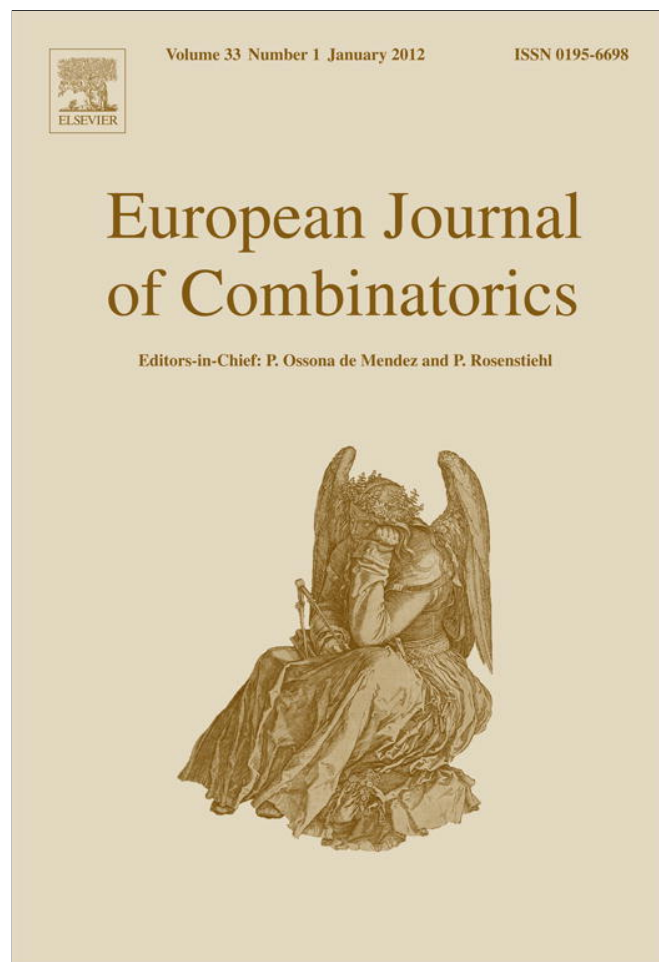


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The median Genocchi numbers, q -analogues and continued fractions

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ABSTRACT

It has been shown recently that the normalized median Genocchi numbers are equal to the Euler characteristics of the degenerate flag varieties. The q -analogues of the Genocchi numbers can be naturally defined as the Poincaré polynomials of the degenerate flag varieties. We prove that the generating function of the Poincaré polynomials can be written as a simple continued fraction. As an application we prove that the Poincaré polynomials coincide with the q -version of the normalized median Genocchi numbers introduced by Han and Zeng.

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0. Introduction

The Genocchi numbers appear in many different contexts (see e.g. [1,4,5,7,6,14,20]). Probably the most well-known definition uses the Seidel triangle

					155	155			
				17	17	155	310		
		3	3	17	34	138	448		
	1	1	3	6	14	48	104	552	
1	1	1	2	2	8	8	56	56	608

By definition, the triangle is formed by the numbers $g_{k,n}$ (k is the number of a row counted from bottom to top and n is the number of a column from left to right) with $n = 1, 2, \dots$ and $1 \leq k \leq \frac{n+1}{2}$, subject to the relations $g_{1,1} = 1$ and

$$g_{k,2n} = \sum_{i \geq k} g_{i,2n-1}, \quad g_{k,2n+1} = \sum_{i \leq k} g_{i,2n}.$$

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For example, $138 = 56 + 48 + 34$ and $48 = 14 + 17 + 17$. The two sequences of numbers sitting on the edges of the Seidel triangle are called the Genocchi numbers. More precisely, the Genocchi numbers of the first kind are $1, 1, 3, 17, 155, \dots$ and those of the second kind are $1, 2, 8, 56, 608, \dots$. The latter numbers are also referred to as the median Genocchi numbers and are denoted by H_{2n-1} . For example, $H_1 = 1$ and $H_7 = 56$. These numbers are known to be divisible by powers of 2 (see [1,4]): $H_{2n+1} \div 2^n$. The ratios are called the normalized median Genocchi numbers and are denoted by h_n . Thus the first values h_0, h_1, h_2, \dots are as follows:

$$1, 1, 2, 7, 38, 295, 3098, \dots$$

It has been shown recently (see [9]) that the numbers h_n are analogues (“degenerations”) of the numbers $n!$. More precisely, let \mathcal{F}_n be the variety of flags in an n -dimensional space, i.e. \mathcal{F}_n consists of collections of subspaces $(V_1 \subset V_2 \subset \dots \subset V_{n-1})$ of a given n -dimensional space W such that $\dim V_k = k$. It is well known that the Euler characteristics of \mathcal{F}_n is equal to $n!$. Combinatorially, the number $n!$ appears in this context as the number of sequences $(I_1 \subset I_2 \subset \dots \subset I_{n-1})$ of subsets of $\{1, \dots, n\}$ such that $\#I_k = k$. The varieties \mathcal{F}_n have natural degenerations \mathcal{F}_n^a , called the degenerate flag varieties (see [8–11]). The degenerate flag varieties consist of collections $(V_1, V_2, \dots, V_{n-1})$ of subspaces of an n -dimensional vector space subject to certain explicit conditions (see Section 1). It turns out that the Euler characteristic of \mathcal{F}_n^a is equal to the normalized median Genocchi number: $\chi(\mathcal{F}_n^a) = h_n$. Combinatorially this means that the number of sequences $(I_1, I_2, \dots, I_{n-1})$ of subsets of $\{1, \dots, n\}$ such that $\#I_k = k$ and $I_k \subset I_{k+1} \cup \{k+1\}$ is equal to h_n .

We introduce natural q -analogues $h_n(q)$ as the Poincaré polynomials of the degenerate flag varieties. We note that the degenerate flag varieties are singular, but share the following important property with their classical analogues: the varieties \mathcal{F}_n^a can be decomposed into a disjoint union of complex (even-dimensional real) affine cells. Therefore the Poincaré polynomials $P_{\mathcal{F}_n^a}(t)$ are functions of $q = t^2$ (odd powers do not show up). Hence we define $h_n(q) = P_{\mathcal{F}_n^a}(q^{1/2})$. Obviously, one has $h_n(1) = h_n$. Various q -analogues of the Genocchi numbers appear in the literature (see e.g. [15,16,22]). In particular, in [15] Han and Zeng used the q -analogues to give a third proof of the Barsky theorem [1,4].

We give a continued fraction presentation of the generating function of the polynomials $h_n(q)$. Namely, it is convenient to introduce the “reversed” polynomials $\tilde{h}_n(q) = q^{n(n-1)/2} h_n(q^{-1})$. Then we have

Theorem 0.1.

$$\sum_{n \geq 0} \tilde{h}_n(q) s^n = \frac{1}{1 - \frac{s}{1 - \frac{qs}{1 - \frac{\binom{3}{2}_q s}{1 - \frac{q \binom{3}{2}_q s}{1 - \frac{\binom{4}{2}_q s}{1 - \frac{q \binom{4}{2}_q s}{1 - \dots}}}}}}$$

Using this formula, we prove that the $\tilde{h}_n(q)$ coincide with the q version of the normalized median Genocchi numbers introduced by Han and Zeng in [15,16]. We also note that the Viennot formula (see [19,21,4,7]) for the generating function of the median Genocchi numbers H_{2n-1} can be derived by specialization at $q = 1$.

Our paper is organized as follows.

In Section 1 we briefly recall the definitions of the degenerate flag varieties and of the q -analogues of the normalized median Genocchi numbers.

In Section 2 we obtain the continued fraction presentation for the generating function of $\tilde{h}_n(q)$.

1. Combinatorics of the normalized median Genocchi numbers

The normalized median Genocchi numbers h_n , $n = 0, 1, 2, \dots$, form a sequence which starts with 1, 1, 2, 7, 38, 295. These numbers enjoy many definitions (see [1,4,3,14,17,18]). We have shown recently that the numbers h_n are equal to the Euler characteristics of certain (singular) algebraic varieties \mathcal{F}_n^a called the degenerate flag varieties (see [9]). Recall that \mathcal{F}_n^a consists of sequences (V_1, \dots, V_{n-1}) of the subspaces of a given n -dimensional space W with a basis w_1, \dots, w_n , subject to the conditions

$$pr_{k+1}V_k \subset V_{k+1}, \quad k = 1, \dots, n - 2,$$

where $pr_k : W \rightarrow W$ is a projection along w_k : $pr_k(\sum_{i=1}^n c_i w_i) = \sum_{i \neq k} c_i w_i$. One proves that the degenerate flag varieties share the following important property with their classical analogues (see [13]): \mathcal{F}_n^a can be decomposed into a disjoint union of complex (even-dimensional real) cells. Hence we obtain a natural graded analogue of the numbers h_n defined by the Poincaré polynomials of \mathcal{F}_n^a :

$$h_n(q) = P_{\mathcal{F}_n^a}(q^{1/2}),$$

where $P_{\mathcal{F}_n^a}(t)$ is the Poincaré polynomial of \mathcal{F}_n^a and $q^{1/2}$ shows up because all the cells are even-dimensional. The first four polynomials $h_n(q)$ are as follows:

$$\begin{aligned} h_1(q) &= 1, & h_2(q) &= 1 + q, \\ h_3(q) &= 1 + 2q + 3q^2 + q^3, \\ h_4(q) &= 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6. \end{aligned}$$

In general, the degree of $h_n(q)$ is equal to $n(n - 1)/2$. Obviously, $h_n(1) = h_n$.

Let us recall an explicit formula for the polynomials $h_n(q)$ derived in [2] using the geometry of quiver Grassmannians. Let $m \geq n \geq 0$. Then the q -binomial (Gaussian) coefficient $\binom{m}{n}_q$ is defined as

$$\binom{m}{n}_q = \frac{m_q!}{n_q!(m - n)_q!}, \quad m_q! = \prod_{i=1}^m \frac{1 - q^i}{1 - q}.$$

Proposition 1.1. *The Poincaré polynomial of the degenerate flag variety \mathcal{F}_n^a is equal to*

$$\sum_{f_1, \dots, f_{n-1} \geq 0} q^{\sum_{k=1}^{n-1} (k-f_k)(1-f_k+f_{k+1})} \prod_{k=1}^{n-1} \binom{1+f_{k-1}}{f_k}_q \prod_{k=1}^{n-1} \binom{1+f_{k+1}}{f_k}_q, \tag{1.1}$$

(we assume $f_0 = f_n = 0$).

Geometrically, formula (1.1) appears as follows. The varieties \mathcal{F}_n^a can be cut into disjoint pieces, such that each piece is fibered over a product of several Grassmannians with fibers being affine spaces. Since the Poincaré polynomial of a Grassmannian is given by a q -binomial coefficient, we arrive at the formula as above.

2. The generating function and continued fractions

Our goal in this section is to give an explicit continued fraction form of the generating function of the Poincaré polynomials $h_n(q)$ and to prove that they coincide with the q -versions of the normalized median Genocchi numbers defined in [15,16].

We first recall the formalism of the weighted generating functions of Motzkin paths due to Flajolet (see [12]). Let α_n, β_n and $\gamma_n, n \geq 0$, be sequences of complex numbers called weights. For a nonnegative integer k we define $w(k, k) = \gamma_k, w(k, k + 1) = \alpha_k$ and $w(k, k - 1) = \beta_{k-1}$ (if $k \geq 1$). We denote by α_\bullet the whole collection $(\alpha_k)_{k=0}^\infty$ and similarly for β_\bullet and γ_\bullet . The weighted generating function of Motzkin paths is given by the formula

$$F(s; \alpha_\bullet, \beta_\bullet, \gamma_\bullet) = \sum_{n \geq 0} s^n \sum_{\mathbf{f} \in M_n} \prod_{k=0}^{n-1} w(f_k, f_{k+1}).$$

The following result is due to Flajolet [12]: the weighted generating sum of the Motzkin paths is given by the continued fraction

$$F(s; \alpha_\bullet, \beta_\bullet, \gamma_\bullet) = \frac{1}{1 - \gamma_0 s - \frac{\alpha_0 \beta_0 s^2}{1 - \gamma_1 s - \frac{\alpha_1 \beta_1 s^2}{1 - \gamma_2 s - \dots}}}$$

Let us apply this formalism to our situation. Formula (1.1) can be rewritten as follows:

$$h_n(q) = q^{n(n-1)/2} \sum_{\mathbf{f} \in M_n} q^{\sum_{k=1}^{n-1} f_k(f_k - f_{k+1} - 2)} \prod_{k=1}^{n-1} \binom{1 + f_{k-1}}{f_k}_q \prod_{k=1}^{n-1} \binom{1 + f_{k+1}}{f_k}_q. \tag{2.1}$$

We introduce three sequences of weights

$$\alpha_m(q) = q^{-3m} \binom{m+2}{2}_q, \quad \beta_m(q) = q^{-m-1} \binom{m+2}{2}_q, \quad \gamma_m(q) = q^{-2m} \binom{m+1}{1}_q^2$$

and define $w(f_k, f_{k+1})$ using these weights. Then formula (2.1) implies the following lemma.

Lemma 2.1.

$$q^{-n(n-1)/2} h_n(q) = \sum_{\mathbf{f} \in M_n} \prod_{k=0}^{n-1} w(f_k, f_{k+1}). \tag{2.2}$$

In order to use the Flajolet theorem we need to get rid of the factor $q^{n(n-1)/2}$ in (2.2). We introduce the notation

$$\tilde{h}_n(q) = q^{n(n-1)/2} h_n(q^{-1})$$

(note that the degree of $h_n(q)$ is exactly $n(n-1)/2$). Let $\tilde{h}(q, s) = \sum_{n \geq 0} \tilde{h}_n(q) s^n$. We note that

$$\gamma_m(q) = \binom{m+1}{1}_{q^{-1}}^2, \quad \alpha_m(q)\beta_m(q) = q^{-1} \binom{m+2}{2}_{q^{-1}}^2.$$

Using the Flajolet theorem we arrive at the following theorem.

Theorem 2.2. *The generating function $\tilde{h}(q, s)$ can be written as follows:*

$$\tilde{h}(q, s) = \frac{1}{1 - s - \frac{qs^2}{1 - \binom{2}{1}_q^2 s - \frac{q \binom{3}{2}_q^2 s^2}{1 - \binom{3}{1}_q^2 s - \frac{q \binom{4}{2}_q^2 s^2}{1 - \binom{4}{1}_q^2 s - \dots}}}} \tag{2.3}$$

Proof. Follows from Lemma 2.1 and the Flajolet theorem. \square

Corollary 2.3.

$$\tilde{h}(q, s) = \frac{1}{1 - \frac{s}{1 - \frac{\binom{3}{2}_q^s}{1 - \frac{q \binom{3}{2}_q^s}{1 - \frac{\binom{4}{2}_q^s}{1 - \frac{q \binom{4}{2}_q^s}{1 - \dots}}}}} \tag{2.4}$$

Proof. Recall the following formula (see [7, Lemma 2]):

$$\frac{c_0}{1 - c_1s - \frac{c_1c_2s^2}{1 - (c_2+c_3)s - \frac{c_3c_4s^2}{1 - (c_4+c_5)s - \dots}}} = \frac{c_0}{1 - \frac{c_1s}{1 - \frac{c_2s}{1 - \frac{c_3s}{1 - \dots}}}}.$$

Now our corollary follows from **Theorem 2.2**. \square

Recall the q -analogues of the normalized median Genocchi numbers $\bar{c}_n(q)$ introduced by Han and Zeng (see formula (17) in [15]). By definition,

$$\bar{c}_n(q) = \frac{C_n(1, q)}{(1 + q)^{n-1}}, \quad n \geq 1,$$

where the polynomials $C_n(x, q)$ are defined by $C_1(x, q) = 1$ and

$$C_n(x, q) = (1 + qx) \frac{(1 + qx)C_{n-1}(1 + qx, q) - xC_{n-1}(x, q)}{1 + qx - x}, \quad n \geq 2.$$

Corollary 2.4. $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$.

Proof. Formula (18) in [15] gives a continued fraction form of the generating function of the polynomials $\bar{c}_n(q)$. Comparing this formula with (2.4) and using the equations

$$\begin{aligned} \binom{2n}{2}_q &= (1 + q^2 + q^4 + \dots + q^{2n-2})(1 + q + q^2 + \dots + q^{2n-2}), \\ \binom{2n+1}{2}_q &= (1 + q^2 + q^4 + \dots + q^{2n-2})(1 + q + q^2 + \dots + q^{2n}), \end{aligned}$$

we obtain $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$. \square

Finally, we note that specializing to $q = 1$, one derives the continued fraction formulas due to Viennot for the generating functions of the (normalized) median Genocchi numbers (see [4,7,19,21]).

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