix} 0\,3\,1 \\ 5\,2\,4 \end{pmatrix}$, $B = \begin{pmatrix} 7\,3\,6 \\ 1\,5\,0 \end{pmatrix}$.

For this game $NE(\Gamma) = \emptyset$.

For the game ${}_2\Gamma$ there are two Nash equilibria $(2,2),(2,8) \in NE\left({}_2\Gamma\right)$.

For the game ${}_1\Gamma$ there is only one Nash equilibrium $(6,2) \in NE\left({}_1\Gamma\right)$.

In this game, for $i = 1$, $j = 1 : \min_i \min_j a_{ij} = 0$, $\max_i \max_j b_{ij} = 7$. According to Assertion 2 and 4, it follows that: for the game ${}_2\Gamma$ the first row does not contain Nash equilibria and for the game ${}_1\Gamma$ the 1^{st}, 2^d, 3^d, 4^{th} rows do not contain Nash equilibria. The extended matrices for the game ${}_2\Gamma$ are:

$\widetilde{A} = \begin{pmatrix} 0\;0\;0\;3\;3\;3\;1\;1\;1 \\ 5\;\underline{\mathbf{2}}\;4\;5\;2\;4\;5\;\underline{\mathbf{2}}\;4 \end{pmatrix}$, $\widetilde{B} = \begin{pmatrix} 7\;7\;7\;3\;3\;3\;6\;6\;6 \\ 1\;\underline{\mathbf{5}}\;0\;1\;5\;0\;1\;\underline{\mathbf{5}}\;0 \end{pmatrix}$.

The extended matrices for the game ${}_1\Gamma$ are:

$$\overline{A}=\begin{pmatrix}0&3&1\\0&3&4\\0&2&1\\0&2&4\\5&3&1\\5&\underline{\mathbf{3}}&4\\5&2&1\\5&2&4\end{pmatrix},\ \overline{B}=\begin{pmatrix}7&3&6\\7&3&0\\7&5&6\\7&5&0\\1&3&6\\1&\underline{\mathbf{3}}&0\\1&5&6\\1&5&0\end{pmatrix}.$$

Example 4 (For Assertions 1 and 3).
Consider the game Γ defined by:

$$A=\begin{pmatrix}7&3&6\\1&5&0\end{pmatrix};\ B=\begin{pmatrix}0&3&1\\4&2&5\end{pmatrix}.$$

For this game $NE(\Gamma)=\emptyset$, and for the informational extended games there are some solutions $(1,4),(1,6)\in NE\left({}_2\Gamma\right),\ (3,2)\in NE\left({}_1\Gamma\right)$.

In this game, for $i=1,\ j=1:\ \max\limits_i\max\limits_j a_{ij}=7,\ \min\limits_i\min\limits_j b_{ij}=0$. According to Assertions 1 and 3, it follows that: for the game ${}_2\Gamma$ the 1^{st}, 2^{d}, 3^{d} columns do not contain Nash equilibria and for the game ${}_1\Gamma$ the first column does not contain Nash equilibria. The extended matrices for the games ${}_2\Gamma$ and ${}_1\Gamma$ are, respectively:

$$\begin{array}{l}\widetilde{A}=\begin{pmatrix}7&7&7&\underline{\mathbf{3}}&3&\underline{\mathbf{3}}&6&6&6\\1&5&0&1&5&0&1&5&0\end{pmatrix}\\ \widetilde{B}=\begin{pmatrix}0&0&0&\underline{\mathbf{3}}&3&\underline{\mathbf{3}}&1&1&1\\4&2&5&4&2&5&4&2&5\end{pmatrix}\end{array}\ \overline{A}=\begin{pmatrix}7&3&6\\7&3&0\\7&\underline{\mathbf{5}}&6\\7&5&0\\1&3&6\\1&3&0\\1&5&6\\1&5&0\end{pmatrix},\overline{B}=\begin{pmatrix}0&3&1\\0&3&5\\0&\underline{\mathbf{2}}&1\\0&2&5\\4&3&1\\4&3&5\\4&2&1\\4&2&5\end{pmatrix}.$$

Example 5 (For Assertions 2 and 4). Consider the game Γ defined by:

$$A=\begin{pmatrix}1&0\\0&2\\4&0\end{pmatrix},\ B=\begin{pmatrix}2&6\\3&1\\1&4\end{pmatrix},\ NE(\Gamma)=\emptyset.$$

The extended matrices for the game ${}_2\Gamma$ are: $\widetilde{A}=\begin{pmatrix}1&1&1&1&0&\underline{\mathbf{0}}&0&0\\0&0&2&2&0&\underline{\mathbf{0}}&2&2\\4&0&4&0&4&\underline{\mathbf{0}}&4&0\end{pmatrix}$, $\widetilde{B}=\begin{pmatrix}2&2&2&2&6&\underline{\mathbf{6}}&6&6\\3&3&1&1&3&\underline{\mathbf{3}}&1&1\\1&4&1&4&1&\underline{\mathbf{4}}&1&4\end{pmatrix}$.

In this game, for the pairs $(i^*,j^*):(1,2),(2,1),(3,2)$ we have $\min\limits_i\min\limits_j a_{ij}=0$, and for each row $\max\limits_j b_{ij}=b_{i^*j^*}$, but because each of rows from the matrix A contains the minimum element $a_{12}=0$, for the 6^{th} column the conditions from assertions 2 and 4 do not hold, and $(1,6),(2,6),(3,6)\in NE\left({}_2\Gamma\right)$. $\square$

4. Main results. The algorithm of Nash equilibria determination

4.1. The generation of the extended matrices

For the generation of the extended matrices $\overline{A}$ and $\overline{B}$ (or the $\widetilde{A}$ and the $\widetilde{B}$, respectively) we can use the next methods.

The first method is based on the representation of decimal numbers in the base which represent the number of rows or the number of columns in the initial matrices.

For the game ${}_1\Gamma$ we need to represent the numbers $0, 1, \ldots, (m^n - 1)$ in the base m with n components: $N_m = (C_0C_1 \ldots C_{n-1})_m$, where $C_j \in \{0, 1, \ldots, m-1\}$, $j = \overline{0, n-1}$, that is $\left(C_0m^0 + C_1m^1 + \ldots + C_{n-1}m^{n-1}\right) = N_{10}$. Each of these numbers N_m represented in the base m will correspond to one column in the extended matrix.

Then for the elements from column j it must to replace:

$0 \to a_{1j}, 1 \to a_{2j}, \ldots, i \to a_{(i+1)j}, \ldots, (m-1) \to a_{mj}$ (similarly for the matrix B).

For the game ${}_2\Gamma$ it must to represent the numbers $0, 1, \ldots, (n^m - 1)$ in the base n with m components: $N_n = (C_0C_1 \ldots C_{m-1})_n$, where $C_i \in \{0, 1, \ldots, n-1\}$, $i = \overline{0, m-1}$, that is $\left(C_0n^0 + C_1n^1 + \ldots + C_{m-1}n^{m-1}\right) = N_{10}$. Each of these numbers N_n represented in the base n will correspond to one row into the extended matrix.

Then for the elements from the row i it must to replace:

$0 \to a_{i1}, 1 \to a_{i2}, \ldots, j \to a_{i(j+1)}, \ldots, (n-1) \to a_{in}$ (similarly for the matrix B).

The second method consists in assigning two numbers to each of the elements from the initial matrices. One of these numbers represents the number of blocks (series) formed by this element, and the second number represents the length of the block (that is, the number of repetitions of this element in the block).

Denote by $nrbl$ the number of blocks for some element a_{ij} (b_{ij}) and by L the length of each of blocks (the number of repetitions of this element in the block).

Thus for the game ${}_2\Gamma$ we assign to each element from the row i: (n^{i-1}) blocks (series), each of them with the length (n^{m-i}).

So for all elements a_{ij}, b_{ij}, $i = \overline{1, m}, j = \overline{1, n}$ we can determine the indices of columns k of this element in the extended matrix. Thus for the element from the row i and from the column j and for all $nrbl = \overline{1, n^{i-1}}$, $L = \overline{1, n^{m-i}}$, we can calculate the number k by:

$$k = n \cdot n^{m-i} \cdot (nrbl - 1) + (j-1) \cdot n^{m-i} + L. \tag{1}$$

In such mode we can construct the extended matrices $\widetilde{A}$ and $\widetilde{B}$: $\widetilde{A}[i,k] = A[i,j]$, $\widetilde{B}[i,k] = B[i,j]$.

Similarly, for the game ${}_1\Gamma$ we assign to each element from the column j: (m^{j-1}) blocks (series) each of them with the length (m^{n-j}).

Thus for all elements $\forall i = \overline{1, m}, j = \overline{1, n}$, we can determine the indices of the rows k of this element in the extended matrix.

In such mode for the element from the row i and from the column j and for all $nrbl = \overline{1, m^{j-1}}$, $L = \overline{1, m^{n-j}}$ we can calculate the number k by:

$$k = m \cdot m^{n-j} \cdot (nrbl - 1) + (i-1) \cdot m^{n-j} + L. \tag{2}$$

In such mode, we can construct the extended matrices $\overline{A}$ and $\overline{B}$ (for each determined k): $\overline{A}[k,j] = A[i,j]$, $\overline{B}[k,j] = B[i,j]$.

Remark. These two different methods may be used independently. Using it we can construct the extended matrices entirely or partly. If the initial matrices are very big, we can use these methods for partial construction of the extended matrices. Thus the first method may be used when we need to construct only one row (for the informational extended game ${}_1\Gamma$), or only one column (for the game ${}_2\Gamma$), and the second method may be used when we need to determine the position of some element in the extended matrix, i.e. the index of the row (in the game ${}_1\Gamma$) or the index of the column (in the game ${}_2\Gamma$, respectively).

Example 6. (The generation of the extended matrices).
Consider the game Γ defined by:

$A = \begin{pmatrix} 0\,3\,1 \\ 5\,2\,4 \end{pmatrix}$, $B = \begin{pmatrix} 7\,3\,6 \\ 1\,5\,0 \end{pmatrix}$ $m = 2$, $n = 3$.

For the **first method:**

For the game ${}_1\Gamma$ the matrices have the dimension $\left[2^3 \times 3\right]$. We construct the 5^{th} row from the extended matrix $\overline{A}$:

$4_{10} = (100)_2$, next we do the substitution with corresponding elements and we obtain the 5^{th} row with elements (5,3,1).

In the same mode we can construct the 8^{th} row: $7_{10} = (111)_2$ and we obtain the row (1,5,0) from the extended matrix $\overline{B}$.

For the game ${}_2\Gamma$ the matrices have the dimension $\left[2 \times 3^2\right]$. We construct the 6^{th} column:

$5_{10} = (12)_3$, next we do the substitution with corresponding elements and we obtain the 6^{th} column: (3,4) from the extended matrix $\widetilde{A}$ and the 6^{th} column (3,0) from the matrix $\widetilde{B}$.

In the same mode we can construct the 9^{th} column: $8_{10} = (22)_3$ and we obtain the columns (1,4) and (6,0) from the extended matrices ($\widetilde{A}$ and $\widetilde{B}$, respectively).

For the **second method:**

For the same game we determine the positions in the extended matrices for the elements $a_{21} = 5$ and $b_{21} = 1$.

For the game ${}_1\Gamma$, the first column will contain one series (2^0 blocks) which will have 2^2 elements; the indices of rows are $k = 5, 6, 7, 8$.

For the game ${}_2\Gamma$ the second row will contain (3^1) series (blocks) and each of them will have one element (i.e. 3^0 elements); so, the indices of columns are $k = 1, 4, 7$. □

4.2. The algorithm for determination of the Nash equilibria

Using these methods we can construct an algorithm for determination of the NE equilibrium. This algorithm does not need the integral construction of the extended matrices, and need only the partial construction of them.

Thus in the case when the dimension of the initial matrices A and B are very big we avoid using a big volume of memory, since the extended matrices will have a bigger dimensions ($[m \times n^m]$ and $[m^n \times n]$, respectively).

The following algorithm can be used for determination of Nash equilibria in the informational extended bimatrix games ${}_1\Gamma$ and ${}_2\Gamma$.

Algorithm.

Consider the extended game ${}_2\Gamma$.

Using the first method we represent the numbers from 0 to $(n^m - 1)$ in the base n. Each of these representations will correspond to one column in the extended matrix $\widetilde{A}$. For each of these representations it must do the substitutions with the corresponding elements from the initial matrix A.

For each column $j_0 = \overline{1, n^m}$, obtained in such mode, from the extended matrix $\widetilde{A}$ we will do the next operations.

1. We determine the maximum element from this column of the extended matrix $\widetilde{A}$, and the corresponding element with the same indices from the matrix $\widetilde{B}$; let them $\widetilde{a}_{i_0 j_0}$ and $\widetilde{b}_{i_0 j_0}$.

2. We determine the maximum element from the row i_0 in the initial matrix B: let it be $b_{i_0 j^*}$.

3. If $\widetilde{b}_{i_0 j_0} = b_{i_0 j^*}$, then (i_0, j_0) is NE equilibrium for the extended game ${}_2\Gamma$: $(i_0, j_0) \in NE\,({}_2\Gamma)$, and the elements $\widetilde{a}_{i_0 j_0}$ and $\widetilde{b}_{i_0 j_0}$ will be the payoffs values for the first and for the second player respectively.

For the informational extended game ${}_1\Gamma$ we can construct the algorithm in the same mode.

Consider now the extended game ${}_1\Gamma$.

Using the first method we represent the numbers from 0 to $(m^n - 1)$ in the base m. For each of these representations it must do the substitutions with the corresponding elements from the initial matrix B. Each of these representations will correspond to one row in the extended matrix $\overline{B}$.

For each row i_0 $(i_0 = \overline{1, m^n})$ from the matrix $\overline{B}$ (thus obtained) we will do the next operations.

1. We determine the maximum element from this row of the extended matrix $\overline{B}$, and the corresponding element with the same indices from the matrix $\overline{A}$; let them be $\overline{b}_{i_0 j_0}$ and $\overline{a}_{i_0 j_0}$.

2. We determine the maximum element from the column j_0 in the initial matrix A : let's consider this element $a_{i^* j_0}$.

3. If $\overline{a}_{i_0 j_0} = a_{i^* j_0}$, then (i_0, j_0) is NE equilibrium for the extended game ${}_1\Gamma$: $(i_0, j_0) \in NE\,({}_1\Gamma)$, and the elements $\overline{a}_{i_0 j_0}$ and $\overline{b}_{i_0 j_0}$ will be the payoffs values for the first and for the second player respectively.

Example 7. Consider the game Γ defined by:

$$A = \begin{pmatrix} 2 & 5 \\ \underline{\mathbf{4}} & 1 \\ 3 & 7 \end{pmatrix},\ B = \begin{pmatrix} 5 & 9 \\ \underline{\mathbf{2}} & 1 \\ 6 & 4 \end{pmatrix}.$$

This game has only one Nash equilibrium.

We can determine the Nash equilibria without using the extended matrices. For the game ${}_2\Gamma$ we need to represent the numbers from 0 to $8 = 2^3$ in the base 2.

For the first column: 0_{10} =$(0{,}0{,}0)_2$ we do the substitution with the corresponding elements (2,4,3), $\max\{2,4,3\} = 4 = a_{21}$, and the corresponding element b_{21} is the maximum element from the second row from the matrix B, thus it follows that: $(2,1) \in NE({}_2\Gamma)$;

- for the second column : 1_{10} =$(0{,}0{,}1)_2$ the corresponding elements are (2,4,7), for which $\max\{2,4,7\} = 7 = a_{32}$, but the corresponding element b_{32} is not the $\max\{6,4\}$ from the third row of the matrix B, so $(3,2) \notin NE({}_2\Gamma)$;

- for the third column 2_{10}=$(0{,}1{,}0)_2$ for which $\max\{2,1,3\} = 3 = a_{31}$ we have $b_{31} = \max\{6,4\}$, thus $(3,3) \in NE({}_2\Gamma)$;

- for the 5^{th} column 4_{10}=$(1{,}0{,}0)_2$ we have $\max\{5,4,3\} = 5 = a_{12}$ and $b_{12} = \max\{5,9\}$, so it follows that $(1,5) \in NE({}_2\Gamma)$;

- for the 7^{th} column 6_{10} =$(1{,}1{,}0)_2$ we have $\max\{5,1,3\} = 5 = a_{12}$ and $b_{12} = \max\{5,9\}$, so $(1,7) \in NE({}_2\Gamma)$.

If we will build the extended matrices, we will see that for the informational extended game ${}_2\Gamma$ there are only four Nash equilibria.

$$\begin{pmatrix} 2\,2\,2\,2\,\underline{\mathbf{5}}\,5\,\underline{\mathbf{5}}\,5 \\ \underline{\mathbf{4}}\,4\,1\,1\,4\,4\,1\,1 \\ 3\,7\,\underline{\mathbf{3}}\,7\,3\,7\,3\,7 \end{pmatrix}, \qquad \begin{pmatrix} 5\,5\,5\,5\,\underline{\mathbf{9}}\,9\,\underline{\mathbf{9}}\,9 \\ \underline{\mathbf{2}}\,2\,1\,1\,2\,2\,1\,1 \\ 6\,4\,\underline{\mathbf{6}}\,4\,6\,4\,6\,4 \end{pmatrix}.$$

If we need to determine the indices of the columns in the extended matrices in the game ${}_2\Gamma$ for the elements a_{21}, b_{21}, and we know that $(2,1) \in NE(\Gamma)$, we can use relation (f1) from the second method. So in this case indices of columns are $k = 1, 2, 5, 6$, but only one of these columns contains NE equilibrium $(2,1) \in NE({}_2\Gamma)$. □

Remark. In the case when the numbers n^m and m^n are very big this algorithm for determination of NE equilibria for the informational extended games and the generation methods of the extended matrices are more complex. But all these operations can be executed operating with the corresponding numbers represented in the base m or n respectively to the informational extension (${}_1\Gamma$ or ${}_2\Gamma$ respectively).

The operating with numbers represented in the base n.

Consider the informational extended game ${}_2\Gamma$.

For the game ${}_2\Gamma$ the extended matrices will have the dimension $[m \times n^m]$ (by definition).

According to the second method, to each element from the row i we assign two numbers : $nrbl = n^{i-1}$ blocks and each of them has the length $L = n^{m-i}$.

The relation (1) used in the second method for the game ${}_2\Gamma$ can be written in the next form:

$$k = n^{m-i} \cdot (n \cdot nrbl - n + (j-1)) + L. \tag{3}$$

We will represent all numbers from the relation (3) in the base n with m components:

$n = \left(00\ldots 0\underset{2}{1}0\right)_n$;

$n^{i-1} = N_n = \left(0\ldots 0\underset{i}{1}0\ldots \underset{1}{0}\right)_n$, $i = \overline{1,m}$;

$n^{m-i} = N_n = \left(0\ldots 0\underset{m-i+1}{1}0\ldots \underset{1}{0}\right)_n$, $i = \overline{1,m}$;

the number of blocks is determined by: $nrbl = \overline{1, n^{i-1}}$, so

$nrbl = (00\ldots 01)_n, \ldots, \left(0\ldots 0\underset{i}{1}0\ldots \underset{1}{0}\right)_n$;

the length of blocks is determined by: $L = \overline{1, n^{m-i}}$, thus

$L = (00\ldots 01)_n, \ldots, \left(0\ldots 0\underset{m-i+1}{1}0\ldots \underset{1}{0}\right)_n$.

Using the relation (3) all operations can be done, operating with numbers represented in the base n.

Thus, in the relation (3) using the numbers represented in the base n, we determine k.

All arithmetic operations (*,+,-) will be executed in the base n.

Remark. The operation "*" in the base n for one number with other number in the form $\left(0\ldots 0\underset{i+1}{1}0\ldots \underset{1}{0}\right)_n = n^i$ is equivalent to moving to the left with i positions of the components from the first number (so add i zeroes to the right).

Remark. The operations (+,-) for two numbers in the base n are done according to the well-known rules characteristic for the base 10.

Example 8. Consider that the game Γ have matrices of dimension $[6 \times 6]$, i.e. $m = 6$, $n = 6$, and we need to determine the index of the column k for the elements a_{25} and b_{25} in the extended matrices for the game ${}_2\Gamma$ (i.e. $i = 2, j = 5$), $m-i+1 = 5$; it is known that for the number of blocks (series) it holds next $(1 \leqslant nrbl \leqslant n^{i-1} = n)$, so we have $nrbl = (0...01)_6, ..., (0...010)_6$ in the base 6, and $n^{m-i} = (010000)_6$. Consider that $nrbl = 000005$ and $L = (015355)_6$. Using the relation (3), all operations can be done operating with numbers represented in the base 6 :

$$\begin{array}{ll}
000005 & = nrbl \\
*\underline{000010} & = n \\
000050 & \\
+\underline{000004} & = j-1 \\
000054 & \\
-\underline{000010} & = n \\
000044 & \\
*\underline{010000} & = n^{m-i} \\
440000 & \\
+\underline{015355} & = L \\
455355 & = k
\end{array}$$

Thus, we just have obtained one of the indices (represented in the base 6: $k = 455355$) of the columns for the elements a_{25}, b_{25} in the extended matrices for the game ${}_2\Gamma$.

Remark. In this algorithm we can do operations in other order for determination Nash equilibria in the informational extended games ${}_1\Gamma$, ${}_2\Gamma$. Using this modified algorithm, we can determine also the number of Nash equilibria in the games ${}_1\Gamma$, ${}_2\Gamma$, without using the extended matrices. Thus for the game ${}_1\Gamma$, $({}_2\Gamma)$ firstly we determine the maximum payoff for the first (or second) player and the corresponding strategy for this maximum element; then we determine the corresponding combinations for that we obtain the maximum payoff and the corresponding strategy for the second (first) player, respectively.

In this way for the game ${}_1\Gamma$, we can firstly to determine the maximum elements for the first player, and for the corresponding elements we determine if there exist some combinations in the matrix of the second player for that we have Nash equilibria.

4.3. The modified algorithm

For the game ${}_1\Gamma$, we determine the maximum element in each column from the matrix A, i.e. $a_{i_j j} = \max\limits_i \{a_{1j}, a_{2j}, \ldots, a_{mj}\}$, for $\forall j = \overline{1,n}$.

For each element $a_{i_j j}$, $j = \overline{1,n}$ thus obtained, we determine the corresponding elements with the same indices from the matrix $B : b_{i_j j}$, $j = \overline{1,n}$.

For each of these pairs $a_{i_j j}, b_{i_j j}, \left(j = \overline{1,n}\right)$ we determine if these values can be the payoffs for players for some Nash equilibria.

Thus if $\forall k \in X_2 \backslash \{j\}$ $\exists b_{ik} : b_{ik} \leqslant b_{i_j j}$, then the pair $a_{i_j j}, b_{i_j j}$ can be the payoffs for players for some Nash equilibria in the game ${}_1\Gamma$; consider this pair $a_{i^* j^*}, b_{i^* j^*}$.

It is possible that for the pair $a_{i^* j^*}, b_{i^* j^*}$ there are more Nash equilibria.

If we wish to determine how many Nash equilibria there are in the game ${}_1\Gamma$ for the pair $a_{i^* j^*}, b_{i^* j^*}$ we determine the number of elements which there are in each column $k \in X_2 \setminus \{j\}$ from the matrix B for that $b_{ik} \leqslant b_{i^*j^*}$. Denote by n_j, $j = \overline{1,n}$ the number of elements b_{ij} from the column j for that $b_{ij} \leqslant b_{i^*j^*}$, and for j^* we have $n_{(j^*)} = 1$.

Then the number of Nash equilibria corresponding to the payoffs $a_{i^*j^*}$ and $b_{i^*j^*}$, can be determined by:

$$N_{j^*} = n_1 \cdot n_2 \cdot \ldots \cdot n_{(j^*-1)} \cdot 1 \cdot n_{(j^*+1)} \cdot \ldots n_n, \tag{4}$$

And the number of all Nash equilibria in the game ${}_1\Gamma$ can be determined by: $N = \sum\limits_j N_j$.

If the pair of elements $a_{i_j j}, b_{i_j j}$ can be the payoffs, corresponding to some Nash equilibrium in the informational extended game ${}_1\Gamma$, then j will be the strategy for the second player. And because $\overline{X_1} \neq X_1$, we have to determine the strategy for the first player, for which the elements $a_{i^*j^*}, b_{i^*j^*}$ will correspond to one Nash equilibrium.

In this way we determine the elements $b_{i_1 1}, b_{i_2 2}, \ldots, b_{i_j j}, \ldots, b_{i_n n}$, for that $b_{i_k k} \leqslant b_{i_j j}$, $\forall k \in X_2 \setminus \{j\}$.

Then using the indices of the rows of these elements, we can determine the strategy for the first player by:

$$i' = (i_1 - 1)\, m^{n-1} + (i_2 - 1)\, m^{n-2} + \ldots + (i_j - 1)\, m^{n-j} + \ldots + (i_n - 1)\, m^0 + 1. \tag{5}$$

So, the pair i', j is Nash a equilibrium for the informational extended game ${}_1\Gamma : (i', j) \in NE\,({}_1\Gamma)$. Similarly, for the game ${}_2\Gamma$, we can determine the strategy for the second player by:

$$j' = (j_1 - 1)\, n^{m-1} + (j_2 - 1)\, n^{m-2} + \ldots + (j_i - 1)\, n^{m-i} + \ldots + (j_m - 1)\, n^0 + 1, \tag{6}$$

where the indices j_i $\left(i = \overline{1,m}\right)$ are determined by the indices of columns of the elements $b_{ij_i} = \max\limits_j \{b_{i1}, b_{i2}, \ldots, b_{in}\}$, $\forall i = \overline{1,m}$.

Example 9. Consider the game Γ defined by:

$$A = \begin{pmatrix} \underline{\mathbf{9}} & 2 & 6 & 0 \\ 2 & \underline{\mathbf{7}} & 7 & 2 \\ 5 & 4 & \underline{\mathbf{9}} & \underline{\mathbf{5}} \\ 3 & 5 & 4 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 5 & 3 & \underline{\mathbf{9}} \\ \underline{\mathbf{8}} & 2 & 5 & 7 \\ \underline{\mathbf{7}} & 5 & 4 & 1 \\ 2 & 3 & 1 & \underline{\mathbf{4}} \end{pmatrix}.$$

For this game $NE(\Gamma) = \emptyset$. For the informational extended games ${}_1\Gamma$, ${}_2\Gamma$ the extended matrices will have the dimension $[256 \times 4]$ and $[4 \times 256]$, respectively.

For the game ${}_1\Gamma$ we determine the maximum elements in each column from the matrix A, and for the corresponding elements we determine if there are some combinations in the matrix B such that the pair $(a_{i^*j^*}, b_{i^*j^*})$ will be the payoffs for the players.

So, the pair $(a_{11}, b_{11}) = (9, 3)$ will be the payoffs for the players, and the strategy for the second player will be $j^* = 1$.

We determine the combination of elements for which we have NE in the game ${}_1\Gamma : (b_{11}, b_{22}, b_{13}, b_{34}) = (3, 2, 3, 1)$, for that

$i' = (i_1 - 1)\, 4^3 + (i_2 - 1)\, 4^2 + (i_3 - 1)\, 4^1 + (i_4 - 1)\, 4^0 + 1 = 0 + 1 \cdot 4^2 + 0 + 2 \cdot 4^0 + 1 = 19$, so $(19, 1) \in NE\,({}_1\Gamma)$.

For the pair $(a_{11}, b_{11}) = (9, 3)$ we have $\{(19, 1), (31, 1), (51, 1), (63, 1)\} \in NE\,({}_1\Gamma)$.

Similarly, for the pair $(a_{33}, b_{33}) = (9, 4)$ we obtain
$\{(27,3),(28,3),(59,3),(60,3),(219,3),(220,3),(251,3),(252,3)\} \in NE\,({}_1\Gamma)$;
for the pair $(a_{22}, b_{22}) = (7, 2)$ we obtain $\{(223,2)\} \in NE\,({}_1\Gamma)$.

Thus, in the game ${}_1\Gamma$ there are 13 Nash equilibria.

Similarly, for the game ${}_2\Gamma$ we can determine the set of Nash equilibria.

In this case for the pair $(a_{31}, b_{31}) = (5, 7)$ we obtain the follow Nash equilibria: $(3,65)$, $(3,66)$, $(3,67)$, $(3,68)$, $(3,113)$, $(3,114)$, $(3,115)$, $(3,116)$, $(3,193)$, $(3,194)$, $(3,195)$, $(3,196)$, $(3,241)$, $(3,242)$, $(3,243)$, $(3,244)$ in the game ${}_2\Gamma$.

Thus, in the game ${}_2\Gamma$ there are 16 Nash equilibria.

5. Conclusions

The definitions of the informational extended bimatrix games are presented in this paper. The properties of the Nash equilibrium for this type of informational extended bimatrix games are given and two methods for the generation of the extended matrices are described. The algorithm for determination of Nash equilibrium is constructed, using the combination of this two methods for generation of the extended matrices. The algorithm for determination of the Nash equilibria is modified and it is presented in other form for the case when the extended matrices have the dimension too big. The numerical examples for the properties of the Nash equilibria in the informational extended bimatrix games, for the methods of the matrices generation, and for the both algorithms are given.

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Forest Situations and Cost Monotonic Solutions

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Abstract In this paper, we generalize the well-known mountain situations by introducing multiple sources called the forest situations. We deal with the cost sharing problem by introducing the cooperative cost game. We show that the Bird allocation is a special core element of the related cost game corresponding to the forest situation. Further, we give solutions for the cost game corresponding to the forest situation. Finally, we show that these solutions satisfy the cost monotonicity property.

Keywords: forest situations, bird allocation, shapley value, cost monotonicity.

1. Introduction

In a classical mountain situation which is studied in (Moretti et al., 2002) a group of people whose houses lie on mountains surrounding a valley or a part of a coast are considered. They want to be connected to a drainage system, where they have to empty their sewage. It is obvious that the sewage has to be purified before introduction into the environment. So, the sewage has to be collected downhill in a water purifier in the valley or along the coast. Consequently, each player wants to connect his house with a drain pipe to the water purifier.

The problem is the higher costs of direct connection to water purifier and pumping water from the houses at lower heights to the houses at upper heights. Further, being connected to the houses at the same height may be dangerous. Figure 1 illustrates a possible situation.

The network drawn in the Figure 1 is a directed weighted graph, whose vertices are the houses, root is the water purifier and edges are the drain pipes which are allowed to be built. The numbers indicate the cost of building to the corresponding pipe. Sometimes connection from the higher houses to lower houses is impossible. However, it is always possible to connect a house directly with the root.

A mountain situation as described above leads to a connection problem of a directed graph without cycles and with some other properties. A connection situation takes place in the presence of a group of agents, each of which needs to be connected to a source. This connection may be directly or via links to other agents. If links are costly, then the agents prefer to cooperate in order to reduce costs.

In this study, we model mountain situations by introducing multiple sources which is called a forest situation. Further, we use the notion of cooperative games

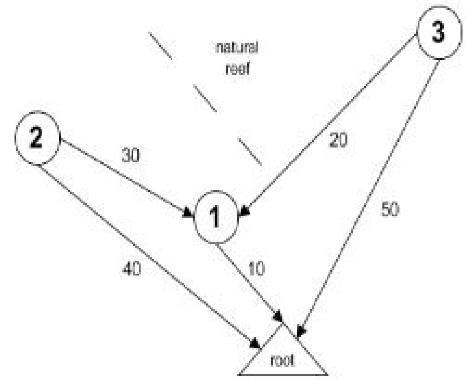

Fig. 1: A possible mountain situation

(Branzei et al., 2005; Tijs, 2003) to tackle the cost sharing problem to a forest situation.

In this context, the distribution of collective gains and costs is the main question to be answered for the individuals and organizations. The theory of cooperative games provides suitable tools for answering this question. Further, cooperative game theory and its solution concepts have had broad applicability in Operational Research, economy, modern finance, climate negotiations and policy, environmental management and pollution control, etc.

The paper is organized as follows. In Section 2, we recall basic notions and facts from graph theory and the theory of cooperative games. In Section 3, the notion of forest situations are introduced. At the same section, an interesting method to find the spanning forest with minimum costs is described. Section 4 deals with the cost sharing problem by introducing the cooperative cost game corresponding to a forest situation. Section 5 gives the Shapley value and the Bird rule which are solutions for the cooperative cost game corresponding to the forest situation. These allocations satisfy interesting cost monotonicity properties.

2. Preliminaries

In this section we give some terminology on graph theory and the theory of cooperative games (Branzei et al., 2005; Diestel, 2000; Moretti et al., 2002; Norde et al., 2001; Tijs, 2003).

A *graph* is a pair $G = < N', E >$ of sets such that $E \subseteq [N']^2$; thus, the elements of E are $2-$element subsets of N'. The elements of N' are the *nodes* of the graph G, the elements of E are its *edges* (or lines). A *complete weighted graph* is a tuple $< N', w >$, where

i) $N' = \{0, 1, ..., n\}$,

ii) $w : E \to \mathbb{R}_+$.

Node 0 is called the *source* and $N = \{1, ..., n\}$ the set of players. Also, for an $l \in E$ the nonnegative number $w(l)$ represents the weight or cost of edge l. A *directed graph* is a pair $< N', E >$ of disjoint sets (of vertices and edges) together with two maps $init : N' \to E$ and $ter : N' \to E$ assigning to every edge e an initial vertex $init(e)$ and a terminal vertex $ter(e)$.

A subset of Γ of E is called a *network*. The cost of network Γ is $w(\Gamma) = \sum_{l \in \Gamma} w(l)$. A *path* from i to j in Γ is a sequence of nodes $i = i_0, i_1, ..., i_k = j$ such that $\{i_s, i_{s+1}\} \in \Gamma$ for every $s \in \{0, ..., k-1\}$. A *network* Γ is a spanning network for S $(S \subseteq N)$ if for every $l \in \Gamma$ we have $l \subseteq S \cup \{0\}$ and if every $i \in S$ there is a path in Γ from i to 0.

A nonempty graph $< N', w >$ is called *connected* if any two of its vertices are linked by a path in $< N', w >$. An acyclic graph, one not containing any cycles, is called a *forest*. A connected forest is called a *tree*. Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the *root* of tree. A tree with fixed root is *a rooted tree*.

A *cooperative (cost) game* in coalitional form is an ordered pair $< N, c >$, where $N = \{1, 2, ..., n\}$ is the set of players, and $c : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $c(\emptyset) = 0$.

Often, we refer to such a game as a *TU (transferable utility)* game, and we identify cooperative cost game $< N, c >$ with its characteristic function c. The family of all games with player set N is denoted by G^N.

Now, we recall that a core allocation of $< N, c >$ is a vector $x \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \textit{efficiency}: \quad & \sum_{i=1}^{n} x_i = c(N), \\ \textit{stability}: \quad & \sum_{i \in S} x_i \le c(S) \text{ for each } S \in 2^N. \end{aligned}$$

The core (Gillies, 953) of $c \in G^N$ is denoted by $\mathcal{C}(N, c)$ and consists of all core allocations.

The subgame of $< N, c >$ with player set $T \in 2^N \setminus \{\emptyset\}$ is the cooperative cost game $< T, c >$, where $c : 2^T \to \mathbb{R}$ is the restriction of $c : 2^N \to \mathbb{R}$.

We call a game $< N, c >$ as *concave* iff

$$c(S) + c(T) \ge c(S \cup T) + c(S \cap T) \quad \forall S, T \in 2^N.$$

We denote by CG^N the class of concave games with player set N. It is well known that a concave game has a non-empty core. In this paper, we focus on the class of concave games.

We call a game $< N, c >$ *monotonic* if $c(S) \le c(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by MG^N the class of monotonic games with player set N. For monotonic games $< N, c >$, $c(T) - c(S)$ is well defined for all $S, T \in 2^N$ with $S \subset T$. Now, we define for each $c \in MG^N$ and each $i \in N$, the marginal contribution of i in the game c by $M_i(N, c) = c(N) - c(N \setminus \{i\})$.

Let $c \in G^N$. A scheme $a = (a_{iS})_{i \in S,\ S \in 2^N \setminus \{\emptyset\}}$ of real numbers is a *population monotonic allocation scheme (pmas)* of c for cost games if

i) $\sum_{i \in S} a_{iS} = c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

ii) $a_{iS} \ge a_{iT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$.

Let $\pi(N)$ be the set of all permutations $\sigma : N \to N$ of N. The set $P^\sigma(i) := \left\{r \in N : \sigma^{-1}(r) < \sigma^{-1}(i)\right\}$ consists of all predecessors of i with respect to the permutation σ.

Let $c \in G^N$ and $\sigma \in \pi(N)$. The marginal contribution vector $m^\sigma(c) \in \mathbb{R}^N$ with respect to σ and has the $i-$th coordinate

$$m_i^\sigma(c) := c(P^\sigma(i) \cup \{i\}) - c(P^\sigma(i))$$

for each $i \in N$.

3. Forest Situations

Consider a tuple given by $< N, \{0_i\}, A, w >$, where $N = \{1, 2, ..., n\}$, is the set of players, $\langle N \cup \{0_i\}, A\rangle$ is a rooted directed graph with $N \cup \{0_i\}$ a set of points (vertices), $A \subset N \times (N \cup \{0_i\})$ a set of arcs, where for $i \in \mathbb{N}$, 0_i is the roots. We assume also that the following conditions F.1 and F.2 hold.

F.1 *(Direction connection possibility)* For each $k \in N$ and $\exists i \in \mathbb{N}$, $(k, 0_i) \in A$.

F.2 *(No cycles)* For each $s \in \mathbb{N}$ and $v_1, v_2, ..., v_s \in N \cup \{0_i\}$ such that $(v_1, v_2) \in A, (v_2, v_3) \in A, ..., (v_{s-1}, v_s) \in A$ we have $(v_s, v_1) \notin A$.

Further, $w : A \to \mathbb{R}$ is a non-negative function on the set of arcs. Next, we introduce the genericity condition:

F.3 *(Genericity condition)* For each $k \in N$ and all $i, j \in N \cup \{0_i\}$, $i \neq j : (k, i) \in A, (k, j) \in A \Longrightarrow w(k, i) \neq w(k, j)$

Notice that, F.3 gives us the possibility to speak of the best connection $b(k)$ of $k \in N$, where

$$b(k) = \underset{i \in N \cup \{0_i\} : (k,i) \in A}{\operatorname{argmin}} w(k, i).$$

We call such a tuple $< N, \{0_i\}, A, w >$ with the properties F.1, F.2 and F.3 *a forest situation.*

Each mountain problem as described in Section 1 leads to the forest situation by introducing multiple sources, where N corresponds to the of agents (houses) in the mountain, 0_i to the purifiers, A to the set of *allowed* connections determined by the *gravity* condition

$$(i, j) \in A \Longrightarrow h(i) > h(j) \tag{3.1}$$

where $h(i)$ is the height of house i) and by *reefs,* etc.. Further $w(i, j)$ describes the cost of connecting i with j via a pipe line. F.1 is demanded and F.2 follows from (3.1).

On other hand, given a forest situation $< N, \{0_i\}, A, w >$ with the properties F.1 and F.2, there exists an *intrinsic height function* $h_0 : N \cup \{0_i\} \to \mathbb{N} \cup \{0_i\}$ such that $(k, l) \in A$ implies $h_0(k) > h_0(l)$. Here, h_0 is defined as follows: for $k \in N \cup \{0_i\}$, $h_0(k)$ is the length of a longest path from k to 0_i.

There are two interesting problems related to such a forest situation. One of them is finding a $0-$connecting subforest $< N \cup \{0_i\}, T >$ of $< N \cup \{0_i\}, A >$, i.e., a subforest connecting each $k \in N$ with 0, with minimum cost; and the other is allocating the connection costs in such a forest among the agent.

This section deals with the first problem; and the next sections deal with the second one.

The next theorem shows that there is a unique optimal forest, connecting all players in N with the root 0_i. This forest corresponds to the situation where each agent $k \in N$ connects himself with his best connection point $b(k) \in N \cup \{0_i\}$.

The proof of the following theorem is straightforward (see (Moretti et al., 2002)).

Theorem 1. *Let* $< N, \{0\}, A, w >$ *be a forest situation. Let* $T = \{(k, b(k)) \mid k \in N\}$*. Then*

(i) $< N \cup \{0_i\}, T >$ *is a* $0-$*connecting subforest of* $< N \cup \{0_i\}, A >$.

(ii) The forest $< N \cup \{0_i\}, T >$ *is the unique* $0-$*connecting subforest with minimum cost.*

Example 1. Figure 2 corresponds to a forest situation $< N, \{0_i\}, A, w >$, where $N = \{1, 2, 3\}, A = \{(1, 0_1), (2, 0), (2, 1), (3, 0_2), (3, 1), (3, 2)\}$. Then the intrinsic height function h_0 is described by $h_0(k) = k$ for each $k \in N$. Since $b(1) = 0_1, b(2) = 1, b(3) = 2$, the tree $\langle N \cup \{0_i\}, T\rangle$ with $T = \{(1, 0_1), (2, 1), (3, 2)\}$ is an optimal $0-$connecting tree with costs $10+15+20 = 45$. The payoff vector $B(N, \{0_i\}, A, w) = (10, 15, 20)$ corresponding to the situation where each player i pays $w(i, b(i))$ will be called the **Bird allocation** (Bird, 1976).

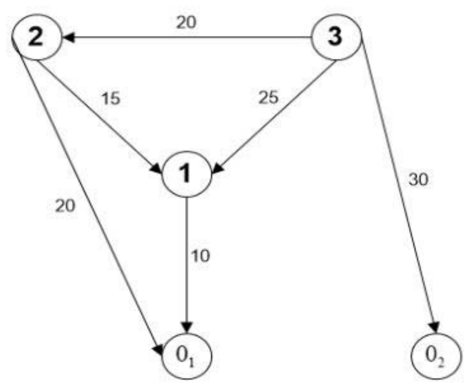

Fig. 2: The forest situation of Example 2

In the next section, we see that the Bird allocation is a special core element of the cost game, corresponding to the forest situaton.

Example 2. Figure 3 corresponds to a forest situation $< N, \{0_i\}, A, w >$, where $N = \{1, 2, 3\}, A = \{(1, 0_1), (2, 0), (2, 1), (3, 0_2), (3, 1), (3, 2)\}$. Then the intrinsic height function h_0 is described by $h_0(k) = k$ for each $k \in N$. Since $b(1) = 0_1, b(2) = 1, b(3) = 0_2$, the forest $\langle N \cup \{0_i\}, T\rangle$ with $T = \{(1, 0_1), (2, 1), (3, 0_2)\}$ is an optimal $0-$connecting tree with costs $10+15+15 = 40$. The payoff vector $B(N, \{0_i\}, A, w) = (10, 15, 15)$ is corresponding to the situation as represented in Figure 3.

Notice that both of the examples illustrated above correspond to a forest situation. In Example 2, player 2 prefers cooperation because of the higher cost of connecting to 0_2, but in Example 3, player 3 does not prefer to cooperate because of the lower cost of connecting to 0_2.

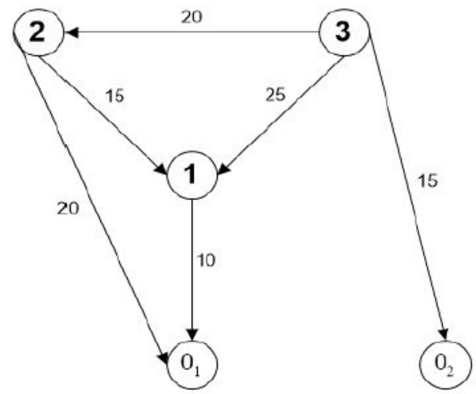

Fig. 3: The forest situation of Example 3

4. Cooperative cost games

In this section, we show that the games introduced for forest situations have non-empty cores. Let $< N, \{0_i\}, A, w >$ be a forest situation. Then the corresponding cost game $< N, c >$ is given by $c(\emptyset) = 0$ and for $T \in 2^N \setminus \{\emptyset\}$ the cost $c(T)$ of coalition T is the cost of the optimal $0-$connecting forest in the forest problem $< T, \{0\}, A(T), w_T >$, where

$$A(T) = \{(i,j) \in A \mid i \in T, j \in T \cup \{0\}\},$$

and $w_T : A(T) \to \mathbb{R}$ is the restriction of $w : A \to \mathbb{R}$ to $A(T)$. For the determination of $c(T)$ only forests are considered which do not contain nodes outside $T \cup \{0\}$. Note that for each $T \in 2^N \setminus \{\emptyset\}$,

$$c(T) = \sum_{k \in T} w(k, b_T(k)),$$

where

$$b_T(k) = \underset{l \in T \cup \{0\} : (k,l) \in A}{\operatorname{argmin}} w(k,l),$$

the cheapest connection point of k in $T \cup \{0\}$. The introduced number $b(k)$ in Section 3 is equal to $b_N(k)$. One core element of $< N, c >$ can be easily described by taking the Bird allocation $B(N, \{0\}, A, w) \in \mathbb{R}^N$ with $B_k(N, \{0\}, A, w) = w(k, b_N(k))$. Then $B(N, \{0\}, A, w)$ is a core element of $< N, c >$, since

$$c(N) = \sum_{k \in N} w(k, b_N(k)) = \sum_{k \in N} B_k(N, \{0\}, A, w)$$

by Theorem 1. Further,

$$c(T) = \sum_{k \in T} w(k, b_T(k)) \geq \sum_{k \in T} w(k, b_N(k)) = \sum_{k \in T} B_k(N, \{0\}, A, w)$$

for each $T \in 2^N \setminus \{\emptyset\}$. This core element corresponds to the situation where the player $b_N(k)$ to which k connects himself does not ask a compensation for this service to k. Further, there are other interesting core allocations, corresponding to the situations where compensation plays a role. In the description of these core elements, the second cheapest connection point of k in $T \cup \{0\}$,

$$s_T(k) = \begin{cases} \underset{l \in (T \cup \{0\}) \setminus \{b_T(k)\} : (k,l) \in A}{\operatorname{argmin}} w(k,l), & \text{if } b_T(k) \neq 0, \\ 0, & \text{if } b_T(k) = 0, \end{cases}$$

plays a role.

Suppose that the player k wants to connect to $b_N(k) \neq 0$ and the player $b_N(k)$ wants to ask a price $p_k \geq 0$ from k for connecting k. The question is which price $b_N(k)$ can ask for his service to k such that k connects with $b_N(k)$ and does not go, e.g., to the second best connection point $s_N(k)$ for a connection. The price should be an element of the closed $[0, w(k, s_N(k)) - w(k, b_N(k))]$. A price p_k larger than $w(k, s_N(k)) - w(k, b_N(k))$ can lead to a connection to $s_N(k)$ and if $s_N(k) \neq 0$ even to a positive compensation for $s_N(k)$, e.g., $\frac{1}{2}(p_k - w(k, s_N(k)) + w(k, b_N(k)))$, and then both players k and $s_N(k)$ are better off. The allocations $(x_1, ..., x_n)$ corresponding to such competitive prices in the given closed turn out to be just the core allocations of the k-connection game $< N, c >$ to be introduced now.

The k-connection game $< N, c >$ is the cooperative cost game with $c_k(s) = 0$ if $k \notin S$ and $c_k(s) = w(k, b_S(k))$ otherwise. Notice that, if $b_N(k) \neq 0$, then

$$M_{b_N(k)}(N, c_k) = c_k(N) - c_k(N \backslash \{b_N(k)\}) = w(k, b_N(k)) - w(k, s_N(k)).$$

It is easy show that the proof of the following theorem.

Theorem 2. *Let $< N, c_1 >, ..., < N, c_n >$ be the connection games corresponding to the forest situation $< N, \{0_i\}, A, w >$ and $< N, c >$ the corresponding cost game. Then,*

(i) $c = \sum_{k=1}^{n} c_k$,

(ii) for every $T \in 2^N \backslash \{\emptyset\}$,

$$\mathcal{C}(T, c_k) = \begin{cases} 0, & \text{if } k \notin T, \\ w(k, b_T(k)) e^k - p(e^{b_T(k)} - e^k), & \text{if } k \in T, b_T(k) \neq 0, \\ w(k, 0) e^k, & \text{if } k \in T, b_T(k) = 0, \end{cases}$$

where

$$0 \leq p \leq w(k, s_T(k)) - w(k, b_T(k)).$$

Here, $e^k \in \mathbb{R}^T$ is the k-th standard basis vector with k-th coordinate 1 *and the other coordinates* 0.

Example 3. Consider again the forest situation in Example 2. The cost game $< N, c >$ corresponds to the situation and the related k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$c(S) =$	10	20	30	25	35	40	45
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	30	0	25	20	20

It is obvious that $c = c_1 + c_2 + c_3$. We have;
$\mathcal{C}(N, c_1) = \{(10, 0, 0)\}$, $\mathcal{C}(N, c_2) = \{(0, 15, 0), (-5, 20, 0)\}$, and
$\mathcal{C}(N, c_3) = \{(0, 0, 20), (0, -5, 25)\}$.

Example 4. Consider again the forest situation in Example 3. The cost game $< N, c >$ corresponds to the situation and the related k-connection games are given in the next table:

$S =$	(1)	(2)	(3)	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$
$c(S) =$	10	20	15	25	25	35	40
$c_1(S) =$	10	0	0	10	10	0	10
$c_2(S) =$	0	20	0	15	0	20	15
$c_3(S) =$	0	0	15	0	15	15	15

It is obvious that $c = c_1 + c_2 + c_3$. We have;
$\mathcal{C}(N, c_1) = \{(10, 0, 0)\}$, $\mathcal{C}(N, c_2) = \{(0, 15, 0), (-5, 20, 0)\}$, and
$\mathcal{C}(N, c_3) = \{(0, 0, 15)\}$.

5. Cost monotonic solutions of the forest situations: Shapley value and the Bird rule

Now, we turn to the second basic question in this paper: "How to allocate the connection costs in such a forest among the agents?" This question is approached with the aid of solution concepts in cooperative game theory. A solution concept gives an answer to the question of how the rewards (cost savings) obtained when all players in N cooperate should be distributed among the individual players while taking account of the potential rewards (cost savings) of all different coalitions of players.

Monotonicity is a general principle of fair division which states that as the underlying data of a problem change, the solution should change in parallel fashion. It is particularly germane to applications in which allocations are not made once and for all, but are reassessed periodically as new information emerges. This is the case, for example, in dividing the joint benefits or costs of a cooperative enterprise fairly among the partners when the underlying structure of the enterprise is evolving over time. Such a situation can be modelled by a cooperative game. The principle of monotonicity for cooperative games states that if a game changes so that some player's contribution to all coalitions increases or stays the same then the player's allocation should not decrease. There is a unique symmetric and efficient solution concept that is monotonic in this most general sense - the *Shapley value.*

The Shapley value (Shapley, 1953), one of the most interesting one-point solution concepts in cooperative game theory, is introduced and characterized for cooperative games with TU-games with a finite player set and where coalition values are real numbers. Subsequently, it has captured much attention being extended in new game theoretic models and widely applied for solving reward/cost sharing problems in Operations Research (OR) and economic situations. The Shapley value associates to each cooperative TU-game one payoff vector whose components are real numbers.

To be more precise, the Shapley value associates to each game $c \in G^N$ one payoff vector in $x \in \mathbb{R}^N$. For a very extensive and interesting discussion on this value the reader is referred to (Roth, 1988). The first formulation of the Shapley value uses the marginal vectors of a cooperative TU-game.

Definition 1. The **Shapley value** $\Phi(c)$ of a game $c \in G^N$ is the average of the marginal vectors of the game, i.e.,

$$\Phi(c) := \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^{\sigma}(c). \tag{5.1}$$

With the aid of (5.1) one can provide a *probabilistic* interpretation of the Shapley value as follows. Suppose we draw from an urn, containing the elements of $\pi(N)$, a permutation σ (with probability $1/(n!)$). Then we let the players enter a room one by one in the order σ and give each player the marginal contribution created by him. Then, for each $i \in N$, the $i-$th coordinate $\Phi_i(c)$ of $\Phi(c)$ is the expected payoff of player i according to this random procedure.

By using Definition 7, one can rewrite (5.1) obtaining

$$\Phi_i(c) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} (c(P^{\sigma}(i) \cup \{i\}) - c(P^{\sigma}(i))). \tag{5.2}$$

We simply write $c(i)$ instead of $c(\{i\})$ and $c(ij)$ instead of $c(\{i,j\})$ along this paper.

Example 5. Consider again the forest situation in Example 2. In such a situation, $N = \{1,2,3\}$, $c(1) = 10, c(2) = 20, c(3) = 30, c(12) = 25, c(13) = 35, c(23) = 40$ and $c(123) = 45$. Then, the Shapley value is the average of the vectors $(10,15,20)$, $(10,10,25)$, $(5,20,20)$, $(5,20,20)$, $(5,10,30)$ and $(5,10,30)$, i.e.,

$$\Phi(c) = \left(\frac{40}{6}, \frac{85}{6}, \frac{145}{6}\right).$$

On other hand, cost allocation scheme, which coincides with the Shapley value of the cost game, is an example of a population monotonic allocation scheme (pmas), i.e.,

	1	2	3
N	40/6	85/6	145/6
(12)	45/6	105/6	$*$
(13)	45/6	$*$	165/6
(23)	$*$	90/6	150/6
(1)	60/6	$*$	$*$
(2)	$*$	120/6	$*$
(3)	$*$	$*$	180/6

As we can see that the cost allocation rule, which coincides with the Shapley value of the cost game, satisfies cost monotonicity (Kent and Skorin-Kapov, 1997). Here, a cost allocation rule is called *cost monotonic* if the decrease (or increase) in the cost of any arc does not increase (or decrease) the cost of any player. On the contrary, the Bird rule does not satisfy cost monotonicity.

However, the Bird rule, which assigns to each forest situation to the corresponding cost game, satisfies interesting monotonicity property, called cost monotonicity. This can also be explained by the concavity of the game. Recall that, the cooperative cost game corresponding to a forest situation is concave (Tijs, 2003).

Suppose a forest situation $< N, \{0_i\}, A, w >$ changes to $< N, \{0_i\}, A, w' >$, where $w'(i,j) = w(i,j)$ for all $(i,j) \in A \setminus \{(k,l)\}$ and $w'(k,l) > w(k,l)$. Suppose that B and B' are the corresponding Bird allocations. Then, obviously, $B_i = B'_i$ for all $i \in N \backslash \{k\}$, and $B_k = w(k, b(k)) = B'_k$ if $b(k) \neq l$, while $B'_k > B_k$ if $b(k) = l$. So the Bird rule is cost monotonic. The following examples illustrate this result.

Example 6. Consider again the forest situation in Example 2. The Bird rule assings to the forest situation the allocation $(10, 15, 20)$. If we change the forest situation in this example such that the cost of $(3, 2)$ raises to 40 then we obtain the Bird allocation $B(N, \{0_i\}, A, w) = (10, 15, 25)$. It is easy to show that the Bird rule is cost monotonic.

Now, we give to the Bird allocation scheme for the forest situation in Example 2.

Example 7. Consider again the forest situation in Example 2. In such a situation, $N = \{1, 2, 3\}$, $c(1) = 10, c(2) = 20, c(3) = 30, c(12) = 25, c(13) = 35, c(23) = 40, c(123) = 45$. Then the Bird allocation to the forest situation, looks as follows:

	1	2	3
N	10	15	20
(12)	10	15	$*$
(13)	10	$*$	25
(23)	$*$	20	20
(1)	10	$*$	$*$
(2)	$*$	20	$*$
(3)	$*$	$*$	30

On other hand, we can see that the Bird allocation scheme is an example of a population monotonic allocation scheme (pmas).

6. Conclusion and Outlook

We studied optimal connection problems and related cost sharing problems for forest situations with the properties F.1, F.2 and F.3. In this context, we show that the Bird allocation is a special core element of the related cost game, corresponding to the forest situation. We deal with cost monotonic allocation rules for forest situations. The Bird rule and the Shapley value play here a special role. Further, it is shown that the Bird allocation and the Shapley value are examples of a pmas for the cost game, corresponding to the forest situation.

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Two Approaches for Solving a Group Pursuit Game

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Abstract In this paper we study a game of group pursuit in which players move on a plane with bounded velocities. The game is supposed to be a nonzero-sum simple pursuit game between a pursuer and m evaders acting independently of each other. The case of complete information is considered. Here we assume that the evaders are discriminated. Two different approaches to formalize this pursuit problem are considered: noncooperative and cooperative. In a noncooperative case we construct a Nash equilibrium, and in a cooperative case we construct the core. We proved that the core is not empty for any initial positions of the players.

Keywords: group pursuit game, Nash equilibrium, realizability area, TU-game, core.

1. Introduction

The process of pursuit represents a typical conflict situation. When only two players are involved in the process of pursuit we deal with a classical zero-sum differential pursuit game. These games grew out of the military problems and were developed by Isaaks (1965).

When more than two players participate in a game and the players' objectives are not strictly opposite it is rather reasonable to consider such a game as a nonzero-sum one. This approach for solving a group pursuit problem was introduced in (Petrosjan and Shirjaev, 1981) and further applied in works (Tarashnina, 1998), (Pankratova and Tarashnina, 2004).

It is obvious that players' goals are not always strictly opposed. We want to illustrate how differential games can be used for solving different kind of problems. In this case under "capture" we can understand just meeting of players and delivering some goods or information. In terms, players are not aimed to destruct each other. Moreover, players in a nonzero-sum game may cooperate with each other to get a maximal profit.

We investigate a nonzero-sum group pursuit game using two different approaches. We construct a game in normal form and its TU-cooperative version and find their solutions.

2. Nonzero-sum group pursuit game

In the work we study a game of pursuit in which n players $-$ the pursuer P and evaders $E_1, \dots, E_m$ $-$ move on a plane with constant velocities with the possibility of changing the direction of their motion at each time instant (simple motion). We consider the case of complete information. This means that each player at each time instant $t \geq 0$ knows the moment t and his own as well as all other player's positions. Additionally, we assume that the pursuer uses strategies with discrimination against the evaders. This means that at each instant t the pursuer P knows the vector-speeds chosen by the evaders at that time moment.

The players start their motion at moment $t = 0$ at the initial positions

$$z_P^0 = (x_P^0, y_P^0), \quad z_i^0 = (x_i^0, y_i^0), \quad i = \overline{1, m}.$$

Let α and β_i are velocities of P and E_i $(i = \overline{1, m})$, respectively. Suppose that $\alpha > \max\limits_{i=1,\dots,m} \beta_i$. Denote by $E_i^t = z_i^t = (x_i^t, y_i^t)$ and $P^t = z_P^t = (x_P^t, y_P^t)$ the current positions of evader E_i and pursuer P at the moment $t > 0$, respectively.

The motion of players is described by the following system of differential equations

$$\begin{aligned} &\dot{z}_P = \mathbf{u}_P, \quad \mathbf{u}_P \in U_P, \\ &\dot{z}_i = \mathbf{u}_{E_i}, \ \mathbf{u}_{E_i} \in U_{E_i}, \ i = \overline{1, m}, \end{aligned} \tag{1}$$

with initial conditions

$$z_P(0) = z_P^0, \quad z_i(0) = z_i^0, \quad i = \overline{1, m}, \tag{2}$$

where $z_P, z_1, \dots, z_m \in \mathbf{R}^2$. The vectors $\mathbf{u}_P \in U_P$ and $\mathbf{u}_{E_i} \in U_{E_i}$ are control variables of P and E_i $(i = \overline{1, m})$, respectively. The set of control variables U_P, U_{E_i} have the following forms

$$\begin{aligned} U_P &= \{\mathbf{u}_P = (u_P^1, u_P^2) : (u_P^1)^2 + (u_P^2)^2 = \alpha^2\}, \\ U_{E_i} &= \{\mathbf{u}_{E_i} = (u_{E_i}^1, u_{E_i}^2) : (u_{E_i}^1)^2 + (u_{E_i}^2)^2 = \beta_i^2\}, \quad i = \overline{1, m}. \end{aligned}$$

We need to explain how the players choose their control variables throughout the game according to the incoming information. Define a strategy of the evader as a function of time and current positions of the players. A strategy of player E_i is a function $u_{E_i}(t, z_P^t, z_1^t, \dots, z_m^t)$ with values in U_{E_i}. The evaders use piecewise open-loop strategies. Denote by $\mathcal{U}_{E_i}$ the set of admissible strategies of player E_i, $i = \overline{1, m}$.

A strategy of player P is a function of time, players' positions and velocity-vectors of the evaders, i.e.

$$u_P(t, z_P^t, z_1^t, \dots, z_m^t, \mathbf{u}_{E_1}^t, \dots, \mathbf{u}_{E_m}^t).$$

That means, that the class of admissible strategies of the pursuer consists of strategies with discrimination (counterstrategies).

The game is played as follows: at the initial time instant the pursuer dictates to the evaders $E_1, \dots, E_m$ a certain behaviour and chooses some pursuit order. In other words, the pursuer fixes some pursuit order and calculates the total pursuit time taking into account that the evaders use the prescribed behaviour. After that, P consequently pursues the evaders according to the chosen order and changes it as soon as any of the evader chooses a direction of motion different from the one dictated by the pursuer. So, the pursuer punishes deviated evader, changing pursuit order by starting the pursuit of defected evader. If group of evaders is deviated then the pursuer punishes anyone of this group.

Let $\mathbf{\Pi}$ be the set of all possible orders. Now we define a notion of a punishment strategy of the pursuer.

Definition 1. We say that the triple $u_P^\pi = \langle \pi, u_P, p \rangle$ is a punishment strategy of pursuer P with

- $\pi(z_P^0, z_1^0, \ldots, z_m^0, u_{E_1}, \ldots, u_{E_m})$ is a pursuit order chosen by the pursuer at the initial instant $t = 0$ for some fixed strategy profile of the evaders $u_{E_1}, \ldots, u_{E_m}$;
- $u_P(t, z_P^t, z_1^t, \ldots, z_m^t, \mathbf{u}_{E_1}^t, \ldots, \mathbf{u}_{E_m}^t)$, $t \geq 0$, is a pursuit strategy of P that consists in consequent pursuit of the evaders according to the chosen order;
- $p = p(t, \mathbf{u}_{E_1}^t, \ldots, \mathbf{u}_{E_m}^t)$ is an element of punishment that consists in changing the pursuit order at the moment t by starting the pursuit of defected evader in case any of the evaders chooses a direction of motion different from $(\mathbf{u}_{E_1}^t, \ldots, \mathbf{u}_{E_m}^t)$ dictated by the pursuer.

Denote by $\mathcal{U}_P = \{u_P^\pi\}_{\pi \in \mathbf{\Pi}}$ the set of punishment strategies of the pursuer.

Evader E_i is considered caught if the positions of P and E_i coincide at some time instant. We say that the game is over if the pursuer captures all the evaders.

Let $\pi = \{1, \ldots, i, \ldots, m\}$ be a pursuit order chosen by pursuer P.

Denoting by K_P the payoff function of P, and by K_{E_i} the payoff function of evader E_i, $i = \overline{1,m}$, we have

$$K_{E_i}(u_P^\pi, u_{E_1}, \ldots, u_{E_i}, \ldots, u_{E_m}) = \sum_{k \leq i,\ k=\overline{1,m}} T_k^\pi, \tag{3}$$

where T_k^π is the time spent by the pursuer for capture of the evader E_k $(k = \overline{1,m})$ minus time according to the pursuit order $\pi \in \mathbf{\Pi}$. Here i is a number of the evader E_i in the pursuit order $\pi = \{1, \ldots, i, \ldots, m\}$ and k $(k \leq i)$ is a number of the evader which is pursued before E_i inclusively.

The payoff of P is defined as the negative value of the payoff of evader E_i that is caught last. Thus,

$$K_P(u_P^\pi, u_{E_1}, \ldots, u_{E_m}) = -T^\pi, \tag{4}$$

where $T^\pi = \sum_{k=1}^{m} T_k^\pi$ is the total pursuit time, and π is the chosen pursuit order.

So, we define the nonzero-sum pursuit game as a normal form game as follows

$$\Gamma(z_P^0, z_1^0 \ldots, z_m^0) = \langle N, \{\mathcal{U}_i\}_{i \in N}, \{K_i\}_{i \in N} \rangle, \tag{5}$$

where $N = \{P, E_1, \ldots, E_m\}$ is the set of players, $\mathcal{U}_i$ is the set of admissible strategies of player i and K_i is a payoff function of player i $(i \in N)$, defined by (3) and (4). Each game depends on a choice of the initial positions of the players. Let us fix the players' initial positions and consider the game $\Gamma(z_P^0, z_1^0, \ldots, z_m^0)$.

3. Nash equilibrium in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$

In nonzero-sum games there is a number of solution concepts that are based on some additional assumptions for players' behaviour and structure of the game. One of them is the well-known concept of Nash equilibrium. In considered game there exists a whole family of Nash equilibria that includes some which are extremely adverse to the evaders' interests, and some which are favorable for them, as well as all intermediate equilibria. Different kind of Nash equilibria in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$ were constructed in (Petrosjan and Shirjaev, 1981), (Tarashnina, 1998), (Pankratova and Tarashnina, 2004). In this game we consider the extremely odd Nash equilibrium that is the most disadvantageous for the evaders among all the equilibria.

The strategy set of pursuer P consists of u_P^π corresponding to the pursuit order $\pi \in \mathbf{\Pi}$. The pursuer aims to minimize the total pursuit time and each evader wants to avoid his own capture as long as possible and does not care about the other evaders. Denote by π^* the pursuit order which minimizes the total pursuit time and by $u_P^{\pi^*}$ the corresponding strategy of the pursuer.

Let $E_i^{j'}$ be the evader who is currently pursued, $j' \in \{1, \dots, m\}$. E_i^j is the j-th in the line of pursuit evader among the ones not yet caught, $j \in \{1, \dots, m\}$, $j > j'$.

Now let us describe two types of behaviours of evader E_i $(i = \overline{1, m})$:

- $E_i^{j'}$, $j' \in \{1, \dots, m\}$, uses behaviour $[u_{E_i}^{j'}]$ that prescribes to move along the straight line connecting his own and the pursuer's current positions in the direction from P (to the current capture point $N^{j'}$).
- E_i^j, $j \in \{1, \dots m\}$, uses behaviour $[u_{E_i}^j]$ that prescribes to move along the straight line to the capture point of the currently pursued evader $E_i^{j'}$, $j > j'$, namely, to the current capture point $N^{j'}$, where $N^{j'} = P^{T_{j'}^\pi}$.

It is obvious that throughout the game at some moment $t_{E_i} > 0$ each evader E_i changes its type from E_i^j into $E_i^{j'}$. So, the strategy $u_{E_i}^*(t, \cdot)$ of evader E_i $(i = \overline{2, m})$ can be describe as

$$u_{E_i}^*(t, \cdot) = \begin{cases} [u_{E_i}^j], & 0 \le t < t_{E_i}, \\ [u_{E_i}^{j'}], & t \ge t_{E_i}. \end{cases}$$

During the game each evaders, accept E_1, consequently applies both types of behaviours. The player E_1 uses just type $[u_{E_i}^{j'}]$, i.e. his strategy is $u_{E_1}^*(t, \cdot) = [u_{E_1}^{j'}]$, $t \ge 0$.

Suppose that $T^0 = 0$, $N^0 = P^0 = (0, 0)$.

The following theorem (Tarashnina, 1998) defines the conditions that support the described Nash equilibrium $(u_P^{\pi^*}, u_{E_1}^*, \dots, u_{E_m}^*)$ in the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$.

Theorem 1. *In the game $\Gamma(z_P^0, z_1^0, \dots, z_m^0)$ in case the conditons*

$$T^{\pi^*} = \min_{\pi \in \mathbf{\Pi}} T^\pi, \tag{6}$$

and

$$\frac{\alpha - \beta_i}{\alpha - \beta_{i-1}} \left| N^{i-2} E_{i-1}^{T_{i-2}} \right| + \left| N^{i-1} E_i^{T_{i-1}} \right| > \left| N^{i-2} E_i^{T_{i-2}} \right|, \quad i = \overline{2, m}. \tag{7}$$

hold for all $i = \overline{1, m}$ *there exists a Nash equilibrium that is constructed as follows:*

1. *E_i $(i = \overline{1,m})$ chooses the strategy $u^*_{E_i}$ that dictates to him*
   - *according to the behaviour $[u^{j'}_{E_i}]$ to move along the straight line connecting current positions $E^{j'}_i$ and P at the moment T_{i-1} in the direction from P if $i = j'$, where $j' \in \{1, ..., m\}$;*
   - *according to the behaviour $[u^{j}_{E_i}]$ to move along the straight line to the capture point of the currently pursued evader $E^{j'}_i$, $j' \in \{1, ..., m\}$, $j > j'$, i.e. to point $N^{j'}$, where $N^{j'} = P^{T_{j'}}$, if $i = j$, where $j > j'$.*
2. *P chooses the strategy $u^{\pi^*}_P$ that minimizes the total pursuit time if each E_i $(i = \overline{1,m})$ adheres to the strategy $u^*_{E_i}$, and P changes the pursuit order as soon as any of the evaders E^j_i $(i > j')$ that are not yet caught deviates from the strategy $u^*_{E_i}$ and pursuits the deviated evader the first.*

Now we introduce the notion of a realizability area. For this purpose we associate with each evader E_i an area Ω_i of initial positions of evaders that support the Nash equilibrium and refer to it as the realizability area of the punishment strategy of the pursuer with respect to evader E_i, $(i = \overline{1,m})$.

In words, area Ω_i is the set of all E_i's initial positions such that, when there, evader E_i has to adhere the strategy $u^*_{E_i}$ dictated to him by the pursuer.

Definition 2. The punishment strategy of pursuer P is called realizable with respect to evader E_i, if the life time of evader E_i, $i \in \{2, \dots, m\}$, in case E_i adheres to the strategy $u^*_{E_i}$ is larger then if E_i deviates, i. e. inequality (7) holds for fixed i.

Definition 3. The punishment strategy of pursuer P is called realizable in the game $\Gamma(z^0_P, z^0_1, \dots, z^0_m)$ if inequality (7) holds for all $i = \overline{2,m}$.

In (Pankratova and Tarashnina, 2004) some illustrative examples for constructing of realizability areas of the punishment strategy are have been presented.

4. Cooperative pursuit game $\Gamma_v(z^0_P, z^0_1, \dots, z^0_m)$

Let us suppose that the players in the game can form a coalitions. Construct a cooperative game between pursuer P and evaders E_1, ..., E_m in assumption the players use the strategies described in the previous paragraph without the threat of punishment.

Assume that utility of any player is transferable.

Let $2^{\mathbf{N}}$ be the set of all subsets of $\mathbf{N}$. The function $v\colon 2^N \to \mathbf{R}^1$ with the following two properties

1. $v(\emptyset)$=0, where $\emptyset$ is an empty set,
2. $v(\mathbf{S} \cup \mathbf{R}) \geq v(\mathbf{S}) + v(\mathbf{R})$ for all $\mathbf{R}, \mathbf{S} \subset \mathbf{N}$ with $\mathbf{S} \bigcap \mathbf{R} = \emptyset$,

is called *the characteristic function* of the game . Condition 2 is a superadditivity property.

For any coalition $\mathbf{S} \subset \mathbf{N}$ we define the characteristic function as follows

$$v(\mathbf{S}) = \max_{u_{\mathbf{S}}} \min_{u_{\mathbf{N}\setminus\mathbf{S}}} \sum_{i \in \mathbf{S}} K_i\big(u_{\mathbf{S}}, u_{\mathbf{N}\setminus\mathbf{S}}\big),$$

where $u_{\mathbf{S}}$ and $u_{\mathbf{N}\setminus\mathbf{S}}$ are vectors of admissible strategies of the coalitions $\mathbf{S}$ and $\mathbf{N}\setminus\mathbf{S}$, respectively. Using this approach, we construct the characteristic function v for the game $\Gamma(z^0_P, z^0_1, \dots, z^0_m)$.

Consider an arbitrary permutation π of the ordered set of indexes $M = \{1, 2, \ldots, m\}$. With this permutation we associate a substitution k_π, i.e. $k_\pi : M \to M$. This means that $k \in M$ goes to $k_\pi \in M$ in permutation π.

The characteristic function of the game has the form

$$v(\{E_{i_1}\}; z_P^0, z_1^0, \ldots, z_m^0) = \min_{\pi \in \mathbf{\Pi}} \left\{ \sum_{k_\pi \le i_1} T_{k_\pi}^\pi \right\},\ i_1 = \overline{1, m}.$$

$v(\{E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \ldots, z_m^0) = \min_{\pi \in \mathbf{\Pi}} \left\{ \sum_{k_\pi \le i_1} T_{k_\pi}^\pi + \sum_{k_\pi \le i_2} T_{k_\pi}^\pi \right\}$, where $i_1, i_2 = \overline{1, m}$, $i_1 \ne i_2$.

$v(\{E_{i_1}, E_{i_2}, E_{i_3}\}; z_P^0, z_1^0, \ldots, z_m^0) = \min_{\pi \in \mathbf{\Pi}} \left\{ \sum_{k_\pi \le i_1} T_{k_\pi}^\pi + \sum_{k_\pi \le i_2} T_{k_\pi}^\pi + \sum_{k_\pi \le i_3} T_{k_\pi}^\pi \right\}$, where $i_1, i_2, i_3 = \overline{1, m}$, $i_1 \ne i_2 \ne i_3$.

...

$$v(\{E_1, \ldots, E_m\}; z_P^0, z_1^0, \ldots, z_m^0) = \min_{\pi \in \mathbf{\Pi}} \left\{ T_{i_1}^\pi + \sum_{k_\pi = i_1}^{i_2} T_{k_\pi}^\pi + \ldots + \sum_{k_\pi = i_1}^{i_{m-1}} T_{k_\pi}^\pi + \sum_{k_\pi = i_1}^{i_m} T_{k_\pi}^\pi \right\}.$$

$$v(\{P\}; z_P^0, z_1^0, \ldots, z_m^0) = \max_{\pi \in \mathbf{\Pi}} \{-T^\pi\}.$$

$$v(\{P, E_{i_1}\}; z_P^0, z_1^0, \ldots, z_m^0) = \max_{\pi \in \mathbf{\Pi}} \left\{ -T^\pi + \sum_{k_\pi \le i_1} T_{k_\pi}^\pi \right\} = 0, \text{ where } i_1 = \overline{1, m}.$$

$v(\{P, E_{i_1}, E_{i_2}\}; z_P^0, z_1^0, \ldots, z_m^0) = \max_{\pi \in \mathbf{\Pi}} \left\{ -T^\pi + \sum_{k_\pi \le i_1} T_{k_\pi}^\pi + \sum_{k_\pi \le i_2} T_{k_\pi}^\pi \right\}$, where $i_1, i_2 = \overline{1, m}$, $i_1 \ne i_2$.

...

$v(\{P, E_1, \ldots, E_m\}; z_P^0, z_1^0, \ldots, z_m^0) =$

$$= \max_{\pi \in \mathbf{\Pi}} \left\{ -T^\pi + T_{i_1}^\pi + \sum_{k=i_1}^{i_2} T_k^\pi + \ldots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi + \sum_{k=i_1}^{i_m} T_k^\pi \right\} =$$

$$= \max_{\pi \in \mathbf{\Pi}} \left\{ T_{i_1}^\pi + \sum_{k=i_1}^{i_2} T_k^\pi + \ldots + \sum_{k=i_1}^{i_{m-1}} T_k^\pi \right\}.$$

For simplicity denote by

$$
\begin{aligned}
&\hat{T}_{i_1} = \min_{\pi\in\mathbf{\Pi}}\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi}\},\\
&\hat{T}_{i_1 i_2} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T^\pi_{k_\pi}\right\},\\
&\hat{T}_{i_1 i_2 i_3} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T^\pi_{k_\pi} + \sum_{k_\pi\le i_3} T^\pi_{k_\pi}\right\},\\
&\vdots\\
&\hat{T}_{i_1 i_2\dots i_m} = \min_{\pi\in\mathbf{\Pi}}\left\{T^\pi_{i_1} + \sum_{k_\pi=i_1}^{i_2} T^\pi_{k_\pi} + \ldots + \sum_{k_\pi=i_1}^{i_{m-1}} T^\pi_{k_\pi} + \sum_{k_\pi=i_1}^{i_m} T^\pi_{k_\pi}\right\},\\
&\tilde{T} = \max_{\pi\in\mathbf{\Pi}}\{-T^\pi\},\\
&\tilde{T}_{i_1} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^\pi + \sum_{k_\pi\le i_1} T^\pi_{k_\pi}\right\},\\
&\tilde{T}_{i_1 i_2} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^\pi + \sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T^\pi_{k_\pi}\right\},\\
&\tilde{T}_{i_1 i_2 i_3} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^\pi + \sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T_{k_\pi} + \sum_{k_\pi\le i_3} T^\pi_{k_\pi}\right\},\\
&\vdots\\
&T^* = \max_{\pi\in\mathbf{\Pi}}\left\{T^\pi_{i_1} + \sum_{k_\pi=i_1}^{i_2} T^\pi_{k_\pi} + \ldots + \sum_{k_\pi=i_1}^{i_{m-1}} T_{k_\pi}\right\}.
\end{aligned}
\tag{8}
$$

The characteristic function v can be described in the following form

$$
\begin{aligned}
&v(\{P\}; z^0_P, z^0_1, \ldots, z^0_m) = \tilde{T},\\
&v(\{E_{i_1}\}; z^0_P, z^0_1, \ldots, z^0_m) = \hat{T}_{i_1}, \quad i_1 = \overline{1,m},\\
&v(\{P, E_{i_1}\}; z^0_P, z^0_1, \ldots, z^0_m) = \tilde{T}_{i_1} = 0,\ i_1 = \overline{1,m},\\
&v(\{E_{i_1}, E_{i_2}\}; z^0_P, z^0_1, \ldots, z^0_m) = \hat{T}_{i_1 i_2},\ i_1, i_2 = \overline{1,m},\ i_1 \ne i_2,\\
&v(\{P, E_{i_1}, E_{i_2}\}; z^0_P, z^0_1, \ldots, z^0_m) = \tilde{T}_{i_1 i_2},\ i_1, i_2 = \overline{1,m},\ i_1 \ne i_2,\\
&v(\{E_{i_1}, E_{i_2}, E_{i_3}\}; z^0_P, z^0_1, \ldots, z^0_m) = \hat{T}_{i_1 i_2 i_3},\ i_1, i_2, i_3 = \overline{1,m},\ i_1 \ne i_2 \ne i_3,\\
&v(\{P, E_{i_1}, E_{i_2}, E_{i_3}\}; z^0_P, z^0_1, \ldots, z^0_m) = \tilde{T}_{i_1 i_2 i_3},\ i_1, i_2, i_3 = \overline{1,m},\ i_1 \ne i_2 \ne i_3,\\
&\vdots\\
&v(\{E_1, \ldots, E_m\}; z^0_P, z^0_1, \ldots, z^0_m) = \hat{T}_{1\dots m},\\
&v(\{P, E_1, \ldots, E_m\}; z^0_P, z^0_1, \ldots, z^0_m) = T^*.
\end{aligned}
\tag{9}
$$

Here and then we will use following designation $v(\{E_{i_1}, \ldots, E_{i_k}\}; z^0_P, z^0_1, \ldots, z^0_m) = v(E_{i_1}, \ldots, E_{i_k})$ and $v(\{P, E_{i_1}, \ldots, E_{i_k}\}; z^0_P, z^0_1, \ldots, z^0_m) = v(P, E_{i_1}, \ldots, E_{i_k})$

Definition 4. The pair $\langle N, v(S; z^0_P, z^0_1, \ldots, z^0_m), S \subset N\rangle$, where N is the set of players, and v is the characteristic function defined by (8)-(9) is called a cooperative pursuit game in characteristic function form and denoted by $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$.

Example 1. let us construct the characteristic function for a pursuit game with a pursuer and three evaders according to formulas (8) and (9). Let $\alpha = 1$ and $\beta_i = \dfrac{1}{2}$, $i = 1, 2, 3$.

Fix the initial positions of the players: $P^0 = (0,0)$, $E_1^0 = (1,0)$, $E_2^0 = (-2,4)$, $E_3^0 = (5,7)$. Note that for the chosen initial positions of the players the punishment strategy of P is realizable. First of all, we compose a table with the players' payoffs for all pursuit orders $\pi_i \in \{1,\dots,6\}$.

Table 1: The players' payoffs for different pursuit orders.

Payoff	$\pi_1 = \{1,2,3\}$	$\pi_2 = \{1,3,2\}$
K_{E_1}	$T_1^{\pi_1} = 2$	$T_1^{\pi_2} = 2$
K_{E_2}	$T_{12}^{\pi_1} = 2 + 9{,}31 = 11{,}31$	$T_{132}^{\pi_2} = 2 + 13{,}23 + 11{,}34 = 26{,}57$
K_{E_3}	$T_{123}^{\pi_1} = 2 + 9{,}31 + 9{,}09 = 20{,}4$	$T_{13}^{\pi_2} = 2 + 13{,}23 = 15{,}23$
K_P	$T_P^{\pi_1} = 20{,}4$	$T_P^{\pi_2} = 26{,}57$
	$\pi_3 = \{2,3,1\}$	$\pi_4 = \{2,3,1\}$
K_{E_1}	$T_{21}^{\pi_3} = 8{,}94 + 9{,}92 = 18{,}86$	$T_{231}^{\pi_4} = 8{,}94 + 9{,}7 + 9{,}73 = 28{,}37$
K_{E_2}	$T_2^{\pi_3} = 8{,}94$	$T_2^{\pi_4} = 8{,}94$
K_{E_3}	$T_{213}^{\pi_3} = 8{,}94 + 9{,}92 + 26{,}03 = 44{,}89$	$T_{23}^{\pi_4} = 8{,}94 + 9{,}7 = 18{,}64$
K_P	$T_P^{\pi_3} = 44{,}89$	$T_P^{\pi_4} = 28{,}37$
	$\pi_5 = \{3,1,2\}$	$\pi_6 = \{3,2,1\}$
K_{E_1}	$T_{31}^{\pi_5} = 17{,}2 + 16{,}08 = 33{,}28$	$T_{321}^{\pi_6} = 17{,}2 + 14{,}04 + 27{,}23 = 58{,}47$
K_{E_2}	$T_{312}^{\pi_5} = 17{,}2 + 16{,}08 + 33{,}52 = 66{,}8$	$T_{31}^{\pi_6} = 17{,}2 + 14{,}04 = 31{,}24$
K_{E_3}	$T_3^{\pi_5} = 17{,}2$	$T_3^{\pi_6} = 17{,}2$
K_P	$T_P^{\pi_5} = 66{,}8$	$T_P^{\pi_6} = 58{,}47$

The characteristic function, according to formulas (8) and (9), has the following form

$$
\begin{aligned}
&v(P) = \max\{-20{,}4;\ -26{,}57;\ -44{,}89;\ -28{,}37;\ -66{,}8;\ -58{,}47\} = -20{,}4,\\
&v(E_1) = \min\{2;\ 2;\ 18{,}86;\ 28{,}37;\ 33{,}28;\ 58{,}47\} = 2,\\
&v(E_2) = \min\{11{,}31;\ 26{,}57;\ 8{,}94;\ 8{,}94;\ 66{,}8;\ 31{,}24\} = 8{,}94,\\
&v(E_3) = \min\{20{,}4;\ 15{,}23;\ 44{,}89;\ 18{,}64;\ 17{,}2;\ 17{,}2\} = 15{,}23,\\
&v(P,E_1) = \max\{-20{,}4 + 2;\ -26{,}57 + 2;\ -44{,}89 + 18{,}86;\\
&\qquad -28{,}37 + 28{,}37;\ -66{,}8 + 33{,}28; -58{,}47 + 58{,}47\} = 0,\\
&v(P,E_2) = \max\{-20{,}4 + 11{,}31;\ -26{,}57 + 26{,}57;\ -44{,}89 + 8{,}94;\\
&\qquad -28{,}37 + 8{,}94;\ -66{,}8 + 66{,}8;\ -58{,}47 + 31{,}24\} = 0,\\
&v(P,E_3) = \max\{-20{,}4 + 20{,}4;\ -26{,}57 + 15{,}23;\ -44{,}89 + 44{,}89;\\
&\qquad -28{,}37 + 18{,}64;\ -66{,}8 + 17{,}2;\ -58{,}47 + 17{,}2\} = 0,\\
&v(E_1,E_2) = \min\{2 + 11{,}31;\ 2 + 26{,}57;\ 18{,}86 + 8{,}94;\\
&\qquad 28{,}37 + 8{,}94;\ 33{,}28 + 66{,}8;\ 58{,}47 + 31{,}24\} = 13{,}31,\\
&v(E_1,E_3) = \min\{2 + 20{,}4;\ 2 + 15{,}23;\ 18{,}86 + 44{,}89;\\
&\qquad 28{,}37 + 18{,}64;\ 33{,}28 + 17{,}2;\ 58{,}47 + 17{,}2\} = 17{,}23,\\
&v(E_2,E_3) = \min\{11{,}31 + 20{,}4;\ 26{,}57 + 15{,}23;\ 8{,}94 + 18{,}64;\\
&\qquad 8{,}94 + 44{,}89;\ 66{,}8 + 17{,}2;\ 31{,}24 + 17{,}2\} = 27{,}58,\\
&v(P,E_1,E_2) = \max\{-20{,}4 + 2 + 11{,}31;\ -26{,}57 + 2 + 26{,}57;\\
&\qquad -44{,}89 + 18{,}86 + 8{,}94; -28{,}37 + 28{,}37 + 8{,}94;\\
&\qquad -66{,}8 + 33{,}28 + 66{,}8;\ -58{,}47 + 58{,}47 + 31{,}24\} = 33{,}28,\\
&v(P,E_1,E_3) = \max\{-20{,}4 + 2 + 20{,}4;\ -26{,}57 + 2 + 15{,}23;\\
&\qquad -44{,}89 + 18{,}86 + 44{,}89;\ -28{,}37 + 28{,}37 + 18{,}64;\\
&\qquad -66{,}8 + 33{,}28 + 17{,}2; -58{,}47 + 58{,}47 + 17{,}2\} = 18{,}86,
\end{aligned}
$$

$$\begin{aligned}
v(P,E_2,E_3) &= \max\{-20,4+11,31+20,4;\ -26,57+26,57+15,23;\\
&\qquad -44,89+8,94+44,89; -28,37+8,94+18,64;\\
&\qquad -66,8+66,8+17,2; -58,47+31,24+17,2\} = 17,2,\\
v(E_1,E_2,E_3) &= \min\{2+11,31+20,4;\ 2+26,57+15,23;\\
&\qquad 18,86+8,94+44,89; 28,37+8,94+18,64;\\
&\qquad 33,28+66,8+17,2;\ 58,47+31,24+17,2\} = 33,71,\\
v(P,E_1,E_2,E_3) &= \max\{-20,4+2+11,31+20,4;\\
&\qquad -26,57+2+26,57+15,23;\ -44,89+18,86+8,94+44,89;\\
&\qquad -28,37+28,37+8,94+18,64;\ -66,8+33,28+66,8+17,2;\\
&\qquad -58,47+58,47+31,24+17,2\} = 50,48.
\end{aligned}$$

Finelly, we construct a cooperative pursuit game in the characteristic function form. That is

$v(P) = -20,4,$
$v(E_1) = 2,$
$v(E_2) = 8,94,$
$v(E_3) = 15,23,$
$v(P,E_1) = 0,$
$v(P,E_2) = 0,$
$v(P,E_3) = 0,$
$v(E_1,E_2) = 13,31,$
$v(E_1,E_3) = 17,23,$
$v(E_2,E_3) = 27,58,$
$v(P,E_1,E_2) = 33,28,$
$v(P,E_1,E_3) = 18,86,$
$v(P,E_2,E_3) = 17,2,$
$v(E_1,E_2,E_3) = 33,71,$
$v(P,E_1,E_2,E_3) = 50,48.$

On this example we can see that the characteristic function of the game is superadditive.

Further we proof that it is true for any number of evaders and any initial positions of the players.

Theorem 2. *In the game $\Gamma_v(z_P^0, z_1^0, \ldots, z_m^0)$ the characteristic function v that is constructed by formulas (8) and (9) is superadditive.*

Proof. In order to prove the theorem we have to show that inequality

$$v(S) + v(T) \le v(S \cup T)$$

holds for all coalitions S, $T \subset N$, $S \cap T = \emptyset$.

In fact, the following inequalities are fulfilled.

$$\begin{aligned}
&\hat{T}_{i_1} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi}\right\} \le \hat{T}_{i_1 i_2} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T^\pi_{k_\pi}\right\} \le \hat{T}_{i_1 i_2 i_3} =\\
&= \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \sum_{k_\pi\le i_2} T^\pi_{k_\pi} + \sum_{k_\pi\le i_3} T^\pi_{k_\pi}\right\} \le \ldots \le \hat{T}_{i_1\ldots i_{m-1}} =\\
&= \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \ldots + \sum_{k_\pi\le i_{m-1}} T^\pi_{k_\pi}\right\} \le\\
&\le \hat{T}_{i_1\ldots i_m} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\le i_1} T^\pi_{k_\pi} + \ldots + \sum_{k_\pi\le i_m} T^\pi_{k_\pi}\right\}.
\end{aligned}$$

It can be easily shown that

$$-\tilde{T} = \max_{\pi\in\mathbf{\Pi}}\{-T^{\pi}\} \le \tilde{T}_{i_1 i_2} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_2} T^{\pi}_{k_\pi}\right\} \le \ldots \le$$
$$\le \tilde{T}_{i_1\ldots i_{m-1}} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \ldots + \sum_{k_\pi \le i_{m-1}} T^{\pi}_{k_\pi}\right\} \le$$
$$\le T^{*} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + T^{\pi}_{i_1} + \sum_{k=i_1}^{i_2} T^{\pi}_{k} + \ldots + \sum_{k=i_1}^{i_{m-1}} T^{\pi}_{k} + \sum_{k=i_1}^{i_m} T^{\pi}_{k}\right\}.$$

So, we have

$$\hat{T}_{i_1} \le \hat{T}_{i_1 i_2} \le \hat{T}_{i_1 i_2 i_3} \le \ldots \le \hat{T}_{i_1\ldots i_{m-1}} \le \hat{T}_{i_1\ldots i_m}, \tag{10}$$
$$-\tilde{T} \le \tilde{T}_{i_1 i_2} \le \tilde{T}_{i_1 i_2 i_3} \le \ldots \le \tilde{T}_{i_1 i_2\ldots i_{m-1}} \le T^{*}. \tag{11}$$

Let $S = \{P\}$ and $T = \{E_{i_1}\}$. Since $\tilde{T} = \min_{\pi\in\mathbf{\Pi}}\{T^{\pi}\} \ge \hat{T}_{i_1}$, we have $v(P)+v(E_{i_1}) = -\tilde{T} + \hat{T}_{i_1} \le 0 = v(P, E_{i_1}),\ i_1 = \overline{1,m}$.

For $S = \{E_{i_1}\}$ and $T = \{E_{i_2}\}$ we have $v(E_{i_1}) + v(E_{i_2}) = \hat{T}_{i_1} + \hat{T}_{i_2} \le \hat{T}_{i_1 i_2} = v(E_{i_1}, E_{i_2}),\ i_1, i_2 = \overline{1,m},\ i_1 \ne i_2$. This follows from (10).

For $S = \{P\}$ and $T = \{E_{i_1}, E_{i_2}\}$ we have $v(P) + v(E_{i_1}, E_{i_2}) = -\tilde{T} + \hat{T}_{i_1 i_2} =$

$$= \max_{\pi\in\mathbf{\Pi}}\{-T^{\pi}\} + \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_2} T^{\pi}_{k_\pi}\right\} =$$

$$= -T^{\pi^*} + \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_2} T^{\pi}_{k_\pi}\right\} \le -T^{\pi^*} + \sum_{k_{\pi^*} \le i_1} T^{\pi^*}_{k_{\pi^*}} + \sum_{k_{\pi^*} \le i_2} T^{\pi^*}_{k_{\pi^*}} \le$$

$$\le \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_2} T^{\pi}_{k_\pi}\right\} = \tilde{T}_{i_1 i_2} = v(P, E_{i_1}, E_{i_2}),$$

$i_1, i_2 = \overline{1,m}$, $i_1 \ne i_2$.

Now consider $S = \{E_{i_1}\}$ and $T = \{P, E_{i_2}\}$. Then

$$v(E_{i_1}) + v(P, E_{i_2}) = \hat{T}_{i_1} + \tilde{T}_{i_2} = \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_1} T^{\pi}_{k_\pi}\right\} + \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum_{k_\pi \le i_2} T^{\pi}_{k_\pi}\right\} =$$

$$= \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_1} T^{\pi}_{k_\pi}\right\} - T^{\pi^*} + \sum_{k_{\pi^*} \le i_2} T^{\pi^*}_{k_{\pi^*}} \le \sum_{k_{\pi^*} \le i_1} T^{\pi^*}_{k_{\pi^*}} - T^{\pi^*} + \sum_{k_{\pi^*} \le i_2} T^{\pi *}_{k_{\pi *}} \le$$

$$\le \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum_{k_\pi \le i_1} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_2} T_{k_\pi}\right\} = \tilde{T}_{i_1 i_2},\ i_1, i_2 = \overline{1,m},\ i_1 \ne i_2.$$

For $S = \{P, E_{i_1}\}$ and $T = \{E_{i_2}, E_{i_3}\}$ we have

$$v(P, E_{i_1}) + v(E_{i_2}, E_{i_3}) = \tilde{T}_{i_1} + \hat{T}_{i_2 i_3} = \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi} + \sum k_\pi \le i_1 T^{\pi}_{k_\pi}\right\} +$$

$$+ \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_2} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_3} T^{\pi}_{k_\pi}\right\} = -T^{\pi^*} + \sum_{k_{\pi^*} \le i_1} T^{\pi^*}_{k_{\pi^*}} + \min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi \le i_2} T^{\pi}_{k_\pi} + \sum_{k_\pi \le i_3} T^{\pi}_{k_\pi}\right\} \le$$

$$\le -T^{\pi^*} + \sum_{k_{\pi^*} \le i_1} T^{\pi^*}_{k_{\pi^*}} + \sum_{k_{\pi^*} \le i_2} T^{\pi^*}_{k_{\pi^*}} + \sum_{k_{\pi^*} \le i_3} T^{\pi^*}_{k_{\pi^*}} \le$$

$$\leq \max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi}+\sum_{k_\pi\leq i_1}T^{\pi}_{k_\pi}+\sum_{k_\pi\leq i_2}T^{\pi}_{k_\pi}+\sum_{k_\pi\leq i_3}T^{\pi}_{k_\pi}\right\}=$$

$$=\tilde{T}_{i_1i_2i_3}=v(P,E_{i_1},E_{i_2},E_{i_3}),\ i_1,i_2,i_3=\overline{1,m},\ i_1\neq i_2\neq i_3.$$

Now consider two coalitions each of which includes only evaders: $\mathbf{E}_l=\{E_{i_1},\ldots,E_{i_l}\}$ and $\mathbf{E}_s=\{E_{j_1},\ldots,E_{j_s}\}$, $\mathbf{E}_l\bigcap\mathbf{E}_s=\emptyset$, $i_k,j_q=\overline{1,m}$, $i_k\neq j_q$, $i_1\neq\ldots\neq i_l$, $j_1\neq\ldots\neq j_s$, $k=\overline{1,l}$ è $q=\overline{1,s}$. For this coalitions we get

$$v(E_{i_1},\ldots,E_{i_l})+v(E_{j_1},\ldots,E_{j_s})=\hat{T}_{i_1\ldots i_l}+\hat{T}_{j_1\ldots j_s}=$$

$$=\min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\leq i_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq i_l}T^{\pi}_{k_\pi}\right\}+\min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\leq j_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq j_s}T^{\pi}_{k_\pi}\right\}\leq$$

$$\leq\min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\leq i_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq i_l}T^{\pi}_{k_\pi}+\sum_{k_\pi\leq j_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq j_s}T^{\pi}_{k_\pi}\right\}=$$

$$=\hat{T}_{i_1\ldots i_l j_1\ldots j_s}=v(E_{i_1},\ldots,E_{i_l},E_{j_1},\ldots,E_{j_s}).$$

It remains to consider the coalitions $S=\{P,E_{i_1},\ldots,E_{i_l}\}$ and $T=\{E_{j_1},\ldots,E_{j_s}\}$, $i_k,\ j_q=\overline{1,m}$, $i_k\neq j_q$, $i_1\neq\ldots\neq i_l$, $j_1\neq\ldots\neq j_s$, $k=\overline{1,l}$ and $q=\overline{1,s}$.
Then

$$v(P,E_{i_1},\ldots,E_{i_l})+v(E_{j_1},\ldots,E_{j_s})=\tilde{T}_{i_1\ldots i_l}+\hat{T}_{j_1\ldots j_s}=$$

$$=\max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi}+\sum_{k_\pi\leq i_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq i_l}T^{\pi}_{k_\pi}\right\}+\min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\leq j_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq j_s}T^{\pi}_{k_\pi}\right\}=$$

$$=-T^{\pi^*}+\sum_{k_{\pi^*}\leq i_1}T^{\pi^*}_{k_{\pi^*}}+\ldots+\sum_{k_{\pi^*}\leq i_l}T^{\pi^*}_{k_{\pi^*}}+\min_{\pi\in\mathbf{\Pi}}\left\{\sum_{k_\pi\leq j_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq j_s}T^{\pi}_{k_\pi}\right\}\leq$$

$$\leq-T^{\pi^*}+\sum_{k_{\pi^*}\leq i_1}T^{\pi^*}_{k_{\pi^*}}+\ldots+\sum_{k_{\pi^*}\leq i_l}T^{\pi^*}_{k_{\pi^*}}+\sum_{k_{\pi^*}\leq j_1}T^{\pi^*}_{k_{\pi^*}}+\ldots+\sum_{k_{\pi^*}\leq j_s}T^{\pi^*}_{k_{\pi^*}}\leq$$

$$\leq\max_{\pi\in\mathbf{\Pi}}\left\{-T^{\pi}+\sum_{k_\pi\leq i_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq i_l}T^{\pi}_{k_\pi}+\sum_{k_\pi\leq j_1}T^{\pi}_{k_\pi}+\ldots+\sum_{k_\pi\leq j_s}T^{\pi}_{k_\pi}\right\}=$$

$$=\tilde{T}_{i_1\ldots i_l j_1\ldots j_s}=v(P,E_{i_1},\ldots,E_{i_l},E_{j_1},\ldots,E_{j_s}).$$

Finally, we consider $S=\{P\}$ and $T=\{E_{i_1},E_{i_2},\ldots,E_{i_m}\}$. Hence,
$v(P)+v(E_{i_1},E_{i_2},\ldots,E_{i_m})=-\tilde{T}+\hat{T}_{i_1i_2\ldots i_m}=$

$$\leq\max_{\pi\in\mathbf{\Pi}}\{-T^{\pi}\}+\min_{\pi\in\mathit{\Pi}}\left\{T^{\pi}_{i_1}+\sum_{k=i_1}^{i_2}T^{\pi}_{k}+\ldots+\sum_{k=i_1}^{i_{m-1}}T^{\pi}_{k}+\sum_{k=i_1}^{i_m}T^{\pi}_{k}\right\}=$$

$$\leq -T^{\pi^*} + \min_{\pi\in\Pi}\left\{T^{\pi}_{i_1} + \sum_{k=i_1}^{i_2} T^{\pi}_{k} + \ldots + \sum_{k=i_1}^{i_{m-1}} T^{\pi}_{k} + \sum_{k=i_1}^{i_m} T^{\pi}_{k}\right\} \leq$$

$$\leq -T^{\pi^*} + T^{\pi^*}_{i_1} + \sum_{k=i_1}^{i_2} T^{\pi^*}_{k} + \ldots + \sum_{k=i_1}^{i_{m-1}} T^{\pi^*}_{k} + \sum_{k=i_1}^{i_m} T^{\pi^*}_{k} \leq$$

$$\leq \max_{\pi\in\Pi}\left\{-T^{\pi} + T^{\pi}_{i_1} + \sum_{k=i_1}^{i_2} T^{\pi}_{k} + \ldots + \sum_{k=i_1}^{i_{m-1}} T^{\pi}_{k} + \sum_{k=i_1}^{i_m} T^{\pi}_{k}\right\} =$$

$$= T^* = v(P, E_1, \ldots, E_m).$$

This completes the proof.

It follows from the superadditivity of v that it is profitable for the players to form the maximal coalition N and obtain the maximal total payoff that is possible in the game.

There exist various methods for distribution of the total payoff between the players in a cooperative TU-game. In our paper we consider the core as a solution concept of the game.

5. The core in the game $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$

Let us describe the imputation set in the game $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$. Denote by $\xi = (\xi_P, \xi_{E_1}, \ldots, \xi_{E_m})$ an imputation in the game. The imputation set is defined as follows

$$E_v(z^0_P, z^0_1, \ldots, z^0_m) = \left\{\xi : \xi_{E_i} \geq \hat{T}_i, i \in \overline{1,m}, \xi_P \geq -\tilde{T}; \sum_{i\in N} \xi_i = T^*\right\}. \qquad (12)$$

From (Bondareva, 1963) and (Shapley, 1967) follows the result. For an imputation ξ to belong to the core of the game $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$ it is necessary and sufficient that the following system of inequalities holds

$$\begin{cases} \xi_P \geq \tilde{T}, & \\ \xi_{E_{i_1}} \geq \hat{T}_{i_1}, & i_1 = \overline{1,m}, \\ \xi_P + \xi_{E_{i_1}} \geq \tilde{T}_{i_1} = 0, & i_1 = \overline{1,m}, \\ \xi_{E_{i_1}} + \xi_{E_{i_2}} \geq \hat{T}_{i_1 i_2}, & i_1, i_2 = \overline{1,m},\ i_1 \neq i_2, \\ \xi_P + \xi_{E_{i_1}} + \xi_{E_{i_2}} \geq \tilde{T}_{i_1 i_2}, & i_1, i_2 = \overline{1,m},\ i_1 \neq i_2, \\ \xi_{E_{i_1}} + \xi_{E_{i_2}} + \xi_{E_{i_3}} \geq \hat{T}_{i_1 i_2 i_3}, & i_1, i_2, i_3 = \overline{1,m},\ i_1 \neq i_2 \neq i_3, \\ \xi_P + \xi_{E_{i_1}} + \xi_{E_{i_2}} + \xi_{E_{i_3}} \geq \tilde{T}_{i_1 i_2 i_3}, & i_1, i_2, i_3 = \overline{1,m},\ i_1 \neq i_2 \neq i_3, \\ \ldots & \\ \xi_P + \xi_{E_{i_1}} + \ldots + \xi_{E_{i_{m-1}}} \geq \tilde{T}_{i_1 \ldots i_{m-1}}, & i_1, \ldots, i_{m-1} = \overline{1,m},\ i_1 \neq \ldots \neq i_{m-1}, \\ \xi_{E_{i_1}} + \ldots + \xi_{E_{i_m}} \geq \hat{T}_{i_1 \ldots i_m}. & \end{cases} \qquad (13)$$

Denote by $C_v(z^0_P, z^0_1, \ldots, z^0_m)$ the core of the game $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$.

The following theorem holds.

Theorem 3. *In the cooperative pursuit game $\Gamma_v(z^0_P, z^0_1, \ldots, z^0_m)$ there exists the non-empty core for any initial positions of the players.*

Proof. First of all, we show that any imputation from the core satisfies the system (13). Suppose that imputation $\xi = (\xi_P, \xi_{E_1}, \ldots, \xi_{E_m})$ belongs to the core $C_v(z_P^0, z_1^0, \ldots, z_m^0)$. We have to show that system (13) is combined.

Summing the inequalities of system (13), we obtain

$$\begin{aligned}
&(1 + C_m^1 + C_m^2 + \ldots + C_m^{m-1}) \cdot (\xi_P + \xi_{E_1} + ... + \xi_{E_m}) \geq \tilde{T} + \sum_{k=1}^{m} \hat{T}_k + \\
&+ \sum_{k=1}^{m} \tilde{T}_k + \underbrace{(\hat{T}_{12} + \hat{T}_{13} + \ldots + \hat{T}_{m-1,m})}_{C_m^2} + \underbrace{(\tilde{T}_{12} + \tilde{T}_{13} + \ldots + \tilde{T}_{m-1,m})}_{C_m^2} + \\
&\quad + \underbrace{(\hat{T}_{123} + \hat{T}_{124} + \ldots)}_{C_m^3} + \underbrace{(\tilde{T}_{123} + \tilde{T}_{134} + \ldots)}_{C_m^3} + \\
&\quad + \underbrace{(\hat{T}_{1234} + \hat{T}_{1235} + \ldots)}_{C_m^4} + \underbrace{(\tilde{T}_{1234} + \tilde{T}_{1234} + \ldots)}_{C_m^4} + \\
&\qquad + \ldots + \underbrace{(\hat{T}_{12\ldots m-1} + \hat{T}_{12\ldots m-2m} + \ldots)}_{C_m^{m-1}} + \underbrace{(\tilde{T}_{12\ldots m-1} + \tilde{T}_{12\ldots m-2m} + \ldots)}_{C_m^{m-1}} + \\
&\qquad\qquad + \underbrace{\hat{T}_{12\ldots m}}_{C_m^m}.
\end{aligned} \tag{14}$$

Let us consider of the left part of inequality (14). Taking into account $v(N) = T^*$ and $1 + C_m^1 + \ldots + C_m^{m-1} = 2^m - 1$, we have

$$\begin{aligned}
&(2^m - 1) \cdot T^* \geq \sum_{k=1}^{m} \hat{T}_k + \sum_{k=1}^{m} \tilde{T}_k \\
&+ \underbrace{(\hat{T}_{12} + \hat{T}_{13} + \ldots + \hat{T}_{m-1,m})}_{C_m^2} + \underbrace{(\tilde{T}_{12} + \tilde{T}_{13} + \ldots + \tilde{T}_{m-1,m})}_{C_m^2} \\
&\quad + \underbrace{(\hat{T}_{123} + \hat{T}_{124} + \ldots)}_{C_m^3} + \underbrace{(\tilde{T}_{123} + \tilde{T}_{134} + \ldots)}_{C_m^3} + \\
&\qquad + \underbrace{(\hat{T}_{1234} + \hat{T}_{1235} + \ldots)}_{C_m^4} + \underbrace{(\tilde{T}_{1234} + \tilde{T}_{1234} + \ldots)}_{C_m^4} + \\
&+ \ldots + \underbrace{(\hat{T}_{12\ldots m-1} + \hat{T}_{12\ldots m-2m} + \ldots)}_{C_m^{m-1}} + \underbrace{(\tilde{T}_{12\ldots m-1} + \tilde{T}_{12\ldots m-2m} + \ldots)}_{C_m^{m-1}} - \\
&\qquad\qquad + \tilde{T} + \hat{T}_{1\ldots m}.
\end{aligned} \tag{15}$$

Let us consider the following of values

- $\hat{T}_{j_i}$ and $\tilde{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_m}$ ($m = C_m^1$ pairs);
- $\tilde{T}_{j_i}$ and $\hat{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_m}$ ($m = C_m^{m-1}$ pairs);
- $\hat{T}_{j_i j_k}$ and $\tilde{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_{k-1} j_{k+1} \ldots j_m}$ (C_m^2 pairs);
- $\tilde{T}_{j_i j_k}$ and $\hat{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_{k-1} j_{k+1} \ldots j_m}$ (C_m^{m-2} pairs);
- $\ldots$
- $\tilde{T}$ and $\hat{T}_{1\ldots m}$ (1 pair).

By superadditivity of the game, we get

$$\begin{aligned}
&\hat{T}_{j_i} + \tilde{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_m} \le T^* \\
&\tilde{T}_{j_i} + \hat{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_m} \le T^* \\
&\hat{T}_{j_i j_k} + \tilde{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_{k-1} j_{k+1} \ldots j_m} \le T^* \\
&\tilde{T}_{j_i j_k} + \hat{T}_{j_1 \ldots j_{i-1} j_{i+1} \ldots j_{k-1} j_{k+1} \ldots j_m} \le T^* \\
&\ldots \\
&\tilde{T} + \hat{T}_{1 \ldots m} \le T^*.
\end{aligned}$$

The right side of (15) can be estimated as follows:

$$\begin{aligned}
&\underbrace{((\hat{T}_1 + \tilde{T}_{23\ldots m}) + \ldots + (\hat{T}_m + \tilde{T}_{12\ldots m-1}))}_{C_m^1} + \\
&+ \underbrace{((\hat{T}_{12} + \tilde{T}_{34\ldots m}) + \ldots + (\hat{T}_{m-1,m} + \tilde{T}_{12\ldots m-2}))}_{C_m^2} + \\
&+ \underbrace{((\hat{T}_{123} + \tilde{T}_{4\ldots m}) + \ldots + (\hat{T}_{m-1m-2m} + \tilde{T}_{1\ldots m-3}))}_{C_m^3} + \ldots + \\
&+ \underbrace{((\tilde{T}_{123} + \hat{T}_{4\ldots m}) + \ldots + (\tilde{T}_{m-2m-1m} + \hat{T}_{1\ldots m-3}))}_{C_m^{m-3}} + \\
&+ \underbrace{((\tilde{T}_{12} + \hat{T}_{34\ldots m}) + \ldots + (\tilde{T}_{m-1,m} + \hat{T}_{12\ldots m-2}))}_{C_m^{m-2}} + \\
&+ \underbrace{(\tilde{T}_1 + \hat{T}_{23\ldots m} + \ldots \hat{T}_m + \hat{T}_{12\ldots m-1})}_{C_m^{m-1}} + \\
&+(\tilde{T} + \hat{T}_{1,2,\ldots,m}) \le \\
&\le (C_m^1 + C_m^2 + C_m^3 + \ldots + C_m^{m-1} + C_m^m) \cdot T^*.
\end{aligned} \tag{16}$$

It remans to show that the following inequality holds

$$(2^m - 1) \cdot T^* \ge (C_m^1 + C_m^2 + C_m^3 + \ldots + C_m^{m-1} + C_m^m) \cdot T^*.$$

If the last inequality is true then (16) is fulfilled. Taking into account that $C_m^1 + C_m^2 + C_m^3 + \ldots + C_m^{m-1} + C_m^m = 2^m - 1$, we get

$$(2^m - 1) \cdot T^* \ge (2^m - 1) \cdot T^*.$$

It is obvious that the last inequality holds for any m. So, inequality (15) is satisfied. This means that inequality (14) is also satisfied for any initial positions of the players. Hence, system (13) is combined.

It remains to show that any vector satisfying the system (13) is an imputation of the game $\Gamma_v(z_P^0, z_1^0, \ldots, z_m^0)$. Indeed, it can be easily checked that the vector $\eta^0 = (T^* - \sum_{i=1}^{m} \hat{T}_i, \hat{T}_1, \hat{T}_2, \ldots, \hat{T}_{m-1}, \hat{T}_m)$ satisfies system (13) and is an imputation of this game. This completes the proof.

6. Conclusion

The considered cooperative and noncooperative approaches to investigation of group pursuit games with one pursuer and m-evaders give us various interesting solutions and allow to look at the same problem from different points of view. This paper extends an application area of group pursuit games.

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Entering of Newcomer in the Perturbed Voting Game

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Abstract The new class of voting games, in which the number of players and their power indexes are changing coherently, is considered. As a power index Shapley–Shubik value is taken. The following problem is considered: how to find a minimal investment, which guarantees the given value of the Shapley–Shubik power index for the newcomer. This value depends on the distribution of weights of players before entering of newcomer and on the capital that can be used to purchase shares of weights from different players.

Keywords: voting game, Shapley–Shubic value, profitable investment, perspective coalitions, veto-player, Monte–Carlo method.

1. Introduction

The solution of voting game was formulated by Shubik on the basis of the Shapley value and was called the Shapley–Shubik value (Shapley and Shubik, 1954; Hu, 2006; Shapley and Shubik, 1969).

In the modernized or extended game for the newcomer it is natural to minimize the capital to purchase the shares from other players to enter the voting game aiming to receive as a result of cooperation a given income as his component of Shapley–Shubik value. This value depends on the distribution of weights of players before entering the newcomer and on the capital that can be used to purchase shares of weights from different players.

In section 2. a description of voting game and its extension is given, in section 3. the essantional propositions are proved and the problem is formulated — the problem of minimization of capital to reach a given component of the Shapley–Shubik value for $(n+1)$-st player, in section 4. the method and algorithm for solving the problem, and also an example illustrating the realization of algorithm is proposed.

2. Voting games

2.1. Essential definitions

Definition 1. Weight a_i of the player i — share which belongs to the player i and satisfies the following condition:

$$\begin{cases} \sum\limits_{i=1}^{n} a_i = 1, \\ a_i \geqq 0, \qquad i = 1, \dots, n, \end{cases} \tag{1}$$

where $N = \{1, \dots, n\}$ — the set of players, $a = (a_1, \dots, a_n)$ — distribution of players weights.

Definition 2. Perturbed voting game is defined as:
$\Gamma = [\omega, a_1, \dots, a_n]$, where ω — quota in the voting game — share of total capital, such that if the sum of weights of players in the coalition is strictly greater than this share, the coalition is winning, otherwise the coalition is losing. Also in the pertrubed voting game the following definition is used: "veto-player in the coalition S" — player, without which the coalition S is losing, and with him is winning.

Note 1. Perturbed voting game differs from the usual voting game, by the fact that the coalition is winning if the sum of weights of players in the coalition is greater when the quota. It is appropriate to consider the perturbed voting game, because of properties of characteristic function. In the perturbed voting game characteristic function is super additive, and in the initial voting game may be not:

Example 1. Let $\Gamma = [0.5; 0.5, 0.5]$ ($\omega = 0.5,\ a_1 = 0.5,\ a_2 = 0.5$), then

$$v(\{1\}) = 1,\ v(\{2\}) = 1,\ v(\{1,2\}) = 1$$

But the super additivity is not satisfied, because:

$$v(\{1,2\}) < v(\{1\}) + v(\{2\})$$

And for the perturbed voting game:

$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{1,2\}) = 1$$

Super additivity is satisfied:

$$v(\{1,2\}) \geqq v(\{1\}) + v(\{2\})$$

In what follows by voting game we shall understande the pertrubed voting game.

The component of the Shapley–Shubik value of i-th player is known as:

$$\varphi_i = \sum_{\substack{S \ni i: \\ S \in W, \{S \setminus i\} \notin W}} \frac{(|S|-1)!(n-|S|)!}{n!}, \tag{2}$$

where $W = \{S : \sum_{i \in S} a_i > \omega\}$, i.e. W — the set of winning coalitions in the voting game Γ, a_i satisfies (1).

Note 2. It is known from the definition (2), that the summation in (2) is taken only over the coalitions in which i-th player is a veto-player.

2.2. Extension of the voting game

The n player voting game Γ is considered. Consider the extended $(n+1)$ player voting game with the newcomer. Let M be the capital of $(n+1)$-st player, defined as a weight, that $(n+1)$-st player posses. Suppose that $M \in (0, 1]$.

Definition 3. Investment — vector $\alpha = (\alpha_1, \dots, \alpha_n)$, characterizing the parts of shares purchased by $(n+1)$-th player from other players and satisfying:

$$\begin{cases} \sum\limits_{i=1}^{n} \alpha_i \leqq M, \\ 0 \leqq \alpha_i \leqq a_i,\ i = 1, \dots, n, \end{cases} \tag{3}$$

(here α_i — part of share, purchased by $(n+1)$-st player from i-th player).

To join the voting game Γ, the newcomer is implementing the investment α. Then player i will get the weight $(a_i - \alpha_i)$ in the extended voting game Γ' for all $i = 1, \ldots, n$ and $(n+1)$-st player will get the weight $\sum_{i=1}^{n} \alpha_i$. Then the new vector of weights wil be:

$$a' = (a'_1, \ldots, a'_i, \ldots, a'_n, a'_{n+1}) = \\ = (a_1 - \alpha_1, \ldots, a_i - \alpha_i, \ldots, a_n - \alpha_n, \sum_{i=1}^{n} \alpha_i).$$

The extended voting game Γ' has the form:

$$\Gamma' = [\omega, a'_1, \ldots, a'_i, \ldots, a'_n, a'_{n+1}] = \\ = [\omega, a_1 - \alpha_1, \ldots, a_i - \alpha_i, \ldots, a_n - \alpha_n, \sum_{i=1}^{n} \alpha_i].$$

Definition 4. The set of possible investments $I = I(a, M)$ — the set of investments, satisfying (3) for given weights of players $a = (a_1, \ldots, a_n)$ and capital M of $(n+1)$-st player.

Define a set of coalitions, such that in each of them player $(n+1)$ can be a veto-player in the extended voting game Γ'. The condition for investment α of the newcomer to be a veto-player in coalition $S \cup \{n+1\}$, where $S \subseteq N$ is following:

$$\begin{cases} \sum\limits_{i \in S} a'_i + a'_{n+1} = \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i > \omega, \quad \alpha \in I \\ \sum\limits_{i \in S} a'_i = \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{4}$$

Note 3. First inequality in (4) shows, that with the $(n+1)$-st player coalition S from the voting game Γ will become a winning coalition in the voting game Γ'. Second inequality in (4) shows, that coalition S from the voting game Γ become a losing coalition in the voting game Γ'.

Based on formula (2), it is clear, that by each coalition, in which $(n+1)$-st player is a veto-player the incremention of $(n+1)$-st component of the Shapley–Shubik value in the voting game Γ' is provided:

$$\frac{(|S+1|-1)!(n+1-|S+1|)!}{(n+1)!} = \frac{|S|!(n-|S|)!}{(n+1)!}. \tag{5}$$

Definition 5. The set of coalitions, for which the set of solutions of (4), i.e. the set of vectors α, satisfying the system (4), is not empty, is called the set of perspective coalitions (PC).

Note 4. The subset of coalitions $S \subseteq PC$, which satisfies (4), is uniquely defined for a given investment α. This subset defines the component of the Shapley–Shubik value of $(n+1)$-st player. Denote this subset as αPC. We also say that subset of a set PC corresponds to the given value h of component of the Shapley–Shubik value of $(n+1)$-st player, if in all coalitions belonging to this subset $(n+1)$-st player will be a veto-player and the sum of all increments (4) is equal to h. This subset is called $SPC(h)$.

Compute now the $(n+1)$-st component of the Shapley–Shubik value in extended voting game, using notions introduced above. For this, let
W' — set of winning coalitions in the game Γ',
$S' \subseteq \{1, \ldots, n, n+1\}$ — coalition in the game Γ'.
Then:

$$\varphi_{n+1} = \sum_{\substack{S' \ni \{n+1\}: \\ S' \in W', \{S' \setminus \{n+1\}\} \notin W'}} \frac{(|S'|-1)!(n+1-|S'|)!}{(n+1)!} =$$
$$= \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha) = \sum_{S \in \alpha PC} \frac{|S|!(n-|S|)!}{(n+1)!}, \quad (6)$$

where $S = S' \setminus \{n+1\}$, set PC is defined by (4).

$$K_S(\alpha) = \begin{cases} 1, \text{ if } \alpha \in I : \alpha \text{ satisfies (4) for } S \subseteq N \\ 0, \text{ in other cases} \end{cases}$$

or $K_S(\alpha) = 0$, for $S : S \in PC \setminus \alpha PC$ and $K_S(\alpha) = 1$, for each $S : S \in \alpha PC$.

Note 5. It is obvious that function K_S takes non-zero values only for coalitions S, for which i-th player in the extended voting game Γ' is a veto-player, which follows from the definition of the Shapley–Shubik value.

In (Petrosyan, 2013) it was proved that:

Proposition 1. *The function $\varphi_{n+1}(\alpha)$, where $\alpha \in I$, takes finite set of values.*

Definition 6. Profitable Investment (PI) — the set of investments $\alpha \in I$, such that for each of them the component of the Shapley–Shubik value of $(n+1)$-st player takes its maximum value in the voting game Γ'.

Denote by α^* any investment which belongs to the set PI and denote by $\alpha^* PC$ the subset of the set PC which correspond to α^*, then:

$$\max_{\alpha \in I} \varphi_{n+1} = \max_{\alpha \in I} \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha) =$$
$$= \sum_{S \in PC} \frac{|S|!(n-|S|)!}{(n+1)!} \cdot K_S(\alpha^*) = \sum_{S \in \alpha^* PC} \frac{|S|!(n-|S|)!}{(n+1)!} = \varphi^*_{n+1} = F(M).$$

Note 6. Obviously, the function $\varphi^*_{n+1} = F(M)$ depends on M. The function $F(M)$ will be called the Bellman function.

In (Petrosyan, 2013), the following problem was considered:

The distribution of weights $a = (a_1, \ldots, a_n)$, quota ω, capital of $(n+1)$-st player are given. It is necessary to define the set PI and corresponding component of the Shapley–Shubik value of $(n+1)$-st player.

3. Problem of finding a minimal investment, which guarantees the given value of the Shapley–Shubik power index for the newcomer

Use the notation introduced in the previous paragraph: $F(M)$ — maximum value of φ_{n+1} for a given capital M. Let $h \in [0, 1]$ be a desired component of the Shapley–Shubik value of $(n+1)$-st player. We need some properties of function $F(M)$.

3.1. Properties of function F(M)

Proposition 2. *Function $F(M)$ — non decreasing function.*

The following statement is given without proof, because it follows from the definition.

Proposition 3. *Function $F(M)$ — finite-valued function.*

Proof. Consider the function $\varphi_{n+1}(\alpha)$ for the value of capital $M = 1$. This function takes all the values that the function $F(M)$ may take, where $M \in [0, 1]$. According to Proposition (1): the function $\varphi_{n+1}(\alpha)$ takes a finite number of values over the set $\sum_{i=1}^{n} \alpha_i \leqq 1$. Consequently, the function $F(M)$ with $M \in [0, 1]$ takes a finite number of values.

Note 7. Function $F(M)$ — step function.

Example 2. Two person voting game is given:

$$\Gamma = [0.5; 0.5, 0.5], \ (\omega = 0.5; \ a_1 = 0.5, \ a_2 = 0.5).$$

The voting game is extended by the third player. It is necessary to find the set of values for the capital M, to get a given numerical value of component of the Shalpey–Shubik value for the third player, which is equal to $\frac{1}{3}$ in the extended voting game:

$$\Gamma' = [0.5; 0.5 - \alpha_1, 0.5 - \alpha_2, \sum_{i=1}^{2} \alpha_i].$$

The given component of the Shapley–Shubik value in the voting game Γ' is achieved by incrementing (5) for the coalition $S_1 = \{1\}$ and $S_2 = \{2\}$ of Γ. Write the system (4) for each of coalitions:

$$S_1 : \begin{cases} \sum\limits_{i \in S_1} a_i + \sum\limits_{i \notin S_1} \alpha_i = 0.5 + \alpha_1 > 0.5 \\ \sum\limits_{i \in S_1} a_i - \sum\limits_{i \in S_1} \alpha_i = 0.5 - \alpha_1 \leqq 0.5 \end{cases} \to \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in (0, 1 - \alpha_1] \end{cases} \tag{7}$$

$$S_2 : \begin{cases} \sum\limits_{i \in S_2} a_i + \sum\limits_{i \notin S_2} \alpha_i = 0.5 + \alpha_2 > 0.5 \\ \sum\limits_{i \in S_2} a_i - \sum\limits_{i \in S_2} \alpha_i = 0.5 - \alpha_2 \leqq 0.5 \end{cases} \to \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in (0, 1 - \alpha_1] \end{cases} \tag{8}$$

Possible values of function $F(M)$ are:

$$F(M) = \begin{cases} 0, \ M = 0 \\ \frac{1}{3}, \ M \in (0, 0.5) \\ \frac{1}{2}, \ M = 0.5 \\ 1, \ M > 0.5 \end{cases}$$

Here, the value $\frac{1}{3}$ is attained for $M \in (0, 0.5)$, because the investment which corresponds to this capital is a solution of systems (8), (7) and the third player becomes a veto-player with the capital $M = 0.5$ in the coalition $\{1, 2\}$. When capital $M > 0.5$ the value function $F(M)$ is equal to 1 (see (Petrosyan, 2013)).

Note 8. It is seen from the example that minimum of capital M for fixed component of the Shapley–Shubik value of third player may not be achieved.

Example 3. Two person voting game is given:

$$\Gamma = [0.5; 0.6, 0.4].$$

The voting game is extended by the third player. It is necessary to find a set of values for the capital M, to get a given numerical value of component of the Shapley–Shubik value for the third player, which is equal to $\frac{1}{6}$ in the extended voting game:

$$\Gamma' = [0.5; 0.6 - \alpha_1, 0.4 - \alpha_2, \sum_{i=1}^{2} \alpha_i].$$

Given component of the Shapley–Shubik value in the voting game Γ' is achieved by incrementing (5) for the coalition $S_1 = \{1\}$ of Γ. Write the system (4) for the coalition:

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i = 0.6 > 0.5 \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i = 0.6 - \alpha_1 \leqq 0.5 \end{cases} \rightarrow \begin{cases} \alpha_1 \in [0, 1 - \alpha_2] \\ \alpha_2 \in [0, 1 - \alpha_1] \end{cases} \tag{9}$$

Possible values of function $F(M)$ are:

$$F(M) = \begin{cases} 0, \ M = 0 \\ \frac{1}{6}, \ M = 0.1 \\ \frac{1}{3}, \ M \in (0.1, 0.5) \\ \frac{1}{2}, \ M = 0.5 \\ 1, \ M > 0.5 \end{cases}$$

Here, the value $\frac{1}{6}$ is achieved for $M = 0.1$, because the investment which corresponds to this capital is a solution of system (9), the third player becomes a veto-player with the capital $M \in (0, 0.5)$ in the coalition $\{2\}$ and in the coalition $\{1, 2\}$ with the capital $M = 0.5$. When capital $M > 0.5$ the value function $F(M)$ is equal to 1 (see (Petrosyan, 2013)).

Note 9. The set of values for M for a fixed component of the Shapley–Shubik value of third player can be a singleton.

3.2. Propositions

Derive the necessary condition for the existence of a "minimal capital" of $(n+1)$-st player to reach the desirable component of the Shapley–Shubik value h. Since $F(M)$ is a step function the solution M' of the equation $F(M) = h$ may not exist.

What is understood by "minimal capital" in this case? It is necessary to consider different cases:

1. The solution M' of the equation $F(M) = h$ exists. Then two cases are possible:
   - $\inf\{M : F(M) = h\}$ is attained and is equal to $M^* = \min\{M : F(M) = h\}$, then M^* — minimal capital.

   - $M^* = \inf\{M : F(M) = h\}$ is not attained, then the definition of minimal capital is to be considered with ε — accuracy ($\varepsilon > 0$), and denoted by M^*_ε, $M^* < M^*_\varepsilon < M^* + \varepsilon$, M^*_ε — ε minimal capital.
2. The solution M' of the equation $F(M) = h$ does not exist, consider $M^* = \inf\{M : F(M)\}$, then two cases are possible:
   - $\min\{M : F(M) > h\}$ is attained and is equal to $M^* = \min\{M : F(M) > h\}$, then M^* — minimal capital.
   - $M^* = \min\{M : F(M) > h\}$ is not attained, then the definition of minimal capital is to be considered with ε — accuracy ($\varepsilon > 0$), and denoted by M^*_ε, $M^* < M^*_\varepsilon < M^* + \varepsilon$, M^*_ε — ε minimal capital.

Define $h^* = \min\{F(M) : F(M) \geqq h\}$, if the solution M' of equation $F(M) = h$ exists, then $h^* = h$, but in general $h^* \geqq h$.

Proposition 4. *To reach $M^* = \min\{M : F(M) = h^*\}$ it is necessary that at least one winning coalition must belong to $SPC(h^*)$.*

Proof. In the section 2.2. it was shown that the component of the Shapley–Shubik value of $(n + 1)$-st player depends on coalitions of original game Γ, in which the entered — $(n+1)$-st player can become a veto-player in the extended game Γ', i.e. depends on set PC. Consequently, the function φ_{n+1} and $F(M)$ depend on set PC. Each component of the Shapley–Shubik value, and hence $F(M)$ corresponds to one or more subsets of PC (due to the fact that the increment for the component of the Shapley–Shubik value (5) depends only on the dimension of perspective coalition).

Consider one of sets $SPC(h^*)$. Consider one coalition from $SPC(h^*)$. Derive the conditions for which the minimum of $\sum\limits_{i=1}^{n} \alpha_i$ is attained on the subset defined by inequality (4) for coalition S.

$$\begin{cases} \sum\limits_{i\in S} a_i + \sum\limits_{i\notin S} \alpha_i > \omega \\ \sum\limits_{i\in S} a_i - \sum\limits_{i\in S} \alpha_i \leqq \omega \end{cases} \tag{10}$$

Analyze the inequalities:

1. First inequality. Note that $\min\{\sum\limits_{i\notin S} \alpha_i : \sum\limits_{i\in S} a_i + \sum\limits_{i\notin S} \alpha_i > \omega\}$ is reached and equals zero only if $\sum\limits_{i\in S} a_i > \omega$, i.e. when the coalition S is a winning coalition in the original game Γ.

2. Second inequality. Note that $\min\{\sum\limits_{i\in S} \alpha_i : \sum\limits_{i\in S} a_i - \sum\limits_{i\in S} \alpha_i \leqq \omega\}$ is reached and equals $\sum\limits_{i\in S} a_i - \omega$.

Conditions for minimum of $\sum\limits_{i\notin S} a_i$ and $\sum\limits_{i\in S} a_i$ under condition (10) are obtained. But

$$\sum_{i=1}^{n} \alpha_i = \sum_{i\notin S} \alpha_i + \sum_{i\in S} \alpha_i. \tag{11}$$

Using the independence of two terms in (11), we obtain the condition for a minimum of $\sum_{i=1}^{n} \alpha_i = M$ under (10). The condition is that S is a winning coalition. Note that the set of investments which determines the value of component of the Shapley–Shubik value of $(n+1)$-st player equal to h^* is the intersection of solution sets of inequalities for each of coalitions, corresponding to the value h^*. In this case at least one of the systems of inequalities (4) corresponds to a winning coalition in the original game Γ. Indeed, if this is not the case, then $\sum_{i=1}^{n} \alpha_i$ will not reach its minimum value (see (10)) on the set of investments described above . Hence it follows, that if $M^* = \min\{M : F(M) = h^*\}$ is reached, then there is at least one coalition $S \in SPC(h^*)$, which is winning.

Derive sufficient conditions for the existence of a minimum capital of $(n+1)$-st player to reach the component of the Shapley–Shubik value equal to h^*. Consider the following sets:

Set IPC — set of investments α, for which $(n+1)$-st component of the Shapley–Shubik value takes the same value equal to h^*. This set is given by a system of inequalities (4). Define now the set $IPC(\varepsilon)$, where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ and k — number of losing coalitions of the game Γ — set of investments α, for which component of the Shapley–Shubik value is h^*, but structure of inequalities (4) changed. For all losing coalitions in the original game Γ it takes the form:

$$\begin{cases} \sum\limits_{i \in S_j} a_i + \sum\limits_{i \notin S_j} \alpha_i \geqq \omega + \varepsilon_j, \ \ j = 1, \ldots, k \\ \sum\limits_{i \in S_j} a_i - \sum\limits_{i \in S_j} \alpha_i \leqq \omega \end{cases}$$

where S_j — losing coalition in the game Γ, $\varepsilon_j > 0$ — increment, corresponding to each losing coalition in the game Γ.

Proposition 5. *To achieve* $M^* = \min\{M : F(M) = h^*\}$ *it is sufficient that* $\exists \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) : \forall \varepsilon_j > 0, j = 1, \ldots, k$ *such that:*

$$\inf\{\sum_{i=1}^{n} \alpha_i : \alpha \in IPC\} = \inf\{\sum_{i=1}^{n} \alpha_i : \alpha \in IPC(\varepsilon)\} \tag{12}$$

Proof. Assume that (12) holds. Then the set of investments α, which corresponds to the $(n+1)$-st component of the Shapley–Shubik value equal to h^*, for each losing coalition in the original game Γ is defined by a system of inequalities :

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i \geqq \omega + \varepsilon_j, \ \ j = 1, \ldots, k \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{13}$$

And each winning coalition in the original game Γ is defined by a system of inequalities:

$$\begin{cases} \sum\limits_{i \in S} a_i + \sum\limits_{i \notin S} \alpha_i > \omega, \text{ ãäå} \sum\limits_{i \in S} a_i > \omega \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \iff \begin{cases} \sum\limits_{i \notin S} \alpha_i \geqq 0 \\ \sum\limits_{i \in S} a_i - \sum\limits_{i \in S} \alpha_i \leqq \omega \end{cases} \tag{14}$$

If only losing coalitions in the original game Γ belong to the subset of PC, which corresponds to component of $(n+1)$-st Shapley–Shubik value equal to h^*, then $M^* = \min\{M : F(M) = h^*\}$ will be achieved (from the system (13)), but according to 3.2. $SPC(h^*)$ must belong at least one winning coalition in the original game Γ.

Consider system (14), for it $M^* = \min\{M : F(M) = h^*\}$ will be achieved, as the coalition S — winning coalition in the voting game Γ. Consequently, $M^* = \min\{M : F(M) = h^*\}$ will be achieved for all coalitions from $SPC(h^*)$, generally.

3.3. Statement of the problem

Distribution of weights $a = (a_1, \ldots, a_n)$ and the quota ω, are given. It is necessary to define a capital of $(n+1)$-st player M^* (M^*_ε), which corresponds as close as possible to the given $(n+1)$-st component of the Shapley–Shubik value, i.e. h^*.

4. Monte – Carlo method to find the approximate optimal solution of the problem

The method consists in the generation of random vector $\xi = (\xi_1, \ldots, \xi_i, \ldots, \xi_n)$, where i-th component have a uniform probability distribution on the interval $[0, a_i]$. To each $\xi = \alpha$ corresponds the component of the Shapley–Shubik value of $(n+1)$-st player: φ_{n+1}. Choose the obtained values of φ_{n+1} close to a given h: h^*. Further compute α^*, such that $\sum\limits_{i=1}^{n} \alpha_i^*$ is minimal. Then $\sum\limits_{i=1}^{n} \alpha_i^* = M^*$ — will be approximate solution of problem.

Define a set COI, which is necessary for realization the Monte–Karlo method:

$$COI = \{\alpha : \alpha_i \in [0, a_i]\}.$$

The following statement is without proof, since it is obvious:

Proposition 6. *For any* $a = (a_1, \ldots, a_n)$*, satisfying* (1)*, and for every* M *the following inclusition holds:*

$$I \subseteq COI.$$

4.1. Algorithm for finding approximate minimal capital

This section contains an algorithm for finding capital M^*.

1. Sample of R random vectors with a uniform probability distribution over the set COI is generated.
2. For each vector the extended voting game Γ' is formed, to each game Γ' corresponds $(n+1)$-st component of the Shapley–Shubik value.
3. The approximate $(n+1)$-st component of the Shapley–Shubik value to the value of h and the corresponding investments α is selected.
4. From the resulting sample of investments select the investment, for which $\sum\limits_{i=1}^{n} \alpha_i$ is minimal — α^*. Then the approximate solution of the problem $M^* = \sum\limits_{i=1}^{n} \alpha_i^*$ is obtained.

Example 4. The numerical simulation by Monte–Carlo method for the perturbed three person voting game is performed:

$$[0.5; 0.3, 0.6, 0.1].$$

Voting game extends by fourth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 4-th player $h = 0.1$, i.e. $h^* = \frac{1}{6} = 0.167$, in the extended voting game is achieved:

$$[0.5; 0.3 - \alpha_1, 0.6 - \alpha_2, 0.1 - \alpha_3].$$

Set of values for the capital M, which corresponds to the $h^* = 0.167$ was found:

$$F^{-1}(h^*) = \{0.176, 0.161, 0.189, 0.157, 0.101, 0.113, 0.127\} \tag{15}$$

From the resulting set of values is appropriate to select the fifth estimate, which is $\widehat{M}^* = 0.101$.

Example 5. The numerical simulation by Monte–Carlo method for the perturbed five person voting game is performed:

$$[0.5; 0.2, 0.2, 0.2, 0.2, 0.2].$$

Voting game extends by sixth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 6-th player $h = 0.6$, i.e. $h^* = 0.6$, in the extended voting game is achieved:

$$[0.5; 0.2 - \alpha_1, 0.2 - \alpha_2, 0.2 - \alpha_3, 0.2 - \alpha_4, 0.2 - \alpha_5].$$

Set of values for the capital M, which corresponds to the $h^* = 0.6$ was found:

$$F^{-1}(h^*) = \{0.411, 0.354, 0.42, 0.436, 0.423, 0.449, 0.412\} \tag{16}$$

From the resulting set of values is appropriate to select the second estimate, which is $\widehat{M}^* = 0.354$.

Example 6. The numerical simulation by Monte–Carlo method for the perturbed seven person voting game is performed:

$$[0.5; 0.02, 0.11, 0.38, 0.06, 0.08, 0.21, 0.14].$$

Voting game extends by eighth player. It is necessary to find minimal capital M^*, for which the closest to the given component of the Shapley–Shubik value of 8-th player $h = 0.28$, i.e. $h^* = \frac{59}{210} = 0.2809$, in the extended voting game is achieved:

$$[0.5; 0.02 - \alpha_1, 0.11 - \alpha_2, 0.38 - \alpha_3, 0.06 - \alpha_4, 0.08 - \alpha_5, 0.21 - \alpha_6, 0.14 - \alpha_7].$$

Set of values for the capital M, which corresponds to the $h^* = 0.2809$ was found:

$$F^{-1}(h^*) = \{0.312, 0.293, 0.253, 0.233, 0.324, 0.338, 0.328\} \tag{17}$$

From the resulting set of values is appropriate to select the fourth estimate, which is $\widehat{M}^* = 0.233$.

5. Conclusion

In this paper and in (Petrosyan, 2013), a complex of problems aimed to expand a set of players in the voting game is investigated. Conditions of optimal behavior of the newcomer is perfermed. The approach will be probably used for a wider class of cooperative games.

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Strategic Support of Cooperative Solutions in 2-Person Differential Games with Dependent Motions

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Abstract The problem of strategically supported cooperation in 2-person differential games with integral payoffs is considered. Based on initial differential game the new associated differential game (CD-game) is designed. In addition to the initial game it models the players actions connected with transition from the strategic form of the game to cooperative with in advance chosen principle of optimality. The model provides possibility of refusal from cooperation at any time instant t for each player. As cooperative principle of optimality the Shapley value is considered. In the bases of CD-game construction lies the so-called imputation distribution procedure described earlier in (Petrosjan and Zenkevich, 2009). The theorem established by authors says that if at each instant of time along the conditionally optimal (cooperative) trajectory the future payments to each player according to the imputation distribution procedure exceed the maximal guaranteed value which this player can achieve in CD-game, then there exist a Nash equilibrium in the class of recursive strategies first introduced in (Chistyakov, 1981) supporting the cooperative trajectory. In the present paper the results similar to (Chistyakov and Petrosyan, 2011) are obtained without the requirement of independent motions and for the more general type of payoff functions.

Keywords: strong Nash equilibrium, time-consistency, core, cooperative trajectory.

1. Introduction

Similar to (Petrosjan and Zenkevich, 2009; Chistyakov and Petrosyan, 2011) in this paper the problem of strategically support of cooperation in differential 2-person game with prescribed duration T and dependent motions is considered.

$$\frac{dx}{dt} = f(t, x, u^{(1)}, u^{(2)}),\ i \in I = [1, 2], \tag{1}$$

$$x \in R^n, u^{(i)} \in P^{(i)} \subset CompR^{k(i)}, \quad i \in I$$

$$x(t_0) = x_0. \tag{2}$$

The payoffs of players $i \in I = [1, 2]$ have integral form

$$H^{(i)}_{t_0,x_0}(u^{(1)}(\cdot), u^{(2)}(\cdot)) = \int_{t_0}^{T} h^{(i)}\left(t, x(t), u^{(1)}(t), u^{(2)}(t)\right) dt, \tag{3}$$

where $u(\cdot) = (u^{(1)}(\cdot), u^{(2)}(\cdot))$ is a given vector-function of open loop controls, $x(t) = x(t, t_0, x_0, u^{(1)}(\cdot), u^{(2)}(\cdot))$ is the solution of the Cauchy problem (1) with corresponding initial conditions (2) and admissible open loop controls $u^{(1)}(\cdot), u^{(2)}(\cdot)$ of players.

Admissible open loop controls of players $i \in I$ are Lebesgue measurable open loop controls

$$u^{(i)}(\cdot): t \mapsto u^{(i)}(t) \in R^{k(i)}, \quad i \in I = \{1, 2\}$$

such that

$$u^{(i)}(t) \in P^{(i)} \text{ for almost all } t \in [t_0, T], i \in I.$$

It is supposed that the function $f: R \times R^n \times P^{(1)} \times P^{(2)} \to R^n$ is continuous, locally Lipschitz with respect to x and satisfies the following condition: $\exists \lambda > 0$ such, that

$$||f(t, x, u^{(1)}, u^{(2)})|| \leq \lambda(1 + ||x||) \quad \forall x \in R^{k(i)}, \quad \forall u^{(1)} \in P^{(1)}, u^{(2)} \in P^{(2)}.$$

Each of the functions

$$h^{(i)}: R \times R^n \times P^{(1)} \times P^{(2)} \to R, \quad i \in I$$

are also continuous.

For all $t \in R^+$, $x \in R^n$, $\ell \in R^n$

$$\max_{u^{(1)} \in P^{(1)}} \min_{u^{(2)} \in P^{(2)}} \left(< \ell, f(t, x, u^{(1)}, u^{(2)}) > + h^{(1)}(t, x, u^{(1)}, u^{(2)})\right) =$$
$$\min_{u^{(2)} \in P^{(2)}} \max_{u^{(1)} \in P^{(1)}} \left(< \ell, f(t, x, u^{(1)}, u^{(2)}) > + h^{(1)}(t, x, u^{(1)}, u^{(2)})\right)$$

and

$$\max_{u^{(2)} \in P^{(2)}} \min_{u^{(1)} \in P^{(1)}} \left(< \ell, f(t, x, u^{(1)}, u^{(2)}) > + h^{(2)}(t, x, u^{(1)}, u^{(2)})\right) =$$
$$\min_{u^{(1)} \in P^{(1)}} \max_{u^{(2)} \in P^{(2)}} \left(< \ell, f(t, x, u^{(1)}, u^{(2)}) > + h^{(2)}(t, x, u^{(1)}, u^{(2)})\right),$$

here $< \cdot, \cdot >$ is scalar product in R^n.

It is supposed that at each time instant $t \in [t_0, T]$ the players have information about the current position $(t, x(t))$ on the time interval $[t_0, t]$ and use recursive strategies (Chistyakov, 1977; Chistyakov, 1999).

2. Recursive strategies

Recursive strategies were first introduced in (Chistyakov, 1977) for justification of dynamic programming approach in zero sum differential games, known as method of open loop iterations in non regular differential games with non smooth value function. The ε-optimal strategies constructed with the use of this method are universal in the sense that they remain ε-optimal in any subgame of the previously defined differential game (for every $\varepsilon > 0$). Exploiting this property it became possible to prove the existence of ε-equilibrium (Nash equilibrium) in non zero sum differential games (for every $\varepsilon > 0$) using the so called "punishment strategies" (Chistyakov, 1981).

The basic idea is that when one of the players deviates from the conditionally optimal trajectory other players after some small time delay start to play against the deviating player. As result the deviating player is not able to get much more than he could get using the conditionally optimal trajectory. The punishment of the deviating player at each time instant using one and the same strategy is possible because of the universal character of ε-optimal strategies in zero sum differential games.

In this paper the same approach is used to testify the stability of cooperative agreements in the game $\Gamma(t_0, x_0)$ and as in mentioned case the principal argument is the universal character of ε-optimal recursive strategies in specially defined zero sum games $\Gamma_i(t_0, x_0)$, $i \in I = [1, 2]$ associated with the non-zero sum game $\Gamma(t_0, x_0)$.

The recursive strategies lie somewhere in-between piecewise open loop strategies (Petrosyan, 1993) and ε-strategies introduced by B. N. Pshenichny (Pschenichny, 1973). The difference from piecewise open loop strategies consists in the fact that like in the case of ε-strategies of B. N. Pshenichny the moments of correction of open loop controls are not prescribed from the beginning of the game but are defined during the game process. In the same time they differ from ε-strategies of B. N. Pshenichny by the fact that the formation of open loop controls happens in finite number of steps.

Recursive strategies $U_i^{(n)}$ of player i with maximal number of control corrections n is a procedure for the admissible open loop formation by player i in the game $\Gamma(t_0, x_0), (t_0, x_0) \in D$.

At the beginning of the game $\Gamma(t_0, x_0)$ player i using the recursive strategy $U_i^{(n)}$ defines the first correction instant $t_1^{(i)} \in (t_0, T]$ and his admissible open loop control $u^{(i)} = u^{(i)}(t)$ on the time interval $[t_0, t_1^{(i)}]$. Then if $t_1^{(i)} < T$ having the information about state of the game at time instant $t_1^{(i)}$ he chooses the next moment of correction $t_2^{(i)}$ and his admissible open loop control $u^{(i)} = u^{(i)}(t)$ on the time interval $(t_1^{(i)}, t_2^{(i)}]$ and so on. Then whether on k-th step $(k \leq n-1)$ the admissible control will be formed on the time interval $[t_k, T]$ or on the step n player i will end up with the process by choosing at time instant $t_{n-1}^{(i)}$ his admissible control on the remaining time interval $(t_{n-1}^{(i)}, T]$.

3. Associated games and corresponding solutions

For each given state $(t_*, x_*) \in D$ and $i \in I = [1, 2]$ consider zero sum differential game $\Gamma_i(t_*, x_*)$ between player i and $I \backslash \{i\}$ with the same dynamics as in $\Gamma(t_*, x_*)$ and payoff of player i equal to:

$$H_{t_* x_*}^{(i)}(u^{(1)}(\cdot), u^{(2)}(\cdot)) = \int_{t_0}^{T} h^{(i)}(t, x(t), u^{(1)}(t), u^{(2)}(t))dt.$$

The game $\Gamma_i(t_*, x_*)$, $i \in I$, $(t_*, x_*) \in D$, as $\Gamma(t_*, x_*), (t_*, x_*) \in D$ we consider in the class of recursive strategies. Under the above formulated conditions each of the games $\Gamma_i(t_*, x_*)$, $i \in I$, $(t_*, x_*) \in D$ has a value

$$val \Gamma_i(t_*, x_*),$$

and optimal strategies (saddle point).

Consider also the following optimization problem $\Gamma_I(t_*, x_*)$:

$$\max_{u^{(1)}(\cdot), u^{(2)}(\cdot)} \sum_{i=1}^{2} H_{t_0, x_0}^{(i)} \left(u^{(1)}(\cdot), u^{(2)}(\cdot) \right),$$

denoting the resulting maximal value as $v_I(t_0, x_0)$. We suppose that this optimization problem has an optimal open-loop solution.

The corresponding trajectory — solution of (1), (2) on the time interval $[t_0, T]$ we denote by $x_0(\cdot)$ and call "*conditionally optimal cooperative trajectory*". This trajectory may not be necessary unique. Thus on the set D the mapping

$$v(\cdot) : D \to R^3$$

is defined with coordinate functions

$$v_I(\cdot), v_1(\cdot), v_2(\cdot) : D \to R,$$

$$v_i(t_*, x_*) = val\Gamma_i(t_*, x_*), i \in I, v_I(t_*, x_*).$$

This mapping correspond to each state $(t_*, x_*) \in D$ a characteristic function $v(t_*, x_*) : 2^I \to R$ of non zero-sum game $\Gamma(t_*, x_*)$ and thus 2-person classical cooperative game $(I, v(t_*, x_*))$.

Let $E(t_*, x_*) = \{\alpha = (\alpha_1, \alpha_2) : \alpha_i \geq v_i(t_*, x_*), \alpha_1 + \alpha_2 = v_I(t_*, x_*)\}$ be the set of all imputations in the game $(I, v(t_*, x_*))$. Multivalue mapping

$$M : (t_*, x_*) \mapsto M(t_*, x_*) \subset E(t_*, x_*) \subset R^2,$$

$$M(t_*, x_*) \neq \Lambda \quad \forall (t_*, x_*) \in D,$$

is called "*optimality principle*" (defined over the family of games $\Gamma(t_*, x_*)$, $(t_*, x_*) \in D$) and the set $M(t_*, x_*)$ "*cooperative solution of the game $\Gamma(t_*, x_*)$ corresponding to this principle*".

As it follows from (Fridman, 1971) under the above imposed conditions the following Lemma holds.

Lemma 1. *The functions $v_I(\cdot), v_1(\cdot), v_2(\cdot) : D \to R$, are locally Lipschitz.*

Since the solution of the Cauchy problem (1), (2) in the sense of Caratheodory is absolutely continuous, from Lemma 1 it follows.

Theorem 1. *For every solution of the Cauchy problem (1), (2) in the sense of Caratheodory $x(\cdot)$ corresponding to the open loop controls $u(\cdot) = (u^{(1)}(\cdot), u^{(2)}(\cdot))$ functions*

$$\varphi_i : [t_0, T] \to R, \quad i \in I, \quad \varphi_i(t) = v_i(t, x(t)), \varphi_I(t) = v_I(t, x(t))$$

are absolutely continuous functions on the time interval $[t_0, T]$.

As defined let $E(t_*, x_*)$ be the set of imputations in the game $\Gamma(t_*, x_*)$, and let

$$\xi(t_*, x_*) = \{\xi_1(t_*, x_*), \xi_2(t_*, x_*)\} \in E(t_*, x_*).$$

Then we have

$$\xi_i(t_*, x_*) \geq v_i(t_*, x_*).$$

4. Realization of cooperative solutions

The realization of the solution of the game $\Gamma(t_0, x_0)$ we shall connect with the known "*imputation distribution procedure*" (IDP) (Petrosjan and Danilov, 1979; Petrosjan, 1995).

Under IDP of the imputation $\xi(t_0, x_0)$ from the solution $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ along conditionally optimal trajectory $x_0(\cdot)$ we understand such function

$$\beta(t) = (\beta_1(t), \beta_2(t)), \quad t \in [t_0, T], \tag{4}$$

that

$$\xi(t_0, x_0) = \int_{t_0}^{T} \beta(t)dt \tag{5}$$

and

$$\int_{t}^{T} \beta(t)dt \in E(t, x_0(t)) \quad \forall t \in [t_0, T] \tag{6}$$

where $E(t, x_0(t))$ is the set of imputations in the game $(I, v(t, x_0(t)))$.

The IDP $\beta(t), \quad t \in [t_0, T]$ of the imputation $\xi(t_0, x_0) \in M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is called *dynamically stable (time-consistent)* along the conditionally optimal trajectory $x_0(\cdot)$ if

$$\int_{t}^{T} \beta(t)dt \in M(t, x_0(t)) \quad \forall t \in [t_0, T] \tag{7}$$

The solution $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is *dynamically stable (time-consistent)* if for all $\xi(t_0, x_0) \in M(t_0, x_0)$ along at least one conditionally optimal trajectory the dynamically stable IDP exist.

If $M(t, x_0(t)) = E(t, x_0(t))$, $t \in [t_0, T]$, then $M(t, x_0(t)) \neq \emptyset$ ($M(t, x_0(t))$ is the set of imputations in the subgame $\Gamma(t, x_0(t))$ with initial conditions on conditionally optimal cooperative trajectory with duration $T - t$), and $\xi(t, x_0(t)) \in M(t, x_0(t))$ can be selected as absolutely continuous function of t. Then the following theorem holds.

Theorem 2. *For any conditionally optimal trajectory $x_0(\cdot)$ the following IDP of the solution $\xi(t_0, x_0) \in M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$*

$$\beta(t) = -\frac{d}{dt}\xi(t, x_0(t)), \quad t \in [t_0, T], \tag{8}$$

is the dynamically stable IDP along this trajectory. Therefore the solution $M(t_0, x_0)$ of the game $\Gamma(t_0, x_0)$ is dynamically stable.

As $\xi_i(t_0, x_0)$ we can take the Shapley value:

$$\xi_i(t_0, x_0) = Sh_i(t_0, x_0) = v_i(t_0, x_0) + \frac{v_I(t_0, x_0) - \sum\limits_{i=1}^{2} v_i(t_0, x_0)}{2}$$

and for subgame along cooperative trajectory

$$\xi_i(t, x_0(t)) = Sh_i(t, x_0(t)) = v_i(t, x_0(t)) + \frac{v_I(t, x_0(t)) - \sum\limits_{i=1}^{2} v_i(t, x_0(t))}{2}.$$

From Theorem 1 it follows that the function $Sh_i(t, x_0(t))$ is absolutely continuous and thus differentiable along $x_0(t)$. This shows that IDP $\beta(t)$ for $\xi_i(t, x_0(t)) = Sh_i(t, x_0(t))$ can be computed by (8) according to Theorem 2.

5. About the strategically support of the imputation $\xi(t_0, x_0)$

If in the game the cooperative agreement is reached and each player gets his payoff according to the IDP (8), then it is natural to suppose that those who violate this agreement are to be punished. The effectiveness of the punishment (sanctions) comes to question of the existence of Nash Equilibrium in the following differential game $\Gamma^{\xi}(t_0, x_0)$ which differs from $\Gamma(t_0, x_0)$ only by payoffs of players.

The payoff of player i in $\Gamma^{\xi}(t_0, x_0)$ is equal to

$$H^{(\xi,\,i)}_{t_0,x_0}(u(\cdot)) = -\int_{t_0}^{t(u(\cdot))} \frac{d}{dt}\xi_i(t, x_0(t))dt + \int_{t(u(\cdot))}^{T} h^{(i)}(t, x(t, t_0, x_0, u(\cdot)))dt$$

where $t(u(\cdot))$ is the last time instant $t \in [t_0, T]$ for which

$$x_0(\tau) = x(\tau, t_0, x, u(\cdot)) \quad \forall \tau \in [t_0, t].$$

Theorem 3. *In the game $\Gamma^{\xi}(t_0, x_0)$ for each $\varepsilon > 0$ there exist ε-Nash equilibrium with outcomes (payoffs) of players in this equilibrium equal to*

$$\xi(t_0, x_0) = \{\xi_1(t_0, x_0), \xi_2(t_0, x_0)\} \in E(t_0, x_0).$$

The idea of the proof is following. Since $\xi(t_0, x_0)$ belongs to the imputation set of the game $\Gamma(t_0, x_0)$ we have

$$\xi_i(t, x_0(t)) \geq v_i(t, x_0(t)) \quad \forall i \in I \quad \forall t \in [t_0, T] \tag{9}$$

This means that at each time instant $t \in [t_0, T]$ moving along conditionally optimal trajectory $x_0(\cdot)$ no player $i \in I$ can guarantee himself the payoff $[t, T]$ more than according to IDP (8), i.e. more than

$$\int_t^T \beta(\tau)d\tau = -\int_t^T \frac{d}{dt}\xi(\tau, x_0(\tau))d\tau = \xi_i(t, x_0(t))$$

since if player i deviates from cooperative trajectory at some time instant t, this will be immediately seen by his opponent $3-i$ (since both players know $x(t)$ at each time instant t, and deviation of one player will cause the change of $x(t)$) and he will use punishment strategy in the zero-sum game $\Gamma_{3-i}(t, x_0(t))$ (his optimal strategy in zero-sum game $\Gamma_{3-i}(t, x_0(t))$). Therefore, the player i will get no more then $v_i(t+\delta, x_0(t+\delta)) \leq \xi_i(t, x_0(t) + \varepsilon$.

In the same time on the time interval $[t_0, t]$ according to the IDP she already got the payoff equal to

$$\int_{t_0}^t \beta_i(\tau)d\tau = -\int_{t_0}^t \frac{d}{dt}\xi_i(\tau, x_0(\tau))d\tau = \xi_i(t_0, x_0) - \xi_i(t, x_0(t))$$

Consequently no player can guarantee in the game $\Gamma^{\xi}(t_0, x_0)$ the payoff more than $\xi_i(t_0, x_0)$.

According to the cooperative solution $x_0(\cdot)$ but moving always in the game $\Gamma^{\xi}(t_0, x_0)$ along conditionally optimal trajectory each player will get his payoff according to the imputation $\xi(t_0, x_0)$. Thus no player can benefit from the deviation from the conditionally optimal trajectory which in this case is natural to call "equilibrium trajectory".

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A Construction of Preference Relation for Models of Decision Making with Quality Criteria

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Abstract We consider a problem of construction of preference relations for models of decision making with quality criteria. A quality criterion one means as a function from a set of alternatives in some chain (i.e. linearly ordered set). A system of axioms for rule of preferences is given. It is shown that any rule for preferences satisfying these axioms can be presented as a rule for preferences based on some pseudofilter of winning coalitions of criteria. The section 4 contains main results of the article. In particular, necessary and sufficient conditions for transitive and for linear preferences are found. An interpretation of Arrow paradox in terms of filters is given.

Keywords: Rule for preference relations, Axiom for preferences, Pseudofilters and filters of winning coalitions.

1. Introduction

We study a general model of multi-criteria decision making with quality criteria in the form of a system

$$G = \langle A, (q_j)_{j \in J} \rangle \,, \tag{1}$$

where A is an arbitrary set with $|A| \geq 2$ (named a set of *alternatives* or *outcomes*) and $(q_j)_{j \in J}$ are criteria for valuation of these alternatives. Formally every criterion $q_j, j \in J$ can be presented as a function from A in some scale, points of which are results of measurement for criterion q_j. It is well known that any scale has some set of *acceptable transformations* and measurements produce up a some acceptable transformation.

A criterion q_j is called *a quality one* if its scale is some linearly ordered set $\langle C_j, \leq_j \rangle$, i.e. *a chain*. In this case acceptable transformations are all isotonic functions defined on C_j.

In this article, we consider some problems concerning of preference relations for model (1).

Definition 1. A pair $\langle A, \rho \rangle$, where A is an arbitrary set with $|A| \geq 2$ and ρ a reflexive binary relation on A is called *a space of preferences*.

For any $a, a' \in A$ put

$$\begin{aligned} &a \lesssim a' \Leftrightarrow (a, a') \in \rho, \\ &a < a' \Leftrightarrow (a, a') \in \rho, (a', a) \notin \rho, \\ &a \sim a' \Leftrightarrow (a, a') \in \rho, (a', a) \in \rho. \end{aligned} \tag{2}$$

In (2), the sign $\lesssim$ means a preference,$<$ strict preference and $\sim$ indifference between elements a and a'. A preference relation $\lesssim$ is well defined by the pair $(<, \sim)$, namely $\lesssim$ is the union of relations $<$ and $\sim$.

Given a model G in the form (1), one can define a preference relation on the set of alternatives A in different manners. Let K be the class of models of the form (1). We say that a rule R for preferences in the class K is given if for each $G \in K$ some reflexive binary relation $R(G) = \rho$ on the set of alternatives of model G is defined. Indicate some known rules for preferences.

1. The most important rule for preferences is *Pareto-preference* $\lesssim^{Par}$ which is given by the formula

$$a_1 \lesssim^{\text{Par}} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) \leq_j q_j\,(a_2)\,. \tag{3}$$

2. *Strict Slater preference* is defined by

$$a_1 <^{\text{Sl}} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) <_j q_j\,(a_2)\,. \tag{4}$$

In this case, indifference is the identity relation.

3. *Rule of simple majority* can be introduced in the following way. Assume in model G the set of criteria is finite and $|J| = n$. For any alternatives $a, a' \in A$ we denote

$$\begin{aligned} n\,(a, a') &= |j \in J \colon q_j\,(a) \geq_j q_j\,(a')|\,, \\ n^*\,(a, a') &= |j \in J \colon q_j\,(a) >_j q_j\,(a')|\,. \end{aligned}$$

One can define two rules of simple majority M_1 and M_2 by formulas

$$\begin{aligned} &a_1 \gtrsim^{M_1} a_2 \Leftrightarrow n\,(a_1, a_2) \geq n\,(a_2, a_1)\,, \\ &a_1 \gtrsim^{M_2} a_2 \Leftrightarrow n\,(a_1, a_2) \geq n/2. \end{aligned}$$

It is easy to show that M_1 coincides with M_2 for any elements $a, a' \in A$ in the case when all inequalities for $n\,(a, a')$ and $n\,(a', a)$ are strict. In general case these relations are different. Particularly the condition $a >^{M_1} a'$ holds if $n^*\,(a, a') > n^*\,(a', a)$ and the condition $a >^{M_2} a'$ if $n^*\,(a, a') > n/2$.

4. *Rule of α-majority* is defined as follows. Fix a real number $\alpha > 1/2$. For any $a, a' \in A$ put $a > a' \Leftrightarrow n\,(a, a') \geq r$ where $r = \alpha n$, if αn is integer and $r = [\alpha n] + 1$ otherwise. The indifference relation can be given here by two manners: a) $a \sim a'$ if and only if neither $a > a'$ nor $a' > a$; b) $a \sim a'$ if and only if $a = a'$.

In this article, we study a construction of preference relation with help of some family of criteria, indices of which form so-called pseudofilter. Remark that pseudofilter is a certain generalization of well known conception of filter which is made use in algebra, mathematical logic and topology (see Birkhoff, 1967; Kelley, 1957; Kuratowski and Mostowski, 1967). Using some properties of filters, we indicate an interpretation of Arrow paradox.

2. Axioms for rules of preference relations

We now state axioms for a rule R of preferences in the class K defined above.

(*A*1) *Axiom of independence.* Consider two models $G = \langle A, (q_j)_{j \in J} \rangle$ and $G^1 = \left\langle B, (q_j^1)_{j \in J} \right\rangle$ of class K. Suppose for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalence

$$q_j\,(a_1) \leq_j q_j\,(a_2) \Leftrightarrow q_j^1\,(b_1) \leq_j q_j^1\,(b_2)$$

holds ($j \in J$). Then the equivalence $a_1 \lesssim^{\rho} a_2 \Leftrightarrow b_1 \lesssim^{\rho^1} b_2$ is truth (*we denote by* $\rho = R(G), \rho^1 = R(G^1)$).

Axiom of independence means that the preference between two alternatives in any model of class K is well defined by the set of criteria under which one of them is more preferential than another and does not depend on comparison of these alternatives with other alternatives of the model.

$(A2)$ *Axiom of monotony.* Consider two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \langle A, (q_j^1)_{j\in J}\rangle$ of class K. Fix two elements $a_1, a_2 \in A$ and assume for any $j \in J$ the following implication

$$q_j(a_1) \leq_j q_j(a_2) \Rightarrow q_j^1(a_1) \leq_j q_j^1(a_2)$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 \lesssim^{\rho^1} a_2$.

Axiom of monotony states that the preference between two alternatives in models of class K is increasing under an enlargement the set of corresponding criteria.

$(A3)$ *Axiom of strict monotony.* Consider two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \langle A, (q_j^1)_{j\in J}\rangle$ of class K. Fix two elements $a_1, a_2 \in A$ and suppose for any $j \in J$ the following implication

$$q_j(a_1) \leq_j q_j(a_2) \Rightarrow q_j^1(a_1) <_j q_j^1(a_2)$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 <^{\rho^1} a_2$.

Remark 1. Formally, axioms $(A2)$ and $(A3)$ are independent one from another since $(A3)$ has more strong assumption but more strong consequence also.

$(A4)$ *Axiom for absence of attachment.* Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there exist two models $G = \langle A, (q_j)_{j\in J}\rangle$ and $G^1 = \langle A, (q_j^1)_{j\in J}\rangle$ of class K such that conditions

$$a_1 \lesssim^{\rho} a_2 \text{ and } \neg\left(a_1 \lesssim^{\rho^1} a_2\right)$$

hold.

We now show that a rule R for preferences satisfying axioms $(A1)$ – $(A4)$ can be defined for models of the form

$$G_Q = \langle A, (\sigma_j)_{j\in J}\rangle \tag{5}$$

where σ_j is some linear quasi-order on A. Indeed, let $G = \langle A, (q_j)_{j\in J}\rangle$ be a model of class K. Put

$$J_{(a_1,a_2)} = \{j \in J\colon q_j(a_1) \leq_j q_j(a_2)\}.$$

It follows from axiom $(A1)$ that for any fix elements $a_1, a_2 \in A$, truth of assertions $a_1 \lesssim^{\rho} a_2$ (where $\rho = R(G)$) is well defined by the subset $J_{(a_1,a_2)}$. Define a linear quasi-ordering σ_j on A by the formula

$$a \leq^{\sigma_j} a' \Leftrightarrow q_j(a) \leq_j q_j(a').$$

It is evident that subsets $J_{(a_1,a_2)}$ can be presented in the form

$$J_{(a_1,a_2)} = \{j \in J\colon a_1 \leq^{\sigma_j} a_2\}. \tag{6}$$

Thus any rule R for preferences in the class K can be given as a mapping which for each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ some reflexive preference relation $R(G_Q) = \rho$ on the set A assigns. By this reason, sometimes we will consider the class K as a class of models of the form (5). Axioms $(A1)$ – $(A4)$ in this case can be written as follows.

$(A1)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Denote by $R(G_Q) = \rho, R\left(G_Q^1\right) = \rho^1$. Assume for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalence

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$

holds for each $j \in J$. Then the equivalence $a_1 \lesssim^{\rho} a_2 \Leftrightarrow b_1 \lesssim^{\rho^1} b_2$ is truth also.

$(A2)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Fix elements $a_1, a_2 \in A$ and assume for any $j \in J$ the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2$$

holds. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 \lesssim^{\rho^1} a_2$.

$(A3)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and any $j \in J$ the following implications

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2$$

hold. Then the condition $a_1 \lesssim^{\rho} a_2$ implies the condition $a_1 <^{\rho^1} a_2$.

$(A4)^*$ Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then there exist two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K such that conditions

$$a_1 \lesssim^{\rho} a_2 \text{ and } \neg\left(a_1 \lesssim^{\rho^1} a_2\right)$$

hold.

We now indicate some consequences of axioms $(A1)^*$ – $(A4)^*$.

Corollary 1. *Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalences*

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$
$$a_2 \leq^{\sigma_j} a_1 \Leftrightarrow b_2 \leq^{\sigma_j^1} b_1$$

hold for each $j \in J$. Then the equivalences

$$\begin{aligned} a_1 \sim^{\rho} a_2 &\Leftrightarrow b_1 \sim^{\rho^1} b_2 \\ a_1 <^{\rho} a_2 &\Leftrightarrow b_1 <^{\rho^1} b_2 \end{aligned} \tag{7}$$

are truth also.

For the proof put

$$\begin{aligned} J^+_{(a_1,a_2)} &= \{j \in J\colon a_1 <^{\sigma_j} a_2\} \\ J^+_{(a_2,a_1)} &= \{j \in J\colon a_2 <^{\sigma_j} a_1\} \\ J^0_{(a_1,a_2)} &= \{j \in J\colon a_1 \sim^{\sigma_j} a_2\}\,. \end{aligned} \tag{8}$$

Assumption of corollary 1 means $J_{(a_1,a_2)} = J_{(b_1,b_2)}$ and $J_{(a_2,a_1)} = J_{(b_2,b_1)}$. Hence

$$J^0_{(a_1,a_2)} = J_{(a_1,a_2)} \cap J_{(a_2,a_1)} = J_{(b_1,b_2)} \cap J_{(b_2,b_1)} = J^0_{(b_1,b_2)}.$$

We obtain $J^0_{(a_1,a_2)} = J^0_{(b_1,b_2)}$ that is the first equivalence in (7). It follows from the assumption of corollary 1 and the equality $J^0_{(a_1,a_2)} = J^0_{(b_1,b_2)}$ that $J^+_{(a_1,a_2)} = J^+_{(b_1,b_2)}$ that is the second equivalence in (7).

Corollary 2 (Pareto optimality). *For each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K following inclusions hold:*

$$\bigcap_{j\in J} \sigma_j \subseteq R(G_Q) \subseteq \bigcup_{j\in J} \sigma_j. \tag{9}$$

Proof (of corollary 2). Fix a pair $(a_1, a_2) \in \bigcap_{j\in J} \sigma_j$. By axiom $(A4)^*$ there exists a family of linear quasi-orders $\left(\sigma^1_j\right)_{j\in J}$ on A such that $(a_1, a_2) \in R\left(G^1_Q\right)$ where $G^1_Q = \left\langle A, (\sigma^1_j)_{j\in J}\right\rangle$. For arbitrary $j \in J$ the following implication

$$a_1 \leq^{\sigma^1_j} a_2 \Rightarrow a_1 \leq^{\sigma_j} a_2$$

holds (since the conclusion of this implication is truth). Put $\rho = R(G_Q)$ and $\rho^1 = R\left(G^1_Q\right)$. According to axiom $(A2)^*$ the condition $a_1 \lesssim^{\rho^1} a_2$ implies the condition $a_1 \lesssim^{\rho} a_2$. Because the first condition is truth by assumption, we have that the second condition is truth also. Thus the first inclusion in (9) is proved. To prove the second inclusion, fix a pair $(a_3, a_4) \notin \bigcup_{j\in J} \sigma_j$. By axiom $(A4)^*$ there exists a family of linear quasi-orders $\left(\sigma^2_j\right)_{j\in J}$ on A such that $(a_3, a_4) \notin R\left(G^2_Q\right)$ where $G^2_Q = \left\langle A, (\sigma^2_j)_{j\in J}\right\rangle$. For arbitrary $j \in J$ the following implication

$$a_3 \leq^{\sigma_j} a_4 \Rightarrow a_3 \leq^{\sigma^2_j} a_4 \tag{10}$$

holds (since the condition of this implication is false). Assume $(a_3, a_4) \in R(G_Q)$. Then using (10) we receive by axiom $(A2)^*$ $(a_3, a_4) \in R\left(G^2_Q\right)$ in contradiction with our assumption. Hence $(a_3, a_4) \notin R(G_Q)$ and the implication

$$(a_3, a_4) \notin \bigcup_{j\in J} \sigma_j \Rightarrow (a_3, a_4) \notin R(G_Q) \tag{11}$$

is shown. It remains to note that (11) is equivalent to the second inclusion in (9).

3. Pseudofilters and filters

In this section we will study a notion of pseudofilter which can be used for construction of some rule of preferences in models of the form (1).

Definition 2. Let J be an arbitrary set. A family W of its subsets is called *a pseudofilter* over J if it satisfies the following conditions:

(PF1) *Nonemptiness:* $W \neq \emptyset$;
(PF2) *Majorant stability:* $S \in W, T \supseteq S \Rightarrow T \in W$;
(PF3) *Anticomplement:* $S \in W \Rightarrow S' \notin W$.

Let us note some consequences of these axioms.

(C1) $J \in W$.

(C2) $\emptyset \notin W$.

(C3) $S, T \in W \Rightarrow S \cap T \neq \emptyset$.

Indeed, by (PF1) there exists a subset $S \subseteq J$ with $S \in W$. By (PF2) we have $J \in W$, i.e. (C1). Using (C1) and (PF3) we obtain (C2). Prove (C3). Suppose $S \cap T = \emptyset$ then $T \subseteq S'$ and by (PF2) we obtain $S' \in W$. Because $S \in W$ that contradict (PF3).

Example 1. A game in the form of characteristic function can be given as a pair $\langle J, v\rangle$ where J is an arbitrary set (named a set of *players*) and v is a function which any subset $S \subseteq J$ assigns a real number $v(S)$. In the game theoretical terminology, any subset $S \subseteq J$ is called a *coalition.* The characteristic function v is said to be *superadditive* if for any subsets $S, T \subseteq J$ with $S \cap T = \emptyset$ the inequality

$$v(S) + v(T) \leq v(S \cup T) \tag{12}$$

holds. A game $\langle J, v\rangle$ is called *prime one* if values of the function v are 0 and 1 only. The following assertion is noted by Herve Moulin (Moulin, 1981).

Lemma 1. *Let $\langle J, v\rangle$ be a prime game and W be a family of winning coalitions (i.e. coalitions $S \subseteq J$ with $v(S) = 1$). The characteristic function v is superadditive if and only if W satisfies conditions (PF2) and (PF3).*

Proof (of lemma 1). Let v be superadditive. Check (PF2). Suppose $S \in W$ and $T \supseteq S$. Put $T_1 = T \cap S'$. Since $S \cap T_1 = \emptyset$ and $S \cup T_1 = T$, by using (12) we have $v(S) + v(T_1) \leq v(T)$. Because $S \in W$, we obtain $v(S) = 1$ and $v(T) \geq 1$ i.e. $T \in W$. Check now (PF3). Suppose $S \in W$ and $S' \in W$ for some coalition $S \subseteq J$. Then by using (12) we have $v(J) \geq v(S) + v(S') = 1 + 1 = 2$, i.e. $v(J) \geq 2$, that is impossible. Necessity is proved.

To prove the sufficiency consider two coalitions $S, T \subseteq J$ with $S \cap T = \emptyset$. The case $S, T \notin W$ is trivial. In the opposite case according the condition (C3) we can put $S \in W, T \notin W$. Then by (PF2) we have $S \cup T \in W$ hence the left and the right parts of (12) are equal to 1 and (12) holds. □

A prime game $\langle J, v\rangle$ is said to be *trivial*, if $v(S) = 0$ for all coalitions $S \subseteq J$. Obviously, a prime game is non-trivial if and only if $W \neq \emptyset$ i.e. when axiom (PF1) holds. Then using Lemma 1, we obtain

Lemma 2. *Let $\langle J, v\rangle$ be a prime game and W be a family of its winning coalitions. A game G is non-trivial with superadditive characteristic function v if and only if W is pseudofilter.*

We now consider some questions concerning a construction of pseudofilters. First of all note an important connection between the notion of pseudofilter and the notion of filter; the last is made use in various branches of algebra, mathematical logic and topology.

Definition 3. Let J be an arbitrary set. A nonempty family F of subsets J is called a *filter* over J if the following conditions hold:

(F1) $S \in F, T \in F \Rightarrow S \cap T \in F$;

(F2) $S \in F, T \supseteq S \Rightarrow T \in F$;

(F3) $\emptyset \notin F$.

Lemma 3.

1. *Any filter is a pseudofilter.*
2. *A pseudofilter is a filter if and only if it is stable under intersection of its subsets.*

Proof (of lemma 3). 1. Let F be a filter. Then axioms (PF1), (PF2) evidently hold. Check (PF3). Assume $S, S' \in F$. Then by (F1) we have $\emptyset = S \cap S' \in F$ that contradicts (F3).

2. If a pseudofilter W is a filter the required condition holds (see (F1)). Conversely let W be a pseudofilter for which axiom (F1) holds. Axiom (F2) is equivalent to (PF2). Axiom (F3) is a consequence of (PF1) and (PF2) (see (C2)). □

We now consider some method for construction of pseudofilters. Let J be an arbitrary set and B a family of its subsets. We denote by $M(B)$ the family of all oversets for sets belonging to B:

$$M(B) = \{T \subseteq J \colon (\exists S \in B)\, S \subseteq T\}.$$

Definition 4. Let W be a pseudofilter over J and B a non empty family of some sets belonging to W i.e. $B \subseteq W$. We say that B forms *a base* of the pseudofilter W if $M(B) = W$.

Remark that any pseudofilter W has a base (for example $B = W$) and psedofilter is well defined by any its base. A base B_0 is called the *smallest base* of pseudofilter W, if $B_0 \subseteq B$ for any base B. In the case the set J is finite, each pseudofilter W has the smallest base consisting of all minimal (under inclusion) subsets of W.

Lemma 4. *Let J be an arbitrary set and B some family of its subsets. Then*

1. *B forms a base of some pseudofilter over J if and only if the following condition holds*
$$S \in B, T \in B \Rightarrow S \cap T \neq \emptyset; \tag{13}$$
2. *B forms the smallest base of some pseudofilter over J if and only if the condition (13) and the following condition*
$$S \in B, T \in B, S \subseteq T \Rightarrow S = T \tag{14}$$
holds.

Proof (of lemma 4). 1. Let B be a base of some pseudofilter. Because (13) holds in each pseudofilter (see (C3)) it holds for any its subset. Conversely, let B be some family of subsets of J for which (13) holds. Put $W = M(B)$ and show that W is a pseudofilter. Axioms (PF1) and (PF2) are evident. Check (PF3). Fix $T \in W$, i.e. $T \supseteq S$ where $S \in B$. Suppose $T' \in W$ i.e. $T' \supseteq S_1$ for some $S_1 \in B$. Then $S \cap S_1 \subseteq T \cap T' = \emptyset$ hence $S \cap S_1 = \emptyset$ that is contradiction with (13). Thus W is pseudofilter and B is its base.

2.The necessity of condition (13) have shown above. To prove (14) remark that the smallest base of pseudofilter W consists of all minimal subsets of W hence the condition (14) for smallest base holds. Let us prove sufficiency. Put $W = M(B)$. It is shown that W is pseudofilter and B is its base. We need to prove that B is the set of all minimal subsets of W. Indeed, fix $S_0 \in B$. Assume for $T \in W$ the inclusion $T \subseteq S_0$ holds. We need to check the equality $T = S_0$. By definition of mapping $M(B)$ we have $T \supseteq S$ for some $S \in B$. Then $S \subseteq T \subseteq S_0$ hence $S \subseteq S_0$ and by (14) $S = S_0$. Thus $T \supseteq S_0$ and because the inclusion $T \subseteq S_0$ also holds we obtain the equality $T = S_0$.

It remains to prove that each minimal subset of W belongs to B. Indeed let subset $T_1 \in W$ be a minimal in W. We have $T_1 \supseteq S_1$ where $S_1 \in B$. The strict inclusion $T_1 \supset S_1$ is impossible and we obtain $T_1 = S_1 \in B$. □

4. Rules for preferences based on pseudofilters of winning coalitions

Consider the class K of models $G = \langle A, (q_j)_{j \in J} \rangle$ of the form (1). Associate with each model $G \in K$ a model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ where σ_j is a linear quasi-order on A defined by

$$a \leq^{\sigma_j} a' \Leftrightarrow q_j(a) \leq_j q_j(a'). \tag{15}$$

It is shown above that we can consider K as a class consisting of models of the form G_Q. The aim of this section is to introduce a fairly general rule for preferences in class K satisfying to some natural axioms. We solve this problem in the following manner.

Definition 5. Let W be a pseudofilter over J. Subsets belonging to W are called *winning coalitions of criteria* (briefly, *winning coalitions*).We now define a rule R_W for preferences in the class K which any model $G \in K$ assigns a binary preference relation $R_W(G) = R_W(G_Q) = \rho_W$ on A given by the formula:

$$a \lesssim^{\rho_W} a' \Leftrightarrow \{j \in J \colon a \leq^{\sigma_j} a'\} \in W. \tag{16}$$

The rule given by definition 5 is called *a rule defined by pseudofilter W*.

Example 2. Put $W = \{J\}$ (obviously, W is a pseudofilter). Then preference relation ρ_W coincides with Pareto-preference.

Example 3. Fix a real number $\alpha > 1/2$. Let $r = \alpha n$ if αn is integer and $r = [\alpha n] + 1$ otherwise (where $n = |J|$). Now put $W = \{S \subseteq J \colon |S| \geq r\}$ (it is easy to show that W is a pseudofilter). Then preference relation ρ_W coincides with rule of α–majority, see section 1.

Remark 2. Because $J \in W$ for any pseudofilter (see (C1), section 3), a preference relation ρ_W is reflexive always. But axiom of transitivity for ρ_W need not be holds. For example, preference relation for rule of α–majority is not transitive in general case.

It follows from the definition 5

Corollary 3. *Fix a models G_Q of class K and let for some $a, a' \in A$ the condition $\{j \in J \colon a <^{\sigma_j} a'\} \in W$ holds. Then $a <^{\rho_W} a'$ holds.*

Proof (of corollary 3). We have $T = \{j \in J \colon a \leq^{\sigma_j} a'\} \supseteq \{j \in J \colon a <^{\sigma_j} a'\} = S$. Since $S \in W$, by axiom (PF2) we obtain $T \in W$ hence $a \lesssim^{\rho_W} a'$ holds. On the other hand $\{j \in J \colon a \geq^{\sigma_j} a'\} = \{j \in J \colon a <^{\sigma_j} a'\}' = S' \notin W$ hence by definition 5 the condition $a' \lesssim^{\rho_W} a$ does not hold. Thus we obtain $a <^{\rho_W} a'$. □

We now state the following important result.

Theorem 1. *Any rule for preferences in class K defined by a pseudofilter W satisfies axioms (A1)*–(A4)*.*

Proof (of theorem 1). We need to check these axioms for rule (16).

$(A1)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle B, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Denote by $R_W(G_Q) = \rho_W, R_W\left(G_Q^1\right) = \rho_W^1$. Suppose for elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following equivalences

$$a_1 \leq^{\sigma_j} a_2 \Leftrightarrow b_1 \leq^{\sigma_j^1} b_2$$

hold for each $j \in J$. Then $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = \left\{j \in J \colon b_1 \leq^{\sigma_j^1} b_2\right\}$ hence conditions

$$\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \in W \text{ and } \left\{j \in J \colon b_1 \leq^{\sigma_j^1} b_2\right\} \in W$$

are equivalent and by (16) conditions $a_1 \lesssim^{\rho_W} a_2$ and $b_1 \lesssim^{\rho_W^1} b_2$ are equivalent also.

$(A2)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Fix elements $a_1, a_2 \in A$ and let for every $j \in J$ the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2$$

holds. Then we have

$$S = \{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \subseteq \left\{j \in J \colon a_1 \leq^{\sigma_j^1} a_2\right\} = T.$$

By (16) the condition $a_1 \lesssim^{\rho_W} a_2$ means $S \in W$; using the inclusion $S \subseteq T$ and axiom (PF2) we obtain $T \in W$, that is $a_1 \lesssim^{\rho_W^1} a_2$.

$(A3)^*$ Consider two models $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ and $G_Q^1 = \left\langle A, (\sigma_j^1)_{j\in J}\right\rangle$ of class K. Assume for elements $a_1, a_2 \in A$ and all $j \in J$ following implications

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2 \tag{17}$$

hold. Then as above we obtain that $a_1 \lesssim^{\rho_W} a_2$ implies $a_1 \lesssim^{\rho_W^1} a_2$. On the other hand, the condition $a_1 \lesssim^{\rho_W} a_2$ means that $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \in W$ then by axiom (PF3) $U = \{j \in J \colon a_1 \leq^{\sigma_j} a_2\}' \notin W$.

It follows from (17) that $\left\{j \in J \colon a_1 <^{\sigma_j^1} a_2\right\}' \subseteq \{j \in J \colon a_1 \leq^{\sigma_j} a_2\}' = U \notin W$.

Then we have

$$V = \left\{j \in J \colon a_2 \leq^{\sigma_j^1} a_1\right\} = \left\{j \in J \colon a_1 <^{\sigma_j^1} a_2\right\}' \subseteq U \notin W$$

and by axiom (PF2) $V \notin W$, i.e. the condition $a_2 \lesssim^{\rho_W^1} a_1$ does not hold. Thus the assumption $a_1 \lesssim^{\rho_W} a_2$ implies $a_1 <^{\rho_W^1} a_2$ which was to be proved.

$(A4)^*$ Let A be an arbitrary set. Fix two elements $a_1, a_2 \in A$ with $a_1 \neq a_2$. Consider two families of linear quasi-orders $(\sigma_j')_{j\in J}$ and $(\sigma_j'')_{j\in J}$ on A such that for any $j \in J$ conditions $a_1 <^{\sigma_j'} a_2$ and $a_2 <^{\sigma_j''} a_1$ hold. Then we have

$$\left\{j \in J \colon a_1 \leq^{\sigma_j'} a_2\right\} = J \in W \text{ and } \left\{j \in J \colon a_1 \leq^{\sigma_j''} a_2\right\} = \emptyset \notin W.$$

Put $\rho_W' = R_W\left(\left\langle A, (\sigma_j')_{j\in J}\right\rangle\right)$ and $\rho_W'' = R_W\left(\left\langle A, (\sigma_j'')_{j\in J}\right\rangle\right)$. According with (16) we obtain that the condition $a_1 \lesssim^{\rho_W'} a_2$ holds and the condition $a_1 \lesssim^{\rho_j''} a_2$ does not hold which completes the proof of Theorem 1. $\square$

We now state the converse of Theorem 1.

Theorem 2. *Fix a family of scales* $\left\langle C_j, (\leq_j)_{j\in J}\right\rangle$ *for measurement of quality criteria. Let* R *be a rule for preferences which every models* $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ *of class* K *assigns some reflexive preference relation* $R(G_Q) = \rho$ *on* A *and for* R *axioms (A1)* – (A4)* hold. Then there exists a pseudofilter* W *over* J *such that* $R = R_W$.

Proof (of theorem 2). Let us define a family W of winning coalitions of criteria in the following manner. For any subset $S \subseteq J$, the condition $S \in W$ means that there exists a model $\overline{G}_Q = \left\langle \overline{A}, (\overline{\sigma}_j)_{j\in J}\right\rangle$ of class K and elements $\overline{a}_1, \overline{a}_2 \in \overline{A}$ such that

$$\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2 \text{ and } \left\{j \in J\colon \overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2\right\} = S \tag{18}$$

(we denote by $\overline{\rho} = R\left(\overline{G}_Q\right)$).

Further we define a rule R_W for preferences in class K and write $R_W(G) = R_W(G_Q) = \rho_W$ by setting for any $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K and every $a_1, a_2 \in A$

$$a_1 \lesssim^{\rho_W} a_2 \Leftrightarrow \{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \in W.$$

As the first step, we show the equality $R_W = R$. It suffices to prove that for each model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K the equivalence

$$a_1 \lesssim^{\rho} a_2 \Leftrightarrow \{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \in W. \tag{19}$$

holds. In fact, the implication $\Rightarrow$ in (19) is truth by definition of family W. Conversely, suppose the right part of (19) holds. Then there exists a model $\overline{G}_Q = \left\langle \overline{A}, (\overline{\sigma}_j)_{j\in J}\right\rangle$ of class K and elements $\overline{a}_1, \overline{a}_2 \in \overline{A}$ such that

$$\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2 \text{ and } \left\{j \in J\colon \overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2\right\} = \{j \in J\colon a_1 \leq^{\sigma_j} a_2\}.$$

Then conditions $a_1 \leq^{\sigma_j} a_2$ and $\overline{a}_1 \leq^{\overline{\sigma}_j} \overline{a}_2$ are equivalent for any $j \in J$; by axiom $(A1)^*$ the propositions $a_1 \lesssim^{\rho} a_2$ and $\overline{a}_1 \lesssim^{\rho^1} \overline{a}_2$ are equivalent also and because $\overline{a}_1 \lesssim^{\overline{\rho}} \overline{a}_2$ is truth we obtain that $a_1 \lesssim^{\rho} a_2$ is truth.

It remains to be shown that W is a pseudofilter. Check axioms (PF1)–(PF3).

(PF1) Let A be an arbitrary set with $|A| \geq 2$. Fix two elements $a_1, a_2 \in A$. For any $j \in J$ let σ_j be a linear quasi-order on A with $a_1 \leq^{\sigma_j} a_2$. Then condition $(a_1, a_2) \in \bigcap\limits_{j\in J} \sigma_j$ holds and using Corollary 2 we obtain $a_1 \lesssim^{\rho} a_2$; since $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} = J$ then by (18) $J \in W$, i.e. $W \neq \emptyset$.

(PF2) Suppose $S \in W$ and $T \supseteq S$. We need to prove $T \in W$. In fact, by (18) there exists a model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K and elements $a_1, a_2 \in A$ such that $a_1 \lesssim^{\rho} a_2$ and $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} = S$. Consider a family of linear quasi-orders $\left(\sigma_j^1\right)_{j\in J}$ on A defined as follows. For $j \in T \cap S'$, the quasi-order σ_j^1 the condition $a_1 \leq^{\sigma_j^1} a_2$ satisfies and $\sigma_j^1 = \sigma_j$ for other $j \in J$. Then

$$\left\{j \in J\colon a_1 \leq^{\sigma_j^1} a_2\right\} = (T \cap S') \cup S = T. \tag{20}$$

Let us show the following implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 \leq^{\sigma_j^1} a_2. \tag{21}$$

Indeed, for $j \in T \cap S'$ the implication (21) holds since its consequence is truth; in other cases the condition and the consequence of (21) are equivalent. Denote by $G_Q^1 = \langle A, (\sigma_j^1)_{j\in J}\rangle$, $R(G_Q) = \rho, R(G_Q^1) = \rho^1$. Since $a_1 \lesssim^\rho a_2$ then by axiom $(A2)$* and (21) we have $a_1 \lesssim^{\rho^1} a_2$; using (20) and (18) we obtain $T \in W$.

(PF3) Suppose $S \in W$ i.e. there exists a model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K and elements $a_1, a_2 \in A$ such that $a_1 \lesssim^\rho a_2$ and $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} = S$. Assume $S' \in W$. Consider a family $(\sigma_j^1)_{j\in J}$ of linear quasi-orders on A satisfying $\left\{j \in J\colon a_1 <^{\sigma_j^1} a_2\right\} = S$. Then for any $j \in J$ the implication

$$a_1 \leq^{\sigma_j} a_2 \Rightarrow a_1 <^{\sigma_j^1} a_2. \tag{22}$$

is truth. Put $G_Q^1 = \langle A, (\sigma_j^1)_{j\in J}\rangle$, $R(G_Q) = \rho, R(G_Q^1) = \rho^1$. Using (22) and the condition $a_1 \lesssim^\rho a_2$ we obtain by axiom $(A3)$* the condition $a_1 <^{\rho^1} a_2$. On the other hand, since

$$\left\{j \in J\colon a_2 \leq^{\sigma_j^1} a_1\right\} = \left\{j \in J\colon a_1 <^{\sigma_j^1} a_2\right\}' = S' \in W$$

we have $a_2 \lesssim^{\rho_W^1} a_1$; because $R_W = R$ we obtain $a_2 \lesssim^{\rho^1} a_1$ in contradiction with condition $a_1 <^{\rho^1} a_2$ proved above. □

To conclude this section we consider a construction of rules for preferences based on filters of winning coalition.

Theorem 3. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ of class K the preference relation $\rho_W = R_W(G_Q)$ is transitive if and only if the pseudofilter W is a filter.*

Proof (of theorem 3). Necessity. Suppose W is not a filter then by Lemma 3 there exist subsets $S, T \in W$ such that $S \cap T \notin W$. Put $A = \{a_1, a_2, a_3\}$ and for every $j \in J$ let us define a linear order relation σ_j as follows:

$$\begin{aligned}
&a_3 <^{\sigma_j} a_1 <^{\sigma_j} a_2 \text{ for all } j \in S \cap T';\\
&a_1 <^{\sigma_j} a_2 <^{\sigma_j} a_3 \text{ for all } j \in S \cap T;\\
&a_2 <^{\sigma_j} a_3 <^{\sigma_j} a_1 \text{ for all } j \in T \cap S';\\
&a_3 <^{\sigma_j} a_1 \text{ for all } j \in J \cap (S \cup T)'.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} \supseteq (S \cap T') \cup (S \cap T) = S \in W;\\
&\{j \in J\colon a_2 \leq^{\sigma_j} a_3\} \supseteq (S \cap T) \cup (T \cap S') = T \in W;\\
&\{j \in J\colon a_1 \leq^{\sigma_j} a_3\} = S \cap T \notin W.
\end{aligned} \tag{23}$$

According with Definition 5 and using (23) and axiom (PF2) we obtain $a_1 \lesssim^{\rho_W} a_2$, $a_2 \lesssim^{\rho_W} a_3$ but the condition $a_1 \lesssim^{\rho_W} a_3$ does not hold.

Sufficiency. Let $G_Q = \langle A, (\sigma_j)_{j\in J}\rangle$ be any model of class K. Put $R_W(G_W) = \rho_W$. Suppose $a_1 \lesssim^{\rho_W} a_2, a_2 \lesssim^{\rho_W} a_3$. Then by definition 5 we have $\{j \in J\colon a_1 \leq^{\sigma_j} a_2\} = S \in W$, $\{j \in J\colon a_2 \leq^{\sigma_j} a_3\} = T \in W$ hence by axiom (F2) $S \cap T \in W$. Obviously, $S \cap T \subseteq \{j \in J\colon a_1 \leq^{\sigma_j} a_3\}$ and by axiom (F2) we obtain $\{j \in J\colon a_1 \leq^{\sigma_j} a_3\} \in W$, i.e. $a_1 \lesssim^{\rho_W} a_3$ which was to be proved. □

We now consider the condition of linearity of preference relations. It connects with condition of maximality for filters. Recall that a filter W over J is a maximal one (or *ultrafilter*) if it satisfies the condition

$$\text{either } S \in W \text{ or } S' \in W \text{ for every } S \subseteq J. \tag{24}$$

Theorem 4. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K the preference relation $\rho_W = R_W(G_Q)$ is linear if and only if the pseudofilter W the condition (24) satisfies.*

Proof (of theorem 4). Necessity. Assume (24) does not hold for pseudofilter W then there exists a subset $S \subseteq J$ such that $S \notin W$ and $S' \notin W$. Consider a model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K where $A = \{a_1, a_2\}$ and linear quasi-orders $(\sigma_j)_{j \in J}$ the following conditions satisfy:

$$a_1 <^{\sigma_j} a_2 \text{ for each } j \in S;$$
$$a_2 <^{\sigma_j} a_1 \text{ for each } j \in S'.$$

Then $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} = S \notin W$ and $\{j \in J \colon a_2 \leq^{\sigma_j} a_1\} = S' \notin W$. Hence by definition5 both conditions $a_1 \lesssim^{\rho_W} a_2$ and $a_2 \lesssim^{\rho_W} a_1$ are false, i.e. the preference relation ρ_W is not linear.

Sufficiency. Let $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ be an arbitrary model of class K. Put $R_W(G_Q) = \rho_W$. Fix two elements $a_1, a_2 \in A$ and suppose the condition $a_1 \lesssim^{\rho_W} a_2$ does not hold, i.e. $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\} \notin W$. Then by assumption of Theorem 4 $\{j \in J \colon a_1 \leq^{\sigma_j} a_2\}' \in W$, i.e. $\{j \in J \colon a_2 <^{\sigma_j} a_1\} \in W$; since $\{j \in J \colon a_2 <^{\sigma_j} a_1\} \subseteq \{j \in J \colon a_2 \leq^{\sigma_j} a_1\}$ by axiom (PF2) we obtain $\{j \in J \colon a_2 \leq^{\sigma_j} a_1\} \in W$, that is $a_2 \lesssim^{\rho_W} a_1$. Thus the relation ρ_W is linear. $\square$

It follows from Theorem 3 and Theorem 4

Corollary 4. *Let R_W be a rule for preferences in class K which based on pseudofilter W. Then for every model $G_Q = \langle A, (\sigma_j)_{j \in J} \rangle$ of class K the preference relation $\rho_W = R_W(G_Q)$ is a linear quasi-order if and only if the pseudofilter W is an ultrafilter.*

It follows from results of this section an interpretation of Arrow paradox in terms of filters. In fact, any rule for preferences in class models K which leads to linear quasi-order can be given by some ultrafilter. Since the set of criteria J is finite, every filter W over J is a principal one and a principal ultrafilter consists of all subsets which contain some fix element $j^* \in J$; namely this element j^* is called *a dictator* in terms of Arroy.

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Existence of Stable Coalition Structures in Three-person Games

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Abstract Cooperative games with coalition structures are considered and the principle of coalition structure stability with respect to cooperative solution concepts is determined. This principle is close to the concept of Nash equilibrium. The existence of a stable coalition structure with respect to the Shapley value and the equal surplus division value for the cases of two- and three-person games is proved. We also consider a specific model of cooperative cost-saving game among banks as an application. In the model, the characteristic function assigning the cost-saving game has a special form. For the model the software product is developed and illustrative examples are provided.

Keywords: coalition structure, stability, Shapley value, equal surplus division value

1. Introduction

Many conflict problems which allow cooperation among players can be modeled with the help of cooperative TU-games. The basic idea of cooperation is that if all players form the unique grand coalition, they immediately start to behave in the interests of this coalition, i. e. try to maximize the grand coalition payoff. The next step of cooperative game theory is to find a proper allocation of the achieved payoff using a priori chosen solution concept. Some of the most commonly known single-valued solution concepts in practice are the Shapley value (Shapley, 1953), the equal surplus division value or the ES-value (Driessen and Funaki, 1991) and the nucleolus (Schmeidler, 1969). If we do not take into account axiomatic properties of the solution concepts and do not compare them, the first two solution concepts have some advantage over the nucleolus: they have explicit formulas which significantly simplify computations.

If we allow cooperation among players in the game, it is naturally to suggest that not only grand coalition but smaller ones should be formed. It is a common situation in politics because of the difficulty of joining all politicians together and, moreover, forcing them to behave in the interests of the unique grand coalition. To describe the model more particulary with these assumptions, we use the theory of games with coalition structure.

In games with coalition structure one coalition might be more preferable for a player than others. That is why it is reasonable to find a coalition structure in which each player does not have any benefit deviating from his coalition. We call this coalition structure stable. The general idea of stability is based on comparing players' payoffs but not coalition payoffs. Some ideas of stability concepts of coalition structures are introduced in (Haeringer, 2001; Tiebout, 1956; Hart and Kurz, 1983). The stable coalition structure must satisfy three basic assumptions proposed in (Carraro, 1997). More specifically, it must be (i) internally stable, i. e. each player looses if he leaves his coalition becoming a singleton, (ii) externally stable, i. e. each player-singleton looses if he joins any coalition or another singleton, and, finally, (iii) intracoalitionally stable, i. e. each player from a coalition looses if he leaves his coalition and joins another one. Here we may find some similarities with the Nash equilibrium concept. There exist papers in which the stability of a coalition structure is investigated in a strategic way assuming that coalitions play the Nash equilibrium, and then payoff of each coalition is allocated by the Shapley value (Petrosyan and Mamkina, 2006). In the present paper we follow the idea of Aumann and Dreze (Aumann and Dreze, 1974) supposing that the characteristic (value) function is given. We consider the Shapley value as well as the equal surplus division value (the ES-value) as solution concepts and examine the stability of coalition structures with respect to these two solution concepts.

The paper has the following structure. In Section 2. the setting of the game with coalition structure is considered. Single-valued solution concepts like the Shapley value and the ES-value are provided. The definition of the stable coalition structure with respect to the single-valued solution concept is introduced. In Section 3. it is proved that for at least two and three-person games there always exists at least one stable coalition structure in terms of the stability concept. In Section 4. a specific model of bank cooperation is proposed. In this setting a cost-saving game with the characteristic function of a special form is constructed. With the help of a developed software product for the specific model, one can easily extract stable coalition structures with respect to the Shapley value and the ES-value. Section 4. also contains the description and screenshots of the product.

2. Game with coalition structure

2.1. Definitions

In a classical setting, a cooperative game is determined by a tuple (N, v) where N is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function defined for every nonempty set $S \subset N$ called coalition. In this setting one may suggest that grand coalition N should be formed and then players from N allocate their total payoff $v(N)$ according to some solution concept. Unlike classic assumption (Owen, 1995), we suppose that the characteristic function might not be supperadditive, i. e. there exist at least two disjoint coalitions $S, T \subset N$ such as $v(S \cup T) < v(S) + v(T)$. Therefore, in general not only the grand coalition but smaller ones can be formed. It can take place when some players get larger payoff if they form a smaller coalition. Therefore, we allow formation of not only grand coalition, and consider games with coalition structure.

Definition 1. Coalition structure π is a partition $\{B_1, \dots, B_m\}$ of the set N, i. e. $B_1 \cup \dots \cup B_m = N$, and $B_i \cap B_j = \emptyset$ for all $i, j = 1, \dots, m, i \neq j$.

Denote a game with player set N, characteristic function v and coalition structure π by (N, v, π).

Definition 2. A profile $x^\pi = (x_1^\pi, \dots, x_n^\pi) \in \mathbb{R}^n$ is a payoff distribution in the game (N, v, π) with coalition structure π if the efficiency condition, i. e. $\sum_{i \in B_j} x_i^\pi = v(B_j)$ holds for all coalitions $B_j \in \pi$, $j = 1, \dots, m$.

Definition 3. A payoff distribution x^π is an allocation in the game (N, v, π) with coalition structure π if the individual rationality condition, i. e. $x_i^\pi \geq v(\{i\})$ holds for any player $i \in N$.

Denote the coalition partition $\pi_{-B_i} = \pi \setminus B_i \subset \pi$ by π_{-B_i}, and the coalition which contains player $i \in N$ by $B(i) \in \pi$.

In the game (N, v, π) with coalition structure $\pi = \{B_1, \dots, B_m\}$ we can choose any cooperative solution concept from the classical cooperative game theory for payoff distribution calculation. If we choose the Shapley value $\phi^\pi = (\phi_1^\pi, \dots, \phi_n^\pi)$, its components are calculated as follows:

$$\phi_i^\pi = \sum_{S \subseteq B(i), i \in S} \frac{(|B(i)| - |S|)!(|S| - 1)!}{|B(i)|!} [v(S) - v(S \setminus \{i\})] \tag{1}$$

for any $i \in N$. As an alternative solution concept, we use the ES-value:

$$\psi_i^\pi = v(\{i\}) + \frac{v(B(i)) - \sum\limits_{j \in B(i)} v(\{j\})}{|B(i)|} \tag{2}$$

for any $i \in N$.

2.2. Stable coalition structures

The determination of stable coalition structures is an actual problem. Here we use an approach which takes into account the player's payoff as a member of his coalition. Therefore, the player compares his payoff according to the current coalition structure with the payoffs that he can obtain if he deviates from his coalition and other players do not deviate. So, he can change coalition structure becoming a singleton or joining another coalition from the current coalition structure. And if any player cannot increase his payoff by the way describing above, the coalition structure is stable. Define this principle as follows:

Definition 4. Coalition structure $\pi = \{B_1, \dots, B_m\}$ is said to be stable with respect to a single-valued cooperative solution concept if for any player $i \in N$ the inequality

$$x_i^\pi \geq x_i^{\pi'} \text{ holds for all } B_j \in \pi \cup \emptyset,\ B_j \neq B(i).$$

Here x^π and $x^{\pi'}$ are two payoff distributions calculated according to the chosen cooperative solution concept for games (N, v, π) and (N, v, π') with coalition structures π, π' respectively, where $\pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$.

The stability concept from Definition 4 is similar to the Nash equilibrium concept. Consider stable coalition structure π and calculate player i's payoff according to the some cooperative solution concept like the Shapley value. Now imagine that player i has the following set of strategies: to stay in a current coalition, to become a singleton or to join any other existing coalition in the coalition structure. If each player compares his payoff x_i^π, $i \in N$ with all the possible payoffs that he can obtain using one of the above mentioned strategies (when all other players do not deviate) and finds out that he cannot get larger payoff, then the current players' strategies form the Nash equilibrium. In other words, the current coalition structure is stable with respect to the chosen cooperative solution concept.

As single-valued cooperative solution concepts we can consider concepts as the Shapley value (Shapley, 1953), nucleolus (Schmeidler, 1969), the equal surplus division value (Driessen and Funaki, 1991).

In Definition 4 we make the following assumption which seems to be natural. If player $i \in B(i)$ leaves coalition $B(i)$, coalition $B(i) \setminus \{i\}$ does not break, and is still the part of a new coalition structure, so player i can join any existing coalition in the current coalition structure without any restrictions or become a singleton.

3. Existence of stable coalition structures

3.1. Transformation of characteristic function

Assume that coalition structure π is stable with respect to a single-valued solution concept and $x^\pi = (x_1^\pi, \dots, x_n^\pi)$ is the allocation calculated according to this solution concept.

Construct new characteristic function $u(\cdot)$ by a transformation of the function $v(\cdot)$ as follows:

$$u(S) = v(S) + \sum_{i \in S} c_i, \quad S \subseteq N,$$

and setting $u(\{i\}) = 0$ for all $i \in N$, constants c_i can be defined below. From the equation $u(\{i\}) = v(\{i\}) + c_i$ conclude that $c_i = -v(\{i\})$, for all $i \in N$. Therefore,

$$u(S) = v(S) - \sum_{i \in S} v(\{i\}), \quad S \subseteq N \tag{3}$$

Following (Petrosyan and Zenkevich, 1996), there is a mapping that each pair $(v(\cdot), x^\pi)$ corresponds to a pair $(u(\cdot), y^\pi)$, where components of allocation y^π are defined by

$$y_i^\pi = x_i^\pi - v(\{i\}), \quad i \in N \tag{4}$$

and function $u(\cdot)$ is defined by (3).

Lemma 1. *If in game (N, v, π) coalition structure $\pi = \{B_1, \dots, B_m\}$ is stable with respect to a single-valued solution concept with allocation x^π, then in game (N, u, π) coalition structure π is also stable with respect to the same solution concept with an allocation y^π and vice versa. Here $u(\cdot)$ and y^π are defined by equations* (3) *and* (4) *respectively.*

Proof. If π is stable with respect to a single-valued solution concept with an allocation x^π, then $x_i^\pi \geq x_i^{\pi'}$ for all $B_j \in \pi \cup \emptyset$, $B_j \neq B(i)$. Here x^π and $x^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, v, π) and (N, v, π') respectively, and $\pi' = \{B(i) \setminus \{i\}, B_j \cup \{i\}, \pi_{-B(i) \cup B_j}\}$. Using (4) the stability condition can be rewritten as:

$$y_i^\pi + v(\{i\}) \geq y_i^{\pi'} + v(\{i\}) \;\text{ or }\; y_i^\pi \geq y^{\pi'}.$$

Here y^π and $y^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, u, π) and (N, u, π') respectively. It means that coalition structure π is also stable in modified game (N, u, π).

On the other hand, if π is stable with respect to a solution concept with allocation y^π in modified game (N, u, π), then $y_i^\pi \geq y_i^{\pi'}$ for all $B_j \in \pi \cup \emptyset$, $B_j \neq B(i)$. Here y^π and $y^{\pi'}$ are two allocations calculated according to the solution concept for games (N, u, π) and (N, u, π') respectively. Using (4) the stability condition can be rewritten as:

$$x_i^\pi - v(\{i\}) \geq x_i^{\pi'} - v(\{i\}) \;\text{ or }\; x_i^\pi \geq x^{\pi'}.$$

Here x^π and $x^{\pi'}$ are two allocations calculated according to the same solution concept for games (N, v, π) and (N, v, π') respectively. We obtain that coalition structure π is also stable in game (N, v, π).

These both facts prove the lemma. □

3.2. Stable coalition structures in two-person games

Following Lemma 1 it is sufficient to consider two-person cooperative games with characteristic function determined by the following way: $v(\{1,2\}) = c$ and $v(\{1\}) = v(\{2\}) = 0$.

In the case of the two-person game there are two possible coalition structures: $\pi_1 = \{\{1,2\}\}$, $\pi_2 = \{\{1\},\{2\}\}$. It is obvious that the Shapley value and the ES-value coincide and are calculated by formulas:

$$\phi_1^{\pi_1} = \phi_2^{\pi_1} = \psi_1^{\pi_1} = \psi_2^{\pi_1} = c/2,$$

$$\phi_1^{\pi_2} = \phi_2^{\pi_2} = \psi_1^{\pi_2} = \psi_2^{\pi_2} = 0.$$

Proposition 1. *In the game (N, v, π) where $N = \{1,2\}$ there always exists stable coalition structure with respect to the Shapley value and the ES-value.*

Proof. It is obvious that if $c < 0$, then coalition structure π_2 is stable with respect to the Shapley value and the ES-value. If $c > 0$, then coalition structure π_1 is stable with respect to the Shapley value and the ES-value. And, finally, if $c = 0$, both coalition structures π_1 and π_2 are stable with respect to the Shapley value and the ES-value. □

3.3. Stable coalition structures with respect to the Shapley value in three-person games

Following Lemma 1, in case of three-person game it is sufficient to consider characteristic function $v(\cdot)$ defined like this: $v(\{1,2,3\}) = c$, $v(\{1,2\}) = c_3$, $v(\{1,3\}) = c_2$, $v(\{2,3\}) = c_1$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$. The Shapley values calculated for all possible coalition structures are represented in Table 1.

Table 1: The Shapley value for a three-person coalition game

π	ϕ_1^π	ϕ_2^π	ϕ_3^π
$\{\{1,2,3\}\}$	$(2c - 2c_1 + c_2 + c_3)/6$	$(2c - 2c_2 + c_1 + c_3)/6$	$(2c - 2c_3 + c_1 + c_2)/6$
$\{\{1,2\},\{3\}\}$	$c_3/2$	$c_3/2$	0
$\{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$
$\{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$
$\{\{1\},\{2\},\{3\}\}$	0	0	0

Table 2: The "Stable if" conditions

π	"Stable if" condition
$\pi_1 = \{\{1,2,3\}\}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 2c \\ 2c \\ 2c \end{pmatrix}$
$\pi_2 = \{\{1,2\},\{3\}\}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_3 = \{\{1,3\},\{2\}\}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_4 = \{\{1\},\{2,3\}\}$	$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2c \end{pmatrix}$
$\pi_5 = \{\{1\},\{2\},\{3\}\}$	$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Notice that if $c_i \leq 0$, $i = 1,2,3$, then coalition structure $\{\{1\},\{2\},\{3\}\}$ is stable with respect to the Shapley value for any c.

Consider the case when $c_1, c_2, c_3 \geq 0$ and $c \geq 0$. Obviously, coalition structure π_5 is not stable with respect to the Shapley value. Using Table 2 and Fig. 1, we can observe that solutions of the first four systems of inequalities cover the first octant. Here assuming that $c \geq 0$, region I is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_1 is stable with respect to the Shapley value; region II is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_2 is stable with respect to the Shapley value; region III is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_3 is stable with respect to the Shapley value, and, finally, region IV is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ where coalition structure π_4 is stable with respect to the Shapley value.

Now consider the case when $c_1, c_2, c_3 \geq 0$ and $c < 0$. In this case, additionally, the coalition structure π_1 is always unstable. Using the analysis similar to the analysis in the previous case and Fig. 2, we can see that solutions of the 2^{nd}, 3^{rd} and 4^{th} systems of inequalities from Table 2 cover the first octant. Here assuming that $c < 0$, region II is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region III is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_3 is stable with respect to the Shapley value; and, finally, region IV is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_4 is stable with respect to the Shapley value.

When $c_1 < 0$, $c_2, c_3 \geq 0$, and $c \geq 0$ using Fig. 3 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, coalition structures π_4 and π_5 are always unstable with respect to the Shapley value. Here region I is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_1 is stable with respect to the Shapley value; II is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region III is the

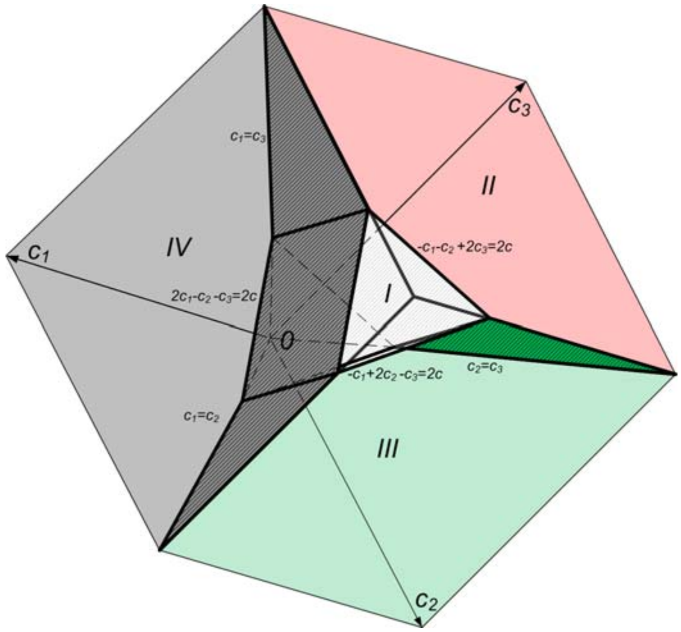


Fig. 1: Case when $c_1, c_2, c_3 \geq 0$ and $c \geq 0$

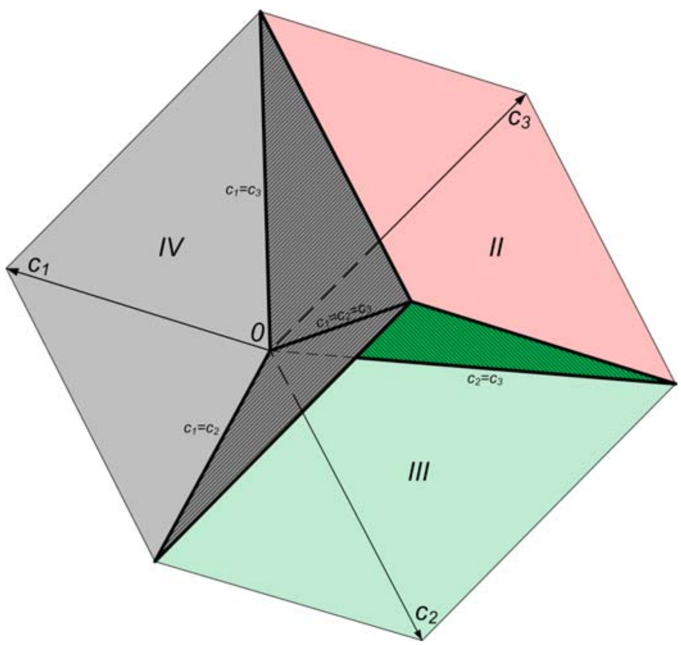


Fig. 2: Case when $c_1, c_2, c_3 \geq 0$ and $c < 0$

set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. coalition structure π_3 is stable with respect to the Shapley value.

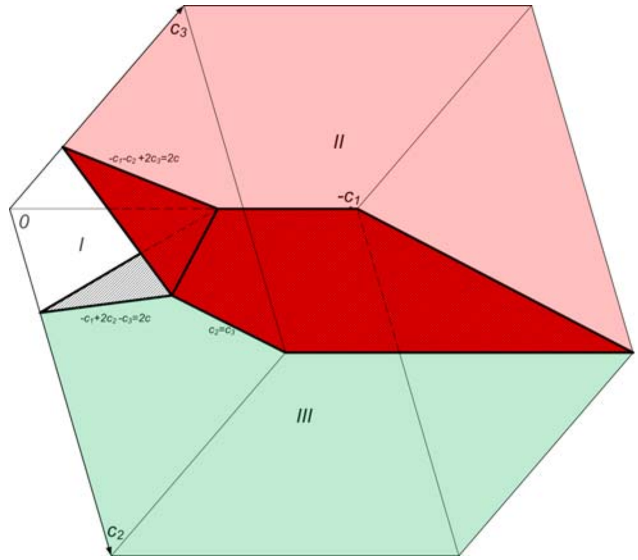


Fig. 3: $c_1 < 0$, $c_2, c_3 \geq 0$, and $c \geq 0$

When $c_1 < 0$, $c_2, c_3 \geq 0$, and $c < 0$ using Fig. 4 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, in this case coalition structures π_1, π_4 and π_5 are always unstable with respect to the Shapley value. Here

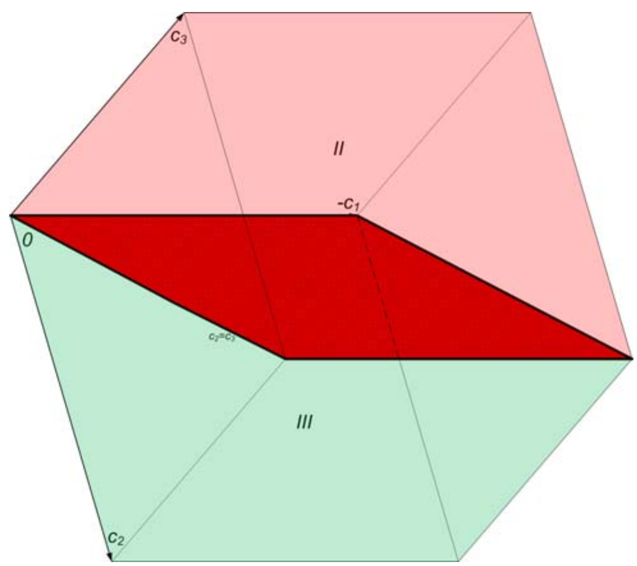


Fig. 4: $c_1 < 0$, $c_2, c_3 \geq 0$, and $c < 0$

region *II* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is stable with respect to the Shapley value; region *III* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. the coalition structure π_3 is stable with respect to the Shapley value.

When $c_1, c_2 < 0$, $c_3 \geq 0$, and $c \geq 0$ using Fig. 5 we conclude that systems of inequalities from Table 2 also cover the octant. Obviously, in this case coalition structures π_3, π_4 and π_5 are always unstable with respect to the Shapley value.

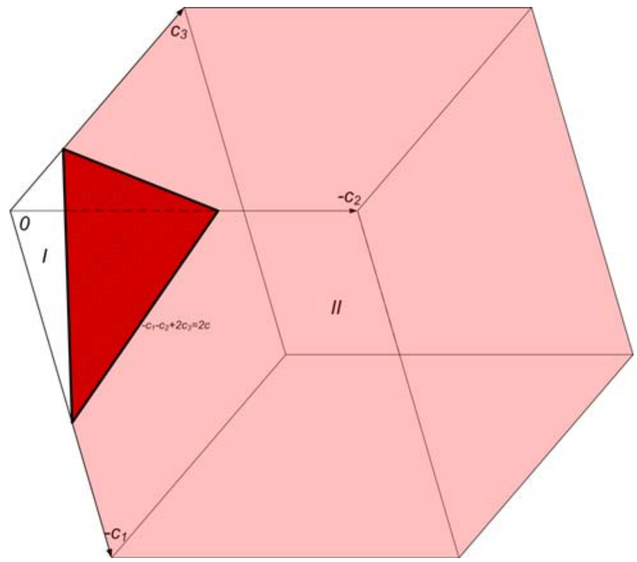


Fig. 5: $c_1, c_2 < 0$, $c_3 \geq 0$, and $c \geq 0$

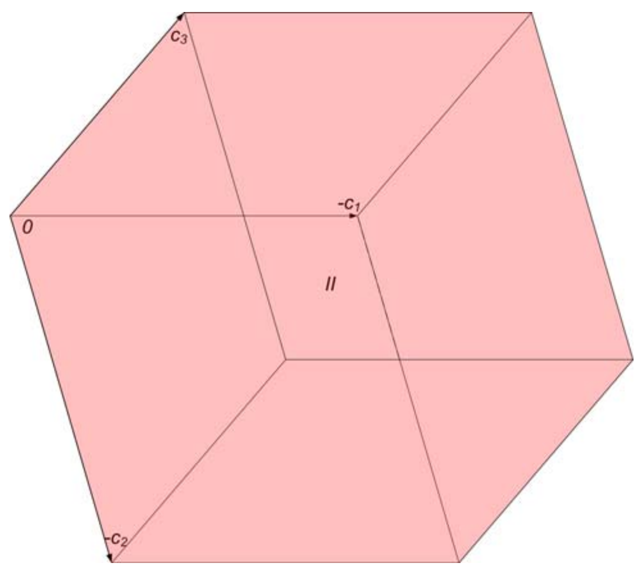


Fig. 6: $c_1, c_2 < 0$, $c_3 \geq 0$, and $c < 0$

Here region I is the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_1 is stable with respect to the Shapley value; II is the set $\{c_1 < 0, c_2 <$

$0, c_3 \geq 0\}$, and $c \geq 0$ s. t. the coalition structure π_2 is stable with respect to the Shapley value.

When $c_1, c_2 < 0$, $c_3 \geq 0$, and $c < 0$ using Fig. 6 we conclude that systems of inequalities from Table 2 and also cover the octant. Here region *II*, i. e. the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c < 0$ covers the octant, and coalition structure π_2 is unique stable with respect to the Shapley value.

For brevity, we omit the cases with another possible values of c_1, c_2, c_3 and c. The analysis for another cases is very similar to the one described above. In any possible cases the systems of inequalities from Table 2 cover an octant and it can be divided into the regions where always exists at least one stable coalition structure. The case when more than one stable coalition structures exist is also possible. For example, consider the case $c_1, c_2, c_3 \geq 0$ and $c \geq 0$. If we add the condition $c_1 = c_3$, then from Fig. 1 the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0, c_1 = c_3\}$, and $c \geq 0$ represents the region where both coalition structures π_2 and π_4 are stable with respect to the Shapley value. This region corresponds to the border between regions *II* and *IV*.

Therefore, the previous analysis proves the following proposition.

Proposition 2. *In three-person coalition game (N, v, π) there always exists a stable coalition structure with respect to the Shapley value.*

3.4. Stable coalition structures with respect to the ES-value in three-person games

Table 3 contains the components of the ES-values calculated for all possible coalition structures. We can notice that if $c \geq 0$ coalition structure π_1 is stable. And if $c_i < 0$, $i = 1, 2, 3$, then coalition structure π_5 is stable with respect to the ES-value.

Table 3: The ES-value for a three-person coalition game and "Stable if" conditions

π	ψ_1^π	ψ_2^π	ψ_3^π	"Stable if" condition
$\pi_1 = \{\{1,2,3\}\}$	$c/3$	$c/3$	$c/3$	$c \geq 0$
$\pi_2 = \{\{1,2\},\{3\}\}$	$c_3/2$	$c_3/2$	0	$\begin{cases} c_3 \geq \max\{0, c_1, c_2\} \\ c \leq 0 \end{cases}$
$\pi_3 = \{\{1,3\},\{2\}\}$	$c_2/2$	0	$c_2/2$	$\begin{cases} c_2 \geq \max\{0, c_1, c_3\} \\ c \leq 0 \end{cases}$
$\pi_4 = \{\{1\},\{2,3\}\}$	0	$c_1/2$	$c_1/2$	$\begin{cases} c_1 \geq \max\{0, c_2, c_3\} \\ c \leq 0 \end{cases}$
$\pi_5 = \{\{1\},\{2\},\{3\}\}$	0	0	0	$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Consider the case when $c < 0$ and $c_i \geq 0$, $i = 1, 2, 3$. Using Table 3, stability of coalition structures π_2, π_3 and π_4 can be proved when $c_3 \geq \max\{c_1, c_2\}$, $c_2 \geq \max\{c_1, c_3\}$ and $c_1 \geq \max\{c_2, c_3\}$ respectively. All these three inequalities cover the first octant, and the graphic solution is the same as in Fig. 2. In this case region *II* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_2 is stable with respect to the ES-value; region *III* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_3 is stable with respect to the ES-value; and, finally, region *IV* is the set $\{c_1 \geq 0, c_2 \geq 0, c_3 \geq 0\}$ s. t. coalition structure π_4 is stable with respect to the ES-value.

When $c_1 < 0, c_2, c_3 \geq 0$ from Table 3 we conclude that we have the same graphic solution as in Fig. 4 and systems of inequalities also cover the octant. Here region *II* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is stable with respect to the ES-value; *III* is the set $\{c_1 < 0, c_2 \geq 0, c_3 \geq 0\}$, and $c < 0$ s. t. the coalition structure π_3 is stable with respect to the ES-value.

And, finally, when $c_1, c_2 < 0$, $c_3 \geq 0$ from Table 3 we conclude that we have the same graphic solution as in Fig. 6 and systems of inequalities also cover the octant. Here region *II* is the set $\{c_1 < 0, c_2 < 0, c_3 \geq 0\}$, and $c < 0$ s. t. coalition structure π_2 is the unique stable with respect to the ES-value.

For brevity, in case of the ES-value we also omit the cases with another possible values of c_1, c_2, c_3 and c. The analysis for another cases is very similar to the one described above. In any possible case the systems of inequalities from Table 3 cover an octant and it can be divided into the regions where always exists at least one stable coalition structure.

Proposition 3. *In three-person coalition game (N, v, π) there always exists at least one stable coalition structure with respect to the ES-value.*

4. One specific model of bank cooperation

4.1. Problem statement

In this section we consider a model of bank cooperation for cost reduction. Let $N = \{1, \ldots, n\}$ be a set of banks which operate in a region, and banks from $A \subseteq N$ have ATMs in the region. Here we consider the simple case when banks are supposed to be focused on the cost reduction of cash withdrawal using ATMs (Bjorndal et al., 2004; Parilina, 2007; Parilina and Sedakov, 2012).

For bank $i \in N$, let $n_i > 0$ be a number of transactions, $k_i > 0$ be a number of ATMs owned by bank $i \in A$ and $k_j = 0$ for $j \in N \setminus A$. These parameters may be different for all banks, while three other parameters $0 < \alpha < \beta < \gamma$ are the same. Here α is bank transaction costs for a single cash withdrawal using his ATMs, β is bank transaction costs for a single cash withdrawal using the ATMs of another bank if both banks have an agreement allowing their clients to withdraw cash from their ATMs without any additional fees. Finally, bank transaction costs are equal to γ in any other cases.

There are two additional assumptions: (i) if a bank has ATMs, clients use only them for cash withdrawal and (ii) if two or more banks consolidate their ATMs in one network, clients choose ATMs for cash withdrawal from the network with equal probabilities.

Taking into account the notations and assumptions, one can calculate transaction costs of coalition $S \subseteq N$ if all its members consolidate their ATMs in one network:

$$c(S) = \begin{cases} \alpha \sum\limits_{i \in S} \dfrac{k_i}{k(S)} n_i + \beta \sum\limits_{i \in S} \left(1 - \dfrac{k_i}{k(S)}\right) n_i, & \text{if } S \cap A \neq \emptyset, \\ \gamma n(S), & \text{if } S \cap A = \emptyset. \end{cases} \tag{5}$$

Here $n(S) = \sum_{i \in S} n_i$, and $k(S) = \sum_{i \in S} k_i$.

Using the expression (5) for costs of coalition S we can define a characteristic function for the cost-saving game:

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S) = \tag{6}$$
$$= \begin{cases} (\gamma - \beta) \sum\limits_{i \in S \setminus A} n_i - (\beta - \alpha) \sum\limits_{i \in S \cap A} \left(1 - \dfrac{k_i}{k(S)}\right) n_i, & \text{if } S \cap A \neq \emptyset, \\ 0, & \text{if } S \cap A = \emptyset. \end{cases}$$

Value $v(S)$, the worth of coalition $S \subseteq N$, represents the costs that coalition S saves if all members of S consolidate their ATMs in one network. Therefore, it is interesting to find stable coalition structures with respect to the Shapley value and the ES-value for this specified characteristic function.

Notice that for $v(\cdot)$ defined by (6), $v(\{i\}) = 0$, i. e. any single bank saves nothing by itself. Moreover, the ES-value calculated for a coalition structure π coincides with the equal division value (the ED-value):

$$\psi_i^\pi = \frac{v(B(i))}{|B(i)|}, \quad \text{for all } i \in N \text{ and } B(i) \in \pi. \tag{7}$$

4.2. Program realization

To simplify numerical calculations, for the specific model of bank cooperation a software product is developed using `C#`. In particular, it allows to find all possible coalition structures for a given set of players, calculate payoff distributions according to the Shapley value or the ES-value and check coalition structures for stability with respect to the payoff distribution rule.

One of the complicated components of the source code is an algorithm for finding coalition structures. It is known that number of different coalition structures $\mathcal{B}(n)$ for n players is the n-th Bell number recursively calculated according to the formula: $\mathcal{B}(n) = \sum_{k=0}^{n-1} C_{n-1}^k \mathcal{B}(k)$ s. t. $\mathcal{B}(0) = 1$, and the value $\mathcal{B}(n)$ increases extremely fast as n increases. So if $\mathcal{B}(3) = 5$, $\mathcal{B}(4) = 15$, $\mathcal{B}(5) = 52$, the number $\mathcal{B}(15)$ exceeds one billion.

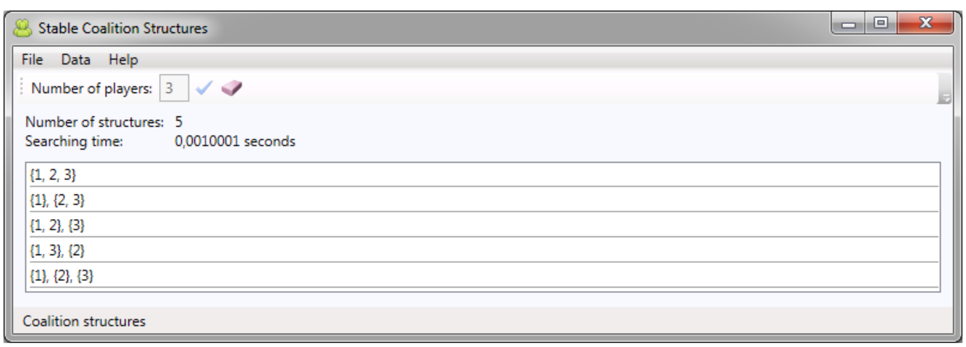


Fig. 7: List of coalition structures for three players

In Fig. 7 there is a screenshot with the number of coalition structures, searching time and a list of all coalition structures for three players. More details regarding to the specified model are presented in the numerical example below.

A recursive algorithm for finding coalition structures is realized in the source code. Knowing coalition structures for one and two players, all coalition structures for three players are found by combining different coalition structures containing one or two players and the structure where all three players belong to the same coalition. More generally, the problem of finding coalition structures for n players can be solved only if the same problem is solved for any number of players less than n. However, recursion is a recourse-costly mechanics. Therefore, the search of coalition structures may require more time if the number of players is large enough.

Implementation of the software product is represented by the following algorithm.

```
# Algorithm for finding coalition structures
 Step 1.1.   Initialize N;
 Step 1.2.   Find n = |N|;
 Step 1.3.   If n = 0 return empty set;
 Step 1.4.   If n = 1 return a player;
 Step 1.5.   If n > 1 find all coalition structures of a form
             {{S}, {π_{-S}}}. Here S is a coalition which contains the
             first element of set N, and π_{-S} is the set of all
             coalition structures for the set N \ S;
             Solve the subproblem for set N \ S (Step 1.2.);
 Step 1.6.   Return all coalition structures found on Step 1.5.;
# Algorithm for payoff distribution computation
 Step 2.1.   Initialize N, α, β, γ, k_i, n_i, i ∈ N;
 Step 2.2.   Choose a cooperative solution concept (the Shapley
             value or the ES-value);
 Step 2.3.   Find coalition structures;
 Step 2.4.   For all coalition structures compute payoff
             distribution according to the chosen cooperative
             solution concept;
# Algorithm for finding stable coalition structures
 Step 3.1.   Choose coalition structure π and calculate the payoff
             distribution;
 Step 3.2.   Fix player i = 1;
 Step 3.3.   do
             {
             Find coalition B(i);
             For i find a set of coalition structures {π'} which
             can be formed if i leaves B(i);
             For each coalition structure π' from the set check the
             stability condition x_i^π ≥ x_i^{π'};
             Once the stability condition fails, π is unstable.
             Otherwise i = i + 1;
             }
             while i ≤ n;
 Step 3.4.   If i = n + 1, π is stable;
```

Example 1. Here we illustrate how the software product works on a numerical example. Let us have three banks, i. e. $N = \{1, 2, 3\}$ and parameters of the game are as follows:

- Costs are $\alpha = 0.5$, $\beta = 1$, $\gamma = 1.5$.
- Number of transactions are $n_1 = 3\,000$, $n_2 = 4\,000$, $n_3 = 6\,000$.
- Number of ATMs are $k_1 = 5$, $k_2 = 3$, $k_3 = 0$.

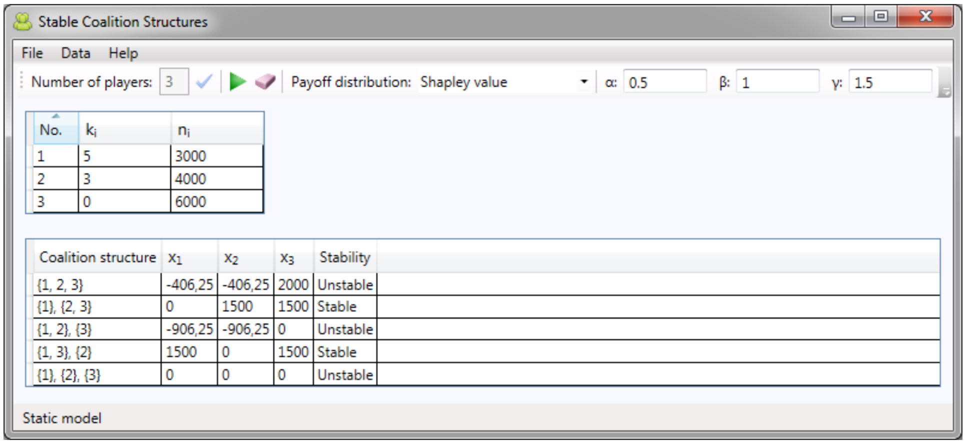


Fig. 8: Stable coalition structures with respect to the Shapley value for Example 1

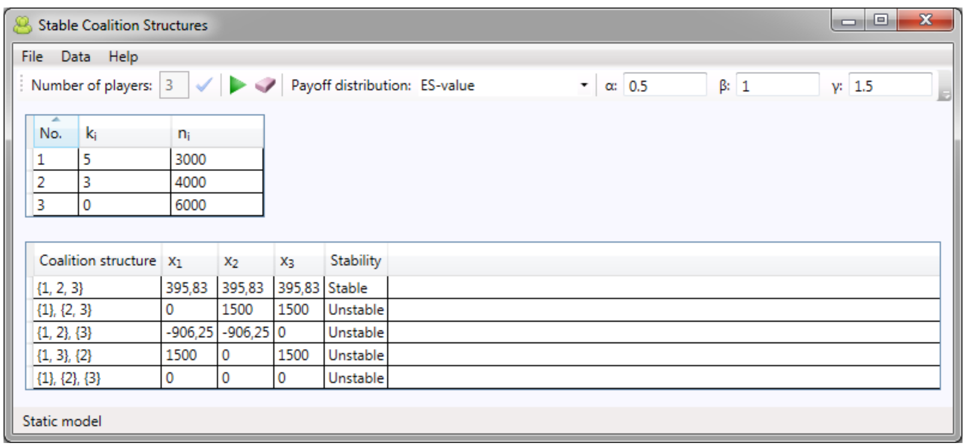


Fig. 9: Stable coalition structures with respect to the ES-value for Example 1

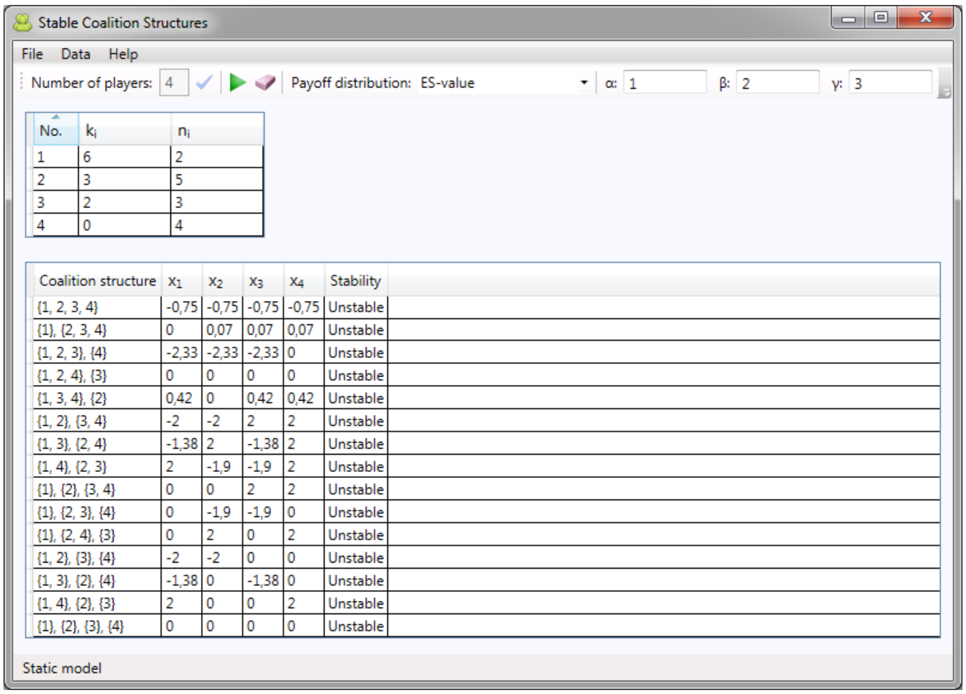
When all required data is entered, the product shows the result. In Fig. 8 payoff distributions calculated according to the Shapley value for all five possible coalition structures are shown. It is also specified whether the coalition structure is stable with respect to the Shapley value or not. Here we may notice that we have two stable coalition structures $\{\{1\},\{2,3\}\}$ and $\{\{1,3\},\{2\}\}$. The corresponding payoff distributions are $(0, 1500, 1500)$ and $(1500, 0, 1500)$.

The similar result for the ES-value is presented in a screenshot in Fig. 9. In this case we obtain the unique stable coalition structure $\{\{1,2,3\}\}$ with respect to the ES-value with payoff distribution $(395.83, 395.83, 395.83)$.

Example 2. Consider the game with the set of players $N = \{1, 2, 3, 4\}$ and parameters of the game are as follows:

- Costs are $\alpha = 1$, $\beta = 2$, $\gamma = 3$.
- Number of transactions are $n_1 = 2$, $n_2 = 5$, $n_3 = 3$, $n_4 = 4$.
- Number of ATMs are $k_1 = 6$, $k_2 = 3$, $k_3 = 2$, $k_4 = 0$.

In this example there are no stable coalition structures with respect to the ES-value as we can see in Fig. 10.



No.	k_i	n_i
1	6	2
2	3	5
3	2	3
4	0	4

Coalition structure	x_1	x_2	x_3	x_4	Stability
{1, 2, 3, 4}	-0,75	-0,75	-0,75	-0,75	Unstable
{1}, {2, 3, 4}	0	0,07	0,07	0,07	Unstable
{1, 2, 3}, {4}	-2,33	-2,33	-2,33	0	Unstable
{1, 2, 4}, {3}	0	0	0	0	Unstable
{1, 3, 4}, {2}	0,42	0	0,42	0,42	Unstable
{1, 2}, {3, 4}	-2	-2	2	2	Unstable
{1, 3}, {2, 4}	-1,38	2	-1,38	2	Unstable
{1, 4}, {2, 3}	2	-1,9	-1,9	2	Unstable
{1}, {2}, {3, 4}	0	0	2	2	Unstable
{1}, {2, 3}, {4}	0	-1,9	-1,9	0	Unstable
{1}, {2, 4}, {3}	0	2	0	2	Unstable
{1, 2}, {3}, {4}	-2	-2	0	0	Unstable
{1, 3}, {2}, {4}	-1,38	0	-1,38	0	Unstable
{1, 4}, {2}, {3}	2	0	0	2	Unstable
{1}, {2}, {3}, {4}	0	0	0	0	Unstable

Fig. 10: Stable coalition structures with respect to the ES-value for Example 2

Let us consider the Shapley value as a cooperative solution concept for this example. There are three stable coalition structures with respect to this concept: $\{\{1\}, \{2\}, \{3, 4\}\}$, $\{\{1\}, \{2, 4\}, \{3\}\}$ and $\{\{1, 4\}, \{2\}, \{3\}\}$. The corresponding players' payoffs are $(0, 0, 2, 2)$, $(0, 2, 0, 2)$ and $(2, 0, 0, 2)$.

5. Conclusion

We considered the problem of stability of coalition structures with respect to the some cooperative solution concepts, i. e. the Shapley value and the ES-value. The approach to define stable coalition structure is similar to the approach of the definition of the Nash equilibrium for non-cooperative games. This approach seems to be natural when the problem of possible players' deviation is considered.

It is important for our analysis that two cooperative solution concepts considered in the paper are single-valued, otherwise, the definition of coalition structure stability is needed to be improved and extended to the multi-valued case.

Example 2 shows that stable coalition structures with respect to the ES-value do not always exist for more than three players. The open question of the work is the existence of stable coalition structures for more than three players with respect to the Shapley value. This result has not been proved yet.

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The Algorithm of Finding Equilibrium in the Class of Fully-mixed Strategies in the Logistics Market with Big Losses *

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Abstract The problem of customer optimal behavior in the service market where two service company operate to handling customer orders is considered. Each company has its own method of forming final cost of service order. The main peculiarity of considering problem is the presence of big customer losses if the lead time of fulfillment its order become very large. In this paper we formulate and prove the theorem for finding optimal strategies for players behavior when choosing a service provided with non-linearity of the loss function.

Keywords: game-theoretical approach, optimal behavior, probability modeling, construction market, nonlinear penalty, n persons game, Nash equilibrium, the fully-mixed strategies.

1. Introduction

At present days more and more gaining global popularity problems are associated with searching the optimal behavior of the player in the market, minimizing the overall cost and time of turnover. To solve such problems is widely used the game-theoretic approach. The widespread problems of the buyer in the market looking for a service provider to perform the customer order continuously take much interest.

The present work is based on (Bure and Sergeeva, 2011) and (Bure and Sergeeva, 2012).

An important feature of the problem is the possibility of the client to incur heavy costs if the duration of work exceeds a certain pre-defined limit. We consider the optimal choice of the client in terms of cost minimization. Costs consist of direct client costs orders for the scheme which sets by the provider specific losses and penalties which are charged to the client for delay in delivery of work. In this case under penalty meant extra money that is paid to the contractor if the nature of the work is too difficult. Each contractor shall determine its own policy formation of the final price. Costs consist of fixed clients and temporary component.

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2. The problem statement

Lets consider service market with two providers. Each provider has its own fundamentally different pricing policy.

Customers try to choose the service way under minimizing total costs of the order. An important feature of discussed service scheme is the existence of feasibility for customer to take big losses if the full time spending on service will be more than some definite limit. So each customer take into account not only the cost of service but also the time of order duration. The economic explanation of big losses existence will discuss later.

Denote by τ_1 - the full time of customer service fulfillment in selecting the first service provider which consists of two components $\tau_1 = \tau_{11} + \tau_{12}$, where τ_{11} - waiting time of the order , τ_{12} - the service time by first provider. $\tau_2 = \tau_{22}$ is the full time of customer service fulfillment in selecting the second service provider which contains only service time τ_{22} since waiting time of service is zero. The parameters τ_1, τ_2 are random variables.

Processing times of client service described by exponential distribution with density functions

$$f_1(t) = \frac{1}{\mu_1} e^{-\frac{1}{\mu_1} t},\ t > 0,$$

$$f_2(t) = \frac{1}{\mu_2} e^{-\frac{1}{\mu_2} t},\ t > 0,$$

where μ_1 and μ_2 are intensities of the service.

Let c_1 - the cost of customer order fulfilment by first provider, it is fixed and does not depend on the duration of the customer order. Assume further that c_2 - the cost of customer order fulfilment by second provider depending on the duration of customer service: $c_2 = c_{21} + c_{22}\tau_{22}$, where c_{21} - fixed price charged for customer order, c_{22} - the cost per unit customer service time by second provider.

Each client losing time that could be used for the completion or delivery while waiting for their order fulfillment. In addition to the cost of customer order fulfillment denote by r the specific losses incurred by the client while waiting for the order. It is a time associated with missed opportunities under choosing this particular contractor. Then we can determine the total loss associated with the expectation of the order. Which will be determined by the following formulas:

$$r\tau_1 = r(\tau_{11} + \tau_{12}),$$

$$r\tau_2 = r\tau_{22}.$$

These expressions will be used late for describing the total loss function.

3. The problem of big losses

We consider the optimal choice of the client in terms of cost minimization. Costs consist of direct client costs orders for the scheme which sets by the provider specific losses and penalties which are charged to the client for delay in delivery of work. In this case under penalty meant extra money that is paid to the contractor if the nature of the work is too difficult. It is prescribed in the contract with the contractor. If it appears that under the objective reasons additional time for work through is no fault of the contractor, for example due to renegotiation of the project, from a determined point of time the customer start to pay for the time on other rates, i.e. pay a fine. The customer is an intermediate in the overall chain of interaction and has an obligation to its clients. Therefore, in order delays he incurs all the costs. In this case we are interested only in the loss of a client as we are looking for his best behavior and the sanctions that are applied at the same time to the provider we are not interested.

The value of penalty will founded as follows. Lets fix T and introduce the indicator:

$$I(t,T) = \begin{cases} 1, t \geq T, \\ 0, t < T. \end{cases}$$

Denote by R_1, R_2 the penalties which customer starts to pay in excess of the service time of more than T_1 for the first and T_2 for the second provider respectively. We assume that R_1 and R_2 large enough.

Denote $J_1 = EI\{\tau_i^{(1)}, T_1\},\ J_2 = EI\{\tau_i^{(2)}, T_2\}$ as expected value of indicators.

Now it is possible to calculate the full loss of clients to service for each provider respectively:

$$\tilde{Q}_1 = r\tau_1 + c_+ R_1 J_1,$$

$$\tilde{Q}_2 = (r + c_{22})\tau_{22} + c_{21} + R_2 J_2.$$

Then the average customer losses for services provided by different providers are determined by the following expectations:

$$Q_1 = E\tilde{Q}_1 = r(E\tau_{11} + E\tau_{12}) + c_1 + R_1 J_1,$$

$$Q_2 = E\tilde{Q}_2 = (r + c_{22})E\tau_{22} + c_{21} + R_2 J_2.$$

4. The game-theoretical model

Game-theoretical approach and probabilistic modeling are more appropriate for solving this problem. Lets formulate this problem in terms of game theory.

$\Gamma =< N, \{p_i\}_{i\in N}, \{H_i\}_{i\in N} >$ - the non-antagonistic game in normal form where $N = \{1, \ldots, n\}$ is the set of players,

$\{p_i\}_{i\in N}$ is the set of strategies, $p_i \in [0, 1]$, where p_i is the probability that player i choose the first provider,

$\{H_i\}_{i\in N}$ is the set of payoff functions.

$$H_i = -(p_i Q_{1i} + (1 - p_i)Q_{2i}) = -(p_i(Q_{1i} - Q_{2i}) + Q_{2i}).$$

The aim of each customer is to minimize his payoff function by choosing the optimal decision on the construction market.

Before we formulate the main statement about finding the optimal customer behavior we have to determine following definition which can also be fined in (Vorobev, 1985).

Definition 1. The strategies for which the probabilities of selection of each of provider are strictly positive, i.e. $p_i > 0,\ 1 - p_i > 0,\ i = 1, \ldots, n$, are called *fully-mixed strategies.*

5. The Nash equilibria for problem with big losses

The following statement establishes the points of Nash equilibria for some conditions which cover all possible situations.

Theorem 1. *There exists a unique point of equilibrium* $(p_1, \ldots, p_n)$, $i = 1, \ldots, n$ *in the game* Γ *defined as follows.*

The following situations are possible:

1. *if* $r((k+1)\mu_1 + \frac{1}{2}\mu_1(n-1)) - (r + c_{22})\mu_2 + R_1 J_1(k) - R_2 J_2 + c_1 - c_{21} < 0$, *then there exists a unique point of equilibrium in the game* Γ*:* $p_i^* = 1$, $i = 1, \ldots, n$ *which means that player* i *choose the first service provider;*
2. *if* $r(k+1)\mu_1 - (r + c_{22})\mu_2 + R_1 J_1(k) - R_2 J_2 + c_1 - c_{21} > 0$, *then there exists a unique point of equilibrium in the game* Γ*:* $p_i^* = 0$, $i = 1, \ldots, n$ *which means that player* i *choose the second service provider;*
3. *and the last if* $r(k+1)\mu_1 \le (r + c_{22})\mu_2 + R_1 J_1(k) - R_2 J_2 + c_{21} - c_1 \le r((k+1)\mu_1 + \frac{1}{2}\mu_1(n-1))$, *then in the class of fully-mixed strategies there exists a unique point of equilibrium* Γ*:* $(p_1^*, \ldots, p_n^*)$, $p_i^* = \dfrac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - R_1 J_1 + R_2 J_2 - c_1 + c_{21})}{r\mu_1(n-1)}$, $i = 1, \ldots, n$,

where $k = 0$ *if no customers on service and in the line at first provider,* $k = 1$ *if there is one customer on service at provider and no customers in line,* $k > 1$ *if there are one customer on service and some customers in line at provider.*

Proof. If m players including the player i choose first provider then player i occupy any of m places in line for service with probability $\frac{1}{m}$ according (Bure, 2002). Conditional expectation waiting time before service player i without the service time players already in service by first provider provided that l players of the m proceed player i:

$$\sum_{l=0}^{m-1} l\mu_1 \frac{1}{m} = \frac{1}{m}\mu_1 \sum_{l=0}^{m-1} l = \frac{1}{m}\mu_1 \frac{m(m-1)}{2} = \frac{1}{2}\mu_1(m-1) \tag{1}$$

Let $P_r(l)$ be the probability of event that r players from set of l players choose the first provider. Then using (1) we can find:

$$\sum_{m=1}^{n} \frac{1}{2}\mu_1(m-1)P_{m-1}(n-1) = \sum_{m=0}^{n-1} \frac{1}{2}\mu_1 m P_m(n-1) \tag{2}$$

Now we can use that expression (2) to determine conditional mean time till order complete for the first provider

$$t_{1i} = k\mu_1 + \frac{1}{2}\mu_1 \sum_{m=1}^{n}(m-1)P_{m-1}(n-1) + \mu_1 = k\mu_1 + \frac{1}{2}\mu_1 \sum_{l=0}^{n-1} lP_l(n-1) + \mu_1,\ i = 1, \ldots, n.$$

If customer choose the second provider he doesn't have to wait the beginning of service because he comes to service immediately. So conditional mean time till order complete for the second provider defined as

$$t_{2i} = \mu_2,\ i = 1, \ldots, n.$$

Lets show that vector $(p_1^*, \ldots, p_n^*)$ is really the point of equilibrium. Assume that $p_1 = \ldots = p_{i-1} = p_{i+1} = \ldots = p_n = p$ than under this assumption and using the Bernoulli scheme for Binomial distribution we can easily find the expression according (Feller, 1984)

$$\sum_{m=0}^{n-1} mC_{n-1}^m p^m (1-p)^{n-1-m} = p(n-1).$$

The next step is determination the mean values $J_1 = EI\{\tau_1, T_1\} = P\{\tau_1 > T_1\}$ and $J_2 = EI\{\tau_2, T_2\} = P\{\tau_2 > T_2\}$.

Since the first provider serve all clients one by one from queue then the duration of their service can be described by Erlang distribution which is the Gamma-distribution with an integer value of the shape parameter. We assume that there are $k+1$ clients in the system. Given a $\tau_1 = \sum_{i=1}^{k+1} \tau_{1i}$ where τ_{1i} is the client i service time. Then τ_1 distributed under Gamma-distribution $G(\frac{1}{\mu_1}, k+1)$ with density function $f_{G(\frac{1}{\mu_1},k+1)}(t) = \begin{cases} (\frac{1}{\mu_1})^{k+1} \frac{t^k e^{-\frac{t}{\mu_1}}}{\Gamma(k+1)}, & t > 0 \\ 0, & t \leq 0. \end{cases}$

Let's proceed $J_1(k)$ by induction.

When $k = 1$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^2 \frac{te^{-\frac{t}{\mu_1}}}{\Gamma(2)} dt = \Big(\frac{T_1}{\mu_1} + 1\Big) e^{-\frac{T_1}{\mu_1}}.$$

When $k = 2$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^3 \frac{t^2 e^{-\frac{t}{\mu_1}}}{\Gamma(3)} dt = \frac{1}{\Gamma(3)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^2 + 2\frac{T_1}{\mu_1} + 2\Big).$$

When $k = 3$

$$J_1(k) = \int_{T_1}^{\infty} \Big(\frac{1}{\mu_1}\Big)^4 \frac{t^3 e^{-\frac{t}{\mu_1}}}{\Gamma(4)} dt = \frac{1}{\Gamma(4)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^3 + 3\Big(\frac{T_1}{\mu_1}\Big)^2 + 6\frac{T_1}{\mu_1} + 6\Big).$$

So the general expression for $J_1(k+1)$ is:

$$J_1(k+1) = \int_{T_1}^{\infty} f_{G(\frac{1}{\mu_1},k+1)}(t) dt = \frac{1}{\Gamma(k+1)} e^{-\frac{T_1}{\mu_1}} \Big(\Big(\frac{T_1}{\mu_1}\Big)^k + k\Big(\frac{T_1}{\mu_1}\Big)^{k-1} +$$

$$+ k(k-1)\Big(\frac{T_1}{\mu_1}\Big)^{k-2} + \ldots + (k)!\Big(\frac{T_1}{\mu_1}\Big) + (k)!\Big).$$

As the second provider don't have any queue we can define J_2 as follows:

$$J_2 = \int_{T_2}^{\infty} f_2(t) dt = \int_{T_2}^{\infty} \frac{1}{\mu_2} e^{-\frac{1}{\mu_2} t} dt = e^{-\frac{T_2}{\mu_2}}.$$

Then the average customer loss for services by first provider is:

$$Q_{1i} = r(k\mu_1 + \frac{1}{2}\mu_1 p(n-1) + \mu_1) + c_1 + R_1 J_1,$$

and for the second provider:

$$Q_{2i} = (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

Since customer trying to minimize total losses so we will consider the function

$$h_i = p_i Q_{1i} + (1 - p_i) Q_{2i} = p_i (Q_{1i} - Q_{2i}) + Q_{2i}.$$

To analyze this expression we will consider the following term

$$Q_{1i} - Q_{2i} = r(k\mu_1 + \frac{1}{2}\mu_1 p(n-1) + \mu_1) + c_1 + R_1 J_1 - (r + c_{22})\mu_2 - c_{21} - R_2 J_2 =$$

$$= r(k+1)\mu_1 + \frac{1}{2} r\mu_1 p(n-1) - (r + c_{22})\mu_2 - c_{21} + c_1 + R_1 J_1 - R_2 J_2.$$

Three situations are possible:

1. If all players except i choose the first provider, i.e. they choose the strategy $p = 1$ then if $Q_{1i} - Q_{2i} < 0$ player i had to choose the same strategy.
2. If all players except i choose the second provider, i.e. they choose the strategy $p = 0$ then if $Q_{1i} - Q_{2i} > 0$ player i had to choose the same strategy.
3. If the above conditions are not met then if in the class of fully-mixed strategies when all players except i choose strategy

$$p = p_i^* = \frac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - c_1 - R_1 J_1 + c_{21} + R_2 J_2)}{r\mu_1 (n-1)}$$

player i is in the situation when the selection of any strategy leads to same result. Therefore player i can not reduce its losses so so it also does not make sense to deviate from strategy p_i^*.

Since the strategy is the probability so we have to prove $p \in (0, 1)$. Because the next inequality is true

$$r\mu_1(k+1) \leq (r + c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 \leq r\mu_1(\frac{n}{2} + k + \frac{1}{2}))$$

thus we have following expression

$$0 \leq (r+c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 - r\mu_1(k+1) \leq r\mu_1(\frac{n}{2} + k + \frac{1}{2})) - r\mu_1(k+1).$$

By transforming this expression we obtain:

$$0 \leq (r + c_{22})\mu_2 + c_{21} - c_1 + R_2 J_2 - R_1 J_1 - r\mu_1(k+1) \leq \frac{1}{2} r\mu_1(n-1).$$

Given a $\frac{1}{2} r\mu_1(n-1) \neq 0$ thus by dividing both parts of the inequality to this equation we can receive

$$0 \leq \frac{2((r + c_{22})\mu_2 - r(k+1)\mu_1 - c_1 + c_{21}) + R_2 J_2 - R_1 J_1}{r\mu_1(n-1)} \leq 1.$$

Thereby we prove that $p \in (0, 1)$.

At the rest part of the proof we will show the uniqueness of the found point of equilibrium.

Suppose that all customers could choose different strategies so we can not use Bernoulli scheme already. In general, the process of selecting one of the two provider is a sequence of independent events when each player chooses either the first provider or the second. Suppose, in contrast to the previous, that the probabilities $p_i,\ i = 1, \ldots, n$ of the choice the first provider may be different, i.e. strategies of the players are different, therefore, considered sequence of independent events is a Bernoulli scheme. We calculate the expectation of time before the customer service i provided that he has chosen the first provider without customers previously adopted for provider service. To calculate the sum $\sum_{l=0}^{n-1} lP_l(n-1)$ which represent the mean value of amount of players picking the first provider at the set of $n-1$ player without the player i as well as customers came previous to service by first provider we can use the following method. Considered mean value equals to sum of mathematical expectation of the number of success (we mean by success the choice of first provider) in each single test, i.e. each player from the set of $n-1$ thus

$$\sum_{l=0}^{n-1} lP_l(n-1) = \sum_{m\ =1, m\neq i}^{n} p_m.$$

Then the mean time till order complete by the first provider is

$$t_{1i} = k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1,$$

and the mean time till order complete by the second provider is

$$t_{2i} = \mu_2.$$

Hence we have the average customer loss for services by the first provider:

$$Q_{1i} = r\Big(k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1\Big) + c_1 + R_1 J_1,$$

and the average customer loss for services by the second provider:

$$Q_{2i} = (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

So the function of customer i total losses is

$$h_i = p_i(Q_{1i} - Q_{2i}) + Q_{2i}.$$

Let consider the equation

$$Q_{1i} - Q_{2i} = r\Big(k\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1\Big) + c_1 + R_1 J_1 - (r + c_{22})\mu_2 - c_{21} - R_2 J_2 = 0. \tag{3}$$

The following three situations are possible

1. if $r\mu_1(\frac{n}{2}+k+\frac{1}{2})-(r+c_{22})\mu_2+c_1-c_{21}+R_1J_1-R_2J_2<0$ then (3) doesn't have solution on $\sum\limits_{m=1,m\neq i}^{n} p_m$. In this case all players have to choose the strategy $p_i=1$, i.e. they select the first provider.
2. if $r\mu_1(k+1)-(r+c_{22})\mu_2-c_{21}+c_1+R_1J_1-R_2J_2>0$ then (3) doesn't have solution on $\sum\limits_{m=1,m\neq i}^{n} p_m$. In this case all players have to choose the strategy $p_i=0$, i.e. they select the second provider.
3. $r(k+1)\mu_1 \le (r+c_{22})\mu_2+R_1J_1(k)-R_2J_2+c_{21}-c_1 \le r((k+1)\mu_1+\frac{1}{2}\mu_1(n-1))$ that means that both conditions 1. and 2. are violated then the value $\sum\limits_{m=1,m\neq i}^{n} p_m$ defined uniquely as a solution of (3).

All sums $\sum\limits_{m=1,m\neq i}^{n} p_m$ should be equal to each other for every possible $i=1,\ldots,n$, i.e. $\sum\limits_{m=1,m\neq i}^{n} p_m = \sum\limits_{m=1,m\neq j}^{n} p_m,\ \ i\neq j$.

Hence we have

$$p_i = p_j,\ \ i \neq j.$$

According with the above considerations we finally show that the point of equilibrium consists only of the same probabilities for each customer in the class of fully-mixed strategies thus it is coincides with p^*.

The theorem is proved.

Remark 1. Obviously all three conditions based in the Theorem are mutually incompatible and together represent all possible options.

Remark 2. In a situation when the cost of service by both providera are equal for customer then the equilibrium defined in Theorem is unique only in the class of fully-mixed strategies. Generally speaking, in this situation the player does not have to adhere to the strategy of choice for other players.

Lets consider the simple case of two construction companies in the market selecting provider, i.e. the number of players $n=2$.

For the first player the average customer losses for services are calculated as follows:

$$Q_{11} = r(k\mu_1 + \frac{1}{2}\mu_1 p_2 + \mu_1) + c_1 + R_1J_1,$$

$$Q_{21} = (r+c_{22})\mu_2 + c_{21} + R_2J_2.$$

For the second player the average customer losses for services are calculated as follows:

$$Q_{12} = r(k\mu_1 + \frac{1}{2}\mu_1 p_1 + \mu_1) + c_1 + R_1J_1,$$

$$Q_{22} = (r+c_{22})\mu_2 + c_{21} + R_2J_2.$$

Then the deviation of losses functions for both players are

$$Q_{11} - Q_{21} = r(k\mu_1 + \frac{1}{2}\mu_1 p_2 + \mu_1) + c_1 + R_1J_1 - (r+c_{22})\mu_2 - c_{21} - R_2J_2,$$

$$Q_{12} - Q_{22} = r(k\mu_1 + \frac{1}{2}\mu_1 p_1 + \mu_1) + c_1 + R_1 J_1 - (r + c_{22})\mu_2 - c_{21} - R_2 J_2.$$

Lets show that under condition that is subject of consideration $r\mu_1(\frac{3}{2}+k)+c_1 \geq (r + c_{22})\mu_2 + c_{21} \geq r\mu_1(k+1) + c_1$ the existence of different points of equilibrium is possible.

At first consider the situation $(1, 0)$ when the first player comes to first provider with probability equals to 1 and the second player comes to second provider with probability equals to 1. Lets show that this strategy is the point of equilibrium under the condition above. This situation occurs in the game if under the selection of the second player the second provider then the first player better to choose the first provider. And on the contrary, if under the selection of the first player the first provider then the second player better to choose the second provider. Thus the situation $(1, 0)$ is a Nash equilibrium under the following conditions:

$$Q_{11}(p_2 = 0) \leq Q_{21}(p_2 = 0),$$

$$Q_{12}(p_1 = 1) \geq Q_{22}(p_1 = 1),$$

or the same:

$$r\mu_1(k + 1) + c_1 + R_1 J_1 \leq (r + c_{22})\mu_2 + c_{21} + R_2 J_2,$$

$$r\mu_1(k + \frac{3}{2}) + c_1 + R_1 J_1 \geq (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

This condition is equals to the third condition from the Theorem.

Now consider the situation $(0, 1)$ when the first player comes to first provider with probability equals to 1 and the second player comes to second provider with probability equals to 1. Lets show that this strategy is the point of equilibrium under the condition above. This situation occurs in the game if under the selection of the second player the first provider then the first player better to choose the second provider. And on the contrary, if under the selection of the first player the second provider then the second player better to choose the first provider. Thus the situation $(0, 1)$ is a Nash equilibrium under the following conditions:

$$Q_{11}(p_2 = 1) \geq Q_{21}(p_2 = 1),$$

$$Q_{12}(p_1 = 0) \leq Q_{22}(p_1 = 0),$$

or the same:

$$r\mu_1(k + \frac{3}{2}) + c_1 + R_1 J_1 \geq (r + c_{22})\mu_2 + c_{21} + R_2 J_2,$$

$$r\mu_1(k + 1) + c_1 + R_1 J_1 \leq (r + c_{22})\mu_2 + c_{21} + R_2 J_2.$$

This condition is equals to the third condition from the Theorem too.

Therefor if the third condition from the Theorem is true then the following strategies are the points of equilibrium:

$$(1, 0),\ (0, 1),$$

$$\Big(\frac{2((r + c_{22})\mu_2 - r(k + 1)\mu_1 - c_1 + c_{21}) + R_1 J_1 - R_2 J_2}{r\mu_1(n - 1)}),$$

$$1-\frac{2((r+c_{22})\mu_2-r(k+1)\mu_1-c_1+c_{21}+R_1J_1-R_2J_2)}{r\mu_1(n-1)}\Big).$$

Thus we conclude that if not to content oneself with the class of fully-mixed strategies, in a situation where the player does not care which of the provider apply there may be several Nash equilibrium. This is the case when depending on the number of customers already in services the player may be advantageous to apply in the first and the second provider, i.e. when the first two conditions of Theorem are not satisfied.

Generally speaking, this observation can be formulated for the case $n = 3, 4, \ldots$, but on the structure of reasoning they will be similar, so we will not give them.

Remark 3. Lets consider the special case of the construction market when in the market there is only one client select the service between the two providers. Obviously, the player just need to calculate the expected cost of service in each of the firms with knowledge that he was the only one of its customer service and choose the lowest cost.

In this case the loss functions described:

$$Q_1 = r\mu_1(k+1) + c_1 + R_1J_1$$

$$Q_2 = (r + c_{22})\mu_2 + c_{21} + R_2J_2.$$

The situations are possible:

1. If $r\mu_1 + c_1 + R_1J_1 < (r + c_{22})\mu_2 + c_{21} + R_2J_2$ then customer choose the first provider for service
2. If $r\mu_1 + c_1 + R_1J_1 > (r + c_{22})\mu_2 + c_{21} + R_2J_2$ then customer choose the second provider for service
3. If $r\mu_1 + c_1 + R_1J_1 = (r + c_{22})\mu_2 + c_{21} + R_2J_2$ then the client does not care which of the firms choose to serve him then he is likely to be any contact either the first or the second provider.

5.1. Conclusion

Throughout the paper, we have defined the problem of customer behavior in the construction market of two service providers. The game-theoretical approach and probabilistic modelling used as a way of representing such an issue. The two providers are the service companies in the construction market which provide repairs and cosmetic finishing works for clients. Each of provider has its own scheme of customer order fulfillment and own cost policy. There is introduced the class of fully-mixed strategies. The theorem which determine points of Nash equilibrium under three possible cases is formulated and proved. The optimal behavior of customers in terms of fully-mixed strategies is found.

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Polar Representation of Shapley Value: Nonatomic Polynomial Games*

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Abstract The paper deals with polar representation formula for the Shapley value, established in (Vasil'ev, 1998). Below, we propose a new, simplified proof of the formula for nonatomic polynomial games. This proof relies on the coincidence of generalized Owen extension and multiplicative Aumann-Shapley expansion for polynomial games belonging to pNA (Vasil'ev, 2009). The coincidence mentioned makes it possible to calculate Aumann-Shapley expansion in a straightforward manner, and to complete new proof of the polar representation formula for nonatomic case by exploiting the generalized Owen integral formula, established in (Aumann and Shapley, 1974).

Keywords: Shapley value, nonatomic polynomial game, generalized Owen extension, polar form, polar representation formula.

1. Introduction

The paper deals with the polar representation formula for the Shapley value, established under rather general assumptions in (Vasil'ev, 1998). In order to simplify a proof of this formula for some special classes of games, we continue our investigation on the generalized Owen extension for regular polynomial games started in (Vasil'ev, 2009). Main attention is paid to the nonatomic cooperative games. Our approach is based on the principal result from (Vasil'ev, 2009), demonstrating that the above-mentioned generalized Owen extension coincides with the multiplicative Aumann-Shapley expansion for some types of nonatomic games, including polynomial games from pNA. This coincidence makes it possible to calculate the Aumann-Shapley expansion in a straightforward manner by applying the corresponding generalized Owen extension. To complete new proof of the polar representation formula for the Shapley value of nonatomic homogeneous game we exploit the famous generalized Owen integral formula from (Aumann and Shapley, 1974), given in terms of the multiplicative Aumann-Shapley expansion.

2. Generalized Owen Extension for Regular Polynomial Games

Below, some main constructions of an explicit definition of the generalized Owen extension, introduced in the paper, are given (for the sake of brevity, we restrict ourselves to the case of regular polynomial games).

Let (Q, d) be an arbitrary nonempty metric compactum with distance function d. Denote by B its Borel σ-algebra and consider a collection $\mathcal{V} = \mathcal{V}(Q)$ of set functions

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$v : B \to \mathbf{R}$ satisfying the requirement $v(\emptyset) = 0$. As usual, a triplet $\Gamma = (Q, B, v)$ with $v \in \mathcal{V}$ is said to be a cooperative game (with elements of Q being players, and elements of B treated as their coalitions). Remind, that we pay attention, mostly, to the case of infinite cooperative games (when Q is an infinite set).

To characterize cooperative games under considerations in more details, we introduce first some technical notations and definitions (most of them, including vector lattice terms, can be found in more details in (Vasil'ev, 1998)). Fix $S \in B$ and denote by $H(S)$ a set of finite B-measurable partitions of S. Put $H = \cup_{S\in B} H(S)$. For any $\eta = \{S_i\}_{i\in\Omega} \in H$ with $|\Omega| = m$, and $v \in \mathcal{V}$ denote by $v(\eta) = v(\{S_i\}_{i\in\Omega})$ a polynomial m-difference, defined by the formula

$$v(\eta) := \sum_{\omega\subseteq\Omega} (-1)^{|\Omega|-|\omega|} v(\cup_{i\in\omega} S_i), \tag{1}$$

where, as usual, $|\omega|$ denotes the number of elements of a finite set ω.

Remark 1. Directly from (1) it follows that polynomial differences satisfy the recursion formula

$$\begin{aligned} &v(\{S_1, \dots, S_{m-1}, S_m, S_{m+1}\}) = v(\{S_1, \dots, S_{m-1}, S_m \cup S_{m+1}\}) \\ &-v(\{S_1, \dots, S_{m-1}, S_m\}) - v(\{S_1, \dots, S_{m-1}, S_{m+1}\}), \quad m \geq 2, \end{aligned}$$

with $v(\{S_1\}) = v(S_1)$, and $v(\{S_1, S_2\}) = v(S_1 \bigcup S_2) - v(S_1) - v(S_2)$. Note also, that for any $S \in B$ and $\eta = \{S_i\}_{i=1}^m \in H(S)$, an equality

$$v(S) = \sum_{\omega\subseteq\Omega_\eta} v(\eta^\omega) \tag{2}$$

is valid, where $\Omega_\eta := \{1, \dots, m\}$, and $\eta^\omega := \{S_i\}_{i\in\omega}$ for any $\omega \subseteq \Omega_\eta$.

Recall (Vasil'ev, 1975a), that *polynomial variation* $||v||_0$ of $v \in \mathcal{V}$ is defined by the formula

$$\|v\|_o := \sup \Big\{ \sum_{\omega\subseteq\Omega} |v(\eta^\omega)| \;\Big|\; \eta = \{S_i\}_{i\in\Omega} \in H(Q) \Big\}$$

with $v(\eta^\omega)$ determined as above. We say that a function $v \in \mathcal{V}$ is of bounded polynomial variation if $||v||_0 < \infty$. Put

$$V = V(Q) := \{v \in \mathcal{V} \mid \|v\|_o < \infty\}$$

and define a cone $V_+ = V_+(Q)$ of positive elements of V in order to equip the collection V with the structure of a vector lattice. Recall (Vasil'ev, 1975a), that a game $v \in \mathcal{V}$ is said to be *totally positive* if $v(\eta) \geq 0$ for any $\eta \in H$. A cone of positive elements, mentioned above, is taken to be a convex cone of the totally positive games:

$$V_+ = V_+(Q) := \{v \in \mathcal{V} \mid v(\eta) \geq 0 \text{ for any } \eta \in H\}.$$

It is not very hard to verify that $V_+ \subseteq V$ and partial order $u \geq_0 v \iff u - v \in V_+$, induced by V_+ (along with the norm of polynomial variation $\| \cdot \|_0$), endows V with the structure of Banach vector lattice. To be exact (Vasil'ev, 1998), V is norm complete and Dedekind complete vector lattice with the norm $\|\cdot\|_0$ compatible with partial order $\geq_0$: monotone order convergence $v_n \downarrow 0$ ($v_n \uparrow \infty$) implies monotone norm convergence $\|v_n\|_0 \downarrow 0$ ($\|v_n\|_0 \uparrow \infty$).

Following notations of the vector lattice theory (Aliprantis and Border, 1994), for any function $v \in V$, denote by $v^+ = v \vee 0$, $v^- = -v \vee 0$, and $|v| = -v \vee v$ the positive, negative, and total variations of v, respectively (as usual, $u \vee w :=$ sup $\{u, w\}$, and $u \wedge w :=$ inf $\{u, w\}$ with respect to the partially ordered vector space $(V, \geq_0)$. Let $\mathcal{F}$ be the collection of all closed subsets of Q. The basic type of games we are going to deal with is given by the following definition.

Definition 1 (Vasil'ev, 1975a). A game $v \in V$ is said to be *regular*, if its total variation $|v|$ meets the requirement:

$$|v|(\{S_i\}_1^m) = \sup \left\{ |v|(\{F_i\}_1^m) \,\middle|\, F_i \subseteq S_i, F_i \in \mathcal{F},\ i = 1, \ldots, m \right\}$$

for any partition $\eta = \{S_i\}_1^m \in H$. A set of regular games is denoted by $rV = rV(Q)$.

Definition 2 (Vasil'ev, 1975a). A game $v \in rV$ is called a (regular) *polynomial game of order* n, if all the polynomial $n+1$-differences of v are equal to zero:

$$v(\{S_i\}_1^{n+1}) = 0 \quad \text{for any} \quad \{S_i\}_1^{n+1} \in H.$$

Denote by $rV^n = rV^n(Q)$ a space of all regular polynomial games of order n, and put

$$rpV := \cup_{n=1}^{\infty} rV^n.$$

We say that v is a (regular) *polynomial game*, if v belongs to rpV.

Passing on directly to the generalization of the Owen multilinear extension, we introduce first a concept of integration with respect to polynomial set function. To this end fix some $v \in rV^n$, and construct an extension of v to the n-th symmetric power $B^{[n]}$ of algebra B. In turn, to introduce definition of $B^{[n]}$, we recall (Vasil'ev, 1975a), that the n-th symmetric power $S^{[n]}$ of a coalition $S \in B$ is given by the formula

$$S^{[n]} = \{\tau \subseteq S \,\big|\, |\tau| \leq n\},$$

where, as before, we denote by $|\tau|$ the number of elements of τ.

Definition 3 (Vasil'ev, 1975a). The *n-th symmetric power* $B^{[n]}$ *of an algebra* B is the smallest algebra that includes the collection $\{S^{[n]} \,\big|\, S \in B\}$.

By applying a description of $B^{[n]}$, given in (Vasil'ev, 1975a), one can prove that there exists a unique additive set function $\lambda_v : B^{[n]} \to \mathbf{R}$, satisfying the requirement: $\lambda_v(S^{[n]}) = v(S)$ for any $S \in B$. Moreover, by taking into account regularity of v and compactness of Q one can establish that there exists a unique σ-additive extension μ_v of λ_v to the smallest σ-algebra $\sigma B^{[n]}$ that includes $B^{[n]}$ (for more details, see (Vasil'ev, 1975a)). Interestingly to note that σ-algebra $\sigma B^{[n]}$ admits rather simple description.

Proposition 1 (Vasil'ev, 1975a). *Algebra $\sigma B^{[n]}$ coincides with the Borel σ-algebra of the compact metric space $(Q^{[n]}, d^{[n]})$, where $d^{[n]}$ is the Hausdorff metric*

$$d^{[n]}(\tau, \tau') := \min \left\{\epsilon \,\middle|\, \tau \subseteq \tau'_\epsilon, \tau' \subseteq \tau_\epsilon\right\}$$

with τ_ϵ, τ'_ϵ to be ϵ-neighborhoods of τ, $\tau' \in Q^{[n]}$.

Let now f be an arbitrary element of the vector space $I(Q, B)$ of bounded B-measurable functions, defined on Q. We introduce a *polynomial extension* $f_\rho^{[n]}$ of the function f to $Q^{[n]}$, determined by the formula

$$f_\rho^{[n]}(\tau) := \prod_{t \in \tau} f(t), \quad \tau \in Q^{[n]}.$$

It is not very hard to verify that for any $f \in I(Q, B)$ its polynomial extension belongs to the vector space $I(Q^{[n]}, \sigma B^{[n]})$ of bounded $\sigma B^{[n]}$-measurable functions, defined on $Q^{[n]}$. Hence, for any $f \in I(Q, B)$ its extension $f_\rho^{[n]}$ is a μ_v-integrable function. Consequently, for any $v \in rV^n$, a functional $P_v : \; I(Q, B) \to \mathbf{R}$, given by the formula

$$P_v(f) := \int f_\rho^{[n]} d\mu_v, \qquad f \in I(Q, B), \tag{3}$$

is well defined.

Remark 2. Certainly, apart from $f_\rho^{[n]}$, some other extensions of $f \in I(Q, B)$ may be of interest. For example, extensions $f_{\max}^{[n]}(\tau) = \max\{f(t) \mid t \in \tau\}$, and $f_\sigma^{[n]}(\tau) = \sum_{t \in \tau} f(t)/|\tau|$ proved to be very useful in description of the Shapley functional (see (Vasil'ev, 1998; Vasil'ev, 2001)) and support function of the core of a convex game ((Vasil'ev, 2006) and (Vasil'ev and Zuev, 1988)), respectively.

Now we are in position to introduce one of the main concept of the paper.

Definition 4 (Vasil'ev, 1998). For any $v \in rV^n$, the functional P_v, defined by formula (3), is said to be a *generalized Owen extension* of a cooperative game v.

It can easily be checked that in case Q is finite we have that the generalized Owen extension of any cooperative game v coincides with its classical Owen multilinear extension, given in (Owen, 1972). As to the infinite set of players, we just mention several most important properties of the functional P_v. To this end we need one more fundamental concept.

Definition 5 (Vasil'ev, 1975a). A game $v \in rV^n$ is said to be *a homogeneous regular game of order* n if it belongs to the disjoint complement of $rV^{n-1} : |v| \wedge |u| = 0$ for any $u \in rV^{n-1}$. Denote by $rV^{(n)} = rV^{(n)}(Q)$ a space of all homogeneous regular games of order n ($rV^0 = rV^{(0)} := \{0\}$).

Proposition 2 (Vasil'ev, 1975a). *For any $n \geq m$ subspace $rV^{(m)}$ is a band in rV^n.*

From Proposition 2 it follows that by the well-known Riesz theorem (see, e.g., (Aliprantis and Border, 1994)), for any $n \geq m$, the space $rV^{(m)}$ is a projection band in rV^n. Consequently, for any $n \geq m$ and $v \in rV^n$ there exists a projection $v_{(m)}$ of v on $rV^{(m)}$, defined by the formula

$$v_{(m)} := \sup \left\{u \in rV^{(m)} \middle| v^+ \geq_0 u\right\} - \sup \left\{u \in rV^{(m)} \middle| v^- \geq_0 u\right\} \tag{4}$$

(for more details concerning the homogeneous components $v_{(m)}$ of v, given by (4), see (Vasil'ev, 1998)).

To present several useful properties of the generalized Owen extension P_v, we introduce first some additional functional spaces, associated with cooperative games of bounded polynomial variation. First, put $I = I(Q, B)$, and denote by $\mathcal{U}(I)$ the set of continuous functionals $l : I \to \mathbf{R}$ such that $l(0) = 0$. Recall (Vasil'ev, 1998), that I supposed to be endowed with the standard norm

$$\|f\|_\infty = \sup\{ \, |f(t)| \mid t \in Q\}, \quad f \in I.$$

Following (Frechet, 1910) we introduce *a polynomial m-difference $l(\{f_1, \ldots, f_m\})$ of the functional $l \in \mathcal{U}(I)$ with respect to* $f_1, \ldots, f_m \in I$ by the formula

$$l\,\big(\{f_1, \ldots, f_m\}\big) = \sum_{\omega \subseteq \{1,\ldots,m\}} (-1)^{m-|\omega|}\; l\Big(\sum_{i \in \omega} f_i\Big).$$

Denote by $U_+(I)$ the cone of *totally positive functionals* (Vasil'ev, 1998) belonging to $\mathcal{U}(I)$:

$$U_+(I) := \{l \in \mathcal{U}(I) \mid l\,\big(\{f_1, \ldots, f_m\}\big) \geq 0 \quad \text{for any } m \geq 1 \text{ and } f_1, \ldots, f_m \in I_+\}$$

(with $I_+ := \{f \in I|\; f(t) \geq 0,\; t \in Q\}$). Further, put $U(I) = U_+(I) - U_+(I)$, and recall the definitions of polynomial and homogeneous polynomial functionals from $U(I)$.

Definition 6 (Frechet, 1910). An element $l \in U(I)$ is said to be *a polynomial functional of order n*, if $l\,\big(\{f_1, \ldots, f_n, f_{n+1}\}\big) = 0$ for any $f_1, \ldots, f_n, f_{n+1} \in I$. For any $n \geq 1$, denote by $\mathcal{P}^n(I)$ the space of all polynomial functionals of order n, defined on I.

Definition 7 (Hille and Phillips, 1957). An element $l \in U(I)$ is said to be *a homogeneous polynomial functional of order n*, if $l \in \mathcal{P}^n(I)$, and $l(\lambda f) = \lambda^n l(f)$ for any $\lambda \in \mathbf{R}$ and $f \in I$. For any $n \geq 1$, by $\mathcal{P}^{(n)}(I)$ denote the space of all homogeneous polynomial functionals of order n, defined on I.

Below, we apply notations: $rV_+^n = rV^n \cap V_+$, and $\mathcal{P}_+^n(I) = \mathcal{P}^n(I) \cap U_+(I)$. Put

$$\mathcal{P}(I) = \cup_{n=1}^\infty \mathcal{P}^n(I).$$

An element $p \in \mathcal{P}(I)$ is said to be *a polynomial functional*. Finally, as usual, by χ_S we denote the indicator function of coalition $S \in B$: $\chi_S(t) = 1$ whenever $t \in S$, and $\chi_S(t) = 0$ otherwise.

In the notations, given above, the most important properties of the generalized Owen extension we use in the sequal are as follows.

Theorem 1 (Vasil'ev, 1998). *Generalized Owen extension P_v is a continuous polynomial functional on $(I, \|\cdot\|_\infty)$ having the properties*

($\mathcal{P}$.1) $P_v(\chi_S) = v(S)$ *for any* $S \in B$;

($\mathcal{P}$.2) $P_v \in \mathcal{P}^n_+(I)$ *for any* $v \in rV^n_+$;

($\mathcal{P}$.3) $P_v \in \mathcal{P}^{(n)}(I)$ *for any* $v \in rV^{(n)}$;

($\mathcal{P}$.4) $|P_v(f)| \leq \sum_{m=1}^n \|v_{(m)}\|_o \|f\|^m_\infty$ *for any* $f \in I$.

Remark 3. By applying argumentation, similar to that employed for the proof of Theorem 1 one can demonstrate the following useful properties of the generalized Owen extension P_v :

($\mathcal{P}$.5) $P_{\alpha u + \beta w} = \alpha P_u + \beta P_w$ for any $\alpha, \beta \in \mathbf{R}$ and $u, w \in rpV$;

($\mathcal{P}$.6) $P_{u \cdot w} = P_u \cdot P_w$ for any $u, w \in rpV$

with $u \cdot w$ and $P_u \cdot P_w$ to be pointwise products of set functions u, w and functionals P_u, P_w, respectively.

3. Axiomatization of Generalized Owen Extension

In this section, like in (Aumann and Shapley, 1974), we assume for simplicity that $Q = [0, 1]$, and, respectively, B is the Borel σ-algebra of the unit interval $[0, 1]$. Recall (Aumann and Shapley, 1974), that by $\|\cdot\|$ we denote the variation norm

$$\|v\| := \inf\{u(Q) + w(Q) \mid v = u - w,\ u, w \in \mathcal{M}\}$$

with $\mathcal{M}$ to be a cone of increasing set functions from $\mathcal{V}$. One of the most important vector spaces investigated in (Aumann and Shapley, 1974) is pNA being the closure (w.r.t. the variation norm $\|\cdot\|$) of linear span of powers μ^k with $k \geq 1$ and μ to be any nonnegative nonatomic measure defined on B. Below, to mitigate argumentation, we restrict our study to the space

$$rpNA := rpV \cap pvNA$$

with $pvNA$ to be the closure (w.r.t. the norm of polynomial variation $\|\cdot\|_0$, defined in Sect. 2) of linear span of powers μ^k, $k \geq 1$, where μ is any nonnegative nonatomic measure defined on B. Note, that due to the inequalities $\|v\| \leq \|v\|_0$, $v \in \mathcal{V}$, we have inclusion $rpNA \subseteq pNA$. Nevertheless, these spaces are not very far from each other: obviously, the closure of $rpNA$ w.r.t. the variation norm $\|\cdot\|$ coincides with pNA.

Slightly modifying definitions from (Aumann and Shapley, 1974) we say that a functional $l : I \to \mathbf{R}$ is increasing if $l(0) = 0$ and $l(f) \geq l(g)$ whenever $f \geq g$ with $f, g \in I_+$. We denote a cone of all the increasing functionals by $\mathcal{M} = \mathcal{M}(I)$, and put

$$\mathcal{B} = \mathcal{M} - \mathcal{M}.$$

An element $l \in \mathcal{B}$ is said to be a functional of *bounded variation;* its norm $\|l\|$ is defined by the formula:

$$\|l\| = \inf\{m(\chi_Q) + n(\chi_Q) \mid l = m - n,\ m, n \in \mathcal{M}\}.$$

Finally, put $U = U_+ - U_+$ and for any functional $p \in U$ denote by $\|p\|_0$ its *polynomial variation norm*

$$\|p\|_0 = \inf\{q(\chi_Q) + r(\chi_Q) \mid p = q - r,\ q, r \in U_+\}.$$

Due to the obvious inclusion $U_+ \subseteq \mathcal{M}$ we have that $U \subseteq \mathcal{B}$ and, besides, $\|l\| \leq \|l\|_0$ for any $l \in U$. Moreover, for the Aumann-Shapley multiplicative expansion $v^* \in \mathcal{B}$ it holds: $\|v^*\| = \|v\|$ for any $v \in pNA$ (Aumann and Shapley, 1974). Hence, for any $v \in rpNA$ we get: $\|P_v\| \leq \|P_v\|_0 \leq P_{v^+}(\chi_Q) + P_{v^-}(\chi_Q) = \|v\|_0$ (the last equality follows from ($\mathcal{P}$.1) and definition of the norm $\|\cdot\|_0$). Summarizing, we obtain

$$\|P_v\| \leq \|v\|_0 \quad \text{for any } v \in rpNA. \tag{5}$$

By applying the same argumentation as in (Vasil'ev, 2009), one can show that (5) makes it possible to give an axiomatization of the generalized Owen extension P_v, based on the well-known axiomatic characterization of multiplicative expansion of nonatomic cooperative games, proposed in (Aumann and Shapley, 1974). Recall, that the expansion mentioned was aimed at the generalization of the famous Owen integral formula (Owen, 1972) to the case of nonatomic cooperative games. It was already mentioned in Sect. 1 that this integral formula plays a crucial role in the new proof of polar representation of the Shapley value for nonatomic homogeneous game. Therefore, axiomatic description of the Aumann-Shapley expansion is closely related to the main problem of our paper. Slightly modifying corresponding definitions from (Aumann and Shapley, 1974), we recall that Aumann-Shapley multiplicative expansion $v^* = \varphi(v)$ of a game v is given implicitly, via indicating the properties of the operator φ, which takes $v \in rpNA$ to the functional $\varphi(v) : I \to \mathbf{R}$. In the notations, given above, properties mentioned are as follows (below, as before, $v \cdot w$ and $\varphi(v) \cdot \varphi(w)$ are pointwise products of the corresponding functions):

(Qw.1) $\varphi(v)(\chi_S) = v(S) \quad$ for any $v \in rpNA$ and $S \in B$;

(Ow.2) $\varphi(\alpha v + \beta w) = \alpha\varphi(v) + \beta\varphi(w), \quad \alpha, \beta \in \mathbf{R},\ v,\ w \in rpNA$;

(Ow.3) $\varphi(v \cdot w) = \varphi(v) \cdot \varphi(w), \quad v,\ w \in rpNA$;

(Ow.4) $\varphi(v)(f) = \int f dv, \quad f \in I(Q, B),\ v \in rV^1$;

(Ow.5) $\varphi(v) \in \mathcal{P}_+, \quad v \in rpNA_+$,

with $\mathcal{P}_+(I) := \mathcal{P}(I) \cap U_+(I)$ and $rpNA_+ := rpNA \cap V_+$.

To conclude this section, let us present a version of Theorem 4.1 (Vasil'ev, 2009), following directly from Theorem 1 (Sect. 2), Theorem G (Aumann and Shapley, 1974), inequality (5), and continuity of the operators $v \mapsto v^*,\ v \in pNA$, and $v \mapsto P_v,\ v \in rpNA$, in variation and polynomial variation norms, respectively. Here, as before, we denote by v^* the Aumann-Shapley multiplicative expansion of a game $v \in pNA$.

Theorem 2. *A mapping $\varphi : rpNA \to \mathcal{P}(I)$ satisfies assumptions $(Ow.1) - (Ow.5)$ if and only if $\varphi(v) = P_v$ for any $v \in rpNA$.*

Note,that Remark 3, properties (Ow.3), (Ow.4), and Theorem G on the existence and uniqueness of the multiplicative expansion from (Aumann and Shapley, 1974) implies equalities: $\varphi(\mu^k) = P_{\mu^k}$ for any nonnegative nonatomic measure μ and integer $k \geq 1$. Hence, we have the following consequence of Theorem 2.

Corollary 1. *Aumann-Shapley multiplicative expansion coincides with the generalized Owen extension on $rpNA$.*

4. Polar Form of Homogeneous Game

Let us call to mind first some definitions from (Aumann and Shapley, 1974) and (Vasil'ev, 2009). Note, that a distinctive feature of the notions from (Vasil'ev, 2009) is their orientation to the regular games defined on the Borel σ-algebra of some metric compactum, while the main concepts from (Aumann and Shapley, 1974) are, mostly, adapted to the nonatomic games of bounded variation. Hence, we need more detailed argumentation than sometimes proposed below, in order to properly transfer corresponding results from (Aumann and Shapley, 1974) to the case considered in the paper. Nevertheless, for the sake of brevity, we leave the additions needed for readers.

As usual, a real-valued set function $\psi : B^n \to \mathbf{R}$ is said to be polyadditive, if it is additive with respect to each variable. Further, a polyadditive function $(S_1, \ldots, S_n) \mapsto \psi(S_1, \ldots, S_n)$ is called a *regular polyadditive function,* if it is regular with respect to each variable:

$$\psi(S_1, \ldots, S_i, \ldots, S_n) = \sup\{\psi(S_1, \ldots, F_i, \ldots, S_n) \mid F_i \in \mathcal{F},\ F_i \subseteq S_i\}$$

for any $(S_1, \ldots, S_n) \in B^n$ and $i = 1, \ldots, n$ (as before, $\mathcal{F}$ is the family of closed subsets of Q). In the sequel, we consider *symmetric polyadditive functions* only, i.e. polyadditive functions $\psi : B^n \to \mathbf{R}$ such that for any elements $S_1, \ldots, S_n \in B^n$ it holds:

$$\psi(S_1, \ldots, S_n) = \psi(S_{i_1}, \ldots, S_{i_n})$$

for any permutation $(i_1, \ldots, i_n)$ of the set $\{1, \ldots, n\}$.

We denote by $r\Psi^n_+$ a cone of all the nonnegative regular symmetric polyadditive functions $\psi : B^n \to \mathbf{R}$. Put $r\Psi^n := r\Psi^n_+ - r\Psi^n_+$, and isolate a special subspace $r\Psi^{(n)} \subseteq r\Psi^n$ similar to the space $rV^{(n)}$ of the regular homogeneous set functions. To this end, following (Vasil'ev, 1998), consider the set $H_n(Q)$ of all B-measurable partitions $\eta = \{S_i\}_{i \in \Omega} \in H(Q)$ such that $|\Omega| \geq n$. For any partition $\eta = \{S_i\}_{i \in \Omega} \in H_n(Q)$ (by definition, consisting of not less than n elements) denote by Π^η_n the set of all its ordered n-element subsets $(S_{i_1}, \ldots, S_{i_n})$. Further, fix some $\psi \in r\Psi^n_+$, and define generalized sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ with ψ_η given by the formula

$$\psi_\eta := \sum_{(S_{i_1}, \ldots, S_{i_n}) \in \Pi^\eta_n} \psi(S_{i_1}, \ldots, S_{i_n}).$$

Taking into account nonnegativity and polyadditivity of ψ, it is quite easy to check that the sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ is increasing: $\psi_{\eta'} \geq \psi_\eta$ whenever $\eta' \geq \eta$. Consequently, for any function $\psi \in r\Psi^n_+$ there exists a limit

$$\psi_{(n)}(Q) = \lim_{\eta \in H_n(Q)} \psi_\eta$$

(as in (Vasil'ev, 1998), we suppose that the sequence $\{\psi_\eta\}_{\eta \in H_n(Q)}$ is ordered by the relation: $\eta' \geq \eta$ whenever η' is a refinement of η). Let $\Psi^{(n)}_+$ be the set of all functions $\psi \in r\Psi^n_+$ satisfying the requirement

$$\psi_{(n)}(Q) = \psi(Q, \ldots, Q).$$

Put $r\Psi^{(n)} := r\Psi_+^{(n)} - r\Psi_+^{(n)}$, and recall (Vasil'ev, 1998) that any function $\psi \in r\Psi^{(n)}$ is said to be *a homogeneous polyadditive set function* from $r\Psi^n$.

Now we are ready to present one of the main definitions of the paper (cf. Definition 11 from (Vasil'ev, 1998)).

Definition 8. For any $v \in rV^{(n)}$, a polyadditive set function $\psi_v \in r\Psi^{(n)}$ is said to be *a polar form of the game* v, if ψ_v meets the requirement

$$v(S) = \psi_v(S, \ldots, S) \quad \text{for any} \quad S \in B.$$

Speaking differently, for any $v \in rV^{(n)}$, a polyadditive set function $\psi_v \in r\Psi^{(n)}$ is a polar form of v if the diagonalization of ψ_v (i.e. restriction ψ_v to the diagonal $D = \{(S_1, \ldots, S_n) \in B^n \mid S_1 = \ldots = S_n\}$) coincides with v.

Remark 4. Note, that in case Q is a metric compactum, regularity of polar form ψ_v is equivalent to its countable additivity with respect to each variable S_i (see, for example, (Neveu, 1965)).

The well-known polar existence theorem for the homogeneous polynomial functionals (Hille and Phillips, 1957), together with Theorem 1 made it possible to establish a polar existence theorem for rather general class of homogeneous polynomial set functions (see (Vasil'ev, 1998) and (Vasil'ev, 2001)). By applying regularity and compactness assumptions imposed on v and Q, respectively, one can quite easily derive from (Vasil'ev, 1998) the following modified version of general polar existence theorem (cf. Theorem 5 from (Vasil'ev, 1998)).

Theorem 3. *For any $n \geq 1$ and $v \in rV^{(n)}$, there exists a unique polar form ψ_v of the game v.*

To present one of the main results, relating to the interconnection between the Shapley value and polar form of homogeneous polynomial game, remind first the definition of the modified Shapley value Φ_*, introduced in (Vasil'ev, 1998) (see, also, (Vasil'ev, 2001)), and covering both nonatomic an mixed games. Recall briefly (Vasil'ev, 1998), that a linear operator $\Phi_* : W \to rV^1$, defined on a symmetric subspace $W \subseteq V$, is to be a modified Shapley value on W, if it is an efficient, positive, support preserving, and commuting with any measurable automorphism θ of the measurable space (Q, B). To be precise, denote by $\mathcal{T}$ the set of this automorphisms, and for any $\theta \in \mathcal{T}$ and $v \in rV$ define a composition $\theta \circ v$ by the formula

$$\theta \circ v(S) = v(\theta(S)), \quad S \in B.$$

Further, recall (Aumann and Shapley, 1974) that a linear subspace $W \subseteq V$ is called *symmetric*, if a set function $\theta \circ v$ belongs to W for any $v \in W$ and $\theta \in \mathcal{T}$.

Definition 9 (Vasil'ev, 1975b). *A modified Shapley value* on a symmetric subspace $W \subseteq V$ is a linear operator $\Phi_* : W \to rV^1$, satisfying assumptions

$(Sh.1)$ $\Phi_*(v) \geq_o 0, \quad v \in W_+;$

$(Sh.2)$ $\Phi_*(\theta \circ v) = \theta \circ \Phi_*(v), \quad \theta \in \mathcal{T}, \quad v \in W;$

$(Sh.3)$ $\Phi_*(v)(R) = v(R), \quad R \in \text{Supp } v, \quad v \in W;$

where, as before, rV^1 is a space of all regular additive set functions on B, and W_+ is a "positive part" of $W : W_+ := W \cap V_+$. As to the collection Supp v consisting of the supports of v, it is defined by the formula

$$\text{Supp } v := \{R \in B \mid v(S \cap R) = v(S) \quad \text{for any} \quad S \in B\}.$$

By applying regularity of functions from rpV and compactness of metric space (Q, d), one can prove that rpV itself, and $rV^{(n)}$, rV^n, $n \geq 1$, are symmetric subspaces of V. Further, in (Vasil'ev, 1998) some special construction was proposed that ensures existence of modified value for $rV^{(n)}$, rV^n, and rpV. Hence, combining this fact together with Theorem 3, and making use of corresponding argumentation yielding Theorem 6 from (Vasil'ev, 1998), one can get the following version of the tatter theorem.

Theorem 4. *Let Q be a nonempty metrisable compactum. For any $n \geq 1$ and $v \in rV^{(n)}(Q)$ it holds*

$$\varPhi_*(v)(S) = \psi_v(S, Q, \ldots, Q), \quad S \in B,$$

where, as before, ψ_v is the polar form of a game v.

5. Polar Representation of the Shapley Value: Nonatomic Games

Turn now to the main part of the paper devoted to the short proof of Theorem 4 in case $v \in rpNA = rpNA(Q)$ with $Q = [0, 1]$. Note first, that rather simple argumentation, based on the well-known Aumann-Shapley existence and uniqueness theorem (Theorem A from (Aumann and Shapley, 1974)), proves the coincidence of the modified Shapley value $\varPhi_*$ and classic Shapley value $\varPhi$ on the space $rpNA$. The reason is the coincidence of the modified value $\varPhi_*$ and the Shapley value $\varPhi$ on the linear hull of degrees of nonatomic probabilistic measures. Namely, from the results obtained in (Aumann and Shapley, 1974) and (Vasil'ev, 1998) it follows that $\varPhi_*(\mu^k) = \varPhi(\mu^k) = \mu$ for any $k \geq 1$ and nonatomic probabilistic measure μ on B. Further, let us stress once more that instead of complicated combinatorial consideration applied in general situation (see proof of Theorem 6 from (Vasil'ev, 1998)), we plan to exploit the generalized Owen integral formula established in (Aumann and Shapley, 1974). Due to the coincidence of generalized Owen extension and Aumann-Shapley expansion for games from $rpNA$ (Corollary 1 in Sect. 3) we have that integrand in the generalized integral formula mentioned above is quite easy to calculate. In fact, by Theorem 1 and Corollary 1 this calculation can be reduced to elementary problem of finding directional derivatives of the continuous symmetric multilinear form, generated by the homogeneous polynomial functional P_v.

To justify a version of the generalized Owen integral formula applied below, let us mention first that Theorem G from (Aumann and Shapley, 1974) implies existence and uniqueness of the Aumann-Shapley expansion for the space $rpNA$. Really, the latter follows directly from Theorems 1 and 2 of this paper. As to the generalized integral formula itself (Theorem H from (Aumann and Shapley, 1974)), it holds for $rpNA$ due to the inclusion $rpNA \subseteq pNA$. Therefore, we get the following analog of the Owen integral formula for the homogeneous games from $rpNA$.

Theorem 5. *For any game $v \in rpNA$, and for any coalition $S \in B$, directional derivative*

$$\partial P_v(t,S) := \frac{d}{d\tau} P_v(t\chi_Q + \tau\chi_S),$$

calculated at $\tau = 0$, exists at each point $t \in [0,1]$. Moreover, this derivative is integrable as a function of $t \in [0,1]$. In addition, for any $n \geq 1$, the Shapley value $\Phi : rpNA \cap rV^{(n)} \to rV^1$, and derivatives of P_v in the direction of χ_S satisfy the equalities

$$\Phi(v)(S) = \int_0^1 \partial P_v(t,S)dt, \quad S \in B. \tag{6}$$

Recall (Vasil'ev, 1998), that a symmetric multilinear functional $\widehat{p} : I^n \to \mathbf{R}$ is said to be *a polar form of homogeneous polynomial functional* $p \in \mathcal{P}^{(n)}$ if

$$p(f) = \widehat{p}(f,\ldots,f) \quad \text{for any } f \in I.$$

It is well-known that polar form $\widehat{p}$ exists whenever polynomial functional p is homogeneous and continuous (see, e.g., (Hille and Phillips, 1957)). Note, that due to Theorem 1 polynomial functional P_v is continuous and homogeneous for any $v \in rV^{(n)}$. In fact, taking into account that $\|v\|_0 = \|v_{(n)}\|_0$ for any $v \in rV^{(n)}$, we have by $(\mathcal{P}.4)$: $|P_v(f)| \leq \|v\|_0 \|f\|_\infty^n$ for any $v \in rV^{(n)}$ and $f \in I$. Therefore, P_v is continuous for any $v \in rV^{(n)}$. As to the homogeneity of P_v in case $v \in rV^{(n)}$, it follows directly from the property $(\mathcal{P}.3)$. Hence, for any $v \in rV^{(n)}$ there exists a polar form $\widehat{P}_v$ of the generalized Owen extension P_v and, consequently, for any $v \in rV^{(n)}$ it holds

$$P_v(f) = \widehat{P}_v(f,\ldots,f), \quad f \in I. \tag{7}$$

Keep in mind (7), let us mention that according to (6) to prove Theorem 4 in case $Q = [0,1]$ and $v \in rpNA \cap rV^{(n)}$ it is enough to calculate corresponding directional derivatives of P_v at each point of the diagonal $\{t\chi_Q \mid t \in [0,1]\}$ of the unit supercube $\mathcal{I} = \{f \in I_+ \mid \|f\|_\infty \leq 1\}$, and then to demonstrate that Lebesgue integral of the directional derivative coincides with the marginal value of the polar form of generalized Owen extension P_v. In order to carry out this program we rewrite integrand in (6) in terms of the polar form $\widehat{P}_v$ of the functional P_v :

$$\partial\varphi(t,S) := \lim_{\tau\to 0} \frac{P_v(tQ+\tau S) - P_v(tQ)}{\tau} =$$

$$\lim_{\tau\to 0} \big[\widehat{P}_v(\underbrace{tQ+\tau S,\ldots,tQ+\tau S}_{n}) - \widehat{P}_v(\underbrace{tQ,\ldots,tQ}_{n})\big]/\tau \tag{8}$$

with a standard shortening $S := \chi_S$ when indicator function χ_S is replaced by the set S itself. Since $\widehat{P}_v$ is symmetric and multilinear, under condition $n \geq 2$ we get

$$\widehat{P}_v(tQ+\tau S,\ldots,tQ+\tau S) - \widehat{P}_v(tQ,\ldots,tQ) =$$

$$C_n^1 \widehat{P}_v(\tau S, \underbrace{tQ,\ldots,tQ}_{n-1}) + \sum_{k=2}^n C_n^k \widehat{P}_v(\underbrace{\tau S,\ldots,\tau S}_{k}, \underbrace{tQ,\ldots,tQ}_{n-k}) =$$

$$n\tau t^{n-1}\widehat{P}_v(S,\underbrace{Q,\dots,Q}_{n-1})+\sum_{k=2}^{n}C_n^k\tau^k t^{n-k}\widehat{P}_v(\underbrace{S,\dots,S}_{k},\underbrace{Q,\dots,Q}_{n-k}). \tag{9}$$

For $n=1$ we have that $\widehat{P}_v$ is a linear form. Hence, in this case we obtain

$$\widehat{P}_v(\tau S+tQ)-\widehat{P}_v(tQ)=\tau\widehat{P}_v(S).$$

After dividing the last term in (9) into τ, and calculating the limit under $\tau\to 0$, we obtain by (8)

$$\partial\varphi(t,S)=nt^{n-1}\widehat{P}_v(S,\underbrace{Q,\dots,Q}_{n-1})+\lim_{\tau\to 0}\tau\Big[\sum_{k=2}^{n}C_n^k\tau^{k-2}t^{n-k}\widehat{P}_v(\underbrace{S,\dots,S}_{k},\underbrace{Q,\dots,Q}_{n-k})\Big].$$

Hence, the boundedness of the polynomial

$$\sum_{k=2}^{n}C_n^k\tau^{k-2}t^{n-k}\widehat{P}_v(\underbrace{S,\dots,S}_{k},\underbrace{Q,\dots,Q}_{n-k})$$

(as a function depending on $\tau\in[0,1]$) implies

$$\partial\varphi(t,S)=nt^{n-1}\widehat{P}_v(S,\underbrace{Q,\dots,Q}_{n-1}). \tag{10}$$

By applying (6) and (10), we deduce the required representation for the Shapley value of nonatomic homogeneous cooperative game $v\in rpNA\cap rV^{(n)}$:

$$\Phi(v)(S)=\int_0^1\partial\varphi(t,S)dt=n\widehat{P}_v(S,Q,\dots,Q)\int_0^1 t^{n-1}dt=\widehat{P}_v(S,Q,\dots,Q).$$

Summarizing, we have that Theorems 1 and 5 together with the straightforward calculation of directional derivatives of the generalized Owen extension P_v yields the polar representation of the Shapley value in case $v\in rpNA\cap rV^{(n)}$.

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A Simple Way to Obtain the Sufficient Nonemptiness Conditions for Core of TU Game

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Abstract The system of linear constraints like one that determines the core of TU game is considered. Expressing its basis solutions through characteristic function we obtain a list of sufficient conditions under which the core is nonempty. Some of them are the generalizations of known results.

Keywords: cooperative TU game, core, balancedness, sufficient conditions.

1. Introduction

The core (Gillies, 1953) is the most frequently applied multivalued solution of cooperative game theory. Under the grand coalition advantageous the analysis of any TU game usually begins with verification of core existence. For concrete game this possible to do by means of linear programming problem with constraint set covers the core. The core of TU game is empty if and only if the optimal value of this problem is strictly greater than a grand coalition's weight. If we try to prove the core non-emptiness for certain class of games we need condition, expressed through characteristic function. Such is Bondareva-Shapley balancedness condition (Bondareva, 1963; Shapley, 1967). That condition is equivalent to linear system with entries corresponding to the extreme points of polytope in R^{2^n-1}. The number of extreme points and their explicit representation known only for small n.

There exist more simple but only sufficient conditions. The most known is the convexity of game. The minimal convexity test consists of $\frac{2^n n(n-1)}{8}$ inequalities (Voorneveld and Grahn, 2001). In this paper the simple procedure to generate sufficient nonemptiness conditions for core of TU game is described. For any basis matrix consisting of coalitional characteristic vectors we can obtain sufficient condition defined by a system of 2^n-n-1 linear inequalities. So each condition determines a cone in linear space of all TU games.

The following table illustrates what increases the number of inequalities in conditions mentioned above when n increase.

n	3	4	5	6	7
balancedness condition	5	41	1291	200213	132422035
minimal convexity test	6	24	80	240	672
sufficient condition	4	11	26	57	120

The paper has the following contents. Next section recalls the standard facts of cooperative game theory which are useful later. The balancedness condition, minimal balanced sets and some classes of balanced games are described in third section.

The last section contains the set of sufficient nonemptiness conditions corresponding to special bases and shows that some of them are the generalizations of known results.

2. Preliminaries

A *cooperative TU game* is a pair (N,ν) where $N=\{1,2,\dots,n\}$ is a *player set*, $n\geq 2$, $\nu:2^N\rightarrow\mathbf{R}$ is a set function satisfying $\nu(\emptyset)=0$. Throughout the paper we identify (N,ν) and ν. The class of TU games with player set N will be denoted by G^N. A *payoff vector* for a game $\nu\in G^N$ (ore *allocation*) is a vector $x\in\mathbf{R}^n$. A subset of N is called a *coalition*, $\nu(S)$ expresses the *worth* of coalition S and e^S is the characteristic vector of coalition S, i.e. $(e^S)_i=1$ if $i\in S$, $(e^S)_i=0$ otherwise. Sometimes (for simplicity) we shall write $N\setminus i$ instead of $N\setminus\{i\}$, $\nu(123)$ instead of $\nu(\{1,2,3\})$ and so on. For any $S\in 2^N$ and $x\in\mathbf{R}^n$ let $x(S)=\sum_{i\in S}x_i$ and $x(\emptyset)=0$. The cardinality of coalition S is written as $|S|$. The rank of matrix A is denoted as $rank(A)$. The *dual game* ν^* of $\nu\in G^N$ is determined by

$$\nu^*(S)=\nu(N)-\nu(N\setminus S)\ \ for\ every\ S\subseteq N.$$

A game $\nu\in G^N$ is called:
- *zero-normalized* if $\nu(i)=0$ for all $i\in N$,
- *nonnegative* if $\nu(S)\geq 0$ for all $S\subseteq N$,
- *monotonic* if $\nu(S)\leq\nu(T)$ for all $S\subset T\subseteq N$,
- *N-essential* if $\sum_{i\in N}\nu(i)<\nu(N)$,
- *convex* (*concave*) if

$$\nu(S)+\nu(T)\leq(\geq)\ \nu(S\cup T)+\nu(S\cap T)\ \ for\ all\ S,T\subseteq N.$$

If the grand coalition is formed then players can divide the amount $\nu(N)$. The *order* on N is a bijection $\pi:N\rightarrow N$. The set of all orders $\pi=\{\pi_1,\dots,\pi_n\}$ is given by $\Pi(N)$. The *Weber set* of game $\nu\in G^N$, denoted by $W(\nu)$, is the convex hull in R^n of all π-*marginal vectors*

$$W(\nu)=conv\{m^\pi(\nu)\mid\pi\in\Pi(N)\}$$

were

$$m_i^\pi(\nu)=\nu(S_i^\pi)-\nu(S_{i-1}^\pi),\ S_i^\pi=\{\pi_1,\dots,\pi_i\},\ i\in N.$$

$W(\nu)\neq\emptyset$ for any game $\nu\in G^N$. For the description of other set-valued solution concepts is used the set

$$X(\nu)=\{x\in\mathbf{R}^N\mid x(N)=\nu(N)\}$$

of efficient payoff distributions of $\nu(N)$ named the *preimputation set* and its subsets: the *imputation set*

$$I(\nu)=\{x\in X(\nu)\mid x_i\geq\nu(i),\ i\in N\},$$

the *dual imputation set*

$$I^*(\nu)=\{x\in X(\nu)\mid x_i\leq\nu^*(i),\ i\in N\}.$$

$I(\nu) \neq \emptyset$ iff $\sum_{i\in N} \nu(i) \leq \nu(N)$. If a game $\nu \in G^N$ is N-essential then $I(\nu)$ is an $(n-1)$-dimensional simplex with extreme points $f^i(\nu) \in \mathbf{R}^n$, $i \in N$, where

$$(f^i(\nu))_j = \begin{cases} \nu(j), & i \neq j, \\ \nu(N) - \sum_{k\in N\setminus i} \nu(k), & i = j, \end{cases} \quad \textit{for all } i \in N.$$

$I^*(\nu) \neq \emptyset$ iff $\sum_{i\in N} \nu^*(i) \geq \nu(N)$. In case of strict inequality $I^*(\nu)$ is an $(n-1)$-dimensional simplex with extreme points $g^i(\nu) \in \mathbf{R}^n$, $i \in N$, where

$$(g^i(\nu))_j = \begin{cases} \nu^*(j), & i \neq j, \\ \nu(N) - \sum_{k\in N\setminus i} \nu^*(k), & i = j. \end{cases} \quad \textit{for all } i \in N.$$

The *core* of a game $\nu \in G^N$ is a subset of *core cover* (Branzei and Tijs, 2001) $CC(\nu) = I(\nu) \cap I^*(\nu)$ defined by

$$C(\nu) = \{x \in I(\nu) \mid x(S) \geq \nu(S),\ S \subseteq N\}.$$

Thus $\nu(i) \leq x_i \leq \nu^*(i)$ for all $i \in N$ and $x \in C(\nu)$. $Web(\nu)$, $I(\nu)$, $I^*(\nu)$, $CC(\nu)$ and $C(\nu)$ are the polytopes in $\mathbf{R}^n$. The set of extreme points of polytope $\mathbf{P}$ will be denoted by $ext(\mathbf{P})$.

Two players $i, j \in N$ are *symmetric* in $\nu \in G^N$ if $\nu(S \cup i) = \nu(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. A player $i \in N$ is a *veto player* in $\nu \in G^N$ if $\nu(S) = 0$ for each $S \not\ni i$. The set of veto players is denoted by $Veto(\nu)$.

3. Balancedness

Let

$$\Omega^1 = 2^N \setminus \{\emptyset\}, \quad \Omega^2 = 2^N \setminus \{N, \emptyset\}, \quad \Omega(k_1, k_2) = \{S \in 2^N \mid k_1 \leq |S| \leq k_2\}$$

are the sets of nonempty coalitions, proper coalitions and coalitions with restricted size. It is proved (Bondareva, 1963; Shapley, 1967) that the core of a game $\nu \in G^N$ is nonempty iff

$$\sum_{S\in\Omega^1} \lambda_S \nu(S) \leq \nu(N) \quad \textit{for all } \lambda \in ext(\mathbf{\Lambda}^n) \tag{1}$$

where

$$\mathbf{\Lambda}^n = \{\lambda \in \mathbf{R}_+^{2^n-1} \mid \sum_{S\in\Omega^1} e^S \lambda_S = e^N\}.$$

A game $\nu \in G^N$ satisfying (1) is called *balanced game*. The condition (1) is called *balancedness condition*. In some game theory literature a game is balanced if it have a nonempty core.

A collection $F = \{F\}_{i=1}^m$ of coalitions $F_i \in \Omega^2$ is called *minimal balanced set* if there exists $\lambda \in ext(\mathbf{\Lambda}^n)$ such that $\lambda_S > 0$ for $S \in F$ and $\lambda_S = 0$ otherwise (this definition can be given in alternative form). The vector $\alpha(F) = (\alpha(F_i))_{i=1}^m$ with $\alpha(F_i) = \lambda_{F_i}$ is called *weight vector* for F. In terms of minimal balanced sets a necessary and sufficient condition for nonemptiness of the core of game $\nu \in G^N$ can be written as

$$\sum_{F_i\in F} \alpha(F_i)\nu(F_i) \leq \nu(N) \quad \textit{for all } F \in \mathbf{F}^n, \tag{2}$$

where $\mathbf{F}^n$ denotes a family of all minimal balanced sets on N.

Definition 1. A linear inequality is called *convexity-inequality*, *concavity-inequality*, *union-inequality* ore *balancedness-inequality* if it contains in system defining corresponding property of game ν.

The number of balancedness-inequalities grows very rapidly with n. Some of them define necessary and sufficient nonemptiness conditions for other sets. $I(\nu) \neq \emptyset$ ($I^*(\nu) \neq \emptyset$) iff the corresponding to $\{\{1\}, \ldots, \{n\}\}$ ($\{N \setminus \{1\}, \ldots, N \setminus \{n\}\}$) balancedness-inequality is satisfied. $CC(\nu) \neq \emptyset$ iff a game $\nu \in G^N$ satisfies inequalities corresponding to $\{\{1\}, \ldots, \{n\}\}$, $\{N \setminus \{1\}, \ldots, N \setminus \{n\}\}$ and $\{\{i\}, N \setminus \{i\}\}$, $i \in N$.

Let us list some types of balanced games which characterization yields a sufficient conditions for core existence. A game is T*-simplex* (Branzei and Tijs, 2001) where $T \in \Omega^1$ if its core is a subsimplex of imputation set, i.e. a game is N-essential and

$$C(\nu) = conv\{f^i(\nu) | i \in T\}.$$

A game is *dual-simplex* (Branzei and Tijs, 2001) if its core is a subsimplex of dual imputation set, i.e. $\nu(N) < \sum_{i \in N} \nu^*(i)$ and there is a coalition $T \in \Omega^1$ such that

$$C(\nu) = conv\{g^i(\nu) | i \in T\}.$$

A balanced game satisfies the *CoMa-property* (Hamers et al., 2002) iff the extreme points of its core are marginal vectors, i.e. $ext(C(\nu)) \subseteq ext(Web(\nu))$. The convex games satisfy the CoMa-property because for them $C(\nu) = Web(\nu)$ (Shapley, 1971). The non-convex games that satisfy the CoMa-property are: information games (Kuipers, 1993), assignment games (Hamers et al., 2002), cost spanning tree games (Granot and Huberman, 1981). If a game $\nu \in G^N$ is *permutationally convex* with respect to an order $\pi \in \Pi(N)$ then corresponding marginal vector $m^\pi(\nu)$ is a core allocation (in other words $ext(C(\nu)) \cap ext(Web(\nu)) \neq \emptyset$). But the reverse is not true in general (Velzen et al., 2005).

A game $\nu \in G^N$ is *clan game* with coalition $CL \neq \emptyset$ as clan (Potters et al., 1989) if: it satisfies the *union property*

$$\nu(N) - \nu(S) \geq \sum_{i \in N \setminus S} \nu^*(i) \quad for\ all\ S \subseteq N\ with\ CL \subseteq S;$$

ν and $\nu^*(i)$, $i \in N$, are non-negative; $\nu(S) = 0$ if $CL \not\subseteq S$ (*clan property)*. A game $\nu \in G^N$ $(n \geq 3)$ is called a *big boss game* with player 1 as big boss (Muto, et al., 1988) if: ν is monotonic; $\nu(S) = 0$ for all $S \subset N$ with $1 \notin S$ (*boss property*); $\nu(N) - \nu(S) \geq \sum_{i \in N \setminus S} \nu^*(i)$ for all $S \subseteq N$ with $1 \in S$ (*union property*).

4. Sufficient conditions

Consider the system

$$x(S) \geq \nu(S),\ S \in \Omega^2, \quad -x(N) \geq -\nu(N) \tag{3}$$

differs from one determines the core of game $\nu \in G^N$ that efficiency condition

$$x(N) = \nu(N) \tag{4}$$

is replaced with inequality $x(N) \le \nu(N)$. If the system (3) is non-solvable than the core of game $\nu \in G^N$ is empty. Let $\hat{x}$ is a solution to system (3). If it satisfies (4) than $\hat{x} \in C(\nu)$. Otherwise the payoff vectors x^i, $i \in N$, where

$$x_j^i = \begin{cases} \hat{x}_j, & i \ne j, \\ \nu(N) - \sum_{k \in N \setminus i} \hat{x}_k, & i = j, \end{cases}$$

are the core allocations. The system (3) can be presented in the form

$$Ax \ge \overline{\nu}, \tag{5}$$

where $x \in \mathbf{R}^n$, $\overline{\nu} = (\overline{\nu}(S))_{S \in \Omega^1}$, $\overline{\nu}(S) = \nu(S)$, $S \in \Omega^2$, $\overline{\nu}(N) = -\nu(N)$, A is the $(2^n - 1) \times n$ matrix with row vectors A^S refering to $S \in \Omega^1$. The first $(2^n - 2)$ rows of matrix A are the characteristic vectors e^S of coalitions $S \in \Omega^2$. The last row $A^N = -e^N$ of A corresponds to the grand coalition. Obviously $rank(A) = n$.

Let $B = (b_{ij})_n$ be a basis of A. By transposition of rows the matrix A can be represented in the form $A = (B\ D)^T$ and system (3) becomes

$$Bx \ge \overline{\nu}_B, \quad Dx \ge \overline{\nu}_D,$$

where $\overline{\nu}_B = (\overline{\nu}(S))_{S \in B}$, $\overline{\nu}_D = (\overline{\nu}(S))_{S \in D}$ is a basis, nonbasis partition of variables in the vector $\overline{\nu}$. The system $Bx = \overline{\nu}_B$ determines the unique basis solution $x^B = B^{-1}\overline{\nu}_B$. If x^B satisfies $Dx^B \ge \overline{\nu}_D$ then it is the feasible solution to system (5). The following provides a simple *sufficient condition*. Let $\nu \in G^N$ and B is a basis of matrix A in (5). If ν satisfies

$$D(B^{-1}\overline{\nu}_B) \ge \overline{\nu}_D \tag{6}$$

then $C(\nu) \ne \emptyset$.

The next two examples illustrate the above technique for the most simple bases.

Example 1. Take basis

$$B = \{A^{\{1\}}, \ldots, A^{\{n\}}\}^T$$

with characteristic vectors of single player coalitions as rows. Thus B and B^{-1} are the $n \times n$ identity matrixes, $x^B = (\nu(1), \ldots .\nu(n))$. Condition (6) becomes

$$\nu(S) \le \sum_{i \in S} \nu(i), \quad S \in \Omega^1.$$

It determines the set of N-simplex games. For all $i \in N$ the payoff vector x^i coincides with extreme point $f^i(\nu)$ of imputation set.

Example 2. Basis

$$B = \{A^{\{1\}}, \ldots, A^{\{n-1\}}, A^N\}^T$$

differs from previous one that $A^{\{n\}}$ is replaced on $A^N = -e^N$. The matrixes B, $B^{-1} = (b_{ij}^{-1})_n$ and basis solution x^B are determined by

$$b_{ij} = b_{ij}^{-1} = \begin{cases} 1, & (i = j) \wedge (i \ne n), \\ -1, & i = n, \\ 0, & otherwise. \end{cases} \qquad x_i^B = \begin{cases} \nu(i), & i \ne n, \\ \nu(N) - \sum_{i=1}^{n-1} \nu(i), & i = n, \end{cases}$$

x^B is a core allocation and coincide with extreme point $f^n(\nu)$ of imputation set. Condition (6) becomes

$$\nu(S) \leq \sum_{i\in S} \nu(i) \ if \ n \notin S, \ \nu(S) \leq \nu(N) - \sum_{i\in N\setminus S} \nu(i) \ if \ n \in S, \ S \in \Omega(2, n-1).$$

We obtain the description of subcone of balanced games containing the set of such T-simplex games that $T \ni n$.

Definition 2. Let $F \in \mathbf{F}^n$ is a minimal balanced set and $\beta \in \mathbf{R}$. A linear inequality

$$\sum_{F_i \in F} \alpha(F_i)\nu(F_i) \leq \beta$$

is called *strengthened-balancedness-inequality* if $\beta \leq \nu(N)$.

The next theorem provides an explicit representation the condition (6) for basis

$$B = (A^{N\setminus 1} \ A^{N\setminus 2} \dots A^{N\setminus n})^T \tag{7}$$

consisting of characteristic vectors for all coalitions of size $(n-1)$. The corollary 1 show that this condition consists of balancedness-inequality and strengthened-balancedness-inequalities only.

Theorem 1. *Let $\nu \in G^N$. The following two conditions*

$$\sum_{i\in N} \frac{\nu(N \setminus i)}{n-1} \leq \nu(N), \tag{8}$$

$$\nu(S) \leq \frac{(|S|+1-n)\sum_{i\in S}\nu(N\setminus i) + |S|\sum_{i\in N\setminus S}\nu(N\setminus i)}{n-1}, \quad S \in \Omega(1, n-2) \tag{9}$$

imply that $C(\nu) \neq \emptyset$.

Proof. Consider the basis (7). The matrix B and inverse matrix B^{-1} have the form

$$b_{ij} = \begin{cases} 0, & i = j, \\ 1, & i \neq j, \end{cases} \qquad b_{ij}^{-1} = \begin{cases} \frac{2-n}{n-1}, & i = j, \\ \frac{1}{n-1}, & i \neq j. \end{cases}$$

Therefor, $x^B = B^{-1}\overline{\nu}_B$ is determined by

$$x_i^B = \frac{\sum_{j\in N\setminus i}\nu(N\setminus j) + (2-n)\nu(N\setminus i)}{n-1}, \quad i \in N. \tag{10}$$

Obviously, $x_B(N)$ is equal to the left side of inequality (8). The equality

$$\sum_{i\in S}\sum_{j\in N\setminus i}\nu(N\setminus j) = |S|\sum_{i\in N\setminus S}\nu(N\setminus i) + (|S|-1)\sum_{i\in S}\nu(N\setminus i)$$

implies that $x_B(S)$ is equal to the right side of inequality in (9). Thus condition (6) holds. $\square$

Corollary 1. *The conditions in Theorem 1 consist of one balancedness-inequality and* $2^n - n - 2$ *strengthened-balancedness-inequalities*

$$\frac{\nu(S) + \sum_{i\in S}\nu(N\setminus i)}{|S|} \le x^B(N),\ \ S \in \Omega(1, n-2), \tag{11}$$

with identical right side, where x_B *is determined by (10).*

Proof. Known that $F^1 = \bigcup_{i\in N}\{N\setminus i\}$ is minimal balanced set with weight vector $\alpha(F^1) = (\frac{1}{n-1}, \dots, \frac{1}{n-1})$. Corresponding to F^1 inequality in (2) coincides with (8). Therefore, (8) is balancedness-inequality. Take $S \in \Omega(1, n-2)$. After transformation the right side of inequality in (9)

$$\frac{(|S| + 1 - n)\sum_{i\in S}\nu(N\setminus i) + |S|\sum_{i\in N\setminus S}\nu(N\setminus i)}{n-1} =$$

$$\frac{|S|\sum_{i\in N}\nu(N\setminus i) - (n-1)\sum_{i\in S}\nu(N\setminus i)}{n-1} = \frac{|S|\sum_{i\in N}\nu(N\setminus i)}{n-1} - \sum_{i\in S}\nu(N\setminus i) =$$

$$|S|x^B(N) - \sum_{i\in S}\nu(N\setminus i),$$

we obtain that the system (9) is equivalent with (11). The collection of coalitions $F^1 = \{\{N\setminus S\}\cup(\bigcup_{i\in S}\{i\})\}$ belongs to $\mathbf{F}^n$ because it is the partition of N, $\alpha(F^1) = (1, \dots, 1)$. The complementation gives minimal balanced set

$$F^2 = \{S \cup (\bigcup_{i\in S}\{N\setminus i\})\}$$

with weight vector $\alpha(F^2)$ where

$$\alpha(F_i^2) = \frac{\alpha(F_i^1)}{\sum_{i=1}^{|F^1|}\alpha(F_i^1) - 1} = \frac{1}{|S|}.$$

According to (8), $x^B(N) \le \nu(N)$. In view of definition 2 any inequality in (11) is strengthened-balanced-inequality. The number of such inequalities is equal to $|D| - 1 = 2^n - n - 2$. □

The following theorem show that it is possible to replace all (ore some) inequalities in system (9) by union-inequalities.

Theorem 2. *Let* $\nu \in G^N$. *For any fixed* $r \in \{0, \dots, n-2\}$ *the balancedness-inequality (8) together with strengthened-balancedness-inequalities*

$$\frac{\nu(S) + \sum_{i\in S}\nu(N\setminus i)}{|S|} \le \frac{\sum_{i\in N}\nu(N\setminus i)}{n-1},\quad S \in \Omega(1, r), \tag{12}$$

and union-inequalities

$$\nu(N) - \nu(S) \ge \sum_{i\in N\setminus S}\nu^*(i),\quad S \in \Omega(r+1, n-2), \tag{13}$$

imply than $C(\nu) \ne \emptyset$.

Proof. Of course every inequality in (13) is a union-inequality (for $CL = S$). In view of corollary 1 it is sufficient to prove that from (13) follows

$$\rho_S = |S|\sum_{i\in N}\nu(N\setminus i) - (n-1)(\nu(S) + \sum_{i\in S}\nu(N\setminus i)) \geq 0, \quad S \in \Omega(r+1, n-2)$$

ore

$$|S|\sum_{i\in N\setminus S}\nu(N\setminus i) - (n-1)\nu(S) + (|S|-n+1)\sum_{i\in S}\nu(N\setminus i) \geq 0, \quad S \in \Omega(r+1, n-2).$$

From (8) it follows that

$$(|S|-n+1)\sum_{i\in S}\nu(N\setminus i) \geq (n-|S|-1)\sum_{i\in N\setminus S}\nu(N\setminus i) + (|S|-n+1)(n-1)\nu(N).$$

Two last inequalities implies

$$\frac{\rho_S}{n-1} \geq \sum_{i\in N\setminus S}\nu(N\setminus i) - \nu(S) - (n-|S|-1)\nu(N) =$$

$$\nu(N) - \nu(S) - (\sum_{i\in N\setminus S}(\nu(N) - \nu(N\setminus i)) = \nu(N) - \nu(S) - \sum_{i\in N\setminus S}\nu^*(i).$$

Using (13) we have $\rho_S \geq 0$ for all $S \in \Omega(r, n-2)$. □

Let BAL^n be the cone of balanced games $\nu \in G^N$ and $BAL^n_r \subset BAL^n$ be the subcone generated by conditions in Theorem 2 . Since each inequality in (12) follows from corresponding inequality in (13) and balancedness-inequality (8) then

$$BAL^n_0 \subset BAL^n_1 \subset \ldots \subset BAL^n_{n-2}.$$

The next example show that $BAL^n_0 \neq \emptyset$.

Example 3. Consider 5-person game :
$\nu(i) = 0$ for all $i \in N = \{1,\ldots,5\}$, $\nu(N) = 10$,
$\nu(12) = \nu(13) = \nu(14) = \nu(15) = 2$,
$\nu(23) = \nu(24) = \nu(25) = \nu(34) = \nu(35) = 1$,
$\nu(123) = \nu(124) = \nu(125) = \nu(134) = \nu(135) = 5$,
$\nu(234) = \nu(235) = \nu(345) = 4$,
$\nu(1234) = \nu(1235) = \nu(1245) = \nu(1345) = 8$, $\nu(2345) = 7$.
The core have five extreme points
$ext(C(\nu)) = \{(2,2,2,2,2),(3,1,2,2,2),(3,2,1,2,2),(3,2,2,1,2),(3,2,2,2,1)\}$.
This is monotonic, superadditive, but non-convex even for the grand coalition ($\nu(1234) + \nu(1235) > \nu(N) + \nu(123)$) game. Players marginal contributions to the grand coalition are:
$\nu^*(1) = 3$, $\nu^*(2) = \nu^*(3) = \nu^*(4) = \nu^*(5) = 2$.
Take $r = 0$. Since the players 2-5 are symmetric it is sufficient to verify:
$|S| = 1 \Longrightarrow \nu(N) - \nu(1) \geq 4\nu^*(2), \quad \nu(N) - \nu(2) \geq \nu^*(1) + 3\nu^*(2),$
$|S| = 2 \Longrightarrow \nu(N) - \nu(12) \geq 3\nu^*(3), \quad \nu(N) - \nu(23) \geq \nu^*(1) + 2\nu^*(4),$
$|S| = 3 \Longrightarrow \nu(N) - \nu(123) \geq 2\nu^*(2), \quad \nu(N) - \nu(234) \geq \nu^*(1) + \nu^*(2).$
All inequalities hold. From (10) we obtain

$$x^B = (2.75, 1.75, 1.75, 1.75, 1.75).$$

As $\nu(N) - x^B(N) = 0.25$ then $x^B \notin C(\nu)$. But we have five core allocations associated with basis B:

$x^1 = (3, 1.75, 1.75, 1.75, 1.75)$, $x^2 = (2.75, 2, 1.75, 1.75, 1.75)$,
$x^3 = (2.75, 1.75, 2, 1.75, 1.75)$, $x^4 = (2.75, 1.75, 1.75, 2, 1.75)$,
$x^5 = (2.75, 1.75, 1.75, 1.75, 2)$.

Next theorem provides the sufficient condition corresponding to basis

$$B = ((A^{\{i\}})_{i\in H}, (A^{N\setminus i})_{i\in(N\setminus H)\setminus i^*}, A^N)^T$$

where $H \in (2^N \setminus \{N\}) \setminus \{i^*\}$.

Theorem 3. *Let $\nu \in G^N$. Let also $i^* \in N$, $H \in (2^N \setminus \{N\}) \setminus \{i^*\}$ are fixed and $\Omega^2_H = \{S \in \Omega^2 | \ S \neq \{i\}$ for $i \in H$, $S \neq \{N \setminus i\}$ for $i \in (N \setminus H) \setminus i^*\}$. The following two conditions*

$$\nu(S) \le \sum_{i\in S\cap H} \nu(i) + \sum_{i\in S\setminus H} \nu^*(i), \quad S \in \Omega^2_H, \quad i^* \notin S, \tag{14}$$

$$\nu(S) \le \nu(N) - \sum_{i\in H\setminus S} \nu(i) - \sum_{i\in(N\setminus H)\setminus S} \nu^*(i), \quad S \in \Omega^2_H, \quad i^* \in S, \tag{15}$$

imply that $C(\nu) \neq \emptyset$.

Proof. Consider the vector x^B_H determined by

$$(x^B_H)_i = \begin{cases} \nu(i), & i \in H, \\ \nu^*(i), & i \in (N \setminus H) \setminus i^*, \\ \nu(N) - \sum\limits_{j\in H} \nu(j) - \sum\limits_{j\in(N\setminus H)\setminus i^*} \nu(N \setminus j), & i = i^*. \end{cases}$$

If $i^* \notin S$ then $x^B_H(S)$ coincides with right side of inequality in (14). If $i^* \in S$ then

$$x^B_H(S) = \sum_{i\in S\cap H} \nu(i) + \sum_{i\in S\setminus H\setminus i^*} \nu^*(i) + \nu(N) - \sum_{i\in H} \nu(i) - \sum_{i\in(N\setminus H)\setminus i^*} \nu(N \setminus i)$$

is equals to the right side of of inequality in (15). Since $x^B_H(S) = \nu(S)$ for all $S \in \Omega^1 \setminus \Omega^2_H$ we have $x^B_H \in C(\nu)$. □

Corollary 2. *The system (14)-(15) characterizes such class of TU games that at least one extreme point of imputation set $I(\nu)$ or dual imputation set $I^*(\nu)$ or core cover $CC(\nu)$ belongs to core.*

Proof. Let H and i^* be the same as in Theorem 3. If $H = N \setminus i^*$ then x^B_H is defined by

$$(x^B_H)_i = \begin{cases} \nu(i), & i \in N \setminus i^*, \\ \nu(N) - \sum\limits_{j\in N\setminus i^*} \nu(j), & i = i^*. \end{cases}$$

Therefore $x^B(H) = f^{i^*}(\nu) \in ext(I(\nu))$. If $H = \emptyset$ then

$$(x^B_H)_i = \begin{cases} \nu^*(i), & i \in N \setminus i^*, \\ \nu(N) - \sum\limits_{j\in N\setminus i^*} \nu^*(j), & i = i^*, \end{cases}$$

and $x_H^B = g^{i^*}(\nu) \in ext(I^*(\nu))$. Let mow $H \in \Omega^2 \setminus \{N \setminus i^*\}$. The core cover $CC(\nu)$ is defined by the system

$$x_i \geq \nu(i), \quad x_i \leq \nu^*(i), \; i \in N, \quad x(N) = \nu(N).$$

The vector x_H^B satisfies as equality $n-1$ linearly independent inequalities in this system and $x_H^B(N) = \nu(N)$. So $x_H^B \in ext(CC(\nu))$. □

Now we give example to illustrate the conditions in Theorem 3.

Example 4. Let $\nu \in G^{\{1,2,3\}}$ and (without loss of generality) $i^* = 3$. The conditions in Theorem 3, corresponding to all possible choices for coalition $H \in (2^N \setminus \{N\}) \setminus i^*$ are given in the following table. For each H the inequalities listed in the last column of table are balancedness-inequalities, convexity-inequalities ore concavity-inequalities only.

H	$S \in \Omega_H^2$	inequalities in (14), (15)	transformed inequalities
	$\{3\}$	$\nu(12) \leq \nu(1) + \nu(2)$	
$\{12\}$	$\{12\}$	$\nu(3) \leq \nu(N) - \nu(1) - \nu(2)$	$\nu(1) + \nu(2) + \nu(3) \leq \nu(N)$
	$\{13\}$	$\nu(13) \leq \nu(N) - \nu(2)$	$\nu(2) + \nu(13) \leq \nu(N)$
	$\{23\}$	$\nu(23) \leq \nu(N) - \nu(1)$	$\nu(1) + \nu(23) \leq \nu(N)$
	$\{1\}$	$\nu(1) \leq \nu^*(1)$	similar $H = \{12\}$, $S = \{23\}$
$\emptyset$	$\{2\}$	$\nu(2) \leq \nu^*(2)$	similar $H = \{12\}$, $S = \{13\}$
	$\{3\}$	$\nu(12) \leq \nu^*(1) + \nu^*(2)$	$\nu(12) + \nu(13) + \nu(23) \leq 2\nu(N)$
	$\{12\}$	$\nu(3) \leq \nu(N) - \nu^*(1) - \nu^*(2)$	$\nu(13) + \nu(23) \geq \nu(N) + \nu(3)$
	$\{2\}$	$\nu(2) \leq \nu^*(2)$	similar $H = \{12\}$, $S = \{13\}$
$\{1\}$	$\{3\}$	$\nu(12) \leq \nu(1) + \nu^*(2)$	$\nu(12) + \nu(13) \leq \nu(N) + \nu(1)$
	$\{12\}$	$\nu(3) \leq \nu(N) - \nu(1) - \nu^*(2)$	$\nu(1) + \nu(3) \leq \nu(13)$
	$\{23\}$	similar $H = \{12\}$, $S = \{23\}$	
$\{2\}$		similar $H = \{1\}$	

Consider the game:
$\nu(1) = 1$, $\nu(2) = -1$, $\nu(3) = 2$, $\nu(12) = 0$, $\nu(13) = 6$. $\nu(23) = 4$. $\nu(N) = 6$.
It is monotonic, superadditive, but non-convex even for the grand coalition ($\nu(13) + \nu(23) > \nu(N) + \nu(3)$) game. Players marginal contributions to the grand coalition are: $\nu^*(1) = 2$, $\nu^*(2) = 0$, $\nu^*(3) = 6$. We have

$ext(Web(\nu)) = \{(2,-1,5), (2,2,2), (4,0,2), (1,-1,6), (1,0,5)\}$,
$ext(I(\nu)) = \{(5,-1,2), (1,3,2), (1,-1,6)\}$,
$ext(I^*(\nu)) = \{(0,0,6), (2,-2,6), (2,0,4)\}$,
$ext(C(\nu)) = ext(CC(\nu)) = \{(2,-1,5), (1,0,5), (1,-1,6), (2,0,4)\}$.

For every coalition $T \in \Omega^1$ the given game is not T-simplex and dual simplex because $ext(C(\nu)) \not\subseteq ext(I(\nu))$, $ext(C(\nu)) \not\subseteq ext(I^*(\nu))$. It do not satisfy the CoMa-property because $ext(C(\nu)) \not\subseteq ext(Web(\nu))$. The conditions (14), (15) are satisfied for all $H \in (2^N \setminus \{N\}) \setminus \{3\}$. We have

$x_{\{1,2\}}^B = (1,-1,6) \in ext(I(\nu))$, $\quad x_{\{\emptyset\}}^B = (2,0,4) \in ext(I^*(\nu))$,
$x_{\{1\}}^B = (1,0,5) \in ext(CC(\nu))$, $\quad x_{\{2\}}^B = (2,-1,5) \in ext(CC(\nu))$.

Remark 1. The set of games satisfying (14)-(15) contains T-simplex games, dual simplex games with $T \ni i^*$, zero-normalized monotonic games with $Veto(\nu) \ni i^*$, clan games with $CL \ni i^*$, big boss games with player i^* as big boss.

Remark 2. For basis containing the rows A^N, $A^{\{\pi_1\}}$, $A^{\{\pi_1,\pi_2\}}$,...,$A^{\{\pi_1,\pi_2,\dots,\pi_k\}}$, where $2 \leq k \leq n-1$, the conditions (6) define TU games with some convexity behaviour. For $k = n-1$ we obtain condition that determines such class of games that at least one extreme point $m^{\pi}(\nu)$ of Weber set belongs to core, i.e. permutationally convex games (in particular, convex games and games satisfy the CoMa-property).

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