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Vertex coloring of graphs with few obstructions

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ABSTRACT

We study the vertex coloring problem in classes of graphs defined by finitely many forbidden induced subgraphs. Of our special interest are the classes defined by forbidden induced subgraphs with at most 4 vertices. For all but three classes in this family we show either NP-completeness or polynomial-time solvability of the problem. For the remaining three classes we prove fixed-parameter tractability. Moreover, for two of them we give a 3/2 approximation polynomial algorithm.

1. Introduction

A graph G is *k-colorable* if the vertex set of G can be partitioned into at most k independent sets (color classes). For a fixed value of k, VERTEX-k-COLORABILITY is the problem of deciding whether a given graph G is k-colorable or not. The smallest value of k such that G is k-colorable is called the *chromatic number* of G and is denoted $\chi(G)$.

VERTEX COLORING is the problem of determining $\chi(G)$ and partitioning the vertex set of G into $\chi(G)$ color classes. From a computational point of view, VERTEX COLORING and VERTEX-k-COLORABILITY for any $k \geq 3$ are difficult problems, i.e. they are NP-complete. Moreover, the problems remain NP-complete under substantial restrictions. For instance, VERTEX-3-COLORABILITY (and hence VERTEX COLORING) is NP-complete for triangle-free graphs [21] and for graphs of vertex degree at most four [17], VERTEX-4-COLORABILITY (and hence VERTEX COLORING) is NP-complete for P_7-free graphs [15] and VERTEX-5-COLORABILITY (and hence VERTEX COLORING) is NP-complete for P_6-free graphs [15]. On the other hand, for graphs in some special classes, the problems can be solved in polynomial time. For instance, VERTEX-3-COLORABILITY can be solved for P_6-free graphs [27], VERTEX-4-COLORABILITY can be solved for P_2+P_3-free graphs [12], VERTEX-k-COLORABILITY for any k can be solved for P_5-free graphs [14] and VERTEX COLORING (and hence VERTEX-k-COLORABILITY for any k) can be solved in polynomial time for (triangle, $2P_3$)-free graphs [10] and for perfect graphs.

All classes of graphs mentioned above and all classes we deal with in the present paper possess the property that whenever a graph belongs to a class then every induced subgraph of the graph also belongs to the same class. Such classes are called *hereditary*.

Our goal is systematization of hereditary classes according to the computational complexity of the VERTEX COLORING problem. It is well-known (and not difficult to see) that a class $\mathcal{X}$ of graphs is hereditary if and only if it can be described by a set $\mathcal{Y}$ of forbidden induced subgraphs (obstructions), in which case we write $\mathcal{X} = Free(\mathcal{Y})$. The induced subgraph characterization provides a uniform way to describe hereditary classes and therefore a systematic way to study them. In this paper, we are interested in classes for which the set of obstructions is finite. We call such classes *finitely defined*.

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Of our special interest are the classes defined by forbidden induced subgraphs with at most 4 vertices. To describe the graphs, we use the following notations. By P_n, C_n, K_n and O_n we denote the chordless path, the chordless cycle, the complete graph and the empty (edgeless) graph on n vertices, respectively. Also, $K_n - e$ is the graph obtained from K_n by deleting an edge, and $K_{p,q}$ is the complete bipartite graph with parts of size p and q. By $G_1 + G_2$ we denote the disjoint union of graphs G_1 and G_2, and by kG the disjoint union of k copies of G. In particular, $nK_1 = O_n$, $2K_2 = \overline{C}_4$, $K_2 + 2K_1 = \overline{K_4 - e}$, $C_3 + K_1 = \overline{K_{1,3}}$. The graphs $K_{1,3}$ and $\overline{P_3 + K_1}$ have special names in the literature, they are called *claw* and *paw*, respectively.

Given a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Also, by $L(G)$ we denote the line graph of G, i.e. the graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the respective edges of G share a vertex. It is well-known (see e.g. [13]) that the class of line graphs is hereditary and is defined by 9 forbidden induced subgraphs, one of which is the claw and 8 others contain $K_4 - e$ as an induced subgraph.

The importance of the class of line graphs for our study is due to the fact that VERTEX COLORING restricted to the class of line graphs is equivalent to EDGE COLORING, which is one more important NP-complete problem. From this relationship, we immediately conclude that VERTEX COLORING is NP-complete in the class $Free(K_{1,3})$. More generally, it was shown in [19] that

Theorem 1. *If G is an (not necessarily proper) induced subgraph of P_4 or $P_3 + K_1$, then* VERTEX COLORING *is polynomial-time solvable in the class $Free(G)$. Otherwise, it is NP-complete in $Free(G)$.*

The polynomial-time solvability of VERTEX COLORING in the class $Free(P_4)$ can also be derived from a more general result (Theorem 2 below) proved in [28], since the clique-width of graphs in $Free(P_4)$ is at most 2.

Theorem 2. VERTEX COLORING *is polynomial-time solvable for graphs of bounded clique-width.*

In [9], the family of hereditary classes of graphs defined by forbidden induced subgraphs with at most 4 vertices was completely characterized in terms of bounded or unbounded clique-width. For classes where the clique-width is bounded, we immediately conclude polynomial-time solvability of the VERTEX COLORING problem by Theorem 2. In this paper, we study computational complexity of the problem on classes of unbounded clique-width. According to [9], in the family of classes defined by forbidden induced subgraphs with at most 4 vertices there are seven minimal classes of unbounded clique-width. These are:

$\mathcal{X}^1 = Free(K_{1,3}, C_4, K_4 - e, K_4)$,
$\mathcal{X}^2 = Free(\overline{K_{1,3}}, \overline{C_4}, \overline{K_4 - e}, \overline{K_4})$,
$\mathcal{X}^3 = Free(K_3, C_4)$,
$\mathcal{X}^4 = Free(O_3, \overline{C}_4)$,
$\mathcal{X}^5 = Free(K_4, 2K_2)$,
$\mathcal{X}^6 = Free(C_4, 2K_2)$,
$\mathcal{X}^7 = Free(C_4, O_4)$.

In the present paper, we study VERTEX COLORING in these classes and in their superclasses defined by forbidden induced subgraphs with at most 4 vertices.

A helpful tool for investigating computational complexity of algorithmic problems on finitely defined classes of graphs is the notion of boundary classes. In Section 2, we use this notion to derive a number of NP-completeness results for VERTEX COLORING. Then in Section 3, we present polynomial-time algorithms for the problem in some classes defined by forbidden induced subgraphs with at most 4 vertices. Together, the results of Sections 2 and 3 cover nearly all classes in the family under consideration. For the remaining classes in this family we prove fixed-parameter tractability of VERTEX COLORING in Section 4. Also, for two of the remaining classes we give a 3/2 approximation polynomial algorithm in Section 5. Section 6 concludes the paper with a number of open problems.

2. Boundary classes and NP-completeness results

As we mentioned in the Introduction, the notion of boundary classes is a helpful tool for the analysis of computational complexity of algorithmic problems on finitely defined classes of graphs. This notion can be defined as follows.

Let Π be an algorithmic graph problem. Let us call a hereditary class $\mathcal{X}$ *Π-easy* if Π can be solved for graphs in $\mathcal{X}$ in polynomial time, and *Π-tough* otherwise. A hereditary class $\mathcal{X}$ is called *Π-limit* if there exists a sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \ldots$ of Π-tough classes such that $\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i$.

To illustrate this notion, let us denote by H_i and L_i the graphs represented in Fig. 1. The following lemma can be found, for instance, in [1].

Lemma 1. *The* MAXIMUM INDEPENDENT SET *problem is NP-complete in the class*

$$Free(C_3, C_4, C_5, \ldots, C_p, H_0, H_1, H_2, \ldots, H_p)$$

for each fixed value of p.

From this lemma it follows that the class $Free(C_3, C_4, C_5, \ldots, H_0, H_1, H_2, \ldots)$ is a limit class for the MAXIMUM INDEPENDENT SET problem. Let us denote this class by $\mathcal{S}$. It is not difficult to see that

$\mathcal{S}$ is the class of graphs in which every connected component is of the form $S_{i,j,k}$ represented in Fig. 2(a).

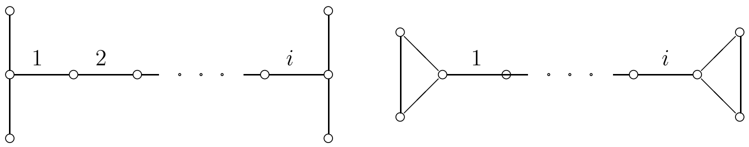

Fig. 1. Graphs H_i (left) and L_i (right).

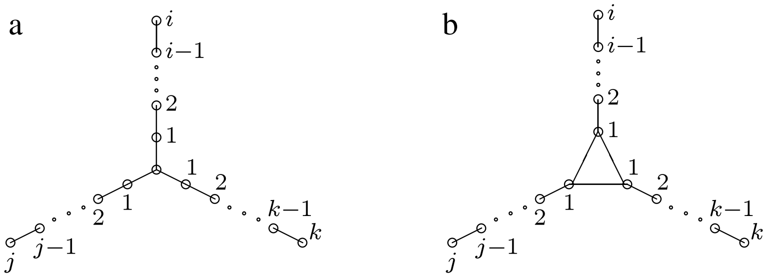

Fig. 2. The graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b).

The class $\mathcal{S}$ and some related classes turned out to be limit for many algorithmic graph problems. For instance, $\mathcal{S}$ is a limit class for the MINIMUM VERTEX COVER problem, since a set $U \subseteq V(G)$ is a vertex cover of G if and only if $V(G) - U$ is an independent set in G. Also, the class of complements of line graphs of graphs in $\mathcal{S}$ is a limit class for VERTEX COLORING. This follows from the results in [29] and can be shown as follows.

Let G be a graph in $Free(K_3)$ and let $H = L(G)$ be its line graph. Every clique K in H corresponds to a set of edges of G incident to the same vertex, say $v(K)$. Let $K^1, \ldots, K^p$ be a set of cliques that partition $V(H)$. Then $\{v(K^1), \ldots, v(K^p)\}$ is a vertex cover of G. This cover is minimum if and only if $K^1, \ldots, K^p$ is a partition of $V(H)$ into minimum number of cliques, or equivalently, if and only if p is the chromatic number of $\overline{H}$. We know that $H \in Free(K_{1,3})$. Also, it is not difficult to see that if $G \in Free(K_3)$, then $H \in Free(K_4 - e)$. This relationship between VERTEX COLORING and MINIMUM VERTEX COVER implies the following conclusion.

Lemma 2. *The* VERTEX COLORING *problem is NP-complete in the class*

$$Free(\overline{K_{1,3}}, \overline{K_4}, \overline{K_4 - e}, \overline{C_4}, \overline{C_5}, \ldots, \overline{C_p}, \overline{L_0}, \overline{L_1}, \overline{L_2}, \ldots, \overline{L_p})$$

for each fixed value of p.

Let us denote the class of line graphs of graphs in $\mathcal{S}$ by $\mathcal{T}$. In other words,

$\mathcal{T}$ is the class of graphs in which every connected component is of the form $T_{i,j,k}$ represented in Fig. 2(b).

In terms of forbidden induced subgraphs the class $\mathcal{T}$ can be described as follows:

$$\mathcal{T} = Free(K_{1,3}, K_4, K_4 - e, C_4, C_5, \ldots, L_0, L_1, L_2, \ldots).$$

By denoting the class of complements of graphs in $\mathcal{T}$ by $\overline{\mathcal{T}}$ we conclude from Lemma 2 that $\overline{\mathcal{T}}$ is a limit class for VERTEX COLORING.

The importance of the notion of limit classes for the study of algorithmic graph problems on finitely defined classes of graphs is due to the following lemma.

Lemma 3. *If a finitely defined class of graphs contains a Π-limit class, then it is Π-tough.*

This "if" statement can be strengthened to an "if and only if" statement by replacing limit classes with *minimal* limit classes. A minimal (with respect to set inclusion) limit class was called in [1] a *boundary class*. This notion was first introduced with respect to the MAXIMUM INDEPENDENT SET problem [1] and then was extended to arbitrary algorithmic graph problems in [2,4]. With this extension, Lemma 3 can be strengthened as follows (see [2,4]).

Theorem 3. *A finitely defined class of graphs is Π-tough if and only if it contains a Π-boundary class.*

According to this theorem, a complete list of Π-boundary classes allows us to efficiently decide whether a finitely defined class of graphs is Π-tough or Π-easy. Unfortunately, up to date, no problem was provided with a complete list of boundary classes. However, even a partial information about boundary classes (i.e. information about specific boundary classes) gives a useful guide for the analysis of computational complexity of algorithmic graph problems on finitely defined classes of graphs. The first boundary class was discovered in [1], where the author proved minimality of the class $\mathcal{S}$ for the MAXIMUM INDEPENDENT SET problem. Also, in [2,3], it was shown that $\mathcal{S}$ and $\mathcal{T}$ are boundary classes for many other algorithmic graph problems. Some classes resembling $\mathcal{S}$ and $\mathcal{T}$ turned out to be boundary for VERTEX-k-COLORABILITY and for EDGE-k-COLORABILITY for all $k \geq 3$ [18,23,24]. The minimality of the class $\overline{\mathcal{T}}$ for VERTEX COLORING was shown in [25].

Below we report two more results about limit classes for VERTEX COLORING and related problems. The minimality of these classes is an open problem. However, even without solving this problem, the reported results can be used to conclude the NP-completeness of VERTEX COLORING in some classes of our interest. The following lemma was proved in [16].

Lemma 4. *For each fixed $k, p \geq 3$,* VERTEX-k-COLORABILITY *is NP-complete for graphs in the class* $Free(C_3, C_4, \ldots, C_p)$.

From this lemma we immediately conclude that for each fixed k, the class of forests is a limit class for VERTEX-k-COLORABILITY. Also, in [16], it was shown that EDGE-3-COLORABILITY is NP-complete for graphs of vertex degree at most 3 in the class $Free(C_3, C_4, \ldots, C_p)$ for each fixed value of p. Translating this result to the language of line graphs, we obtain the following conclusion.

Lemma 5. *For each fixed value of p,* VERTEX-3-COLORABILITY *is NP-complete in the class* $Free(K_{1,3}, K_4 - e, K_4, C_4, C_5, \ldots, C_p)$.

Therefore, the class of forests of vertex degree at most 3 is a limit class for EDGE-3-COLORABILITY.

Now we apply the above results to derive the NP-completeness of VERTEX COLORING in some classes defined by forbidden induced subgraphs with at most 4 vertices.

Theorem 4. *The* VERTEX COLORING *problem is NP-complete in the following classes:* $\mathcal{X}^1 = Free(K_{1,3}, C_4, K_4 - e, K_4)$, $\mathcal{X}^2 = Free(\overline{K_{1,3}}, \overline{C_4}, \overline{K_4 - e}, \overline{K_4})$, $\mathcal{X}^3 = Free(K_3, C_4)$.

Proof. For the first class, the result follows from Lemma 5, for the second, from Lemma 2, and for the third, from Lemma 4. □

3. Polynomial-time results

Since VERTEX COLORING is NP-complete in classes $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$, it is also NP-complete in their superclasses. Therefore, in order to obtain polynomial-time results for classes of unbounded clique-width, we study classes $\mathcal{X}^4, \mathcal{X}^5, \mathcal{X}^6, \mathcal{X}^7$ and their superclasses defined by forbidden induced subgraphs with at most 4 vertices.

Each of the proper superclasses of $\mathcal{X}^5, \mathcal{X}^6, \mathcal{X}^7$ is defined by forbidding exactly one of the following graphs: $K_4, O_4, C_4, 2K_2$. By Theorem 1, the problem is NP-complete in each of these classes. Therefore, we do not need to consider proper superclasses of $\mathcal{X}^5, \mathcal{X}^6, \mathcal{X}^7$.

Graphs in the class $\mathcal{X}^6$ have a simple structure [8,22], which leads to a polynomial-time solution to the VERTEX COLORING problem [26]. In what follows, we analyze complexity of the problem in other classes of the family in question.

3.1. The class $\mathcal{X}^5$

Theorem 5. *The* VERTEX COLORING *problem is polynomial-time solvable in the class* $Free(K_4, 2K_2)$.

Proof. The result follows from two facts. First, the class $Free(2K_2)$ is χ-bounded, i.e. the chromatic number of any graph in this class is bounded by a function of its clique number. More precisely, in [30] it was proved that if $G \in Free(2K_2)$, then $\chi(G) \leq \binom{\omega(G)+1}{2}$, where $\omega(G)$ is the clique number of G, i.e. the size of a maximum clique in G. Since for graphs in $Free(K_4, 2K_2)$ the clique number is at most 3, the chromatic number is at most 6.

The second important property of the class $Free(2K_2)$ is that n-vertex graphs in this class have at most n^2 maximal (with respect to set inclusion) independent sets [11]. Therefore, for each fixed k, VERTEX-k-COLORABILITY can be solved for graphs in $Free(2K_2)$ in polynomial time. Indeed, we may assume without loss of generality that in a proper coloring of $G \in Free(2K_2)$ one of the color classes is a maximal independent set. Therefore, VERTEX-k-COLORABILITY can be solved for G by solving VERTEX-$(k-1)$-COLORABILITY in the graph $G - S$ for each maximal independent set S in G. Since VERTEX-2-COLORABILITY is polynomial-time solvable, the result follows by induction. □

3.2. The class $\mathcal{X}^4$ and its superclasses

The VERTEX COLORING problem is known to be polynomial-time solvable in the class $Free(O_3)$, since it is solvable in the more general class $Free(P_3 + K_1)$. Therefore, it is solvable in the class $\mathcal{X}^4 = Free(O_3, 2K_2)$. Now let $\mathcal{X} = Free(\mathcal{Y})$ be a superclass of $\mathcal{X}^4$ defined by a set $\mathcal{Y}$ of forbidden induced subgraphs with 4 vertices. Each graph in $\mathcal{Y}$ different from $2K_2$ is obtained from O_3 by adding to it one more vertex. There are precisely 4 such extensions: O_4, $K_2 + 2K_1$, $P_3 + K_1$ and $K_{1,3}$. We assume that:

- $P_3 + K_1 \notin \mathcal{Y}$, since the problem is polynomial-time solvable in $Free(P_3 + K_1)$,
- $K_{1,3} \in \mathcal{Y}$, since otherwise $\mathcal{X}$ contains $\mathcal{X}^2$, in which case the problem is NP-complete by Theorem 4,
- $\mathcal{Y} \cap \{2K_2, K_2 + 2K_1, O_4\} \neq \emptyset$, since the problem is NP-complete in the class $Free(K_{1,3})$ (Theorem 1).

3.2.1. The class Free($K_{1,3}$, $2K_2$)

Let G be a graph in the class *Free*($K_{1,3}$, $2K_2$). Without loss of generality we will assume that G is connected, since the VERTEX COLORING problem admits an obvious reduction to connected graphs. For a subset $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U.

Let x be a vertex of maximum degree in G and V_1 its neighborhood. By V_2 we will denote the remaining vertices of the graph. Since G is $2K_2$-free, the graph $G[V_2]$ contains at most one nontrivial (i.e. of size more than 1) connected component. In other words, there is a subset $U \subseteq V_2$ such that $G[U]$ is connected and every vertex of $V_2 - U$ is isolated in $G[V_2]$.

Lemma 6. *The set U is a clique.*

Proof. Assume U is not a clique. Then $G[U]$ has a P_3 induced by three vertices a, b, c (in this order). Suppose V_1 contains a vertex y nonadjacent to b, then y must be adjacent to a (to avoid a $2K_2$ induced by x, y, a, b) and similarly y must be adjacent to c. But then x, y, a, c induce a $K_{1,3}$. This contradiction shows that every vertex of V_1 is adjacent to b. But then the degree of b is strictly greater than the degree of x, contradicting the choice of x. Therefore, U is a clique. □

Lemma 7. *If $|U| \geq 2$, then the graph $G_U = G[U \cup V_1 \cup \{x\}]$ belongs to the class Free(O_3) and $\chi(G_U) = \chi(G)$.*

Proof. Assume G_U contains three pairwise nonadjacent vertices a, b, c. Then, according to Lemma 6 and $K_{1,3}$-freeness of G_U, exactly one of these three vertices, say a, belongs to U and the remaining two vertices belong to V_1. Since $|U| \geq 2$, there exists a vertex $y \in U$ different from a, and this vertex must be adjacent both to b and to c (else a, y, b, x or a, y, c, x induce a $2K_2$). But then y, a, b, c induce a $K_{1,3}$. This contradiction shows that $G_U \in$ *Free*(O_3).

To prove the second part of the lemma, consider an optimal coloring ϕ of the graph G_U. This coloring can be extended to a coloring of G by assigning color $\phi(x)$ to every vertex in $V_2 - U$. This extension is a *proper* coloring of G, since the set $V_2 - U$ is independent, no vertex of this set has a neighbor in U and no vertex of V_1 uses color $\phi(x)$. Therefore, $\chi(G_U) = \chi(G)$. □

Lemma 8. *If $|U| \leq 1$, then $\chi(G) = \chi(G[V_1]) + 1$.*

Proof. Since x is adjacent to each vertex of V_1, we have $\chi(G) \geq \chi(G[V_1]) + 1$. On the other hand, since $|U| \leq 1$, the set $V_2 \cup \{x\}$ is independent, and hence any proper coloring of $G[V_1]$ can be extended to a proper coloring of G by assigning to the vertices of $V_2 \cup \{x\}$ any color not used in V_1. Therefore, $\chi(G) \leq \chi(G[V_1]) + 1$. Combining, we obtain $\chi(G) = \chi(G[V_1]) + 1$. □

Theorem 6. VERTEX COLORING *can be solved for graphs in Free($K_{1,3}$, $2K_2$) in polynomial time.*

Proof. From Lemmas 7 and 8 it follows that VERTEX COLORING can be reduced in polynomial time from graphs in *Free*($K_{1,3}$, $2K_2$) to graphs in *Free*($K_{1,3}$, $2K_2$, O_3). On the other hand, in the class *Free*(O_3) the problem admits a polynomial-time solution [19]. □

3.2.2. The class Free($K_{1,3}$, $K_2 + 2K_1$)

Lemma 9. *If a graph $G \in$ Free($K_{1,3}$, $K_2 + 2K_1$) contains an O_4, then G is edgeless.*

Proof. If $G \in$ *Free*($K_{1,3}$, $K_2 + 2K_1$) contains an O_4, then we extend this O_4 to a maximal (with respect to set inclusion) independent set S and assume there is a vertex $x \notin S$. Due to the maximality of S, x has at least one neighbor in S. If x has at most two neighbors in S, then it also has at least two non-neighbors in S (since $|S| \geq 4$). But then x, its neighbor and two non-neighbors induce a $K_2 + 2K_1$. If x has at least three neighbors in S, then these neighbors together with x induce a $K_{1,3}$. The contradiction in both cases shows that there are no vertices outside S, i.e. G has no edges. □

Corollary 1. VERTEX COLORING *in the class Free($K_{1,3}$, $K_2 + 2K_1$) is polynomially equivalent to the same problem in the class Free($K_{1,3}$, $K_2 + 2K_1$, O_4).*

3.3. Remaining classes

The above discussion reduces the analysis to the following three classes: *Free*($K_{1,3}$, O_4), *Free*($K_{1,3}$, $K_2 + 2K_1$, O_4) and *Free*(C_4, O_4). The first two of them generalize the class *Free*(O_3), where VERTEX COLORING is solvable in polynomial time. A solution to the problem for a graph $G \in$ *Free*(O_3) is based on solving the MAXIMUM MATCHING problem in the complement of G. Therefore, a polynomial algorithm for VERTEX COLORING in *Free*($K_{1,3}$, O_4) and *Free*($K_{1,3}$, $K_2 + 2K_1$, O_4) would generalize the maximum matching algorithm applied to K_3-free graphs. Determining whether such an algorithm exists is a challenging open problem. We leave it for future research. In the present paper, we restrict ourselves to weaker conclusions. In particular, in Section 4 we show that VERTEX COLORING is fixed-parameter tractable in all three classes where polynomial-time solvability of the problem remains an open question. Also, for two of these classes we give a 3/2 approximation polynomial algorithm in Section 5.

4. Fixed-parameter tractability of VERTEX COLORING in some finitely defined classes of graphs

Parameterized complexity is a recent branch of computational complexity theory that provides a framework for a refined analysis of hard algorithmic problems. A problem Π is said to be *fixed-parameter tractable* with respect to a parameter k if it can be solved for n-vertex graphs in time $f(k)n^{O(1)}$, where $f(k)$ is a function of the parameter k only (i.e. independent of n). Therefore, fixed-parameter tractability is a relaxation of polynomial-time solvability.

We parameterize VERTEX COLORING by the number of colors used in the solution. We say that VERTEX COLORING is fixed-parameter tractable if there is an algorithm that decides whether an n-vertex graph G is k-colorable in time $f(k)n^{O(1)}$. We denote the parameterized version of the problem by k-VERTEX COLORING. By $R(k, p)$ we denote the Ramsey number, i.e. the minimum number n such that every n-vertex graph contains either a clique of size k or an independent set of size p.

In the present section, we derive a number of general results on fixed-parameter tractability of VERTEX COLORING in finitely defined classes of graphs (i.e. we do not restrict ourselves to classes defined by forbidden induced subgraphs with at most 4 vertices). We start with an easy observation.

Theorem 7. *For each fixed p, the k-VERTEX COLORING problem is fixed-parameter tractable in the class Free*(O_p).

Proof. If a graph $G \in Free(O_p)$ contains at least $R(k+1, p)$ vertices, then it necessarily has a clique of size $k+1$, in which case it is not k-colorable. Otherwise, the number of vertices of G is at most $R(k+1, p)$, which is a function of k only (since p is a fixed constant). Therefore, the problem is fixed-parameter tractable in the class $Free(O_p)$. □

To extend this theorem to stronger results, let us quote a number of known results about the tree-width of a graph G (denoted $tw(G)$).

Lemma 10 ([20]). *There is a function f_1 such that for any graph G,*

$$tw(G) < f_1(G_1, G_2, \Delta(G)),$$

where G_1 and G_2 are induced subgraphs of G with maximum number of vertices that belong to $\mathcal{S}$ and to $\mathcal{T}$, respectively, and $\Delta(G)$ is the maximum vertex degree in G.

Lemma 11 ([6]). *There is a function f_2 such that* VERTEX COLORING *can be solved for graphs with n vertices and with a tree decomposition of width k in time $O(f_2(k)n)$.*

Lemma 12 ([5]). *Given a graph G, a tree decomposition of width $O(tw(G)\log(tw(G)))$ can be constructed in time polynomial in the number of vertices of G.*

Theorem 8. *For any graph $H \in \mathcal{T}$, the k-VERTEX COLORING problem is fixed-parameter tractable in the class Free*$(K_{1,3}, H)$.

Proof. Let H be a graph in $\mathcal{T}$ and G be a graph in $Free(K_{1,3}, H)$. If G contains a vertex x of degree $R(k, 3)$, then the neighborhood of x contains a clique of size k (since $G \in Free(K_{1,3})$), in which case $\chi(G) > k$, i.e. G is not k-colorable. Therefore, the problem reduces to graphs in $Free(K_{1,3}, H)$ of vertex degree at most $R(k, 3)$. Let us denote this subset of $Free(K_{1,3}, H)$ by $\mathcal{X}$. Since $K_{1,3} \in \mathcal{S}$ and $H \in \mathcal{T}$, we conclude by Lemma 10 that the tree-width of graphs in $\mathcal{X}$ is bounded by a function of k. Together with Lemmas 11 and 12 this implies that k-VERTEX COLORING is fixed-parameter tractable in the class $\mathcal{X}$ and hence in the class $Free(K_{1,3}, H)$. □

Theorem 9. *For each fixed p and s, the k-VERTEX COLORING problem is fixed-parameter tractable in the class Free*$(K_{p,p}, P_s)$.

Proof. It was recently shown in [7] that for each k, p and s there is a number $L = L(k, p, s)$ such that every graph with a (not necessarily induced) path of length L contains either K_k or $K_{p,p}$ or P_s as an induced subgraph. Therefore, if a graph $G \in Free(K_{p,p}, P_s)$ contains a path of length $L(k+1, p, s)$, then it necessarily contains a clique of size $k+1$, in which case it is not k-colorable. Otherwise, the length of a longest path is bounded by a function of k. This implies that the tree-width of G is bounded (see e.g. [16]). Therefore, the problem is fixed-parameter tractable by Lemmas 11 and 12. □

We conclude this section with one more result about fixed-parameter tractability of k-VERTEX COLORING in a finitely defined class of graphs.

Theorem 10. *The k-VERTEX COLORING problem is fixed-parameter tractable in the class Free*$(\overline{K_{1,3}}, \overline{K_4 - e})$.

Proof. The result follows from three facts. First, k-VERTEX COVER (a version of the MINIMUM VERTEX COVER problem parameterized by the solution size) is fixed-parameter tractable for arbitrary graphs. Second, k-VERTEX COVER for a graph $G \in Free(K_3)$ is equivalent to k-VERTEX COLORING in the complement of the line graph of G, as was discussed in Section 2. Third, the line graph of a graph from $Free(K_3)$ belongs to $Free(K_{1,3}, K_4 - e)$. □

Observe that by Theorem 4 VERTEX COLORING is NP-complete in the class $Free(\overline{K_{1,3}}, \overline{K_4 - e})$.

5. 3/2 approximation polynomial algorithm for VERTEX COLORING in the class $Free(K_{1,3}, O_4)$

As we mentioned earlier, the classes $Free(K_{1,3}, O_4)$ and $Free(K_{1,3}, K_2 + 2K_1, O_4)$ generalize the class $Free(O_3)$, where a solution to the VERTEX COLORING problem is based on solving the MAXIMUM MATCHING problem in the complement of the graph. We denote the size of a maximum matching in a graph G by $\mu(G)$ and show that by solving the maximum matching problem in the complement of a $(K_{1,3}, O_4)$-free graph G we find a coloring of G with at most $\frac{3}{2}\chi(G)$ colors. This provides a 3/2 approximation polynomial algorithm for VERTEX COLORING in both classes $Free(K_{1,3}, O_4)$ and $Free(K_{1,3}, K_2+2K_1, O_4)$. We start with the following useful observation.

Lemma 13. *For any graph G in the class $Free(K_{1,3}, O_4)$ there is an optimal coloring in which all color classes contain either at least two vertices or at most two vertices.*

Proof. Since G is O_4-free, in any coloring of G every color class consists of at most 3 vertices. Let $\mathcal{C}$ be an optimal coloring of G. Assume $\mathcal{C}$ contains both a color class $A = \{a\}$ of size 1 and a color class $B = \{b_1, b_2, b_3\}$ of size 3. Since G is claw-free, vertex a must have a non-neighbor $b_i \in B$. By moving b_i to A we obtain a new optimal coloring in which the number of color classes of size 1 and 3 is strictly less than that in $\mathcal{C}$. Applying this procedure as long as possible, we obtain an optimal coloring either with no color classes of size 1 or with no color classes of size 3, as required. □

Theorem 11. *For any graph $G \in Free(K_{1,3}, O_4)$ with n vertices, either $\chi(G) = n - \mu(\overline{G})$ or $\mu(\overline{G}) = \lfloor \frac{n}{2} \rfloor$.*

Proof. For any graph G the following inequality is obvious: $\chi(G) \le n - \mu(\overline{G})$. Moreover, this inequality becomes equality if and only if there is an optimal coloring of G in which all color classes contain at most two vertices.

Assume $\chi(G) \ne n - \mu(\overline{G})$. Then by Lemma 13 there is an optimal coloring $\mathcal{C}$ of G in which all color classes have size 2 or 3. Let k_2 be the number of color classes of size 2 and k_3 be the number of color classes of size 3 in $\mathcal{C}$. Then $2k_2 + 3k_3 = n$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be two arbitrary color classes of size 3 in $\mathcal{C}$. Since G is claw-free, vertex a_1 must have a non-neighbor $b_i \in B$. Therefore, $\{a_2, a_3\}$, $\{a_1, b_i\}$ and $B - \{b_i\}$ are independent sets and hence the union of k_3 color classes of size 3 contains $\lceil \frac{3k_3-1}{2} \rceil$ pairwise disjoint independent sets of size 2. Thus, the number of such sets in G is at least $\lceil \frac{3k_3-1}{2} \rceil + k_2$ and hence

$$\mu(\overline{G}) \ge \left\lceil \frac{3k_3 - 1}{2} \right\rceil + k_2 = \left\lceil \frac{n - 2k_2 - 1}{2} \right\rceil + k_2 = \left\lfloor \frac{n}{2} \right\rfloor.$$

Together with the obvious inequality $\mu(\overline{G}) \le \lfloor \frac{n}{2} \rfloor$ we conclude that $\mu(\overline{G}) = \lfloor \frac{n}{2} \rfloor$, as required. □

One important corollary from Theorem 11 is that it provides a sufficient condition for polynomial-time solvability of VERTEX COLORING in the class $Free(K_{1,3}, O_4)$.

Corollary 2. *Let G be a graph with n vertices in the class $Free(K_{1,3}, O_4)$. If $\mu(\overline{G}) \ne \lfloor \frac{n}{2} \rfloor$, then $\chi(G) = n - \mu(\overline{G})$.*

Also, combining Theorem 11 with the obvious inequality $\chi(G) \ge n/3$ which is valid for any O_4-free graph, we obtain the following conclusion.

Corollary 3. *Let G be a graph with n vertices in the class $Free(K_{1,3}, O_4)$. Then*

$$\frac{2(n - \mu(\overline{G})) - 1}{3} \le \chi(G).$$

This corollary shows that by solving the maximum matching problem in the complement of G we find a coloring of G with at most $\frac{3}{2}\chi(G)$ colors, i.e. this solution provides a 3/2 approximation polynomial algorithm for VERTEX COLORING in the class $Free(K_{1,3}, O_4)$.

6. Conclusion

In this paper, we studied computational complexity of VERTEX COLORING in classes of graphs defined by forbidden induced subgraphs with at most 4 vertices. For all but three classes in this family we showed either NP-completeness or polynomial-time solvability of the problem. For the remaining three classes we proved fixed-parameter tractability. Also, for two of them we gave a 3/2 approximation polynomial algorithm.

The obvious open question is clarifying the computational status of the problem in the following three classes: $Free(K_{1,3}, O_4)$, $Free(K_{1,3}, K_2 + 2K_1, O_4)$ and $Free(C_4, O_4)$. We conjecture that for graphs in $Free(C_4, O_4)$ the problem can be solved in polynomial time. For the other two classes, even conjecturing is not so easy. On the one hand, both of them extend the class $Free(O_3)$ and hence a polynomial-time solution for these two classes can be developed only by a non-trivial generalization of the maximum matching algorithm. On the other hand, if the problem is NP-complete in these two classes, they must contain a new boundary class for the problem, and presently nothing suggests any idea about the structure of such a class. Resolving this issue is a very challenging task.

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