Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 341 (2008) 613–625
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A pointwise selection principle for metric semigroup valued functions

Vyacheslav V. Chistyakov [a,*], Caterina Maniscalco [b]

[a] *Department of Applied Mathematics, State University Higher School of Economics, Bol'shaya Pechërskaya street 25, Nizhny Novgorod 603600, Russia*

[b] *Dipartimento di Matematica e Applicazioni, Universitá di Palermo, Via Archirafi 34, 90123 Palermo, Italy*

Received 4 May 2007

Available online 7 November 2007

Submitted by B. Bongiorno

Abstract

Let $\emptyset \neq T \subset \mathbb{R}$, $(X, d, +)$ be an additive commutative semigroup with metric d satisfying $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in X$, and X^T the set of all functions from T into X. If $n \in \mathbb{N}$ and $f, g \in X^T$, we set $\nu(n, f, g, T) = \sup \sum_{i=1}^{n} d(f(t_i)+g(s_i), g(t_i) + f(s_i))$, where the supremum is taken over all numbers $s_1, \ldots, s_n, t_1, \ldots, t_n$ from T such that $s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \cdots \leqslant s_n \leqslant t_n$. We prove the following pointwise selection theorem: *If a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subset X^T$ is such that the closure in X of the set $\{f_j(t)\}_{j\in\mathbb{N}}$ is compact for each $t \in T$, and*

$$\lim_{n\to\infty}\left(\frac{1}{n}\lim_{N\to\infty}\sup_{j,k\geqslant N,\, j\neq k}\nu(n, f_j, f_k, T)\right) = 0,$$

then it contains a subsequence which converges pointwise on T. We show by examples that this result is sharp and present two of its variants.

Keywords: Joint modulus of variation; Selection principle; Pointwise convergence; Double sequence; Metric semigroup

1. Introduction and main result

Given a nonempty subset T of the set $\mathbb{R}$ of real numbers and a metric space (X, d) with metric d, let X^T be the set of all functions $f : T \to X$ mapping T into X. We are interested in finding conditions on the sequence of functions $\{f_j\} \equiv \{f_j\}_{j=1}^{\infty} \subset X^T$, under which $\{f_j\}$ admits a pointwise convergent subsequence. Recall that $\{f_j\}$ *converges pointwise* (or *everywhere*) *on* T to a function $f \in X^T$ provided $d(f_j(t), f(t)) \to 0$ as $j \to \infty$ for all $t \in T$. If $T = [a, b]$ is an interval and $X = \mathbb{R}$, the classical conditions on $\{f_j\}$ are given by the famous *Helly Selection Theorem* [17]: $\{f_j\}$ is uniformly bounded and each f_j is a monotone function.

* Corresponding author.
E-mail addresses: czeslaw@mail.ru (V.V. Chistyakov), maniscal@math.unipa.it (C. Maniscalco).

0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.10.055

There is a number of generalizations of the Helly Theorem for functions of a real variable: [1,2,4–10], [13, Part III, Section 2], [14,15,18,20,23–25] and references therein. The problem under consideration is motivated by the numerous applications in Analysis of the Helly Theorem as well as its generalizations, e.g., [1,4–6,11–13]. Usually these generalizations rely on the boundedness of certain types of variations for functions from the sequence $\{f_j\}$, which consists of regulated functions (i.e., those having finite left and right limits at all points). However, of interest are conditions having nothing to do with the boundedness of variations or regulated functions as presented, e.g., in [8,18] and [23]. In order to recall one of these conditions, to be generalized in the sequel, we need a definition.

Given $n \in \mathbb{N}$, $f \in X^T$ and $\emptyset \neq E \subset T$, we set

$$\nu(n, f, E) = \sup \sum_{i=1}^{n} d\big(f(t_i), f(s_i)\big) \tag{1}$$

where the supremum is taken over all $2n$ numbers $\{s_i\}_{i=1}^n$, $\{t_i\}_{i=1}^n \subset E$ such that $s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \cdots \leqslant s_{n-1} \leqslant t_{n-1} \leqslant s_n \leqslant t_n$. The sequence $\{\nu(n, f, E)\}_{n=1}^\infty \subset [0, \infty]$ is called the *modulus of variation* of f on E—this notion was introduced by Chanturiya in [3] for $E = T = [a, b]$ and $X = \mathbb{R}$ (the general case was considered in [8]). We note that $\nu(1, f, E)$ is just the *diameter of the set* $f(E)$ (= the image of E under f), also known as the *oscillation* of f on E. Clearly, $\nu(n, f, E)$ is finite for all $n \in \mathbb{N}$ if and only if $\nu(1, f, E) < \infty$ (i.e., when f is bounded on E) and, moreover, $\nu(1, f, E) \leqslant \nu(n, f, E) \leqslant n\nu(1, f, E)$ (for more properties of the modulus of variation see [7,8] and [10]).

A sequence $\{f_j\} \subset X^T$ is said to be *pointwise precompact* (on T) provided the sequence $\{f_j(t)\} \equiv \{f_j(t)\}_{j=1}^\infty$ is precompact (i.e., its closure in X is compact) for all $t \in T$. Given a sequence $\mu : \mathbb{N} \to \mathbb{R}$, the condition $\mu(n)/n \to 0$ as $n \to \infty$ will be written as $\mu(n) = o(n)$ (in E. Landau's notation).

The following is a *pointwise selection principle* for metric space valued functions of a real variable in terms of the modulus of variation [8, Theorem 1].

Theorem A. *Let $\emptyset \neq T \subset \mathbb{R}$ and (X, d) be a metric space. If $\{f_j\} \subset X^T$ is a pointwise precompact sequence satisfying*

$$\mu(n) \equiv \limsup_{j\to\infty} \nu(n, f_j, T) = o(n), \tag{2}$$

then it contains a subsequence which converges pointwise on T to a function $f \in X^T$ such that $\nu(n, f, T) \leqslant \mu(n)$, $n \in \mathbb{N}$.

It is shown in [7–10] that this theorem and its more general counterparts contain as particular cases many Helly-type selection theorems—actually, all those from references above, except [14] (where functions between linearly ordered sets were treated) and [18] and [23] (which were shown in [19] to be independent); for more details see Remark 4 in Section 4.

The aim of this paper is to show that if the metric space (X, d) is equipped with an additional algebraic structure, namely, the addition operation, then condition (2) in Theorem A can be weakened. In order to present our main result in this direction (Theorem 1 below), we review some more definitions.

In what follows the triple $(X, d, +)$ is a *metric semigroup* [6, Section 4], that is, (X, d) is a metric space with metric d, $(X, +)$ is an Abelian semigroup with the addition operation $+$ and d is translation invariant in the sense that $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$. Given $x, y, u, v \in X$, we have

$$d(x, y) \leqslant d(x + u, y + v) + d(u, v), \tag{3}$$

$$d(x + u, y + v) \leqslant d(x, y) + d(u, v). \tag{4}$$

In particular, (4) implies that the addition operation $(x, y) \mapsto x + y$ is a continuous mapping from $X \times X$ into X.

Given $n \in \mathbb{N}$, $f, g \in X^T$ and $\emptyset \neq E \subset T$, we set

$$\nu(n, f, g, E) = \sup \sum_{i=1}^{n} d\big(f(t_i) + g(s_i), g(t_i) + f(s_i)\big) \tag{5}$$

where the supremum is taken in the same manner as in (1). The sequence $\{\nu(n, f, g, E)\}_{n=1}^\infty \subset [0, \infty]$ will be termed the *joint modulus of variation* of f and g on E. We note that (5) is symmetric in f and g, it is equal to zero if

$f = g$, and it is just $\nu(n, f - g, E)$ if $(X, \|\cdot\|)$ is a normed vector space with the generated metric $d(x, y) = \|x - y\|$, $x, y \in X$. Also, if $f, g \in X^T$ are bounded on E, then (5) is finite for all $n \in \mathbb{N}$, for, by virtue of (4) and (5), we have

$$\nu(n, f, g, E) \leqslant \nu(n, f, E) + \nu(n, g, E). \tag{6}$$

More properties of the joint modulus of variation are presented in Lemma 1 below.

Throughout the paper we shall be concerned with double sequences $\alpha_{j,k} \in \mathbb{R}$ for $j, k \in \mathbb{N}$ having the property that $\alpha_{j,j} = 0$ for all $j \in \mathbb{N}$ (see, e.g., condition (7)). Such a sequence is said to be *convergent* to a number $l \in \mathbb{R}$, in symbols, $\lim_{j,k\to\infty} \alpha_{j,k} = l$, provided for each $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $\alpha_{j,k} \in [l - \varepsilon, l + \varepsilon]$ for all $j \geqslant N$ and $k \geqslant N$ with $j \neq k$. Also, we set

$$\limsup_{j,k\to\infty} \alpha_{j,k} = \lim_{N\to\infty} \sup\{\alpha_{j,k}\colon\ j \geqslant N,\ k \geqslant N,\ j \neq k\}.$$

The main result of the paper is the following *pointwise selection principle* for metric semigroup valued functions of a real variable in terms of the joint modulus of variation.

Theorem 1. *Let $\emptyset \neq T \subset \mathbb{R}$ and $(X, d, +)$ be a metric semigroup. Suppose that $\{f_j\} \subset X^T$ is a pointwise precompact sequence of functions such that*

$$\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T) = o(n). \tag{7}$$

Then $\{f_j\}$ contains a subsequence which converges pointwise on T.

This theorem will be proved in the next two sections. Now we note that for functions with values in a metric semigroup condition (2) implies condition (7): in fact, it follows from inequality (6) that

$$\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T) \leqslant 2 \limsup_{j\to\infty} \nu(n, f_j, T).$$

Therefore Theorem 1 extends the class of sequences having pointwise convergent subsequences, but we no longer can infer that the pointwise limits f of these subsequences satisfy regularity conditions such as $\nu(n, f, T) = o(n)$ from Theorem A (see Examples 4 and 2 in Section 4). So, Theorem 1 may be considered as an "irregular" version of Theorem A. In the proof of Theorem 1 we apply the technique similar to that used in the proof of Theorem A (cf. [8]); however, there is a significant difference: instead of the Helly Selection Theorem (which is inapplicable) in Step 2 we apply the Ramsey Theorem from formal logic to double sequences. In this respect Theorem 1 is not a consequence of and is not equivalent to the Helly Theorem. The idea to apply Ramsey's Theorem in the context of pointwise selection principles has appeared in [23] and later on has been extended in [18]. Our application of Ramsey's Theorem and the resulting Theorem 1 are quite different from those exposed in both of these papers (see also Remark 4 in Section 4).

The paper is organized as follows. In Section 2 we prove Theorem 1, except Step 2, present two corollaries and comment on the necessity of condition (7). Section 3 is devoted to the proof of Step 2. Since, at least at first sight, condition (7) may look cumbersome and somewhat involved (especially as it is written in the Abstract), in Section 4 we show by several examples that condition (7) can be effectively verified and that all assumptions in Theorem 1 are sharp. Finally, in Section 5 we give two variants of our selection principle for the almost everywhere convergence as well as for functions with values in a reflexive separable Banach space.

2. Proof of the main result

The properties of the joint modulus of variation needed in the proof of Theorem 1 are gathered in the following Lemma 1—they resemble the corresponding properties of the modulus of variation (1) presented in [7, Lemma 1] and [8, Lemma 2], and so, their immediate proofs are omitted.

Lemma 1. *Given $n, m \in \mathbb{N}$, $f, g \in X^T$ and $\emptyset \neq E \subset T$, we have*:

(a) $\nu(n + m, f, g, E) \leqslant \nu(n, f, g, E) + \nu(m, f, g, E)$;
(b) $\nu(n, f, g, E) \leqslant \nu(n + 1, f, g, E)$;

(c) $\nu(n, f, g, E') \leqslant \nu(n, f, g, E)$ *if* $\emptyset \neq E' \subset E$;
(d) $d(f(t)+g(s), g(t)+f(s)) + \nu(n, f, g, (-\infty, s] \cap E) \leqslant \nu(n+1, f, g, (-\infty, t] \cap E)$ *if* $s, t \in E$ *and* $s \leqslant t$;
(e) $\nu(n+1, f, g, E) \leqslant \nu(n, f, g, E) + \frac{\nu(n+1, f, g, E)}{n+1}$.

We note that if f and g are bounded functions on E (or $\nu(1, f, g, E) < \infty$), the inequality in Lemma 1(e) is equivalent to

$$\frac{\nu(n+1, f, g, E)}{n+1} \leqslant \frac{\nu(n, f, g, E)}{n} \tag{8}$$

where, by virtue of Lemma 1(a), the right-hand side is $\leqslant \nu(1, f, g, E)$ (cf. also (6)), and so, the limit $\lim_{n\to\infty} \nu(n, f, g, E)/n$ exists in $\mathbb{R}^+ = [0, \infty)$.

Moreover, under the conditions of Theorem 1, if $\mu(n)$ designates the left-hand side of (7), then it is immediate from (8) that $\{\mu(n)/n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ is a nonincreasing sequence, and so, assumption (7) in Theorem 1 is quite natural.

Now we are in a position to prove our main result.

Proof of Theorem 1. If there are only finitely many distinct functions in $\{f_j\}$, we may choose a constant subsequence of $\{f_j\}$, and we are done. Otherwise, picking a subsequence of $\{f_j\}$ if necessary, we may assume that all functions in $\{f_j\}$ are *distinct*. Also, if T is at most countable, then, since the set $\{f_j(t)\}$ is precompact in X for all $t \in T$, we may apply the standard diagonal process to extract a subsequence of $\{f_j\}$ which converges pointwise on T. So we assume that T is *uncountable*. The rest of the proof is divided into four steps for clarity.

Step 1. There exists a subsequence of $\{f_j\}$, again denoted by $\{f_j\}$, and a sequence $\gamma : \mathbb{N} \to \mathbb{R}^+$ such that

$$\nu(n, f_j, f_k, T) \leqslant \gamma(n) \quad \text{for all } n, j, k \in \mathbb{N}. \tag{9}$$

In fact, condition (7) implies that its left-hand side, denoted by $\mu(n)$, is finite for all $n \in \mathbb{N}$: for some $n_0 \in \mathbb{N}$ we have $\mu(n) \leqslant n$ if $n \geqslant n_0$ and, by virtue of Lemma 1(b), $\mu(n) \leqslant n_0$ if $1 \leqslant n \leqslant n_0$. It follows that there is $N_0 \in \mathbb{N}$ such that if $j \geqslant N_0$, $k \geqslant N_0$ and $j \neq k$, then $\nu(n, f_j, f_k, T) \leqslant \mu(n) + 1 \leqslant n + 1$ for $n \geqslant n_0$ and, again by Lemma 1(b), $\nu(n, f_j, f_k, T) \leqslant n_0 + 1$ for $1 \leqslant n \leqslant n_0$. In order to get (9), it suffices to denote the subsequence $\{f_{j+N_0-1}\}_{j=1}^{\infty}$ of $\{f_j\}_{j=1}^{\infty}$ again by $\{f_j\}$ (so that condition (7) is still satisfied for $\{f_j\}$) and define γ by $\gamma(n) = n + 1$ if $n \geqslant n_0$ and $\gamma(n) = n_0 + 1$ if $1 \leqslant n \leqslant n_0$.

Step 2. There is a subsequence of $\{f_j\}$ satisfying (9), again denoted by $\{f_j\}$, and for each $n \in \mathbb{N}$ there exists a nondecreasing function $\nu_n : T \to [0, \gamma(n)]$ such that

$$\lim_{j,k\to\infty} \nu\big(n, f_j, f_k, (-\infty, t] \cap T\big) = \nu_n(t) \quad \text{for all } n \in \mathbb{N} \text{ and } t \in T. \tag{10}$$

Since the proof of (10) is unexpectedly lengthy and uses certain ideas from formal logic [21], we postpone it until the next section. Now, taking into account (10), we proceed as follows.

Step 3. Let Q denote an at most countable dense subset of T, and so, $Q \subset T \subset \overline{Q}$ where $\overline{Q}$ is the closure of Q in $\mathbb{R}$. We note that Q contains all points of T which are not limit points for T. By virtue of the monotonicity of each function ν_n from Step 2, the set $Q_n \subset T$ of its points of discontinuity is at most countable, and so, the set $S = Q \cup \bigcup_{n=1}^{\infty} Q_n$ is an at most countable dense subset of T having the property:

$$\text{for each } n \in \mathbb{N} \text{ the function } \nu_n \text{ is continuous on } T \setminus S. \tag{11}$$

Since the set $\{f_j(t)\}$ is precompact in X for all $t \in T$ and $S \subset T$ is at most countable, we may assume with no loss of generality (applying the standard diagonal process and passing to a subsequence of $\{f_j\}$ if necessary) that, for all $s \in S$, $f_j(s)$ converges in X as $j \to \infty$ to a point of X denoted by $f(s)$.

Step 4. Now we are going to show that, given $t \in T \setminus S$, the sequence $\{f_j(t)\}$ is Cauchy. If this is already done, the precompactness of $\{f_j(t)\}$ would imply that it is convergent in X as $j \to \infty$ to a point of X denoted by $f(t)$. This, the argument at the end of Step 3 and equality $T = S \cup (T \setminus S)$ would complete the proof of Theorem 1.

Let us fix $\varepsilon > 0$ arbitrarily. By the definition of $\mu(n)$ in Step 1 and condition (7), we choose and fix a number $n = n(\varepsilon) \in \mathbb{N}$, depending only on ε, such that $\mu(n+1) \leqslant \varepsilon(n+1)$. Because $\{f_j\}$ is a subsequence of the original sequence $\{f_j\}$, we get $\limsup_{j,k\to\infty} \nu(n+1, f_j, f_k, T) \leqslant \mu(n+1)$, which implies the existence of a number $J_0 = J_0(\varepsilon) \in \mathbb{N}$, depending on ε and n and hence only on ε, such that $\nu(n+1, f_j, f_k, T) \leqslant \mu(n+1) + \varepsilon$ for all $j \geqslant J_0$ and $k \geqslant J_0$

with $j \neq k$. By the definition of S and (11), the point t is a limit point for T and a point of continuity of ν_n, and so, the density of S in T yields a point $s = s(\varepsilon) \in S$, depending on ε, t and n, such that $|\nu_n(t) - \nu_n(s)| \leqslant \varepsilon$. Applying (10) we find a number $J_1 = J_1(\varepsilon) \in \mathbb{N}$, depending on ε, n, t and s, such that if $j \geqslant J_1$, $k \geqslant J_1$ and $j \neq k$, then

$$\bigl|\nu\bigl(n, f_j, f_k, (-\infty, t] \cap T\bigr) - \nu_n(t)\bigr| \leqslant \varepsilon \quad \text{and}$$
$$\bigl|\nu\bigl(n, f_j, f_k, (-\infty, s] \cap T\bigr) - \nu_n(s)\bigr| \leqslant \varepsilon.$$

Being convergent, the sequence $\{f_j(s)\}$ is Cauchy, and so, there exists a number $J_2 = J_2(\varepsilon) \in \mathbb{N}$, depending on ε and s, such that $d(f_j(s), f_k(s)) \leqslant \varepsilon$ for all $j \geqslant J_2$ and $k \geqslant J_2$. Assuming that $s < t$ (the case $t < s$ is treated similarly), applying (3) and items (d), (e) and (c) of Lemma 1 and noting that the number $J = \max\{J_0, J_1, J_2\}$ depends only on ε, we get, for all $j \geqslant J$ and $k \geqslant J$ with $j \neq k$,

$$\begin{aligned} d\bigl(f_j(t), f_k(t)\bigr) &\leqslant d\bigl(f_j(t) + f_k(s), f_k(t) + f_j(s)\bigr) + d\bigl(f_j(s), f_k(s)\bigr) \\ &\leqslant \nu\bigl(n+1, f_j, f_k, (-\infty, t] \cap T\bigr) - \nu\bigl(n, f_j, f_k, (-\infty, s] \cap T\bigr) + \varepsilon \\ &\leqslant \nu\bigl(n+1, f_j, f_k, (-\infty, t] \cap T\bigr) - \nu\bigl(n, f_j, f_k, (-\infty, t] \cap T\bigr) \\ &\quad + \bigl|\nu\bigl(n, f_j, f_k, (-\infty, t] \cap T\bigr) - \nu_n(t)\bigr| + \bigl|\nu_n(t) - \nu_n(s)\bigr| \\ &\quad + \bigl|\nu_n(s) - \nu\bigl(n, f_j, f_k, (-\infty, s] \cap T\bigr)\bigr| + \varepsilon \\ &\leqslant \frac{\nu(n+1, f_j, f_k, (-\infty, t] \cap T)}{n+1} + \varepsilon + \varepsilon + \varepsilon + \varepsilon \\ &\leqslant \frac{\nu(n+1, f_j, f_k, T)}{n+1} + 4\varepsilon \\ &\leqslant \frac{\mu(n+1)}{n+1} + \frac{\varepsilon}{n+1} + 4\varepsilon \leqslant 6\varepsilon, \end{aligned}$$

whence the Cauchy property of $\{f_j(t)\}$ follows. □

Remark 1. If $(X, \|\cdot\|)$ is a finite-dimensional normed vector space, the condition of precompactness of sets $\{f_j(t)\}$ at all points $t \in T$ in Theorem 1 can be lightened to the condition $\sup_{j\in\mathbb{N}} \|f_j(t_0)\| = C_0 < \infty$ for some $t_0 \in T$: in fact, by virtue of (9) we have $\nu(1, f_j, f_1, T) \leqslant \gamma(1)$, and so

$$\begin{aligned} \bigl\|f_j(t)\bigr\| &\leqslant \bigl\|(f_j - f_1)(t) - (f_j - f_1)(t_0)\bigr\| + \bigl\|f_1(t)\bigr\| + \bigl\|f_1(t_0)\bigr\| + \bigl\|f_j(t_0)\bigr\| \\ &\leqslant \gamma(1) + \bigl\|f_1(t)\bigr\| + 2C_0, \quad t \in T,\ j \in \mathbb{N}. \end{aligned} \tag{12}$$

If $\dim X = \infty$, the precompactness of $\{f_j(t)\}$ at all $t \in T$ cannot be replaced by the boundedness and closedness even at a single point t_0 (cf. [8, Section 3, Example 1]).

Remark 2. If a sequence $\{f_j\} \subset X^T$ converges uniformly on T to a function $f \in X^T$ (i.e., $\sup_{t\in T} d(f_j(t), f(t)) \to 0$ as $j \to \infty$), then condition (7) is *necessary*, namely,

$$\lim_{j,k\to\infty} \nu(n, f_j, f_k, T) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

This is a consequence of the following straightforward inequality (cf. (4)):

$$\nu(n, f_j, f_k, T) \leqslant 2n\Bigl(\sup_{t\in T} d\bigl(f_j(t), f(t)\bigr) + \sup_{s\in T} d\bigl(f_k(s), f(s)\bigr)\Bigr).$$

Condition (7) is not necessary for the pointwise convergence as is shown in Example 1 from Section 4; however, it is "almost" necessary as can be seen from the next remark.

Remark 3. Let $\emptyset \neq T \subset \mathbb{R}$ be a Lebesgue measurable set having finite measure. If $\{f_j\} \subset X^T$ is a sequence of measurable functions which converges pointwise (or almost everywhere) on T, then, by Egorov's theorem, for each $\varepsilon > 0$ there exists a Lebesgue measurable set $E(\varepsilon) \subset T$ whose Lebesgue measure is $\leqslant \varepsilon$ such that $\{f_j\}$ converges uniformly on $T \setminus E(\varepsilon)$. Applying the observation of Remark 2 with T replaced by $T \setminus E(\varepsilon)$, we get:

$$\lim_{j,k\to\infty} \nu\bigl(n, f_j, f_k, T \setminus E(\varepsilon)\bigr) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Under the assumptions of Theorem 1 we have the following corollary, which is established by applying the standard diagonal process:

Corollary 1. *If $\{f_j\} \subset X^T$ is a sequence such that*

$$\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T \setminus E) = o(n) \quad \textit{for an at most countable } E \subset T$$

or

$$\limsup_{j,k\to\infty} \nu\big(n, f_j, f_k, T \cap [s,t]\big) = o(n) \quad \textit{for all } s,t \in T,\ s \leqslant t,$$

then $\{f_j\}$ contains a subsequence which converges pointwise on T.

In order to formulate one more corollary (and a particular case) of Theorem 1, we introduce two notions related to generalized variations: φ-variation in the sense of N. Wiener and L.C. Young (e.g., [20]) and Λ-variation in the sense of D. Waterman [25].

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous function vanishing only at zero and such that $\varphi(\rho) \to \infty$ as $\rho \to \infty$, and $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. Given $f, g \in X^T$, we set

$$V_\varphi(f,g,T) = \sup \sum_{i=1}^{n} \varphi\big(d\big(f(t_i)+g(s_i), g(t_i)+f(s_i)\big)\big),$$

where the supremum is taken over all $n \in \mathbb{N}$ and $\{s_i, t_i\}_{i=1}^n \subset T$ such that $s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2 \leqslant \cdots \leqslant s_n \leqslant t_n$, and

$$V_\Lambda(f,g,T) = \sup \sum_{i=1}^{n} \frac{d(f(t_i)+g(s_i), g(t_i)+f(s_i))}{\lambda_{\omega(i)}},$$

where the supremum is taken over all n and $\{s_i, t_i\}_{i=1}^n$ as above and all permutations $\omega : \{1,\dots,n\} \to \{1,\dots,n\}$. If g is a constant function, then the quantity $V_\varphi(f,g,T)$ is the usual Wiener–Young φ-variation and $V_\Lambda(f,g,T)$ is the usual Waterman Λ-variation of f on T, and if $\varphi(\rho) = \rho$ and $\lambda_i \equiv 1$, these quantities give the classical notion of the Jordan variation (for more details in this context see, e.g., [8]).

If $n \in \mathbb{N}$, the following estimates hold for the joint modulus of variation (5) in terms of the two quantities above (their proofs are the same as the proofs of the corresponding estimates in [7, p. 27], [8, p. 612 and Example 7] and [16, Theorem 11.17]): if φ is convex, then it admits the continuous inverse φ^{-1} and

$$\nu(n,f,g,T) \leqslant n\varphi^{-1}\left(\frac{1}{n} V_\varphi(f,g,T)\right);$$

if φ is not necessarily convex, then

$$\nu(n,f,g,T) \leqslant \sup\left\{\sum_{i=1}^{n} \varphi_+^{-1}(r_i)\text{: } \{r_i\}_{i=1}^n \subset \mathbb{R}^+ \text{ and } \sum_{i=1}^{n} r_i \leqslant V_\varphi(f,g,T)\right\},$$

where $\varphi_+^{-1}(r) = \max\{\rho \in \mathbb{R}^+\text{: } \varphi(\rho) = r\}$ for $r \in \mathbb{R}^+$, and

$$\nu(n,f,g,T) \leqslant \frac{n}{\sum_{i=1}^{n} 1/\lambda_i} V_\Lambda(f,g,T).$$

Taking into account the above three estimates for the joint modulus of variation and noting that their right-hand sides are $o(n)$ provided the values $V_\varphi(f,g,T)$ and $V_\Lambda(f,g,T)$ are finite, under the assumptions of Theorem 1 we have the following

Corollary 2. *If a sequence $\{f_j\} \subset X^T$ is such that*

$$\limsup_{j,k\to\infty} V_*(f_j, f_k, T) < \infty \quad \textit{with } * = \varphi \quad \textit{or } \Lambda,$$

then condition (7) *holds, and so, $\{f_j\}$ contains a subsequence which converges pointwise on T.*

Similar corollaries can be readily given when more general generalized variations are involved; for more details we refer to [7, Section 6], [9] and [10, Section 6].

3. Proof of claim (10)

In order to prove assertion (10), we need Ramsey's logical theorem [21, Theorem A] which, for the sake of convenience, is recalled below as Theorem B.

Given a nonempty set Γ, $n \in \mathbb{N}$ and an injective function $\sigma : \{1, \dots, n\} \to \Gamma$, the set $\{\sigma(1), \dots, \sigma(n)\}$ is called an *n-combination* of elements of Γ (note that an n-combination may be generated by $n!$ different injective functions). Let Γ^{*n} denote the family of all n-combinations of elements of Γ.

Theorem B. *Suppose Γ is an infinite set, $n, m \in \mathbb{N}$, and $\Gamma^{*n} = \bigcup_{i=1}^{m} C_i$ is a disjoint union of its m nonempty subsets C_i. Then, under the Axiom of Choice, Γ contains an infinite subset Δ such that $\Delta^{*n} \subset C_{i_0}$ for some $i_0 \in \{1, \dots, m\}$.*

This theorem will be applied several times with Γ a subsequence of $\{f_j\}$ and $n = m = 2$.

Proof of (10) will itself be subdivided into steps (i)–(iv).

(i) Let us show that given $n \in \mathbb{N}$ and $t \in T$, there exists a subsequence $\{f_j^{(n,t)}\}$ of $\{f_j\}$, depending on n and t, such that the limit

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n,t)}, f_k^{(n,t)}, (-\infty, t] \cap T\big) \quad \text{exists in } \big[0, \gamma(n)\big]. \tag{14}$$

Let c_0 be the middle point of the interval $[0, \gamma(n)]$ and (cf. (9) and Lemma 1(c)) let C_1^1 be the set of those pairs $\{f_j, f_k\}$ with $j, k \in \mathbb{N}$, $j \neq k$, for which

$$\nu\big(n, f_j, f_k, (-\infty, t] \cap T\big) \in [0, c_0), \tag{15}$$

and C_2^1—the set of those $\{f_j, f_k\}$ with $j, k \in \mathbb{N}$, $j \neq k$, for which the quantity on the left in the inclusion (15) belongs to the interval $[c_0, \gamma(n)]$. If C_1^1 and C_2^1 are nonempty, they are disjoint, and so, by Theorem B, there exists a subsequence $\{f_j^1\}$ of $\{f_j\}$ such that either (i$_1$) $\{f_j^1, f_k^1\} \in C_1^1$ for all $j, k \in \mathbb{N}$, $j \neq k$, or (ii$_1$) $\{f_j^1, f_k^1\} \in C_2^1$ for all $j, k \in \mathbb{N}$, $j \neq k$. If $C_1^1 \neq \emptyset$ and (i$_1$) holds, or if $C_2^1 = \emptyset$, we set $[a_1, b_1] = [0, c_0]$, while if $C_2^1 \neq \emptyset$ and (ii$_1$) holds, or if $C_1^1 = \emptyset$, we set $[a_1, b_1] = [c_0, \gamma(n)]$.

Inductively, if $p \in \mathbb{N}$, $p \geqslant 2$, and a subsequence $\{f_j^{p-1}\}_{j=1}^{\infty}$ of $\{f_j\}$ and an interval $[a_{p-1}, b_{p-1}] \subset [0, \gamma(n)]$ are already chosen, we let c_{p-1} be the middle point of $[a_{p-1}, b_{p-1}]$ and C_1^p be the set of those pairs $\{f_j^{p-1}, f_k^{p-1}\}$ with $j, k \in \mathbb{N}$, $j \neq k$, for which

$$\nu\big(n, f_j^{p-1}, f_k^{p-1}, (-\infty, t] \cap T\big) \in [a_{p-1}, c_{p-1}), \tag{16}$$

and C_2^p—the set of those $\{f_j^{p-1}, f_k^{p-1}\}$ with $j, k \in \mathbb{N}$, $j \neq k$, for which the quantity on the left in the inclusion (16) belongs to $[c_{p-1}, b_{p-1}]$. If the sets C_1^p and C_2^p are nonempty, they are disjoint, and so, applying Theorem B, we obtain a subsequence $\{f_j^p\}_{j=1}^{\infty}$ of $\{f_j^{p-1}\}$ such that either (i$_p$) $\{f_j^p, f_k^p\} \in C_1^p$ for all $j, k \in \mathbb{N}$, $j \neq k$, or (ii$_p$) $\{f_j^p, f_k^p\} \in C_2^p$ for all $j, k \in \mathbb{N}$, $j \neq k$. If $C_1^p \neq \emptyset$ and (i$_p$) holds, or if $C_2^p = \emptyset$, we set $[a_p, b_p] = [a_{p-1}, c_{p-1}]$, while if $C_2^p \neq \emptyset$ and (ii$_p$) holds, or if $C_1^p = \emptyset$, we set $[a_p, b_p] = [c_{p-1}, b_{p-1}]$.

In this way for each $p \in \mathbb{N}$ we have nested intervals $[a_{p+1}, b_{p+1}] \subset [a_p, b_p]$ in $[0, \gamma(n)]$ with $b_p - a_p = \gamma(n)/2^p$ and a subsequence $\{f_j^p\}_{j=1}^{\infty}$ of $\{f_j^{p-1}\}_{j=1}^{\infty}$ (where $\{f_j^0\}_{j=1}^{\infty} = \{f_j\}$) such that $\nu(n, f_j^p, f_k^p, (-\infty, t] \cap T) \in [a_p, b_p]$ for all $j, k \in \mathbb{N}$, $j \neq k$. Let $l \in [0, \gamma(n)]$ be the common limit of a_p and b_p as $p \to \infty$. Denoting the diagonal sequence $\{f_j^j\}_{j=1}^{\infty}$ by $\{f_j^{(n,t)}\}$ we infer that the limit in (14) is equal to l: in fact, given $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$ such that $a_{p(\varepsilon)}, b_{p(\varepsilon)} \in [l - \varepsilon, l + \varepsilon]$ and, since $\{f_j^{(n,t)}\}_{j=p(\varepsilon)}^{\infty}$ is a subsequence of $\{f_j^{p(\varepsilon)}\}_{j=1}^{\infty}$, we find, for all $j, k \geqslant p(\varepsilon)$, $j \neq k$, that

$$\nu\big(n, f_j^{(n,t)}, f_k^{(n,t)}, (-\infty, t] \cap T\big) \in [a_{p(\varepsilon)}, b_{p(\varepsilon)}] \subset [l - \varepsilon, l + \varepsilon].$$

(ii) Let Q be an at most countable dense subset of T (and so, $Q \subset T \subset \overline{Q}$). The set $L_T = \{t \in T\colon (t-\delta, t) \cap T = \emptyset$ for some $\delta > 0\}$ of points from T isolated from the left for T is at most countable (possibly empty), and the same is true for $R_T = \{t \in T\colon (t, t+\delta) \cap T = \emptyset$ for some $\delta > 0\}$. Moreover, $L_T \cap R_T \subset Q$. Then the set $Z = Q \cup L_T \cup R_T$ is an at most countable dense subset of T.

We assert that, given $n \in \mathbb{N}$, there exists a subsequence $\{f_j^{(n)}\}$ of $\{f_j\}$ satisfying (9) and a nondecreasing function $\varphi_n : Z \to [0, \gamma(n)]$ such that

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n)}, f_k^{(n)}, (-\infty, s] \cap T\big) = \varphi_n(s) \quad \text{for all } s \in Z. \tag{17}$$

We may assume that $Z = \{s_p\}_{p=1}^{\infty}$. By step (i), there exists a subsequence $\{f_j^{(n,s_1)}\}$ of $\{f_j\}$, denoted by $\{f_j^{(n)1}\}$, and a number from $[0, \gamma(n)]$, denoted by $\varphi_n(s_1)$, such that

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n)1}, f_k^{(n)1}, (-\infty, s_1] \cap T\big) = \varphi_n(s_1).$$

Inductively, if $p \in \mathbb{N}$, $p \geqslant 2$, and a subsequence $\{f_j^{(n)p-1}\}_{j=1}^{\infty}$ of $\{f_j\}$ is already chosen, we apply step (i) to pick a subsequence $\{f_j^{(n)p}\}_{j=1}^{\infty}$ of $\{f_j^{(n)p-1}\}$ and a number $\varphi_n(s_p) \in [0, \gamma(n)]$ such that

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n)p}, f_k^{(n)p}, (-\infty, s_p] \cap T\big) = \varphi_n(s_p).$$

Then (17) is satisfied for the diagonal sequence $\{f_j^{(n)j}\}_{j=1}^{\infty}$, denoted by $\{f_j^{(n)}\}$. It is clear from Lemma 1(c) that the function φ_n defined by the left-hand side in (17) is nondecreasing on Z.

The assertion (18) in the next step (iii) is, actually, a variant of Helly's selection theorem for specific double sequences.

(iii) Let us prove that, given $n \in \mathbb{N}$, there is a subsequence of $\{f_j\}$ satisfying (9), denoted as in step (ii) by $\{f_j^{(n)}\}$, and a bounded nondecreasing function $\nu_n : T \to [0, \gamma(n)]$ such that

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n)}, f_k^{(n)}, (-\infty, t] \cap T\big) = \nu_n(t) \quad \text{for all } t \in T. \tag{18}$$

We extend the function φ_n, given by (17), from the set Z to the whole $\mathbb{R}$ according to Saks' idea [22, Chapter 7, Section 4, Lemma (4.1)] as follows:

$$\widetilde{\varphi}_n(t) = \sup_{s\in Z,\, s\leqslant t} \varphi_n(s) \quad \text{if } t \in \mathbb{R} \text{ and } (-\infty, t] \cap Z \neq \emptyset$$

and $\widetilde{\varphi}_n(t) = \inf_{s\in Z} \varphi_n(s)$ otherwise. Clearly, the function $\widetilde{\varphi}_n : \mathbb{R} \to \mathbb{R}^+$ is nondecreasing and bounded; moreover, $\widetilde{\varphi}_n(\mathbb{R}) \subset \overline{\varphi_n(Z)} \subset [0, \gamma(n)]$. Therefore, the set $P_n \subset \mathbb{R}$ of points of discontinuity of $\widetilde{\varphi}_n$ is at most countable. Let us show that

$$\lim_{j,k\to\infty} \nu\big(n, f_j^{(n)}, f_k^{(n)}, (-\infty, t] \cap T\big) = \widetilde{\varphi}_n(t) \quad \text{for all } t \in T \setminus P_n, \tag{19}$$

where $\{f_j^{(n)}\}$ is the sequence constructed in (17) of step (ii). Taking into account (17), we may assume that $t \in T \setminus (P_n \cup Z)$. Let $\varepsilon > 0$ be arbitrarily fixed. Since t is a point of continuity of $\widetilde{\varphi}_n$, there exists a $\delta = \delta(n, \varepsilon, t) > 0$ such that

$$\widetilde{\varphi}_n(s) \in \big[\widetilde{\varphi}_n(t) - \varepsilon, \widetilde{\varphi}_n(t) + \varepsilon\big] \quad \text{for all } s \in \mathbb{R} \text{ with } |s - t| \leqslant \delta. \tag{20}$$

Since $t \notin L_T$ and $T \subset \overline{Z}$, we have $\emptyset \neq (t-\delta, t) \cap T \subset (t-\delta, t) \cap \overline{Z}$, and so, there is an $s_1 = s_1(n, \varepsilon, t) \in (t-\delta, t) \cap Z$; similarly, $t \notin R_T$ implies the existence of an $s_2 = s_2(n, \varepsilon, t) \in Z$ with $t < s_2 < t + \delta$. Denoting, for the sake of brevity, the quantity under the limit sign in (19) by $\nu_{n,j,k}(t)$, by (17) we find a number $N = N(n, \varepsilon) \in \mathbb{N}$ such that, for all $j \geqslant N$ and $k \geqslant N$ with $j \neq k$, we have

$$\nu_{n,j,k}(s_1) \in \big[\varphi_n(s_1) - \varepsilon, \varphi_n(s_1) + \varepsilon\big] \quad \text{and} \quad \nu_{n,j,k}(s_2) \in \big[\varphi_n(s_2) - \varepsilon, \varphi_n(s_2) + \varepsilon\big].$$

In view of Lemma 1(c), $\nu_{n,j,k}(s_1) \leqslant \nu_{n,j,k}(t) \leqslant \nu_{n,j,k}(s_2)$, and so, (20) together with equalities $\widetilde{\varphi}_n(s_1) = \varphi_n(s_1)$ and $\widetilde{\varphi}_n(s_2) = \varphi_n(s_2)$ yield

$$\nu_{n,j,k}(t) \in \big[\varphi_n(s_1) - \varepsilon, \varphi_n(s_2) + \varepsilon\big] \subset \big[\widetilde{\varphi}_n(t) - 2\varepsilon, \widetilde{\varphi}_n(t) + 2\varepsilon\big]$$

for all $j, k \geqslant N$, $j \neq k$, which establishes (19).

In order to obtain (18), we note that $T = (T \setminus P_n) \cup (T \cap P_n)$ where the set $T \cap P_n$ is at most countable. Arguing as in step (ii) with Z replaced by $T \cap P_n$, we find a subsequence of $\{f_j^{(n)}\}$, again denoted by $\{f_j^{(n)}\}$, and a nondecreasing function $\psi_n : T \cap P_n \to [0, \gamma(n)]$ such that the limit on the left in (19) is equal to $\psi_n(t)$ for all $t \in T \cap P_n$. Defining $\nu_n : T \to [0, \gamma(n)]$ by $\nu_n(t) = \widetilde{\varphi}_n(t)$ if $t \in T \setminus P_n$ and $\nu_n(t) = \psi_n(t)$ if $t \in T \cap P_n$, we arrive at (18) where, in view of Lemma 1(c), the function ν_n is nondecreasing on T.

(iv) Here we complete the proof of assertion (10). By step (iii), there is a subsequence $\{f_j^{(1)}\}$ of the sequence $\{f_j\}$ satisfying (9) and a nondecreasing function $\nu_1 : T \to [0, \gamma(1)]$ such that

$$\lim_{j,k\to\infty} \nu\big(1, f_j^{(1)}, f_k^{(1)}, (-\infty, t] \cap T\big) = \nu_1(t) \quad \text{for all } t \in T.$$

If $n \in \mathbb{N}$, $n \geqslant 2$, and a subsequence $\{f_j^{(n-1)}\}$ of $\{f_j\}$ is already chosen, by virtue of step (iii) applied to the sequence $\{f_j^{(n-1)}\}$ (instead of $\{f_j\}$ from (iii)), we find a subsequence $\{f_j^{(n)}\}$ of $\{f_j^{(n-1)}\}$ and a nondecreasing bounded function $\nu_n : T \to [0, \gamma(n)]$ such that condition (18) holds. It follows that the diagonal subsequence $\{f_j^{(j)}\}_{j=1}^{\infty}$ of $\{f_j\}$, again denoted by $\{f_j\}$, satisfies property (10). □

4. Examples illustrating the sharpness of Theorem 1

Example 1. Condition (7) is not *necessary* even if $\nu(n, f_j, T) = o(n)$ for all $j \in \mathbb{N}$. Let $\mathcal{D} : \mathbb{R} \to \{0, 1\}$ be the Dirichlet function, i.e., the indicator function of the set $\mathbb{Q}$ of all rational numbers: $\mathcal{D}(t) = 1$ if $t \in \mathbb{Q}$ and $\mathcal{D}(t) = 0$ if $t \in \mathbb{R} \setminus \mathbb{Q}$. Given $t \in \mathbb{R}$ and $j \in \mathbb{N}$, we set $f_j(t) = 1$ if $j!t$ is integer and $f_j(t) = 0$ otherwise. Clearly, f_j converges pointwise on $\mathbb{R}$ to $\mathcal{D}$. If $T = [0, 1]$, we have $\nu(n, f_j, T) = o(n)$ for each $j \in \mathbb{N}$ and $\nu(n, \mathcal{D}, T) = n$ (cf. [8, Section 3, Example 5]), and a straightforward calculation shows that

$$\nu(n, f_j, f_k, T) = \begin{cases} n & \text{if } n \leqslant 2|j! - k!|, \\ 2|j! - k!| & \text{if } n > 2|j! - k!|, \end{cases} \quad n, j, k \in \mathbb{N}.$$

Thus, $\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T) = n \neq o(n)$.

Example 2. Under the assumptions of Theorem 1 we cannot infer, for the limit function f, that $\nu(n, f, T) = o(n)$. In fact, setting $f_j(t) = (1 + (1/j))\mathcal{D}(t)$, $t \in T = [0, 1]$, we find that f_j converges uniformly on T to $\mathcal{D}$, condition (7) is satisfied and $\nu(n, \mathcal{D}, [0, 1]) = n \neq o(n)$. This example shows also that Theorem 1 can be applied to the sequence $\{f_j\}$ while Theorem A is inapplicable; see also Example 4 below.

Theorem 1 cannot be applied to sequences of the form $f_j(t) = (-1)^j\mathcal{D}(t)$ or $f_j(t) = ((-1)^j + (1/j))\mathcal{D}(t)$, $t \in [0, 1]$. However, noting that the restriction of $\mathcal{D}$ to $[0, 1] \setminus \mathbb{Q}$ is the zero function, Theorem 1 can be applied to these sequences in the form of Corollary 1. Now we construct a sequence $\{f_j\}$ having a pointwise convergent subsequence, but $\{f_j\}$ does not satisfy the assumptions of Corollary 1.

Let $[0, 1] = A \cup B$ be a disjoint union of two uncountable dense subsets of $[0, 1]$ (e.g., if $C \subset [0, 1]$ is the usual Cantor ternary set, we may put $A = ([0, 1] \cap \mathbb{Q}) \cup C$ and $B = [0, 1] \setminus A = [0, 1] \setminus (\mathbb{Q} \cup C)$), and let $\mathcal{E} : [0, 1] \to \mathbb{R}$ be given by $\mathcal{E}(t) = 1$ if $t \in A$ and $\mathcal{E}(t) = -1$ if $t \in B$. Set $f_j(t) = (-1)^j\mathcal{E}(t)$, $t \in [0, 1]$. If $E \subset [0, 1]$ is at most countable, $n \in \mathbb{N}$, $\{t_i\}_{i=1}^n \subset A \setminus E$ and $\{s_i\}_{i=1}^n \subset B \setminus E$ are such that $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \leqslant 1$, then for j even and k odd or vice versa we have $|(f_j - f_k)(t_i) - (f_j - f_k)(s_i)| = 4$ for all $i \in \{1, \ldots, n\}$, and so, $\nu(n, f_j, f_k, [0, 1] \setminus E) \geqslant 4n$.

Example 3. Condition (7) in Theorem 1 is essential: the sequence $f_j(t) = \sin(jt)$, $t \in [0, 2\pi]$, $j \in \mathbb{N}$, has no pointwise convergent subsequence and does not satisfy (7). To see the latter, given $n \in \mathbb{N}$ and $k \geqslant 3^{n-1}$, we set

$$s_i^k = \frac{\pi}{2k}\big(2 \cdot 3^{i-1} - 1\big) \quad \text{and} \quad t_i^k = \frac{\pi}{k}\big(2 \cdot 3^{i-1} - 1\big), \quad i = 1, \ldots, n.$$

Clearly, $0 < s_1^k < t_1^k < s_2^k < t_2^k < \cdots < s_n^k < t_n^k < 2\pi$ and, if $j = 2k$, then

$$\left|(f_j - f_k)(t_i^k) - (f_j - f_k)(s_i^k)\right| = 1 \quad \text{for all } i \in \{1, \dots, n\}.$$

Thus, $\nu(n, f_{2k}, f_k, [0, 2\pi]) \geqslant n$, and so, $\limsup_{j,k\to\infty} \nu(n, f_j, f_k, [0, 2\pi]) \geqslant n$.

Example 4. Here we show that if $f_j \in \mathbb{R}^{[0,1]}$, $j \in \mathbb{N}$, is given by

$$f_j(t) = \mathcal{D}(t) + \frac{(-1)^{m+j}}{j} + \sum_{i=1}^{m} \frac{(-1)^i}{i} \quad \text{if } t \in I_m = \left[\frac{m-1}{m}, \frac{m}{m+1}\right), \ m \in \mathbb{N},$$

and $f_j(1) = 1 - \log 2$, then the assumptions of Theorem 1 are fulfilled for $\{f_j\}$ (because $\{f_j\}$ is uniformly bounded and uniformly convergent on [0, 1]), while those of Theorem A are not (see also Remark 4 below). Indeed, in order to see that (2) is not satisfied, we let $0 \leqslant s_1 < t_1 < \cdots < s_n < t_n < 1$ be such that $s_k \in I_{2k-1} \setminus \mathbb{Q}$ and $t_k \in I_{2k} \cap \mathbb{Q}$, $k = 1, \dots, n$. For such k's we get:

$$\begin{aligned} \left|f_j(t_k) - f_j(s_k)\right| &= \left|1 + \frac{(-1)^{2k+j}}{j} + \sum_{i=1}^{2k} \frac{(-1)^i}{i} - \frac{(-1)^{2k-1+j}}{j} - \sum_{i=1}^{2k-1} \frac{(-1)^i}{i}\right| \\ &= \left|1 + \frac{1}{2k} + 2\frac{(-1)^j}{j}\right| \geqslant \frac{1}{3} \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

This gives $\nu(n, f_j, [0, 1]) \geqslant n/3$ for all $n, j \in \mathbb{N}$.

Remark 4. The last example and the observation at the end of Section 1 show that if X is a metric semigroup then Theorem 1 is more general than Theorem A. Since Theorem A implies many selection theorems based on certain notions of generalized variations (see [7–10]), Theorem 1 does as well. Another types of pointwise selection theorems, based on notions of oscillations, were presented in [23] and [18], and it was proved in [19] that Theorem A has no relationship neither with Theorem 1.2 from [23] nor with Theorem 2.1 from [18]. In this respect we note that, along with Example 4, all functional sequences constructed in [19] satisfy the assumptions of Theorem 1. However, the sequence $\{f_j\}$ from Example 4 does not satisfy the assumptions of [23, Theorem 1.2] and [18, Theorem 2.1]: the proof of this is analogous to Step II in the proof of Theorem 3.1 from [19].

Example 5. Here we construct two sequences $\{f_j\} \subset \mathbb{R}^T$ with $T = [0, 1]$, for which condition (7) holds in its full generality with $o(n)$ at the right-hand side as compared with (13) and Example 4 where $o(n) = 0$ at the right in (7).

Let $f_j(t) = 0$ if $0 \leqslant t < 1$ and $f_j(1) = (-1)^j$. Then $\nu(n, f_j, f_k, T) = |(-1)^j - (-1)^k|$, and so

$$\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T) = 2 \quad \text{for all } n \in \mathbb{N}.$$

Now, define f_j as follows: if j is odd, then $f_j = 0$ on T, and if $j = 2p$ is even, then $f_j(1) = -\log 2$ and

$$f_{2p}(t) = \frac{(-1)^{p+m}}{pm} + \sum_{i=1}^{m} \frac{(-1)^i}{i} \quad \text{if } t \in I_m = \left[\frac{m-1}{m}, \frac{m}{m+1}\right), \ m \in \mathbb{N}.$$

We are going to show that, for all $n \in \mathbb{N}$,

$$\frac{1}{4}\sum_{i=1}^{n} \frac{1}{i} \leqslant \limsup_{j,k\to\infty} \nu(n, f_j, f_k, T) \leqslant 4\sum_{i=1}^{n} \frac{1}{i}, \tag{21}$$

where $\sum_{i=1}^{n} 1/i = \gamma + \log n + \alpha_n = o(n)$, $\gamma = 0.57721566490\ldots$ is the Euler constant and $\alpha_n \to 0$ as $n \to \infty$. Let $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{2n-1} \leqslant t_{2n} \leqslant 1$ and $m_i \in \mathbb{N}$ be such that $t_i \in I_{m_i}$, $i \in \{1, \dots, 2n\}$ if $t_{2n} < 1$ and $i \in \{1, \dots, 2n-1\}$ if $t_{2n} = 1$. It is clear for such i's that $m_i \leqslant m_{i+1}$, and with no loss of generality we suppose that $m_{2i-1} < m_{2i}$ for all $i = 1, \dots, n$. In order to prove the right-hand side inequality in (21), we consider three cases (i)–(iii).

(i) If j and k are odd, then $\nu(n, f_j, f_k, T) = 0$.

(ii) Let $j = 2p$ and $k = 2q$ for some $p, q \in \mathbb{N}$. If $t_{2n} < 1$ and $i \in \{1, \dots, n\}$, we have:

$$\begin{aligned}\left|(f_j-f_k)(t_{2i})-(f_j-f_k)(t_{2i-1})\right| &= \left|\frac{(-1)^{p+m_{2i}}}{pm_{2i}}+\sum_{l=1}^{m_{2i}}\frac{(-1)^l}{l}-\frac{(-1)^{q+m_{2i}}}{qm_{2i}}-\sum_{l=1}^{m_{2i}}\frac{(-1)^l}{l}-\frac{(-1)^{p+m_{2i-1}}}{pm_{2i-1}}\right.\\ &\quad\left.-\sum_{l=1}^{m_{2i-1}}\frac{(-1)^l}{l}+\frac{(-1)^{q+m_{2i-1}}}{qm_{2i-1}}+\sum_{l=1}^{m_{2i-1}}\frac{(-1)^l}{l}\right|\\ &\leqslant \left(\frac{1}{m_{2i}}+\frac{1}{m_{2i-1}}\right)\left(\frac{1}{p}+\frac{1}{q}\right)\leqslant\frac{4}{m_{2i-1}}.\end{aligned} \tag{22}$$

If $t_{2n}=1$, then along with (22) for $i\in\{1,\dots,n-1\}$, we have:

$$\begin{aligned}\left|(f_j-f_k)(t_{2n})-(f_j-f_k)(t_{2n-1})\right| &= \left|-\frac{(-1)^{p+m_{2n-1}}}{pm_{2n-1}}-\sum_{l=1}^{m_{2n-1}}\frac{(-1)^l}{l}+\frac{(-1)^{q+m_{2n-1}}}{qm_{2n-1}}+\sum_{l=1}^{m_{2n-1}}\frac{(-1)^l}{l}\right|\\ &\leqslant\frac{1}{m_{2n-1}}\left(\frac{1}{p}+\frac{1}{q}\right)\leqslant\frac{2}{m_{2n-1}}.\end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$\sum_{i=1}^{n}\left|(f_j-f_k)(t_{2i})-(f_j-f_k)(t_{2i-1})\right|\leqslant\sum_{i=1}^{n}\frac{4}{m_{2i-1}}\leqslant 4\sum_{i=1}^{n}\frac{1}{i}. \tag{24}$$

(iii) Let $j=2p$ and k be odd (the case when j is odd and $k=2p$ is considered similarly). Since $\sum_{l=1}^{\infty}(-1)^l/l=-\log 2$ and $f_k=0$, in the case $t_{2n}<1$ we have, for all $i\in\{1,\dots,n\}$,

$$\begin{aligned}\left|(f_j-f_k)(t_{2i})-(f_j-f_k)(t_{2i-1})\right| &= \left|\frac{(-1)^{p+m_{2i}}}{pm_{2i}}+\sum_{l=1}^{m_{2i}}\frac{(-1)^l}{l}-\frac{(-1)^{p+m_{2i-1}}}{pm_{2i-1}}-\sum_{l=1}^{m_{2i-1}}\frac{(-1)^l}{l}\right|\\ &\leqslant\left(\frac{1}{m_{2i}}+\frac{1}{m_{2i-1}}\right)\frac{1}{p}+\left|\sum_{l=1}^{m_{2i}}\frac{(-1)^l}{l}+\log 2\right|+\left|-\log 2-\sum_{l=1}^{m_{2i-1}}\frac{(-1)^l}{l}\right|\\ &\leqslant\frac{2}{m_{2i-1}}+\frac{1}{m_{2i}+1}+\frac{1}{m_{2i-1}+1}\leqslant\frac{4}{m_{2i-1}}.\end{aligned}$$

Analogously, this estimate can be established if $t_{2n}=1$ and $i\in\{1,\dots,n\}$, yielding (24). In this way the right-hand side inequality in (21) follows.

To obtain the left-hand side inequality in (21), we pick arbitrarily $t_i\in I_i$, $i=1,\dots,2n$. If $j=2p$ with $p\geqslant 6$ and k is odd, we have:

$$\begin{aligned}\left|(f_j-f_k)(t_{2i})-(f_j-f_k)(t_{2i-1})\right| &= \left|\frac{(-1)^{p+2i}}{2pi}+\sum_{l=1}^{2i}\frac{(-1)^l}{l}-\frac{(-1)^{p+2i-1}}{p(2i-1)}-\sum_{l=1}^{2i-1}\frac{(-1)^l}{l}\right|\\ &= \left|\frac{(-1)^p}{p}\left(\frac{1}{2i}+\frac{1}{2i-1}\right)+\frac{1}{2i}\right|\\ &\geqslant\frac{1}{2i}-\frac{1}{p}\left(\frac{1}{2i}+\frac{1}{2i-1}\right)\geqslant\frac{1}{4i},\end{aligned}$$

which, after summing over $i=1,\dots,n$, implies (21).

5. Variants of the selection principle

Suppose that in Corollary 1 (p. 618) $E\subset T$ is a set of Lebesgue measure zero. Then, according to Theorem 1, a subsequence of $\{f_j\}$ converges pointwise on $T\setminus E$, that is almost everywhere on T. The following theorem is a *selection principle for the almost everywhere convergence* in terms of the joint modulus of variation and may be considered as a converse to the observation in Remark 3 from Section 2.

Theorem 2. *Let $\emptyset \neq T \subset \mathbb{R}$, $(X, d, +)$ be a metric semigroup and $\{f_j\} \subset X^T$ a sequence of functions which is pointwise precompact almost everywhere on T and satisfies the condition: for each $p \in \mathbb{N}$ there exists a measurable set $E_p \subset T$ of Lebesgue measure $\leqslant 1/p$ such that*

$$\limsup_{j,k\to\infty} \nu(n, f_j, f_k, T \setminus E_p) = o(n).$$

Then $\{f_j\}$ contains a subsequence which converges almost everywhere on T.

Taking into account Theorem 1 and the diagonal process, the proof of Theorem 2 follows the same lines as that of Theorem 6 from [8] with obvious modifications, and so, is omitted.

In order to present one more variant of Theorem 1, let $(X, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K} = (\mathbb{R}$ or $\mathbb{C})$. The notion of the joint modulus of variation (5) is introduced with respect to the usual induced metric $d(x, y) = \|x - y\|$, $x, y \in X$.

A certain geometrical interpretation of $\nu(n, f, g, T)$ can be gained if we note that the value $\|f(t_i) + g(s_i) - g(t_i) - f(s_i)\|$ from (5) is equal to the two times the distance between the "middle" points $(f(t_i) + g(s_i))/2$ and $(g(t_i) + f(s_i))/2$.

Let $(X^*, \|\cdot\|)$ be the dual of X where $\|x^*\| = \sup\{|x^*(x)|\colon x \in X,\ \|x\| \leqslant 1\}$, $x^* \in X^*$. If the bilinear functional $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{K}$ is defined by $\langle x, x^*\rangle = x^*(x)$ for all $x \in X$ and $x^* \in X^*$, then it determines the natural duality between X and X^*. Recall that a sequence $\{x_j\} \subset X$ is said to *converge weakly* in X to an $x \in X$ provided $\langle x_j, x^*\rangle \to \langle x, x^*\rangle$ in $\mathbb{K}$ as $j \to \infty$ for all $x^* \in X^*$, which will be written as $x_j \overset{w}{\to} x$ in X.

Theorem 3. *Let $\emptyset \neq T \subset \mathbb{R}$ and $(X, \|\cdot\|)$ be a reflexive separable Banach space with separable dual $(X^*, \|\cdot\|)$. Suppose that $\{f_j\} \subset X^T$ is such that $\sup_{j\in\mathbb{N}} \|f_j(t_0)\| = C_0 < \infty$ for some $t_0 \in T$ and condition (7) is satisfied. Then $\{f_j\}$ contains a subsequence, again denoted by $\{f_j\}$, such that $f_j(t) \overset{w}{\to} f(t)$ in X for all $t \in T$ and some function $f \in X^T$.*

Due to the separability of X^*, the proof again relies on the diagonal process and Theorem 1 applied to the sequence of $\mathbb{K}$-valued functions $T \ni t \mapsto \langle f_j(t), x^*\rangle \in \mathbb{K}$, $j \in \mathbb{N}$, $x^* \in X^*$, and can be easily adapted to the case under consideration from the proof of Theorem 7 from [8]—we note only that, by virtue of (12), the quantity $C(t) = \sup_{j\in\mathbb{N}} \|f_j(t)\|$ is finite for all $t \in T$ and, given $x^* \in X^*$, we have

$$\bigl|\langle f_j(t), x^*\rangle\bigr| \leqslant \bigl\|f_j(t)\bigr\| \cdot \bigl\|x^*\bigr\| \leqslant C(t)\bigl\|x^*\bigr\|, \quad t \in T,\ j \in \mathbb{N},$$

and

$$\limsup_{j,k\to\infty} \nu\bigl(n, \langle (f_j - f_k)(\cdot), x^*\rangle, T\bigr) \leqslant \bigl\|x^*\bigr\| \limsup_{j,k\to\infty} \nu(n, f_j, f_k, T).$$

As a simple example (cf. Example 3), let $x_j(t) = \sin(jt)$, $t \in [0, 2\pi]$, and $f_j : [0,1] \to X = L^2[0, 2\pi]$ be given by $f_j(s) = x_j$ for all $s \in [0,1]$ (i.e., each f_j is a constant function), $j \in \mathbb{N}$. Then, by Theorem 3 applied to $\{f_j\}$, a subsequence of $\{x_j\}$ converges weakly in X and, since $\{x_j\}$ is weakly Cauchy, $\{x_j\}$ converges weakly in X; clearly, the weak limit of $\{x_j\}$ is zero, which is the well-known classical result.

References

[1] S.A. Belov, V.V. Chistyakov, A selection principle for mappings of bounded variation, J. Math. Anal. Appl. 249 (2) (2000) 351–366.
[2] P.C. Bhakta, On functions of bounded variation relative to a set, J. Aust. Math. Soc. 13 (1972) 313–322.
[3] Z.A. Chanturiya, The modulus of variation of a function and its application in the theory of Fourier series, Dokl. Akad. Nauk SSSR 214 (1974) 63–66 (in Russian); English translation: Soviet Math. Dokl. 15 (1974) 67–71.
[4] V.V. Chistyakov, On mappings of bounded variation, J. Dyn. Control Syst. 3 (2) (1997) 261–289.
[5] V.V. Chistyakov, On the theory of multivalued mappings of bounded variation of one real variable, Mat. Sb. 189 (5) (1998) 153–176 (in Russian); English translation: Sb. Math. 189 (5–6) (1998) 797–819.
[6] V.V. Chistyakov, Selections of bounded variation, J. Appl. Anal. 10 (1) (2004) 1–82.
[7] V.V. Chistyakov, A selection principle for functions of a real variable, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia 53 (1) (2005) 25–43.
[8] V.V. Chistyakov, The optimal form of selection principles for functions of a real variable, J. Math. Anal. Appl. 310 (2) (2005) 609–625.
[9] V.V. Chistyakov, A selection principle for uniform space-valued functions, Dokl. Akad. Nauk 409 (5) (2006) 591–593 (in Russian); English translation: Dokl. Math. 74 (1) (2006) 559–561.

[10] V.V. Chistyakov, A pointwise selection principle for functions of one variable with values in a uniform space, Mat. Tr. 9 (1) (2006) 176–204 (in Russian); English translation: Siberian Adv. Math. 16 (3) (2006) 15–41.
[11] V.V. Chistyakov, A. Nowak, Regular Carathéodory-type selectors under no convexity assumptions, J. Funct. Anal. 225 (2) (2005) 247–262.
[12] V.V. Chistyakov, D. Repovš, Selections of bounded variation under the excess restrictions, J. Math. Anal. Appl. 331 (2) (2007) 873–885.
[13] R.M. Dudley, R. Norvaiša, Differentiability of Six Operators on Nonsmooth Functions and p-Variation (with the Collaboration of Jinghua Qian), Lecture Notes in Math., vol. 1703, Springer-Verlag, Berlin, 1999.
[14] S. Fuchino, Sz. Plewik, On a theorem of E. Helly, Proc. Amer. Math. Soc. 127 (1999) 491–497.
[15] S. Gniłka, On the generalized Helly's theorem, Funct. Approx. Comment. Math. 4 (1976) 109–112.
[16] C. Goffman, T. Nishiura, D. Waterman, Homeomorphisms in Analysis, Math. Surveys Monogr., vol. 54, Amer. Math. Soc., Providence, RI, 1997.
[17] E. Helly, Über lineare Funktionaloperationen, Sitzungsber. Naturwiss. Kl. Kaiserlichen Akad. Wiss. Wien 121 (1912) 265–297.
[18] L. Di Piazza, C. Maniscalco, Selection theorems, based on generalized variation and oscillation, Rend. Circ. Mat. Palermo (2) 35 (1986) 386–396.
[19] C. Maniscalco, A comparison of three recent selection theorems, Math. Bohem. 132 (2) (2007) 177–183.
[20] J. Musielak, W. Orlicz, On generalized variations (I), Studia Math. 18 (1959) 11–41.
[21] F. Ramsey, On a problem of formal logic, Proc. London Math. Soc. (2) 30 (1930) 264–286.
[22] S. Saks, Theory of the Integral, second revised ed., Stechert, New York, 1937.
[23] K. Schrader, A generalization of the Helly selection theorem, Bull. Amer. Math. Soc. 78 (3) (1972) 415–419.
[24] M. Schramm, Functions of Φ-bounded variation and Riemann–Stieltjes integration, Trans. Amer. Math. Soc. 287 (1) (1985) 49–63.
[25] D. Waterman, On Λ-bounded variation, Studia Math. 57 (1) (1976) 33–45.