Stochastic processes with Z_N symmetry and complex Virasoro representations. The partition functions

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Abstract. In a previous Letter [1] we have presented numerical evidence that a Hamiltonian expressed in terms of the generators of the periodic Temperley-Lieb algebra has, in the finite-size scaling limit, a spectrum given by representations of the Virasoro algebra with complex highest weights. This Hamiltonian defines a stochastic process with a Z_N symmetry. We give here analytical expressions for the partition functions for this system which confirm the numerics. For N even, the Hamiltonian has a symmetry which makes the spectrum doubly degenerate leading to two independent stochastic processes. The existence of a complex spectrum leads to an oscillating approach to the stationary state. This phenomenon is illustrated by an example.

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Considering Z_N symmetric representations of the periodic Temperley-Lieb algebra $PTL_L(x)$ [2, 3, 4, 5], we have defined a Hamiltonian as a linear combination of the generators of this algebra. Taking $x = 1$, this Hamiltonian gives the time evolution of a one-dimensional stochastic process. Looking at the finite-size scaling spectra of this Hamiltonian, we have obtained numerical evidence for the appearance of Virasoro representations with complex highest weights. Moreover, the real part of the complex highest weights is smaller than the real highest weights and hence dominate the large time behavior of the systems. This observation was a big surprise and was the main content of a previous Letter [1]. In the present one, we give an analytic derivation of this result and present the partition function for each sector of the model. We also present an application of our results. For N even we show that there is a symmetry in the model which makes the spectrum for any lattice size to be doubly degenerate indicating the presence of a zero fermionic mode. This Letter is basically a continuation of the previous one [1], we did nevertheless our best to make it self-consistent.

The $PTL_L(x)$ algebra has L generators $e_k,\ k = 1, 2, \ldots, L$ satisfying the relations:

$$e_k^2 = xe_k, \qquad e_k e_{k\pm1} e_k = e_k, \qquad [e_k, e_\ell] = 0, \ \ |k-\ell| > 1, \tag{1}$$

with $e_{k+L} = e_k$. We take L even only. We consider two quotients of the algebra:

$$(AB)^N A = A, \tag{2}$$

where

$$A = \prod_{j=1}^{L/2} e_{2j}, \qquad B = \prod_{j=0}^{L/2-1} e_{2j+1}, \tag{3}$$

and

$$ABA = \alpha^2 A \tag{4}$$

with $\alpha = e^{i2\pi r/N},\ r = 0, 1, \ldots, N-1$. One can see that the quotient (4) is a solution of eq.(2) which defines the first quotient. This observation will be crucial in obtaining the partition functions mentioned above.

In order to get the Z_N symmetric representations of (1) and (2), we consider N copies of a one-dimensional periodic system with L sites. Each copy consists of $\binom{L}{L/2}$ configurations of link patterns on a cylinder and n noncontractible loops ($n = 0, 1, \ldots, N-1$) on the same cylinder. This is the vector space in which the generators of the $PTL_L(x)$ algebra act. It has the dimension $N \times \binom{L}{L/2}$. In Fig.1 we show the 6 configurations for $L = 4$ and $n = 2$.

An alternative way to label the states in the vector space is to use the spin representation in which the slopes in the arches are used $+(-)$ for the beginning (ending)

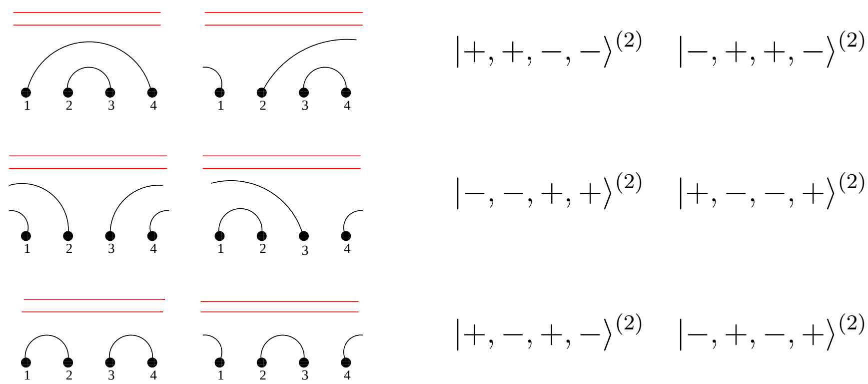

Figure 1. The six link patterns configurations for $L = 4$ sites on a cylinder and two circles without sites (noncontractible loops). The open arches and circles meet behind the cylinder. The corresponding spin presentations of the same link patterns are given on the right.

of an arch. The number of non-contractible loops is indicated by a supplementary label. This notation is also given in Fig.1.

The action of the generators e_k on the link patterns for a given copy n is the same as the one used for the usual (non-periodic) Temperley-Lieb algebra [6] with one exception. If the generator acts on the bond connecting the beginning and the end of an arch having the size of the system L, one obtains a configuration of the copy $n + 1$ (see Fig.2).

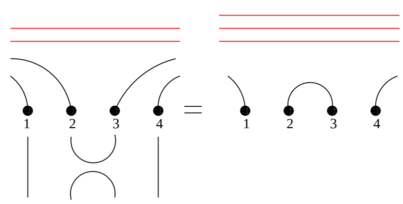

Figure 2. The action of the e_2 generator acting the bond between the sites 2 and 3 which are the end of an arch of the size of the system $L = 4$. A new circle is created on the cylinder and one moves from the copy $n = 2$ to the copy $n = 3$.

In order to obtain the Z_N representation of the $PTL_L(x)$ algebra with the quotient (2) one identifies the copy $n = N$ with the copy $n = 0$.

In what follows, we take the parameter x in the $PTL_L(x)$ algebra equal to one. With this choice the Hamiltonian

$$H = \sum_{k=1}^{L}(1 - e_k) \tag{5}$$

gives the time evolution of a stochastic process. We want to stress that the properties of the spectra which are going to be discussed below, stay valid for any value of x.

Notice that for N even the Z_N symmetric representation decomposes into a pair of identical $\frac{N}{2} \times \binom{L}{L/2}$ -dimensional irreps.‡ To see this, consider two linear transformations

‡ For N odd the Z_N representation is irreducible.

on the Z_N space.

The transformation X_1 acts diagonally in a following way. We start considering configurations with no arches hidden in the back of the cylinder (the first and fifth link patterns in Fig.1). These configurations get a factor of $(-1)^n$. The configurations translated with one lattice unit get a factor $(-1)^{(n+1)}$ (the second and sixth configurations in Fig.1). The next translated configurations get again the factor $(-1)^n$ and so on and so forth.

Transformation X_2 permutes copies as follows:

$$|\ldots\rangle^{(2k)} \leftrightarrow |\ldots\rangle^{(2k+1)}, \qquad k = 0, 1, \ldots, \frac{N}{2} - 1.$$

The two transformations constitute the algebra:

$$(X_1)^2 = (X_2)^2 = \mathrm{id}, \qquad X_1X_2 + X_2X_1 = 0. \tag{6}$$

This algebra has only 2-dimensional equivalent irreducible representations. Since X_1 and X_2 commute with the action of the $PTL_L(x)$ on the Z_N symmetric link patterns, it follows that for N even, the spectrum of H is doubly degenerate. To illustrate this observation, let us take $L = 4$ and $N = 2$. The Hamiltonian splits into two stochastic Hamiltonians having each six states. The first one having the states $|++--\rangle^{(0)}$, $|--++\rangle^{(0)}$, $|+--+-\rangle^{(0)}$, $|-++-\rangle^{(1)}$, $|+--+\rangle^{(1)}$ and $|-+-+\rangle^{(1)}$, the second one having the six states in which the copies 0 and 1 are permuted.

We are interested in the spectra of H in the finite-size scaling limit. Let us keep in mind that in a stochastic process, the energies coincide with the energy gaps since the ground-state energy is zero for any system size. Since H is invariant under translations ($e_k \mapsto e_{k+1(\mathrm{mod} L)}$) and the cyclic rotations Z_N ($|\ldots\rangle^{(n)} \mapsto |\ldots\rangle^{(n+1(\mathrm{mod}\ N))}$), one has $N \times L$ sectors labeled by $p = 0, \pm 1, \pm 2, \ldots$ corresponding to the momenta $P = 2\pi p/L$ and by $r = 0, 1, \ldots, N-1$, labeling the irreps of Z_N. If $E_p^r(q)$, $q = 1, 2, \ldots,$ are the energy levels in the sector (p, r), the scaling dimensions $x_p^r(q)$ are given by $\lim_{L\to\infty}(E_p^r(q)L) = 2\pi v_s x_p^r(q)$, with the sound velocity $v_s = 3\sqrt{3}/2$.

In [1] we have diagonalized numerically H for $N = 3$. We went up to $L = 30$ and looked at the lowest excitations. In the $r = 0$ sector we confirmed the expected value $x_0^0(1) = 0.25$. The surprise came when we looked at the $r = 1$ sector where we found:

$$x_0^1(1) = 0.03905 + 0.08753i, \qquad x_0^1(2) = 0.14908 - 0.11806i, \tag{7}$$

i.e. complex values. For N even we found $E_p^r(q) = E_{p+L/2}^{r+N/2}(q)$ which is a consequence of the symmetry (6).

We present now our new results. In order to obtain the partition function in each sector r, we use the fact that the Z_N representation of the algebra with the quotient (2) can be decomposed into N representations of the quotient (4) [3]. The representations of the quotient (4), in the link patterns vector space, are obtained by considering a single copy but changing the action of the generators when they act on a bond connecting the beginning and the end of an arch of the system size L like in Fig.2. Instead of adding a non-contractible loop, one multiplies the state in the right hand side of the figure by a fugacity $\alpha = \exp(2\pi i r/N)$. It was shown [2] that the quotient (4) admits also a representation in the standard spin 1/2 basis (not to be confused with the one used in Fig.1) and the Hamiltonian (5) can be written in this basis. By performing a similarity transformation, the Hamiltonian is the XXZ quantum chain with a twist:

$$e_k = \sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ + \frac{1}{4}(1-\sigma_k^z\sigma_{k+1}^z) + i\frac{\sqrt{3}}{4}(\sigma_{k+1}^z - \sigma_k^z), \quad k = 1, 2, \ldots, L-1,$$
$$e_L = e^{i2\pi\phi}\sigma_L^+\sigma_1^- + e^{-i2\pi\phi}\sigma_L^-\sigma_1^+ + \frac{1}{4}(1-\sigma_L^z\sigma_1^z) + i\frac{\sqrt{3}}{4}(\sigma_1^z - \sigma_L^z). \tag{8}$$

The twist ϕ is related to the parameter α by the relation:

$$\alpha = 2\cos(\pi\phi). \tag{9}$$

The vector space of the $\binom{L}{L/2}$ link patterns configurations corresponds to the $S^z = \sum_{k=1}^{L}\sigma_k^z = 0$ sector of the spin vector space.

The Hamiltonian (5),(8) is integrable using the Bethe Ansatz and the scaling dimensions (highest weights of Virasoro representations) are known [7]. They are given by the Gaussian model. In the $S^z = 0$ sector they are:

$$x = \frac{3}{4}(s+\phi)^2 - \frac{1}{12} + m + m', \quad p = m - m', \tag{10}$$

where $s, m, m' = 0, \pm 1, \pm 2, \ldots$.

From (9) we see that for N even $\alpha = e^{i2\pi r/N}$ and $\alpha = e^{i2\pi(r+N/2)/N} = -e^{i2\pi r/N}$ give the same value for the twist ϕ and therefore the spectrum of the Hamiltonian (5) is doubly degenerate, in agreement with our previous observation. Moreover, for the sectors r not equal to 0 or $N/2$ the values of ϕ obtained from (9) are complex, henceforth the scaling dimensions (critical exponents) (10) are complex too.

As a check, taking $N = 3$ and $r = 1$, from (9) one gets:

$$\phi = -0.426642 - 0.137279i$$

from which we get using (10) with $s = 0$ and 1

$$x_0^1(1) = 0.03990499 + 0.087853992i; \quad x_0^1(2) = 0.1490874 - 0.11808136i \tag{11}$$

in excellent agreement with the values (7).

We have to stress that although the spectrum of the Z_N symmetric Hamiltonian splits into N sectors, the stochastic process doesn't. The condition of positivity of the wave function describing the probability distribution function, mixes the sectors.

The existence of complex scaling dimensions has consequences on the time behavior of various correlators showing oscillatory phenomena, more so since their real part is smaller than real scaling dimensions of the $r = 0$ sector. To illustrate the phenomenon, we looked at the density of "peaks" $d^{(n)}(t, L)$ in different copies. Those are $+-$ pairs

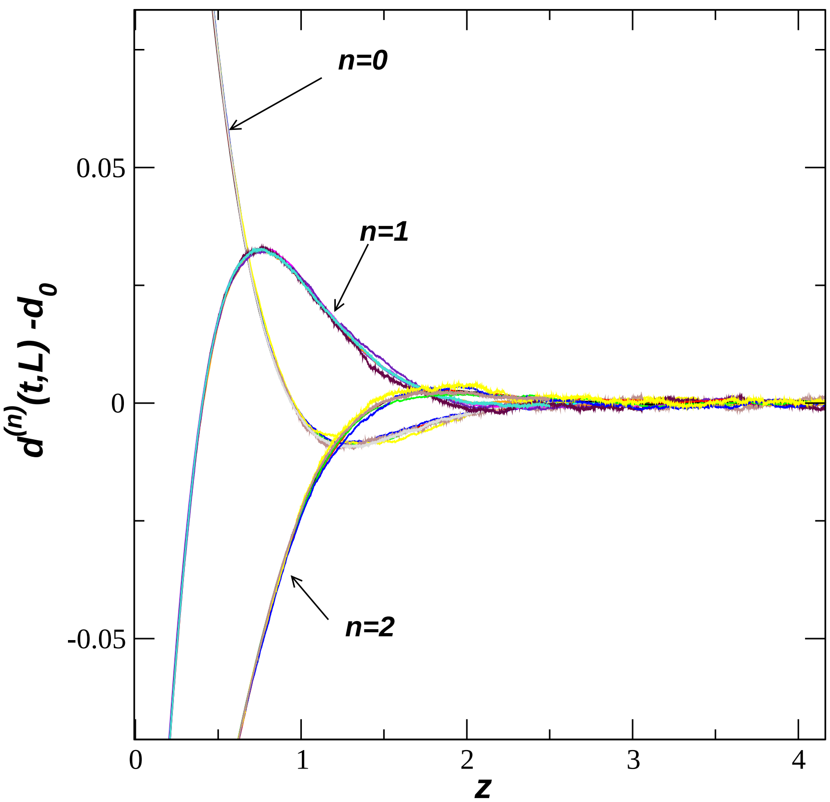

Figure 3. The density of "peaks" $d^{(n)}(z)$ in the copy n as a function of $z = 2\pi v_s t/L$ for $N = 3$, and various lattice sizes $L = 40, 80, 160, 500, 1000, 2000$ and 4000. d_0, the value of the density in the stationary state, is subtracted.

which in the Dyck paths picture of the link patterns [8] correspond to peaks in the paths. This local observable is measured easily in Monte Carlo simulations. The time dependence of the "peaks" in various sectors is determined by the initial conditions. For large values of t and large lattice sizes we expect $d^{(n)}(t, L)$ to be a function of t/L:

$$d^{(n)}(t, L) \;=\; d_0 \;+\; \sum_k A_k \cos\left[\mathrm{Im}(x_0^{(n)}(k))z\right] e^{-\mathrm{Re}(x_0^{(n)}(k))z}, \quad z = \frac{2\pi v_s t}{L}, \quad (12)$$

where d_0 is the density of "peaks" in the stationary state which is the same for each copy n and the A_k's are dependent on the initial conditions. We have computed $d^{(n)}(t, L) - d_0$ using Monte Carlo simulations in the case $N = 3$ for different lattice sizes. The initial state was the configuration $|+, -, +, -, \ldots, +, -\rangle^{(0)}$ in the copy $n = 0$. The results are shown in Fig.3. One can see that, as expected, the densities are dependent on z only.

Encouraged by this observation, we did a fit to the $n = 0$ data (see Fig.4) using the parameterization

$$d^{(0)}(z) \;-\; d_0 \;=\; e^{-az} \frac{\cos b(z - z_0)}{\cos b z_0} \quad (13)$$

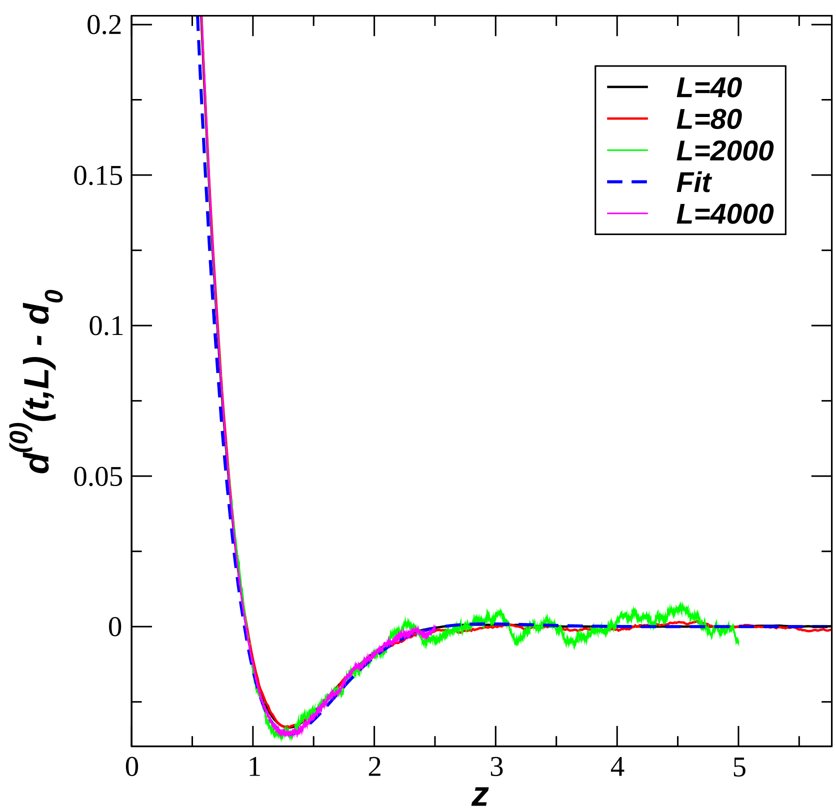

Figure 4. A fit to $d^{(0)}(z) - d_0$ as a function of z, using the parameterization (13) and (14). Monte Carlo data for lattice sizes $40, 80, 2000$ and 4000 were used. The fit is indicated by dashed line.

and obtain:

$$a = 0.1379\,, \quad b = 0.1107\,, \quad z_0 = 0.060\,. \tag{14}$$

These values are compatible with $x_0^1(2)$ given in (11). To our knowledge, it is for the first time that expressions like (13) appear in a conformal invariant theory.

Before closing this Letter, we would like to notice that in a seminal paper Saleur and Sornette [9] have suggested the possible existence of complex critical exponents in non unitary conformal field theories. We have shown that they indeed exist.

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