

24 FACES OF THE BORCHERDS MODULAR FORM Φ_{12}

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ABSTRACT. The fake monster Lie algebra is determined by the Borcherds function Φ_{12} which is the reflective modular form of the minimal possible weight with respect to $O^+(II_{2,26})$. We prove that the first non-zero Fourier–Jacobi coefficient of Φ_{12} in any of 23 Niemeier cusps is equal to the Weyl–Kac denominator function of the affine Lie algebra of the root system of the corresponding Niemeier lattice. This is an automorphic answer (in the case of the fake monster Lie algebra) on the old question of Frenkel and Feingold (1983) about possible relations between hyperbolic Kac–Moody algebras, Siegel modular forms and affine Lie algebras.

1. THE BORCHERDS MODULAR FORM Φ_{12} AND LORENTZIAN AND AFFINE KAC–MOODY ALGEBRAS

In 1983, I. Frenkel and A. Feingold posed a question about possible relations between the simplest hyperbolic Kac–Moody algebra, Siegel modular forms and affine Lie algebras (see [FF]). A Siegel modular form of genus 2 can be considered as a modular form on the orthogonal group $O(2, 3)$. The Weyl–Kac denominator functions of affine Lie algebras are Jacobi modular forms similar to Fourier–Jacobi coefficients of modular forms on $O(2, n)$.

In [GN2] we constructed *the automorphic correction* of the simplest hyperbolic Kac–Moody algebra, i.e. a generalized hyperbolic Kac–Moody super Lie algebra with the same (hyperbolic) real simple roots whose Weyl–Kac–Borcherds denominator function is the classical Igusa cusp form Ψ_{35} of weight 35. (Ψ_{35} is essentially the unique Siegel modular form of odd weight.) The multiplicities of positive roots of this Lorentzian Kac–Moody algebra are given by the Fourier coefficients of a weakly holomorphic Jacobi form of weight 0. The first non-zero Fourier–Jacobi coefficient of Ψ_{35} is equal to the Jacobi modular form $\eta(\tau)^{69}\vartheta(\tau, 2z)$ of weight 35 and index 2 which is not a denominator function of affine type. In the class of Lorentzian Kac–Moody algebras of hyperbolic rank 3 classified in [GN3]–[GN5] there exists an algebra determined by the even Siegel theta-series $\Delta_{1/2}$ whose first Fourier–Jacobi coefficient is equal to $\vartheta(\tau, z)$ which is the denominator function of the simplest affine Lie algebra $\hat{\mathfrak{g}}(A_1)$.

In this paper we analyze the fake monster Lie algebra discovered by R. Borcherds in 1995 (see [B1]). The algebra is determined by the Borcherds modular form Φ_{12} of weight 12 with respect to $O^+(II_{2,26})$ where $II_{2,26}$ is the even integral lattice of signature $(2, 26)$. The modular form Φ_{12} has 24 different Fourier–Jacobi expansions in the 24 one-dimensional cusps (23 Niemeier cusps and the Leech cusp). We prove that Φ_{12} vanishes at all 23 Niemeier cusps and its first non-zero Fourier–Jacobi coefficient is equal to the Weyl–Kac denominator function of the affine Lie algebra of

the root system of the corresponding Niemeier lattice. This result is an automorphic answer on the question of Frenkel and Feingold in the case of the fake monster Lie algebra.

Lorentzian Kac–Moody algebras are hyperbolic analogue of the finite dimensional and affine Lie algebras (see [B1]–[B2] and [GN1]–[GN5]). A Lorentzian Kac–Moody algebra is graded by a hyperbolic root lattice L_1 , its Weyl group is a hyperbolic reflection group and its Weyl–Kac–Borcherds denominator function is a modular form with respect to an orthogonal group of signature $(2, n)$.

The famous example is the *fake monster Lie algebra* \mathfrak{G}_Λ with the root lattice $II_{1,25} \cong U \oplus \Lambda_{24}(-1)$ of signature $(1, 25)$, where $U \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic plane and $\Lambda_{24}(-1)$ is the rescaled Leech lattice, the negative definite even unimodular lattice of rank 24 without vectors of square 2. One can consider the Borcherds modular form

$$\Phi_{12} \in M_{12}(\mathrm{O}^+(II_{2,26}), \det)$$

as the generating function of \mathfrak{G}_Λ because it contains the full information on the generators and relations of the algebra and on the multiplicities of all positive roots. The divisor of Φ_{12} is the union of all rational quadratic divisors determined by -2 -vectors in $II_{2,26}$.

Let L_2 be an even integral lattice with a quadratic form of signature $(2, n)$,

$$\mathcal{D}(L_2) = \{[\mathcal{Z}] \in \mathbb{P}(L_2 \otimes \mathbb{C}) \mid (\mathcal{Z}, \mathcal{Z}) = 0, (\mathcal{Z}, \overline{\mathcal{Z}}) > 0\}^+$$

be the associated n -dimensional bounded symmetric Hermitian domain of type *IV* (here $+$ denotes one of its two connected components). We denote by $\mathrm{O}^+(L_2)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L_2)$ preserving $\mathcal{D}(L_2)$. The domain contains the following *rational quadratic divisors*

$$\mathcal{D}_v = \{[\mathcal{Z}] \in \mathcal{D}(L_2) \mid (\mathcal{Z}, v) = 0\} \cong \mathcal{D}(v_\perp^\vee) \quad \text{where } v \in L_2^\vee, (v, v) < 0$$

and L_2^\vee is the dual lattice. The modular quotient $\Gamma \backslash \mathcal{D}(L_2)$ where Γ is a subgroup of finite index in $\mathrm{O}^+(L_2)$ is a quasi-projective variety. Its Baily–Borel compactification contains only boundary components of dimension 0 and 1 (see [BB]).

Lemma 1.1. *The Baily–Borel compactification of the quasi-projective modular variety $\mathrm{O}^+(II_{2,26}) \backslash \mathcal{D}(II_{2,26})$ is a bouquet of 24 modular curves $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ with the common zero-dimensional cusp.*

Proof. According to [BB] the zero- and one-dimensional boundary components of $\mathrm{O}^+(II_{2,26}) \backslash \mathcal{D}(II_{2,26})$ correspond to the $\mathrm{O}^+(II_{2,26})$ -orbits of the primitive isotropic vectors and totally isotropic planes respectively. There exists only one orbit of primitive isotropic vectors in $II_{2,26}$, i.e. only one zero-dimensional cusp. Let $F < II_{2,26}$ be a primitive totally isotropic sublattice of rank 2. Then $L(-1) = F^\perp/F$ is a negative even unimodular lattice (see [GHS1, §2.3] for more details). There are exactly 24 classes of such lattices. They are the 23 Niemeier lattices $N(R)$ uniquely determined by its root lattice R of rank 24

$$\begin{aligned} & 3E_8, E_8 \oplus D_{16}, D_{24}, 2D_{12}, 3D_8, 4D_6, 6D_4, \\ & A_{24}, 2A_{12}, 3A_8, 4A_6, 6A_4, 8A_3, 12A_2, 24A_1, \\ & E_7 \oplus A_{17}, 2E_7 \oplus D_{10}, 4E_6, E_6 \oplus D_7 \oplus A_{11}, \\ & A_{15} \oplus D_9, 2A_9 \oplus D_6, 2A_7 \oplus D_5, 4A_5 \oplus D_4 \end{aligned}$$

and the Leech lattice Λ_{24} without roots (see [CS, Chapter 18]). In other words, there are 24 different models of the unique (up to isomorphism) even unimodular lattice $II_{2,26}$ of signature $(2, 26)$ corresponding to the one-dimensional cusps

$$II_{2,26} \cong U \oplus U \oplus N(-1)$$

where $N(-1)$ is the negative definite Niemeier lattice $N = N(R)$. (The bilinear form on $N(-1)$ is equal to $-(\cdot, \cdot)_N$.) The corresponding one dimensional boundary components are the modular curves $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. All of them have the common zero-dimensional cusp. \square

In this paper we analyze the Fourier-Jacobi expansions of Φ_{12} . For this end we define the tube domain realizations of $\mathcal{D}(II_{2,26})$ corresponding to one-dimensional cusps. We construct it in more general context. Let

$$L_2 = U \oplus U_1 \oplus L(-1)$$

where $U \cong U_1$ are two hyperbolic planes and L is an even integral positive definite lattice. We fix a basis of the first hyperbolic plane $U = \mathbb{Z}e \oplus \mathbb{Z}f$: $(e, f) = 1$ and $e^2 = f^2 = 0$ in L_2 . Similarly $U_1 \cong \mathbb{Z}e_1 \oplus \mathbb{Z}f_1$. We choose a basis of L_2 of the form (e, e_1, \dots, f_1, f) where \dots denote a basis of $L(-1)$. The one dimensional boundary component related to $L(-1)$ is defined by the isotropic tower $\langle f \rangle \subset \langle f, f_1 \rangle$. Let $[\mathcal{Z}] = [\mathcal{X} + i\mathcal{Y}] \in \mathcal{D}(L_2)$. Then $(\mathcal{X}, \mathcal{Y}) = 0$, $(\mathcal{X}, \mathcal{X}) = (\mathcal{Y}, \mathcal{Y})$ and $(\mathcal{Z}, \bar{\mathcal{Z}}) = 2(\mathcal{Y}, \mathcal{Y}) > 0$. Using the basis $\langle e, f \rangle_{\mathbb{Z}} = U$ we write $\mathcal{Z} = z'e + \tilde{Z} + zf$ with $\tilde{Z} \in L_1 \otimes \mathbb{C}$ where $L_1 = U_1 \oplus L(-1)$ is a hyperbolic sublattice of L_2 . We note that $z \neq 0$. (If $z = 0$, then the real and imaginary parts of \tilde{Z} are two orthogonal vectors of positive norm in the real hyperbolic space $L_1 \otimes \mathbb{R}$ of signature $(1, n+1)$.) Thus $[{}^t\mathcal{Z}] = [(-\frac{1}{2}(Z, Z)_1, {}^tZ, 1)]$ where $(\cdot, \cdot)_1$ is the hyperbolic bilinear form in L_1 . Using the basis $\langle e_1, f_1 \rangle_{\mathbb{Z}} = U_1$ of the second hyperbolic plane we see that $\mathcal{D}(L_2)$ is isomorphic to the tube domain

$$\mathcal{H}(L) = \left\{ Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix}, \tau, \omega \in \mathbb{H}, \mathfrak{z} \in L \otimes \mathbb{C}, (\text{Im } Z, \text{Im } Z)_1 > 0 \right\}$$

where $(\text{Im } Z, \text{Im } Z)_1 = 2\text{Im}(\omega)\text{Im}(\tau) - (\text{Im}(\mathfrak{z}), \text{Im}(\mathfrak{z}))_L$. We note that $\mathcal{H}(L)$ is the complexification of the connected light cone $V^+(L_1) = \{Y \in L_1 \otimes \mathbb{R} \mid (Y, Y) > 0\}^+$. We fix the isomorphism $[\text{pr}] : \mathcal{H}(L) \rightarrow \mathcal{D}(L_2)$ defined by the 1-dimensional cusp L fixed above

$$Z = \begin{pmatrix} \omega \\ \mathfrak{z} \\ \tau \end{pmatrix} \mapsto \text{pr}(Z) = \begin{pmatrix} -\frac{1}{2}(Z, Z)_1 \\ \omega \\ \mathfrak{z} \\ \tau \\ 1 \end{pmatrix} \mapsto [\text{pr}(Z)] \in \mathcal{D}(L_2).$$

The root lattice of the fake monster Lie algebra \mathfrak{G}_Λ is the hyperbolic lattice $\Lambda_{1,25} = U \oplus \Lambda_{24}(-1)$ where Λ_{24} is the Leech lattice. (Since we are working with modular forms we inverse the signature of the root lattice.) The Weyl group $W = W_{-2}(\Lambda_{1,25})$ of \mathfrak{G}_Λ is generated by the reflections in all elements $v \in \Lambda_{1,25}$ with $v^2 = -2$. The group W is discrete in the hyperbolic space $\mathcal{L}(\Lambda_{1,25}) = V^+(\Lambda_{1,25})/\mathbb{R}_{>0}$. The (infinite) set P of the real simple roots of \mathfrak{G}_Λ contains the -2 -vectors which are orthogonal to the walls of a fundamental chamber of W in $\mathcal{L}(S)$. The set P has a Weyl vector ρ . For example, one can take $\rho = e_1$ where e_1 is the first isotropic vector of the basis for U_1 . Then $P = \{v \in \Lambda_{1,25} \mid (v, v) = -2 \text{ and } (\rho, v) = 1\}$.

In [B1] the Borcherds form Φ_{12} was determined in the Leech cusp of the modular group $O^+(II_{2,26})$. In order to describe the Fourier expansion of Φ_{12} (one has only

one Fourier expansion) we need the Ramanujan cusp form Δ of the weight 12

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{m \geq 0} \tau(m) q^m, \quad \Delta^{-1}(\tau) = \sum_{n \geq 0} p_{24}(n) q^{n-1}.$$

Then one has the following two expressions for the Borcherds modular form (see [B1, §10])

$$\begin{aligned} \Phi_{12}(Z) &= \exp(2\pi i(\rho, Z)) \prod_{\alpha \in \Delta_+} (1 - \exp(2\pi i(\alpha, Z)))^{p_{24}(1 - (\alpha, \alpha)/2)} = \\ &= \sum_{w \in W} \det(w) \sum_{m > 0} \tau(m) \exp(2\pi i(w(m\rho), Z)) \end{aligned} \quad (1)$$

where $Z \in \mathcal{H}(\Lambda_{24})$ and $\Delta_+ = \{\alpha \in \Lambda_{1,25} \mid \alpha^2 = -2 \text{ and } (\alpha, \rho) > 0\} \cup (\Lambda_{1,25} \cap V^+(\Lambda_{1,25}) - \{0\})$ is the set of positive roots of \mathfrak{G}_Λ . The last identity between the sum over W and the infinite product over Δ_+ is the *Weyl–Kac–Borcherds identity* for the fake monster Lie algebra \mathfrak{G}_Λ .

We note that the Weyl–Kac denominator function of an affine Lie algebra is a holomorphic Jacobi modular form (see [Ka], [KP]). The denominator identity for the simplest affine Lie algebra $\hat{\mathfrak{g}}(A_1)$ is reduced to the Jacobi triple product identity

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n} \right) q^{\frac{n^2}{8}} r^{\frac{n}{2}} = -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1} r)(1 - q^n r^{-1})(1 - q^n) \quad (2)$$

where $q = e^{2\pi i\tau}$, $r = e^{2\pi iz}$, $z \in \mathbb{C}$ and $\left(\frac{-4}{n} \right)$ is the Kronecker symbol. The last function is the Jacobi theta-series of characteristic $(\frac{1}{2}, \frac{1}{2})$. A question about relations between the affine Lie algebras and the simplest hyperbolic Lie algebra with

the Cartan matrix $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ was posed in 1983 in the paper of I. Frenkel

and A. Feingold [FF]. Theorem 1.2 below is an automorphic answer on a similar question in the case of the fake monster Lie algebra.

The Fourier expansion (1) shows that the value of $\Phi_{12}(Z)$ at the one dimensional cusp determined by the Leech lattice is equal to $\Delta(\tau)$ (see also the Fourier-Jacobi coefficient of index one in (8) below). The root lattice of Λ_{24} is trivial. A Niemeier lattice $N(R)$ is uniquely determined by its root system R . The list of possible R was given in the proof of Lemma 1.1. We note that if $R = R_1 \oplus \dots \oplus R_m$ is reducible, then the Coxeter numbers $h(R_i)$ of all components of the Niemeier root system R are the same. We denote this number by $h(R)$.

Theorem 1.2. *Let $N(R)$ be a Niemeier lattice with a non-empty root system R . The Borcherds modular form Φ_{12} vanishes with order $h(R)$ along the one-dimensional boundary component determined by $N(R)$. The first non-zero Fourier-Jacobi coefficient of Φ_{12} in this cusp is, up to a sign, the Weyl–Kac denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(R)$*

$$\Phi_{12}(\tau, \mathfrak{z}, \omega) = \pm \eta(\tau)^{24} \prod_{v \in R_+} \frac{\theta(\tau, (v, \mathfrak{z}))}{\eta(\tau)} e^{2\pi i h(R)\omega} + \dots$$

where the product is taken over all positive roots of the finite root system R of rank 24. The sign in the formula depends on the choice of the positive roots in R .

We prove this theorem in §3 where we give explicit formulae for the first three Fourier–Jacobi coefficients at all one dimensional cusps including the Leech cusp.

2. JACOBI MODULAR FORMS IN MANY VARIABLES

In this section we discuss Jacobi modular forms of orthogonal type. In the definitions we follow [G1]–[G2] where Jacobi forms were considered as modular forms with respect to a parabolic subgroup of an orthogonal group of signature $(2, n)$. We mentioned in §1 that a one-dimensional cusp of $\mathcal{D}(L_2)$ is defined by a maximal totally isotropic tower $\langle f \rangle \subset \langle f, f_1 \rangle = F < L_2$. A Jacobi modular form in our approach is a modular form with respect to the integral parabolic subgroup $P_F < O^+(L_2)$ (see [G2] and [CG2] for more details). The regular part of P_F is the so-called *Jacobi modular group* $\Gamma^J(L) \cong \mathrm{SL}_2(\mathbb{Z}) \rtimes H(L)$ where $\mathrm{SL}_2(\mathbb{Z})$ acts on the isotropic plane F and $H(L)$ is the Heisenberg group acting trivially on the totally isotropic plane F . The group $H(L)$ is a central extension of $L \times L$ and any element $h \in H(L)$ can be written in the form $h = [\lambda, \mu; \kappa]$ where $\lambda, \mu \in L$, $\kappa \in \frac{1}{2}\mathbb{Z}$ and $\kappa + \frac{1}{2}(\lambda, \mu) \in \mathbb{Z}$. We note that P_F is the product of the Jacobi group and the finite orthogonal group of the positive definite lattice L . Analyzing the holomorphic function $\varphi(\tau, \mathfrak{z})e^{2\pi im\omega}$ on $\mathcal{H}(L)$ modular with respect to the semi-simple part $\mathrm{SL}_2(\mathbb{Z})$ and the unipotent part $H(L)$ of the Jacobi group we obtain the following definition.

Definition 2.1. *A holomorphic (resp. cusp or weak) Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ for L is a holomorphic function*

$$\phi : \mathbb{H} \times (L \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

satisfying the functional equations

$$\begin{aligned} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) &= (c\tau + d)^k \exp\left(\pi i \frac{cm(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right) \phi(\tau, \mathfrak{z}), \\ \phi(\tau, \mathfrak{z} + \lambda\tau + \mu) &= \exp\left(-\pi im((\lambda, \lambda)\tau + 2(\lambda, \mathfrak{z}))\right) \phi(\tau, \mathfrak{z}) \end{aligned}$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and any $\lambda, \mu \in L$ and having a Fourier expansion

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \ell \in L^\vee} f(n, \ell) \exp(2\pi i(n\tau + (\ell, \mathfrak{z}))),$$

where $n \geq 0$ for a **weak** Jacobi form, $N_m(n, \ell) = 2nm - (\ell, \ell) \geq 0$ for a **holomorphic** Jacobi form and $N_m(n, \ell) > 0$ for a **cusp** form.

We denote the space of all holomorphic Jacobi forms by $J_{k,m}(L)$. We use the notation $J_{k,m}^c(L)$, $J_{k,m}^w(L)$ and $J_{k,m}^{wh}(L)$ for the space of cusp, weak or weakly holomorphic Jacobi forms. φ is called *weakly holomorphic* if there exists N such that $\Delta^N(\tau)\varphi(\tau, \mathfrak{z}) \in J_{k,m}^w(L)$.

If $J_{k,m}(L) \neq \{0\}$, then $k \geq \frac{1}{2} \mathrm{rank} L$ (see [G1]). The weight $k = \frac{1}{2} \mathrm{rank} L$ is called *singular*. The denominator function of an affine Lie algebra is a holomorphic Jacobi form of singular weight (see [Ka] and [KP]).

It is known (see [G2, Lemma 2.1]) that $f(n, \ell)$ depends only on the hyperbolic norm $N_m(n, \ell) = 2nm - (\ell, \ell)$ and the image of ℓ in the discriminant group $D(L(m)) = L^\vee/mL$ ($L(m)$ denotes the rescaling of the lattice L by m). Moreover, $f(n, \ell) = (-1)^k f(n, -\ell)$. We note that $J_{k,m}(L) = J_{k,1}(L(m))$ and the space

$J_{k,m}(L)$ depends essentially only on the discriminant form of $L(m)$ (see [G2, Lemma 2.4]).

Example 1. *Jacobi theta-series of singular weight* (see [G1]–[G2]). Let L be an even unimodular lattice of rank $n \equiv 0 \pmod{8}$. Then

$$\vartheta_L(\tau, \mathfrak{z}) = \sum_{\ell \in L} \exp(\pi i(\ell, \ell)\tau + 2\pi i(\ell, \mathfrak{z})) \in J_{\frac{n}{2}, 1}(L).$$

One can also define Jacobi forms of integral or half-integral weights with a character (or multiplier system) of fractional index (see [GN4], [CG2]).

A Jacobi form determines a vector valued modular form related to the corresponding Weil representation (see [G2, Lemma 2.3]). In particular, for $\varphi \in J_{k,1}(L)$ we have

$$\varphi(\tau, \mathfrak{z}) = \sum_{\substack{n \in \mathbb{Z}, \ell \in L^\vee \\ 2n - (\ell, \ell) \geq 0}} f(n, \ell) \exp(2\pi i(n\tau + (\ell, \mathfrak{z}))) = \sum_{h \in D(L)} \phi_h(\tau) \Theta_{L,h}(\tau, \mathfrak{z}), \quad (3)$$

where $h \in D(L) = L^\vee/L$, $\Theta_{L,h}(\tau, \mathfrak{z})$ is the Jacobi theta-series with characteristic h and the components of the vector valued modular forms $(\phi_h)_{h \in D(L)}$ have the following Fourier expansions at infinity

$$\phi_h(\tau) = \sum_{r \equiv -\frac{1}{2}(h,h) \pmod{\mathbb{Z}}} f_h(r) \exp(2\pi i r \tau)$$

with $f_h(r) = f(r + \frac{1}{2}(h, h), h)$. We note that a vector valued modular form depends on the genus of the lattice L , i.e. on the discriminant group $D(L)$. A Jacobi form contains information on the class of L . We shall use this property in order to prove Theorem 1.2. Moreover the Jacobi forms build a natural bigraded ring with respect to weights and indices. This fact is useful for many constructions (see [G3] and [GN4]).

Example 2. If $L = A_1 = \langle 2 \rangle$, then $J_{k,m}(A_1) = J_{k,m}$ is the space of holomorphic Jacobi modular forms of Eichler–Zagier type studied in the book [EZ]. The isomorphism $J_{k,m} \cong J_{k,1}(\langle -2m \rangle)$ was one of the main starting points for the construction of the Jacobi lifting for the paramodular Siegel groups in [G2].

In the context of Borcherds products, reflective Siegel modular forms and the corresponding Lorentzian Kac-Moody algebras (see [GN1]–[GN5]), the basis Jacobi form is the Jacobi theta-series

$$\vartheta(\tau, z) \in J_{\frac{1}{2}, \frac{1}{2}}(v_\eta^3 \times v_H) \quad (\text{see (2)})$$

which is the Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with multiplier system $v_\eta^3 \times v_H$, where v_η is the multiplier system of order 24 of the Dedekind eta-function η and v_H is the binary character of the Heisenberg group (see [GN4, Example 1.5]). In particular, we have

$$\vartheta\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = v_\eta^3(M)(c\tau + d)^{1/2} \exp(-\pi i \frac{cz^2}{c\tau + d}) \vartheta(\tau, z)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\lambda+\mu} \exp(-\pi i(\lambda^2\tau + 2\lambda z)) \vartheta(\tau, z) \quad (\lambda, \mu \in \mathbb{Z}).$$

The next proposition shows the role of $\vartheta(\tau, z)$ in the context of this paper.

Proposition 2.2. *Let*

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \ell \in L^\vee} f(n, \ell) q^n r^\ell \in J_{0,1}^{wh}(L)$$

be a weakly holomorphic Jacobi form of weight 0 and index 1 for the lattice L . The Fourier coefficients $f(0, \ell)$ determine a generalized 2-design in L^\vee . More exactly the following identity is valid

$$\sum_{\ell \in L^\vee} f(0, \ell)(\ell, \mathfrak{z})^2 = 2C(\mathfrak{z}, \mathfrak{z}), \quad \forall \mathfrak{z} \in L \otimes \mathbb{C} \quad (4)$$

where

$$C = \frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell) - \sum_{n > 0, \ell \in L^\vee} f(-n, \ell) \sigma_1(n) = \frac{1}{2 \operatorname{rank} L} \sum_{\ell \in L^\vee} f(0, \ell)(\ell, \ell).$$

Proof. We prove the proposition using the method of automorphic correction of Jacobi forms (see [G3, Lemma 1.10], [G4, Proposition 1.5] and [CG1, (6)]). We consider the quasi-modular Eisenstein series of weight 2

$$G_2(\tau) = -D(\log(\eta(\tau))) = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n) q^n, \quad q = e^{2\pi i \tau}, \quad D = q \frac{d}{dq}$$

where $\sigma_k(m) = \sum_{d|m} d^k$. We define the automorphic correction of the weakly holomorphic Jacobi form φ as follows $\varphi_{cor}(\tau, \mathfrak{z}) = e^{-4\pi^2 G_2(\tau)(\mathfrak{z}, \mathfrak{z})} \varphi(\tau, \mathfrak{z})$. This function transforms like an automorphic function in τ

$$\varphi_{cor}\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = \varphi_{cor}(\tau, \mathfrak{z}).$$

Therefore the sum of the coefficients of order 2 in the Taylor expansion of $\varphi_{cor}(\tau, \mathfrak{z})$ in \mathfrak{z} is a meromorphic $\operatorname{SL}_2(\mathbb{Z})$ -modular form of weight 2 in τ . It is easy to find its zeroth Fourier coefficient. It is equal to

$$(2\pi i)^2 \left(\sum_{\ell \in L^\vee} f(0, \ell)(\ell, \mathfrak{z})^2 + 2(\mathfrak{z}, \mathfrak{z}) \left[-\frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell) + \sum_{n > 0, \ell \in L^\vee} \sigma_1(n) f(-n, \ell) \right] \right).$$

The differential operator D maps the space of modular functions of weight 0 onto the space of meromorphic modular forms of weight 2. In particular the zeroth Fourier coefficient of a meromorphic $\operatorname{SL}_2(\mathbb{Z})$ -modular form of weight 2 is equal to zero. It proves the identity (4) with the first expression for C . After that we can also find the second expression for C acting by the Laplace operator on (4) because $\Delta_3(\ell, \mathfrak{z})^m = m(m-1)(\ell, \ell)(\ell, \mathfrak{z})^{m-2}$ and $\Delta_3(\mathfrak{z}, \mathfrak{z}) = 2 \operatorname{rank} L$. \square

A finite multiset of vectors $\{(\ell; m(\ell))\}$ from L^\vee (one takes every vector $m(\ell)$ times) which satisfies (4) is called *vector system* in [B1, §6]. One can introduce positive and negative elements ($\ell > 0$ and $\ell < 0$) in a vector system using the sign of the scalar product with a vector which is not orthogonal to any vectors in the system. Then every element will be either positive or negative. If L is even integral, then $s(L)$ (respectively, $n(L)$) denotes the positive generator of the integral ideal generated by (λ, μ) (respectively, by (λ, λ)) for λ and μ in L .

Corollary 2.3. *Let φ be as in Proposition 2.2. We assume that all Fourier coefficients $f(0, \ell)$ are integral. Then the function*

$$\psi_\varphi(\tau, \mathfrak{z}) = \prod_{\ell \in L^\vee, \ell > 0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0, \ell)}$$

where the product is taken with respect to a fixed ordering in L mentioned above, transforms like Jacobi form of weight 0 and index C

$$C = \frac{1}{2 \operatorname{rank} L} \sum_{\ell \in L^\vee} f(0, \ell)(\ell, \ell)$$

for L with a character. Let $\vec{B} = \frac{1}{2} \sum_{\ell \in L^\vee, \ell > 0} f(0, \ell) \ell \in \frac{1}{2} L^\vee$. If $C \cdot n(L) \in 2\mathbb{Z}$, then $C \cdot s(L) \in \mathbb{Z}$ and $\vec{B} \in L^\vee$.

Proof. Using the functional equations of $\vartheta(\tau, z)$ we obtain that the theta-product transforms like a Jacobi form of weight 0 and index C with a character if and only if the system $\{(\ell, f(0, \ell))\}$ satisfies (4). This is clear for $\mathrm{SL}_2(\mathbb{Z})$ -transformations. In order to prove the same for the abelian translations $\mathfrak{z} \rightarrow \mathfrak{z} + \lambda\tau + \mu$ one uses the bilinear variant of (4)

$$\sum_{\ell \in L^\vee, \ell > 0} f(0, \ell)(\ell, \mathfrak{z}_1)(\ell, \mathfrak{z}_2) = C(\mathfrak{z}_1, \mathfrak{z}_2), \quad \forall \mathfrak{z}_1, \mathfrak{z}_2 \in L \otimes \mathbb{C}.$$

For any $\lambda, \mu \in L$ we have

$$\psi_\varphi(\tau, \mathfrak{z} + \lambda\tau + \mu) = (-1)^{\sum_{\ell > 0} f(0, \ell)(\ell, \lambda + \mu)} e^{-\pi i C((\lambda, \lambda)\tau + 2(\lambda, \mathfrak{z}))} \psi_\varphi(\tau, \mathfrak{z}).$$

Therefore the restriction of the Jacobi character χ of ψ_φ to the minimal Heisenberg subgroup $H_s(L)$ generated by the elements in $L \times L$ is a binary character. In fact one can prove (see [CG2, §1]) that

$$\chi|_{H(L)}([\lambda, \mu; \kappa]) = e^{\pi i C((\lambda, \lambda) + (\mu, \mu) - (\lambda, \mu) + 2\kappa)}, \quad [\lambda, \mu; \kappa] \in H(L).$$

The properties of the index C follow from [CG2, Proposition 1.1]. The property of \vec{B} reflects the fact that the binary character of $H_s(L)$ is trivial, if $C \cdot n(L) \in 2\mathbb{Z}$. The SL_2 -part of the Jacobi character is equal to $v_\eta^{2A'}$ where $A' = \sum_{\ell \in L^\vee, \ell > 0} f(0, \ell)$. \square

Remark. The theta-product of the corollary is equal to the product function ψ on the page 183 of [B1]. In particular, Corollary 2.3 gives another simple proof of [B1, Theorem 6.5]. Our proof does not use the Poisson summation formula.

3. BORCHERDS PRODUCTS AND JACOBI FORMS

In this section, we write Borcherds automorphic products on $\mathrm{O}^+(L_2)$ in terms of Jacobi modular forms of weight 0 for the even positive definite lattice L . In [B2], the language of vector valued automorphic forms was used. The main theorem of this section is a natural generalization of [GN4, Theorem 2.1] where Siegel modular forms with respect to the symplectic paramodular group Γ_t were considered. This subject is very natural. It was given in my lecture course for Ph.D. students in Lille in 2002/03 without publishing a separate paper (see the chapter 4 in the dissertation of C. Desreumaux (2004) where the corresponding formulations were used). New applications of Borcherds products in algebraic geometry (see [GHS1]–[GHS2], [G5] and [GH]), in string theory (see [CG1] and the references there) and

to the classification theory of Lorentzian Kac–Moody algebras make this subject actual again.

We recall the definition of the stable orthogonal group

$$\tilde{O}^+(L) = \{g \in O^+(L) \mid g|_{L^\vee/L} = \text{id}\}.$$

Theorem 3.1. *Let*

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z}, \ell \in L^\vee} f(n, \ell) q^n r^\ell \in J_{0,1}^{wh}(L)$$

be a weakly holomorphic Jacobi form of weight 0 and index 1 for an even integral positive definite lattice L . We fix an ordering (> 0 and < 0) in the lattice L like in Corollary 2.3. Assume that $f(n, \ell) \in \mathbb{Z}$ if $2n - \ell^2 \leq 0$. Then, the product

$$\mathcal{B}_\varphi(Z) = q^A r^{\vec{B}} s^C \prod_{\substack{n, m \in \mathbb{Z}, \ell \in L^\vee \\ (n, \ell, m) > 0}} (1 - q^n r^\ell s^m)^{f(nm, \ell)},$$

where $Z = (\tau, \mathfrak{z}, \omega) \in \mathcal{H}(L)$, $q = \exp(2\pi i \tau)$, $r^\ell = \exp(2\pi i(\ell, \mathfrak{z}))$, $s = \exp(2\pi i \omega)$, $(n, \ell, m) > 0$ means that $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and $\ell < 0$,

$$A = \frac{1}{24} \sum_{\ell \in L^\vee} f(0, \ell), \quad \vec{B} = \frac{1}{2} \sum_{\ell > 0} f(0, \ell) \ell \in \frac{1}{2} L^\vee, \quad C = \frac{1}{2 \text{rank } L} \sum_{\ell \in L^\vee} f(0, \ell) (\ell, \ell)$$

defines a meromorphic modular form of weight $k = \frac{1}{2} f(0, 0)$ with respect to $\tilde{O}^+(L_2)$ with a character χ (see (6) below). The poles and zeros of \mathcal{B}_φ lie on the rational quadratic divisors defined by the Fourier coefficients $f(n, \ell)$ with $2n - \ell^2 < 0$. In particular, \mathcal{B}_φ is holomorphic if all such coefficients are positive. The explicit formula for the multiplicities is given in (7).

Proof. The product of the theorem is a special case of the Borcherds automorphic products considered in [B2, Theorem 13.3] because the Jacobi form φ determines a vector valued modular form of weight $-\frac{\text{rank } L}{2}$ according (3). The product converges if $Y = \text{Im } Z$ with $(Y, Y) > M$ for a sufficiently large M lies in a fundamental domain of the hyperbolic orthogonal group $\tilde{O}^+(L_1)$ acting on the cone $V^+(L_1)$.

We define the invariants of the automorphic products (the modular group, the weight, the character, the multiplicities of the poles and the zeros, the first several members of the Fourier-Jacobi expansions) in terms of the Fourier coefficients of the Jacobi form φ using a Hecke type representation of the Borcherds products given in [GN4].

The choice of the vector (A, \vec{B}, C) in the definition of \mathcal{B}_φ is motivated by the following identity

$$q^A r^{\vec{B}} s^C \prod_{(n, \ell, 0) > 0} (1 - q^n r^\ell)^{f(0, \ell)} = \eta(\tau)^{f(0, 0)} \prod_{\ell > 0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z})) e^{\frac{2\pi i}{\text{rank } L} (\ell, \ell) \omega}}{\eta(\tau)} \right)^{f(0, \ell)}.$$

The vector (A, \vec{B}, C) is called usually *Weyl vector* of the Borcherds product. According to Proposition 2.2 and Corollary 2.3

$$\tilde{\psi}_{L; C}(Z) = \eta(\tau)^{f(0, 0)} \prod_{\ell > 0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)} \right)^{f(0, \ell)} e^{2\pi i C \omega}$$

is a (meromorphic) Jacobi form of weight $k = \frac{f(0,0)}{2}$ and index C for the lattice L with a character founded in Corollary 2.3 times $v_\eta^{f(0,0)}$.

Next we consider the second term of \mathcal{B}_φ containing the factors with $m > 0$

$$\begin{aligned} \log \left(\prod_{(n,\ell,m), m>0} (1 - q^n r^\ell s^m)^{f(nm,\ell)} \right) &= - \sum_{(n,\ell,m)>0} f(nm,\ell) \sum_{e \geq 1} \frac{1}{e} q^{en} r^{e\ell} s^{em} = \\ &- \sum_{(a,\vec{b},c)>0} \left(\sum_{d|(a,\vec{b},c)} d^{-1} f\left(\frac{ac}{d^2}, \frac{\vec{b}}{d}\right) \right) q^a r^{\vec{b}} s^c = - \sum_{m \geq 1} m^{-1} \tilde{\varphi} | T_-(m)(Z) \end{aligned}$$

where $T_-(m)$ is the Hecke operator defined in [G1] and [G2, (2.12)]. $T_-(m)$ is a Hecke operator of type V_m from [EZ] and it multiplies the index of Jacobi modular forms by m (see [G2, Corollary 2.9]). Therefore we have the following representation of the Borcherds product

$$\mathcal{B}_\varphi(Z) = \tilde{\psi}_{L;C}(Z) \exp \left(- \sum_{m \geq 1} m^{-1} \tilde{\varphi} | T_-(m)(Z) \right) \quad (5)$$

which is similar to [GN4, (2.7)] where we considered the case of Siegel modular forms. Therefore \mathcal{B}_φ transforms like a modular form of weight $k = \frac{f(0,0)}{2}$ with respect to the Jacobi modular group $\Gamma^J(L) < \tilde{\mathcal{O}}^+(L_2)$.

We know (see [G2, §3]) that $\tilde{\mathcal{O}}^+(L_2) = \langle \Gamma^J(L), V \rangle$ where $V : (\tau, \mathfrak{z}, \omega) \rightarrow (\omega, \mathfrak{z}, \tau)$. We can analyze the behavior of the Borcherds product under the V -action using Proposition 2.2. A straightforward calculation shows that

$$\frac{\mathcal{B}_\varphi(V\langle Z \rangle)}{\mathcal{B}_\varphi(Z)} = \frac{q^C s^A}{q^A s^C} \prod_{n>0, m>0, \ell \in L^\vee} \left(\frac{1 - s^{-n} r^\ell q^m}{1 - q^{-n} r^\ell s^m} \right)^{f(-nm,\ell)}.$$

We note that $f(-nm, \ell) = f(-nm, -\ell)$. Therefore the last product is equal to

$$= \frac{q^{C+\sum n f(-nm,\ell)} s^A}{q^A s^{C+\sum n f(-nm,\ell)}} \prod_{n>0, m>0, \ell \in L^\vee} \left(\frac{s^n r^{-\ell} - q^m}{q^n - r^\ell s^m} \right)^{f(-nm,\ell)} = (-1)^{\sum f(-nm,0)}$$

according to the formulae for C from Proposition 2.2. Therefore we have

$$\mathcal{B}_\varphi(\tau, \mathfrak{z}, \omega) = (-1)^D \mathcal{B}_\varphi(\omega, \mathfrak{z}, \tau) \quad \text{where} \quad D = \sum_{n<0} \sigma_0(-n) f(n, 0).$$

This proves that \mathcal{B}_φ transforms like a modular form of weight $\frac{f(0,0)}{2}$ with respect to $\tilde{\mathcal{O}}^+(L_2)$. The character χ is induced by the $\Gamma^J(L)$ -character of the Jacobi form $\psi_{L;C}$ and by the last relation

$$\chi|_{\text{SL}_2} = v_\eta^{24A}, \quad \chi|_{H(L)}([\lambda, \mu; r]) = e^{\pi i C((\lambda, \lambda) + (\mu, \mu) - (\lambda, \mu) + 2r)}, \quad \chi(V) = (-1)^D. \quad (6)$$

Now we calculate the multiplicities of the divisors. The Borcherds product \mathcal{B}_φ has an analytic continuation to $\mathcal{H}(L)$ (see [B1, Theorem 5.1 and Theorem 10.1] and [B2, Theorem 13.3]). The singularities of \mathcal{B}_φ are the solutions of the equations $(1 - q^n r^\ell s^m)^{f(nm,\ell)} = 0$ for (n, ℓ, m) with $2nm - (\ell, \ell) < 0$ and $f(nm, \ell) \neq 0$. The lattice L contains two hyperbolic planes. According to the Eichler criterion (see [G2], [GHS3]) the $\tilde{\mathcal{O}}^+(L_2)$ -orbit of any primitive vector $v \in L_2^\vee$ is uniquely

determined by its norm (v, v) and by its image $v \cong \ell \pmod{L_2}$ in the discriminant group L_2^\vee/L_2 . Therefore there exists

$$w = (0, n, \ell, 1, 0) \in \tilde{\mathcal{O}}^+(L_2)v \quad \text{such that} \quad (v, v) = 2n - (\ell, \ell) < 0, \quad v \cong \ell \pmod{L_2}.$$

In particular, $\tilde{\mathcal{O}}^+(L_2) \cdot \mathcal{D}_v = \tilde{\mathcal{O}}^+(L_2) \langle \{Z \in \mathcal{H}(L) \mid n\tau + (\ell, \mathfrak{z}) + \omega = 0\} \rangle$. The Fourier coefficient $f(n, \ell)$ also depends only on the norm $2n - \ell^2$ and ℓ modulo L . Therefore the multiplicity along the divisor \mathcal{D}_v with the vector v as above is equal to

$$\text{mult } \mathcal{D}_v = \sum_{\substack{d>0 \\ (v,v)=2n-(\ell,\ell) \\ v \equiv \ell \pmod{L_2}}} f(d^2n, d\ell). \quad (7)$$

□

Corollary 3.2. *The representation (5) gives the first terms of the Fourier-Jacobi expansion of \mathcal{B}_φ at the one dimensional cusp defined by the lattice L*

$$\begin{aligned} B_\varphi(\tau, \mathfrak{z}, \omega) &= \psi(\tau, \mathfrak{z}) e^{2\pi i C \omega} \left(1 - \varphi(\tau, \mathfrak{z}) e^{2\pi i \omega} + \frac{1}{2} (\varphi^2(\tau, \mathfrak{z}) - \varphi(\tau, \mathfrak{z}) |T_-(2)) e^{4\pi i \omega} \right. \\ &\quad \left. - \frac{1}{6} (\varphi^3(\tau, \mathfrak{z}) - 3\varphi(\tau, \mathfrak{z}) \cdot (\varphi(\tau, \mathfrak{z}) |T_-(2)) + 2\varphi(\tau, \mathfrak{z}) |T_-(3)) e^{6\pi i \omega} + \dots \right). \end{aligned}$$

The next corollary is evident but it is very useful if one would like to prove that a Jacobi (additive) lifting has a Borcherds product. In this context this property was often used in [GN4].

Corollary 3.3. The Fourier-Jacobi criterion for automorphic products. *Let us assume that a modular form*

$$F(\tau, \mathfrak{z}, \omega) = \varphi_m(\tau, \mathfrak{z}) e^{2\pi i m \omega} + \varphi_{m+1}(\tau, \mathfrak{z}) e^{2\pi i (m+1) \omega} + \dots$$

has Borcherds product expansion. Then

$$F = \mathcal{B}_\varphi \quad \text{where} \quad \varphi(\tau, \mathfrak{z}) = -\frac{\varphi_{m+1}(\tau, \mathfrak{z})}{\varphi_m(\tau, \mathfrak{z})}.$$

3.2. The Fourier-Jacobi expansions of the Borcherds form Φ_{12} . We prove Theorem 1.2 as a corollary of Theorem 2.1. Let $N(R)$ be the Niemeier lattice with the root system R . We put

$$\varphi_{0,N}(\tau, \mathfrak{z}) = \Delta(\tau)^{-1} \vartheta_{N(R)}(\tau, \mathfrak{z}) \in J_{0,1}^{wh}(N(R))$$

where $\vartheta_{N(R)}$ is the Jacobi theta-series of the even unimodular lattice $N(R)$ (see Example 1 of §2). We have

$$\varphi_{0,N}(\tau, \mathfrak{z}) = \sum_{n \geq -1, \ell \in N(R)} f(n, \ell) q^n r^\ell = q^{-1} + 24 + \sum_{v \in R} e^{2\pi i (v, \mathfrak{z})} + \dots$$

The hyperbolic norm of the index of any non-zero Fourier coefficient of the Jacobi theta-series is equal to zero. Therefore, if $f(n, \ell) \neq 0$, then $2n - (\ell, \ell) \geq -2$. Moreover if $2n - (\ell, \ell) = -2$, then $f(n, \ell) = 1$ because $f(n, \ell)$ depends only on the norm of the index. We note that all -2 -vectors in the even unimodular lattice $2U \oplus N(R)(-1)$ build only one orbit with respect to the orthogonal group. According to Theorem 3.1 the Borcherds products $\mathcal{B}_{\varphi_{0,N}}$ vanishes with order one along all rational quadratic divisors \mathcal{D}_v where $v \in 2U \oplus N(R)(-1)$ and $(v, v) = -2$. Therefore $\mathcal{B}_{\varphi_{0,N}}$ is equal, up to a constant, to Φ_{12} according to the Koecher principle. We can

use Corollary 3.2 in order to calculate the first two terms in the Fourier–Jacobi expansion. If $R = \emptyset$, then $N(\emptyset) = \Lambda_{24}$ and

$$\mathcal{B}_{\varphi_{0,\Lambda}} = \Delta(\tau) - \vartheta_{\Lambda_{24}}(\tau, \mathfrak{z})e^{2\pi i\omega} + \frac{1}{2}(\vartheta_{\Lambda_{24}}\varphi_{0,\Lambda} - \Delta(\varphi_{0,\Lambda}|T_-(2)))e^{4\pi i\omega} + \dots \quad (8)$$

The last function is equal to $\Phi_{12}(Z)$ because the two modular forms have the same value at the one dimensional Leech cusp. If R is not empty then for $(\tau, \mathfrak{z}, \omega) \in \mathcal{H}(N(R))$

$$\mathcal{B}_{\varphi_{0,N}}(\tau, \mathfrak{z}, \omega) = \Delta(\tau) \prod_{v \in R_+} \frac{\vartheta(\tau, (v, \mathfrak{z}))}{\eta(\tau)} e^{2\pi i h(R)\omega} - \vartheta_N(\tau, \mathfrak{z}) \prod_{v \in R_+} \frac{\vartheta(\tau, (v, \mathfrak{z}))}{\eta(\tau)} e^{2\pi i (h(R)+1)\omega} + \dots \quad (9)$$

where the product is taken over all positive roots v in the finite root system R . We note that $\text{rank } R = 24$ and the first term is the Weyl–Kac denominator function of the affine Lie algebra $\hat{\mathfrak{g}}(R)$. We note that the Fourier coefficient of the Weyl vector (A, \vec{B}, C) of the Borcherds product is equal to 1. Therefore $\mathcal{B}_{\varphi_{0,N(R)}} = \pm\Phi_{12}$. The sign depends on the choice of the positive roots in the finite root system R . Theorem 1.2 is proved.

3.3. The Jacobi criterion for Borcherds products. In this section we give an illustration of the Jacobi criterion formulated in Corollary 3.3. For this end we find the Borcherds product of the reflective modular form Φ_2 which is the “roof” of the $4A_1$ -tower of reflective modular forms for the root lattices $A_1 < 2A_1 < 3A_1 < 4A_1$ (see [G5]). The last member of this tower for A_1 is the classical Igusa modular form Δ_5 . The details of the Jacobi construction of Φ_2 were given in [CG2, Example 2.4]. The “direct” product of the four theta-series is a Jacobi form of index $\frac{1}{2}$ for the lattice $L = 4A_1$ with a character of order 2

$$\vartheta_{4A_1}(\tau, \mathfrak{z}) = \vartheta(\tau, z_1) \cdot \dots \cdot \vartheta(\tau, z_4) \in J_{2, \frac{1}{2}}(4A_1, v_\eta^{12} \times v_H(4A_1)).$$

According to [CG2, Theorem 2.2] we have

$$\Phi_2(Z) = \text{Lift}(\vartheta_{4A_1}(\tau, \mathfrak{z})) \in M_2(\mathcal{O}^+(2U \oplus 4A_1(-1)), \chi_2)$$

where χ_2 is a character of order 2 of the full orthogonal group. This fundamental modular form of singular weight has the following Fourier expansion

$$\begin{aligned} \Phi_2(Z) &= \sum_{\substack{m \equiv 1 \pmod{2} \\ m \geq 1}} m^{-1} (\tilde{\vartheta}_{4A_1}|_2 T_-^{(2)}(m))(Z) = \\ &= \sum_{\ell=(\ell_1, \dots, \ell_4) \in 4A_1^\vee} \sum_{\substack{n, m \in \mathbb{Z}_{>0} \\ n \equiv m \equiv 1 \pmod{2} \\ 2nm - (\ell, \ell) = 0}} \sigma_1((n, \ell, m)) \left(\frac{-4}{2l_1}\right) \dots \left(\frac{-4}{2l_4}\right) e^{\pi i(n\tau + (\ell, \mathfrak{z}) + m\omega)} \end{aligned}$$

where $\sigma_1((n, \ell, m))$ is the sum of the positive divisors of the greatest common divisor of the vector $(n, \ell, m) \in (U \oplus A_1(-4))^\vee$. According [CG2, Proposition 2.1] and [GN4, (1.27)]

$$3^{-1}\vartheta_{4A_1}|_2 T_-(3)(\tau, \mathfrak{z}) = 3\vartheta_{4A_1}(3\tau, 3\mathfrak{z}) + \frac{1}{3} \sum_{b=0}^2 \vartheta_{4A_1}\left(\frac{\tau + 2b}{3}, \mathfrak{z}\right).$$

The quotient

$$\phi_{0, 4A_1} = -\frac{3^{-1}\vartheta_{4A_1}|_2 T_-(3)}{\vartheta_{4A_1}} \in J_{0,1}^w(4A_1)$$

is a weak Jacobi form of weight 0 and index 1 for the lattice $4A_1$. Its Fourier coefficient $f(n, \ell)$ depends only on $2n - \ell^2$ and $\ell \pmod{4A_1}$. Applying (3) to a weak Jacobi form of index one for $4A_1$ we see that if $f(n, \ell) \neq 0$, then $2n - \ell^2 \geq -2$. Therefore, in order to find all Fourier coefficients with negative hyperbolic norm of their indices we have to calculate only the q^0 -part of the Fourier expansion of $\phi_{0, 4A_1}$. The first term $\frac{\vartheta_{4A_1}(3\tau, 3\mathfrak{z})}{\vartheta_{4A_1}(\tau, \mathfrak{z})}$ of the quotient contains coefficients q^n with positive n . The second term is equal, up to the sign, to

$$\frac{3^{-1} \sum_{b \pmod{3}} \left(q^{\frac{1}{2}} \prod_{n \geq 1} (1 - q^n)^4 \prod_{i=1}^4 (1 - q^n r_i)(1 - q^n r_i^{-1}) \right) \begin{pmatrix} 1 & 2b \\ 0 & 3 \end{pmatrix}}{q^{\frac{1}{2}} \prod_{n \geq 1} (1 - q^n)^4 \prod_{i=1}^4 (1 - q^n r_i)(1 - q^n r_i^{-1})}$$

where $r_i = e^{2\pi i z_i}$ and the matrix denotes the action $\tau \rightarrow \frac{\tau + 2b}{3}$. The straightforward calculation shows that

$$\phi_{0, 4A_1}(\tau, \mathfrak{z}) = r_1 + r_2 + r_3 + r_4 + 4 + r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1} + q(\dots).$$

This part of the Fourier expansion contains all orbits of the Fourier coefficients with negative norm of its indices. Therefore the Borcherds product

$$\mathcal{B}_{\phi_{0, 4A_1}} \in M_2(\tilde{\mathcal{O}}^+(2U \oplus 4A_1(-1)), \chi_2)$$

vanishes with order 1 along the rational quadratic divisors $\tilde{\mathcal{O}}^+(2U \oplus 4A_1(-1))$ -equivalent to $z_i = 0$ ($1 \leq i \leq 4$) which are the walls of the reflections with respect to the (-2) -roots of $4A_1(-1)$. The Jacobi lifting Φ_2 vanishes along these divisors. Therefore $\Phi_2/\mathcal{B}_{\phi_{0, 4A_1}}$ is holomorphic. Analyzing the first Fourier-Jacobi coefficients we get

$$\Phi_2 = \text{Lift}(\vartheta_{4A_1}) = \mathcal{B}_{\phi_{0, 4A_1}} \quad (10)$$

due to the Koecher principle. Φ_2 is a reflective modular form with the simplest possible divisor in the sense of [G5]. Taking its quasi pullbacks on the lattices $A_1 < 2A_1 < 3A_1 < 4A_1$ we get three other reflective modular forms with respect to $\mathcal{O}^+(2U \oplus mA_1(-1))$ for $m = 3, 2$ and 1 . The last one (for $m = 1$) is the Igusa modular form Δ_5 , i.e.

$$\Phi_2(\tau, z_1, z_2, z_3, z_4, \omega) \Big|_{z_2=z_3=z_4=0}^{(quasi \ pullback)} = \Delta_5(\tau, z_1, \omega) \in S_5(\text{Sp}_2(\mathbb{Z}), \chi_2).$$

See [G5] for more details where we analyzed also the towers of reflective modular forms for $D_1 < \dots < D_8$ and $A_2 < 2A_2 < 3A_2$.

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