

## Kobayashi pseudometric on hyperkähler manifolds

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## ABSTRACT

The Kobayashi pseudometric on a complex manifold is the maximal pseudometric such that any holomorphic map from the Poincaré disk to the manifold is distance-decreasing. Kobayashi has conjectured that this pseudometric vanishes on Calabi–Yau manifolds. Using ergodicity of complex structures, we prove this for all hyperkähler manifold with  $b_2 \geq 7$  that admits a deformation with a Lagrangian fibration and whose Picard rank is not maximal. The Strominger–Yau–Zaslow (SYZ) conjecture claims that parabolic nef line bundles on hyperkähler manifolds are semi-ample. We prove that the Kobayashi pseudometric vanishes for any hyperkähler manifold with  $b_2 \geq 7$  if the SYZ conjecture holds for all its deformations. This proves the Kobayashi conjecture for all K3 surfaces and their Hilbert schemes.

## 1. Introduction

The Kobayashi pseudometric on a complex manifold  $M$  is the maximal pseudometric such that any holomorphic map from the Poincaré disk to  $M$  is distance-decreasing (see Section 1.3 for more details and references). Kobayashi conjectured that the Kobayashi pseudometric vanishes for all projective varieties with trivial canonical bundle (see [20, Problems C.1 and F.3]). The conjecture was proved for projective K3 surfaces via the non-trivial theorem in [30] that all projective K3 surfaces are swept out by elliptic curves (see [38, Lemma 1.51]). We prove the conjecture for all K3 surfaces as well as for many classes of hyperkähler manifolds. For an extensive survey on problems of Kobayashi and Lang, we recommend the beautiful survey papers by Voisin [38] and by Demailly [11].

Using density arguments and the existence of Lagrangian fibrations, it was proved in [18] that all known hyperkähler manifolds are Kobayashi non-hyperbolic. Then in [37], this result was generalized to all hyperkähler manifolds with  $b_2 > 3$ . All known examples of hyperkähler manifolds have  $b_2 > 5$  (in fact,  $b_2 \geq 7$ ) and this has been conjectured to be true in general.

We introduce the basics of hyperkähler geometry and Teichmüller spaces in Subsection 1.1. Upper semi-continuity of the Kobayashi pseudometric is discussed in Subsection 1.4. Our main results are in Sections 2 and 3.

For a compact complex manifold  $M$ , the Teichmüller space  $\text{Teich}$  is the space of complex structures up to isotopies. The mapping class group  $\Gamma$ , or the group of ‘diffeotopies’, acts naturally on  $\text{Teich}$ . Complex structures with dense  $\Gamma$ -orbits are called *ergodic* (see Definition 1.17). We show that if the Kobayashi pseudometric on  $M$  vanishes, then the Kobayashi pseudometric vanishes for all ergodic complex structures on  $M$  in the same deformation class (Theorem 2.1). As a corollary, the Kobayashi pseudometric vanishes for all K3 surfaces (Corollary 2.2), and for all ergodic complex structures on a hyperkähler manifold (Theorem 2.3). The Strominger–Yau–Zaslow (SYZ) conjecture predicts that any hyperkähler manifold has a deformation which admits a Lagrangian fibration (more precisely, SYZ conjecture is a special form of Kawamata’s

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abundance conjecture predicting that all parabolic nef line bundles on hyperkähler manifolds are semi-ample; see [32, 36] for more details). Assuming this conjecture to be true, we show the vanishing of the Kobayashi pseudometric for all hyperkähler manifolds with  $b_2 \geq 7$  (which is expected for all hyperkähler manifolds). When, in addition, the complex structure is non-ergodic, we prove that the infinitesimal Kobayashi pseudometric defined by Royden vanishes on a Zariski dense open subset of the manifold (this result is stronger).

We summarize the main results of this article in the following theorems; please see the main body of the paper for details of the definitions and of the proofs.

**THEOREM 1.1.** *Let  $M$  be a compact simple hyperkähler manifold with  $b_2(M) \geq 7$ . Assume that some deformation of  $M$  admits a holomorphic Lagrangian fibration and the Picard rank of  $M$  is not maximal. Then the Kobayashi pseudometric on  $M$  vanishes.*

*Proof.* See Corollary 2.14. □

**REMARK 1.2.** All known examples of hyperkähler manifolds have  $b_2(M) \geq 7$  and can be deformed to one which admits a Lagrangian fibration [18, Claim 1.20]. By the above result, the Kobayashi pseudometric on known manifolds vanishes, unless their Picard rank is maximal.

**THEOREM 1.3.** *Let  $M$  be a compact simple hyperkähler manifold with  $b_2(M) \geq 7$ . Assume that nef line bundles on all deformations of  $M$  are semi-ample, or that  $M$  is projective and admits a holomorphic Lagrangian fibration up to birational equivalence with a smooth base and no multiple fibers in codimension 1. Then the Kobayashi pseudometric on  $M$  vanishes and the infinitesimal pseudometric vanishes on a Zariski open subset of  $M$ .*

*Proof.* See Corollary 3.4 and Theorem 3.1. □

### 1.1. Teichmüller spaces and hyperkähler geometry

We summarize the definition of the Teichmüller space of hyperkähler manifolds, following [35].

**DEFINITION 1.4.** Let  $M$  be a compact complex manifold and  $\text{Diff}_0(M)$  a connected component of its diffeomorphism group (*the group of isotopies*). Denote by  $\text{Comp}$  the space of complex structures on  $M$ , equipped with a structure of Fréchet manifold. We let  $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$  and call it *the Teichmüller space* of  $M$ .

**REMARK 1.5.** In many important cases, such as in the case of Calabi–Yau manifolds [10],  $\text{Teich}$  is a finite-dimensional complex space; usually it is non-Hausdorff.

**DEFINITION 1.6.** Let  $\text{Diff}(M)$  be the group of orientable diffeomorphisms of a complex manifold  $M$ . Consider the *mapping class group*

$$\Gamma := \text{Diff}(M) / \text{Diff}_0(M)$$

acting on  $\text{Teich}$ . The quotient  $\text{Comp} / \text{Diff} = \text{Teich} / \Gamma$  is called *the moduli space* of complex structures on  $M$ . Typically, it is very non-Hausdorff. The set  $\text{Comp} / \text{Diff}$  corresponds bijectively to the set of isomorphism classes of complex structures.

DEFINITION 1.7. A *hyperkähler manifold* is a compact holomorphically symplectic manifold admitting a Kähler structure.

DEFINITION 1.8. A hyperkähler manifold  $M$  is called *simple* if  $\pi_1(M) = 0$  and  $H^{2,0}(M) = \mathbb{C}$ . In the literature, simple hyperkähler manifolds are often called *irreducible holomorphic symplectic manifolds*, or simply an *irreducible symplectic varieties*.

The equivalence between these two notions is based on the following theorem of Bogomolov (via [39]) that motivated this definition.

THEOREM 1.9 [4]. *Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.*

REMARK 1.10. Further on, all hyperkähler manifolds are assumed to be simple,  $\text{Comp}$  is the space of all complex structures of hyperkähler type on  $M$ , and  $\text{Teich}$  its quotient by  $\text{Diff}_0(M)$ .

A simple hyperkähler manifold admits a primitive integral quadratic form on its second cohomology group known as the Beauville–Bogomolov–Fujiki form. We define it using the Fujiki identity given in the theorem below; see [13]. For a more detailed description of the form, we refer the reader to [3, 5].

THEOREM 1.11 (Fujiki, [13]). *Let  $M$  be a simple hyperkähler manifold of dimension  $2n$  and  $\alpha \in H^2(M, \mathbb{Z})$ . Then  $\int_M \alpha^{2n} = cq(\alpha, \alpha)^n$ , where  $q$  is a primitive integral quadratic form on  $H^2(M, \mathbb{Z})$  of index  $(3, b_2(M) - 3)$ , and  $c > 0$  is a rational number.*

REMARK 1.12. Fujiki formula can be used to show that  $\int_M \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{2n}$  is proportional to a sum of  $q(\alpha_{i_1} \alpha_{i_2}) q(\alpha_{i_3} \alpha_{i_4}) \cdots q(\alpha_{i_{2n-1}} \alpha_{i_{2n}})$  taken over all permutations  $(i_1, i_2, \dots, i_{2n})$ . Whenever  $\alpha, \beta \in H^2(M)$  satisfy  $q(\alpha, \alpha) = 0$ , Fujiki formula gives

$$\int_M \alpha^n \cup \beta^n = cq(\alpha, \beta)^n.$$

DEFINITION 1.13. From Theorem 1.11, the form  $q$  is defined uniquely up to a sign, except the case of even  $n$  and  $b_2 \neq 6$ . To fix the sign, we make the additional assumption that  $q(\omega, \omega) > 0$  for every Kähler form  $\omega$ . Such a form  $q$  is called *the Bogomolov–Beauville–Fujiki form* (or the *BBF form*) of  $M$ .

The mapping class group of a hyperkähler manifold can be described in terms of the BBF form as follows.

THEOREM 1.14 [35]. *Let  $M$  be a simple hyperkähler manifold,  $\Gamma$  its mapping class group, and  $\Gamma \xrightarrow{\varphi} O(H^*(M, \mathbb{Z}), q)$  the natural map. Then  $\varphi$  has finite kernel and its image has finite index in  $O(H^*(M, \mathbb{Z}), q)$ .*

DEFINITION 1.15. Let  $\text{Teich}^I$  be a connected component of the Teichmüller space containing  $I \in \text{Teich}$ , and  $\Gamma^I$  the subgroup of the mapping class group preserving  $\text{Teich}^I$ . The group  $\Gamma^I$  is called *the monodromy group* of  $(M, I)$  (see [26]).

REMARK 1.16. In [35], it was shown that  $\Gamma^I$  is a finite index subgroup in  $O(H^*(M, \mathbb{Z}), q)$  independent of  $I$ .

### 1.2. Ergodic complex structures

DEFINITION 1.17. Let  $M$  be a complex manifold,  $\text{Teich}$  its Teichmüller space, and  $I \in \text{Teich}$  a point. Consider the set  $Z_I \subset \text{Teich}$  of all  $I' \in \text{Teich}$  such that  $(M, I)$  is biholomorphic to  $(M, I')$ . Clearly,  $Z_I = \Gamma \cdot I$  is the orbit of  $I$ . A complex structure is called *ergodic* if the corresponding orbit  $Z_I$  is dense in  $\text{Teich}$ .

THEOREM 1.18. *Let  $M$  be a simple hyperkähler manifold or a compact complex torus of dimension  $\geq 2$ , and  $I$  a complex structure on  $M$ . Then  $I$  is non-ergodic if and only if the Neron–Severi lattice of  $(M, I)$  has maximal possible rank. This means that  $\text{rk } NS(M, I) = b_2(M) - 2$  for  $M$  hyperkähler, and  $\text{rk } NS(M, I) = (\dim_{\mathbb{C}} M)^2$  for  $M$  a torus.*

*Proof.* See [37]. □

### 1.3. Kobayashi pseudometric/pseudodistance

Let  $M$  be a complex manifold. Recall that a pseudometric on  $M$  is a function  $d$  on  $M \times M$  that satisfies all the properties of a metric (or distance function) except for the non-degeneracy condition:  $d(x, y) = 0$  only if  $x = y$ . The Kobayashi pseudometric (a.k.a. pseudodistance)  $d_M$  on  $M$  is defined as the supremum of all pseudometrics  $d$  on  $M$  that satisfy the distance-decreasing property with respect to holomorphic maps  $f$  from the Poincaré disk  $(\mathbb{D}, \rho)$  to  $M$ :

$$f^*d \leq \rho \quad \text{or equivalently} \quad d(f(x), f(y)) \leq \rho(x, y) \quad \forall x, y \in \mathbb{D}.$$

Here  $\rho$  denotes the Poincaré metric on  $\mathbb{D}$ .

The following is S. Kobayashi’s standard construction of  $d_M$ . Let

$$\delta_M(p, q) = \inf \{ \rho(x, y) \mid f : \mathbb{D} \rightarrow M \text{ holomorphic, } f(x) = p, f(y) = q \}.$$

Although it does not satisfy the triangle inequality, in general, this is a very useful invariant of the complex structure on  $M$ . For an ordered subset  $S = \{p_1, \dots, p_l\}$  of  $M$ , let

$$\delta_M^S(p, q) = \delta_M(p, p_1) + \delta_M(p_1, p_2) + \dots + \delta_M(p_l, q).$$

Then the triangle inequality is attained by setting

$$d_M(p, q) = \inf \delta_M^S(p, q),$$

where the infimum is taken over all finite-ordered subsets  $S$  in  $M$ .

Royden introduced an infinitesimal version of  $d_M$  as follows. The Kobayashi–Royden Finsler norm on  $TM$  is given, for  $v \in TM$ , by

$$|v|_M = \inf \left\{ \frac{1}{R} \mid f : \mathbb{D} \rightarrow M \text{ holomorphic, } R > 0, f'(0) = Rv \right\}.$$

It is the largest ‘Finsler’ pseudonorm on  $TM$  that satisfies the distance-decreasing property with respect to holomorphic maps from the Poincaré disk and therefore, it is automatically ‘distance-decreasing’ with respect to holomorphic maps. Royden showed that  $| \cdot |_M$  is upper semi-continuous and that  $d_M$  is the integrated version of  $| \cdot |_M$ , see [31]. In particular, this implies the well-known fact that  $d_M$  is a continuous function for a complex manifold  $M$ .

We recall that both the pseudometric and its infinitesimal version are insensitive to removing complex codimension 2 subsets of  $M$ .

**THEOREM 1.19.** *Let  $M$  be a complex manifold and  $Z \subset M$  be a complex analytic subvariety of codimension at least 2. (In fact, the same proof would work for any subset  $Z \subset M$  of Hausdorff codimension at least 3.) Then  $d_{M \setminus Z} = d_M|_{M \setminus Z}$  and  $|\cdot|_{M \setminus Z} = (|\cdot|_M)|_{M \setminus Z}$ .*

*Proof.* Theorems 3.2.19 and 3.5.35 in [21]. □

**COROLLARY 1.20.** *Let  $\tau : M \dashrightarrow M'$  be a birational equivalence of Calabi–Yau manifolds. Suppose that the Kobayashi pseudometric on  $M$  vanishes. Then it vanishes on  $M'$ .*

*Proof.* It is easy to check (see [15, Subsection 4.4]) that the exceptional set of  $\tau$  is a subvariety of codimension at least 2. Then Theorem 1.19 can be applied to obtain that the Kobayashi pseudometric vanishes on  $M$  and  $M'$  (by the distance-decreasing property) whenever it vanishes on the smooth locus of  $\tau$ . □

#### 1.4. Upper semi-continuity

Recall that a function  $F$  on a topological space  $X$  with values in  $\mathbb{R} \cup \{\infty\}$  is upper semi-continuous if and only if  $\{x \in X \mid F(x) < \alpha\}$  is an open set for every  $\alpha \in \mathbb{R}$ . It is upper semi-continuous at a point  $x_0 \in X$  if for all  $\varepsilon > 0$ , there is a neighborhood of  $x_0$  containing  $\{x \in X \mid F(x) < F(x_0) + \varepsilon\}$ . If  $X$  is a metric space, then this is equivalent to

$$\limsup_{t_i \rightarrow t_0} F(t_i) \leq F(t_0),$$

for all sequence  $(t_i)$  converging to  $t_0$ . From its very definition, the infimum of a collection of upper semi-continuous functions is again upper semi-continuous.

We will be interested in the upper semi-continuity of  $d_{M_t}$  and  $|\cdot|_{M_t}$  in the variable  $t$  for a proper smooth fibration  $\pi : \mathcal{M} \rightarrow T$ , that is,  $\pi$  is holomorphic, surjective, having everywhere of maximal rank and connected fibers  $M_t = \pi^{-1}(t)$ . This follows in the standard way as is for the case of  $|\cdot|_M$  by the following result of Siu.

**THEOREM 1.21 [33].** *Let  $f : D \rightarrow M$  be a holomorphic immersion of a Stein manifold  $D$  into a complex manifold  $M$ . Identify  $D$  as the zero section of the normal bundle  $X = f^*TM/TD$  of  $D$  in  $M$ . Then there is a holomorphic immersion of a neighborhood of  $D$  in  $X$  which extends  $f$ .*

Since  $\pi$  is locally differentiably trivial, we may assume that  $\mathcal{M}$  is differentiably a product  $T \times M$  and  $\pi$  its projection to the first factor. One can easily deduce from the above theorem of Siu applied to the graph of a holomorphic map from  $D = \mathbb{D}$  that  $\delta_{J(t)}(p, q)$  and  $|v|_{J(t)}$  are upper semi-continuous with respect to  $p, q \in M, v \in TM$  and  $t \in T$ , where we have replaced the subscript  $M_t$  by its associated complex structure  $J(t)$ . It follows then that  $\delta_{J(t)}^S(p, q)$  is upper semi-continuous with respect to  $p, q$  and  $t$ , and hence so is  $d_{J(t)}(p, q)$ . We have established the following proposition, cf. [40].

**PROPOSITION 1.22.** *Let  $\pi : \mathcal{M} \rightarrow T$  be a proper holomorphic and surjective map having everywhere of maximal rank and connected fibers  $M_t = \pi^{-1}(t)$ . Then  $d_{M_t}$  and  $|\cdot|_{M_t}$  are upper semi-continuous with respect to all variables involved, including  $t$ .*

Although we will not need this, a little reflection will show that one can relax many of the conditions on  $\pi$ . An immediate consequence of the above proposition is the following.

**COROLLARY 1.23.** *For  $M$ , a compact complex manifold, let  $\text{diam}(M)$  be the diameter of  $M$  with respect to  $d_M$ . Then  $\text{diam}(M)$  is upper semi-continuous with respect to the variation of the complex structure on  $M$ .*

*Proof.* We need to show that  $\text{diam}(M_t)$  is upper semi-continuous with respect to  $t$  for a family as given above, that is, for all  $t_0 \in T$  and sequences  $(t_i)$  converging to  $t_0$ ,

$$\limsup_{t_i \rightarrow t_0} \text{diam}(M_{t_i}) \leq \text{diam}(M_{t_0}).$$

If the inequality is false, then after replacing the sequence  $(t_i)$  by a subsequence there is an  $\varepsilon > 0$  such that  $\text{diam}(M_{t_i}) > \text{diam}(M_{t_0}) + \varepsilon$  for all  $i$ . By compactness and the continuity of the pseudometric on each  $M_t$ , there exist  $p_i, q_i$  such that  $\text{diam}(M_{t_i}) = d_{M_{t_i}}(p_i, q_i)$ . Replacing by a further subsequence if necessary, we may assume that the sequences  $(p_i)$  and  $(q_i)$  are convergent. Let  $p, q \in M_{t_0}$  be their respective limit. Then by upper semi-continuity, we have

$$\text{diam}(M_{t_0}) + \varepsilon \leq \limsup_{i \rightarrow \infty} d_{M_{t_i}}(p_i, q_i) \leq d_{M_{t_0}}(p, q) \leq \text{diam}(M_{t_0}).$$

This is a contradiction. □

## 2. Vanishing of the Kobayashi pseudometric

### 2.1. Kobayashi pseudometric and ergodicity

The main technical result of this paper is the following theorem. Recall that an ergodic complex structure  $I$  on  $M$  is one which has a dense  $\text{Diff}(M)$ -orbit in the deformation space of complex structures.

**THEOREM 2.1.** *Let  $M$  be a complex manifold with vanishing Kobayashi pseudometric. Then the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.*

*Proof.* Let  $\text{diam} : \text{Teich} \rightarrow \mathbb{R}^{\geq 0}$  map a complex structure  $I$  to the diameter of the Kobayashi pseudodistance on  $(M, I)$ . By Corollary 1.23, this function is upper semi-continuous. Let  $I$  be an ergodic complex structure. The set of points  $I' \in \text{Teich}$  such that  $(M, I')$  is biholomorphic to  $(M, I)$  is dense, because  $I$  is ergodic. By upper semi-continuity,  $0 = \text{diam}(I) \geq \inf_{I' \in \text{Teich}} \text{diam}(I')$ . □

**COROLLARY 2.2.** *Let  $M$  be a K3 surface. Then the Kobayashi pseudometric on  $M$  vanishes.*

*Proof.* Note that any non-ergodic complex structure on a hyperkähler manifold is projective. Indeed, if the rank of the Picard group is maximal, then the set of rational  $(1, 1)$ -classes is dense in  $H^{1,1}(M)$ , hence the Kähler cone contains a rational class and  $M$  is projective. For all projective  $M$ , one has  $\text{diam}(M) = 0$  (see [38, Lemma 1.51] or [23, Corollary 4.5]). Therefore, Theorem 2.1 implies that  $\text{diam}(M) = 0$  for non-projective complex structures as well. □

The same argument leads to the following result.

**THEOREM 2.3.** *Let  $M$  be a hyperkähler manifold admitting a complex structure with vanishing Kobayashi pseudometric and  $b_2(M) \geq 4$ . Then the Kobayashi pseudometric vanishes for all complex structures  $I$  in the same deformation class.*

*Proof.* The diameter of the Kobayashi pseudometric is upper semi-continuous, by Corollary 1.23. Choose any ergodic complex structure  $J$  on  $M$  (such  $J$  exists because  $b_2(M) > 3$ ). By definition of ergodic complex structures, in any neighborhood of  $I$  one has a complex manifold isomorphic to  $(M, J)$ . By upper semi-continuity, one has  $\text{diam}(M, J) \leq \text{diam}(M, I) = 0$ . Now vanishing of the Kobayashi pseudometric follows from Theorem 2.1.  $\square$

## 2.2. Lagrangian fibrations in hyperkähler geometry

The theory of Lagrangian fibrations on hyperkähler manifolds is based on the following remarkable theorem of Matsushita.

**THEOREM 2.4** [28]. *Let  $M$  be a simple hyperkähler manifold, and  $\varphi : M \rightarrow X$  a surjective holomorphic map, with  $0 < \dim X < \dim M$ . Then the fibers of  $\varphi$  are Lagrangian subvarieties on  $M$ , and the general fibers of  $\varphi$  are complex tori. (These fibers are known to be abelian varieties, see [8, Proposition 3.3].)*

**REMARK 2.5.** Such a map is called a *Lagrangian fibration*. All the known examples of hyperkähler manifolds admit Lagrangian fibrations [18, Claim 1.20].

**DEFINITION 2.6.** A cohomology class  $\eta \in H^2(M, \mathbb{R})$  is called *nef* if it lies in the closure of the Kähler cone; a line bundle  $L$  is nef if  $c_1(L)$  is nef. A nef line bundle  $L$  is *big* if  $\int_M c_1(L)^{\dim_{\mathbb{C}} M} \neq 0$ . A non-trivial nef line bundle  $L$  on a hyperkähler manifold is called *parabolic* if it is not big. From the definition of the BBF form, this is equivalent to  $q(c_1(L), c_1(L)) = 0$ . Lagrangian fibrations are in bijective correspondence with semi-ample parabolic bundles, as follows from Matsushita’s theorem.

**CLAIM 2.7.** *Let  $M$  be a simple hyperkähler manifold, and  $L$  a non-trivial semi-ample bundle on  $M$ . Assume that  $L$  is not ample. Consider the holomorphic map  $\pi : M \rightarrow \text{Proj}(\bigoplus_N H^0(M, L^N))$ . Then  $\pi$  is a Lagrangian fibration. Moreover, every Lagrangian fibration is uniquely determined by a parabolic nef line bundle.*

*Proof.* The first statement of Claim 2.7 is a corollary of Theorem 2.4. Let  $M \xrightarrow{\pi} X$  be a Lagrangian fibration. By Matsushita’s results [28],  $X$  is projective and  $H^*(X) \cong H^*(\mathbb{C}P^n)$ . Denote by  $\eta \in H^2(M, \mathbb{Z})$  the ample generator of  $\text{Pic}(X)$ . Then  $\pi^*\eta = c_1(L)$ , where  $L = \pi^*\mathcal{O}_X(1)$  is a parabolic nef bundle on  $M$ .  $\square$

The *SYZ conjecture* [32, 36] claims that any parabolic nef line bundle on a hyperkähler manifold is semi-ample, that is, it is associated with a Lagrangian fibration. This is true for K3 surfaces (as it follows from the Riemann–Roch formula) and for all deformations of Hilbert schemes of K3 surfaces [2, 27].

Further on, we shall need the following birational version of Matsushita’s theorem on Lagrangian fibrations, which is due to Matsushita–Zhang.

**THEOREM 2.8** [29, Theorem 1.4]. *Let  $X$  be a projective hyperkähler manifold, and  $\overline{BK}(X)$  be the closure of a union of all Kähler cones for all birational models of  $X$ . Consider an effective  $\mathbb{R}$ -divisor  $P \in \overline{BK}(X)$ . Then there exists a birational modification  $\tau : X' \dashrightarrow X$ , where  $X'$  is a projective hyperkähler manifold such that  $\tau^*P$  is nef.*

**THEOREM 2.9.** *Let  $M$  be a projective hyperkähler manifold, and  $L$  a line bundle of Kodaira dimension  $\frac{1}{2} \dim_{\mathbb{C}} M$ . Then there exists a birational modification  $\tau : M' \dashrightarrow M$  from a projective hyperkähler manifold such that  $\tau^*L$  is semi-ample, and induces a Lagrangian fibration as in Claim 2.7.*

*Proof.* Let  $L$  be a nef bundle on a Kähler manifold. Recall that the *numerical Kodaira dimension* of  $L$  is the maximal  $k$  such that  $c_1(L)^k \neq 0$ . The *Kodaira dimension* of  $L$  is the Krull dimension of the ring  $\bigoplus_{\mathbb{N}} H^0(M, L^N)$ .

Consider the modification  $\tau : M' \dashrightarrow M$  produced by the Matsushita–Zhang theorem. Then the numerical dimension of  $\tau^*L$  is equal to  $\frac{1}{2} \dim_{\mathbb{C}} M$ , by [34], and the Kodaira dimension stays the same. As shown in [19, Theorem 1] (see also [1, Proposition 2.8]), whenever the numerical dimension of a nef bundle is equal to its Kodaira dimension, the bundle is semi-ample. Then Theorem 2.9 follows from Claim 2.7. □

This result motivates the following definition.

**DEFINITION 2.10.** Let  $\tau : M' \dashrightarrow M$  be a birational map of hyperkähler manifolds, and  $\mathcal{L}$  a Lagrangian fibration on  $M$ . Then  $\tau^*\mathcal{L}$  is called a *birational Lagrangian fibration on  $M'$* . Its fibers are proper preimages of those fibers of  $\mathcal{L}$  which are not contained in the exceptional locus of  $\tau$ .

### 2.3. Kobayashi pseudometrics and Lagrangian fibrations

The idea to use Theorem 2.11 is suggested by Claire Voisin. We are very grateful to Prof. Voisin for her invaluable help.

**THEOREM 2.11.** *Let  $M$  be a simple hyperkähler manifold admitting two Lagrangian fibrations associated with two non-proportional parabolic classes. Then the Kobayashi pseudometric on  $M$  vanishes.*

*Proof.* Let  $\pi_i : M \rightarrow X_i$ ,  $i = 1, 2$ , be the Lagrangian fibration maps. Since the general fibers of  $\pi_i$  are tori, the Kobayashi pseudometric vanishes on each fiber of  $\pi_i$ . To prove that the Kobayashi pseudometric vanishes on  $M$ , it would suffice to show that a general fiber of  $\pi_1$  intersects all the fibers of  $\pi_2$ .

Let now  $\omega_i$  be an ample class of  $X_i$  lifted to  $M$ , and  $2n = \dim_{\mathbb{C}} M$ . Since  $\omega_1$  and  $\omega_2$  are not proportional, the standard linear-algebra argument, often called the Hodge index formula, implies that  $q(\omega_1, \omega_2) \neq 0$ . Indeed,  $q(\omega_1, \omega_2) = 0$  or else the space  $(H^{1,1}(M, \mathbb{R}), q)$  would contain a two-dimensional isotropic plane while its signature is  $(1, b_2 - 3)$ .

Clearly, the fundamental class  $[Z_i]$  of a fiber of  $\pi_i$  is proportional to  $\omega_i^n$ . Fix the constant multiplier in such a way that  $[Z_i] = \omega_i^n$ . The fibers of  $\pi_1$  intersect that of  $\pi_2$  if  $\int_M [Z_1] \wedge [Z_2] > 0$ . However, Fujiki’s formula (see Remark 1.12) shows that  $\int_M [Z_1] \wedge [Z_2] = Cq(\omega_1, \omega_2)^n > 0$ . This means that  $Z_1$  and  $Z_2$  always intersect. □



Further on, we shall use a birational version of this statement.

**THEOREM 2.12.** *Let  $M$  be a simple hyperkähler manifold admitting a Lagrangian fibration  $A$  and a birational Lagrangian fibration  $B$  associated to two non-proportional parabolic classes. Then the Kobayashi pseudometric on  $M$  vanishes.*

*Proof.* Let  $\tau : M \dashrightarrow M'$  be a birational modification such that  $B$  is the pullback of a Lagrangian fibration on  $M'$ . Since  $M$  and  $M'$  have trivial canonical bundle, the exceptional locus of  $\tau$  has codimension at least 2, hence a general fiber  $L$  of  $B$  is birationally equivalent to a torus outside of this exceptional locus. By Theorem 1.19, the Kobayashi pseudometric on  $L$  vanishes. The same argument as used in Theorem 2.11 shows that  $L$  meets all general fibers of  $A$ , and thus the Kobayashi pseudodistance between any two general points  $x, y$  in  $M$  vanishes. Indeed, take a general fiber  $L$  of  $B$ . Let  $x', y'$  be the points where the fibers of  $A$  associated with  $x, y$  intersect  $L$ . The Kobayashi pseudodistance  $d(x', y')$  vanishes, because it vanishes on  $L$ , and  $d(x, x') = d(y, y') = 0$ , because these are points in the same complex tori.  $\square$

**THEOREM 2.13.** *Let  $M$  be a simple hyperkähler manifold with a Lagrangian fibration  $\varphi : M \rightarrow X$ . Assume  $b_2(M) \geq 7$ . Then  $M$  has a deformation  $M'$  admitting both a Lagrangian fibration and a birational Lagrangian fibration that correspond to different classes  $\eta, \eta' \in H^2(M, \mathbb{Z})$ , respectively. Also  $M'$  is projective.*

*Proof.* Let  $\eta \in H^{1,1}(M)$  be a parabolic nef class associated with  $\varphi$  as in Claim 2.7. Denote by  $\text{Teich}_\eta$  the divisor parameterizing deformations of  $M$  for which  $\eta$  is of type (1,1). Denote by  $L$  the line bundle with  $c_1(L) = \eta$ . We can think of  $L$  as of a holomorphic line bundle on  $(M, I)$  for all  $I \in \text{Teich}_\eta$ .

When  $\text{rk Pic}(M, I) = 1$ , and the Picard group is generated by a non-negative vector, the positive cone is equal to the Kähler cone, as shown in [15]. In [18, Theorem 3.4], the following result was proved. Let  $D_0$  be the set of all  $J \in \text{Teich}_\eta$  such that  $L$  is semi-ample on  $(M, J)$ . Then  $D_0$  is dense in  $\text{Teich}_\eta$ , if it is non-empty. By Matsushita’s theorem, for such  $J$ , the image of the map  $(M, J) \rightarrow \text{Proj}(\bigoplus_N H^0(L^N))$  has dimension  $\frac{1}{2} \dim M$ . When  $J \notin D_0$ , the Kodaira dimension of  $L$  is at least  $\frac{1}{2} \dim_{\mathbb{C}} M$ , by upper semi-continuity. (Note that by Kawamata’s result [19, Theorem 1], the Kodaira dimension of  $L$  cannot exceed its numerical dimension, which is equal to  $\frac{1}{2} \dim M$ .) Since  $\eta$  is nef whenever  $\text{Pic}(M)$  is generated by  $\eta$ , we may assume that for all  $J \notin D_0$ ,  $\text{Pic}(M)$  contains a positive vector. As shown by Huybrechts, [15, Theorem 3.11; 16], for such  $J$  the manifold  $(M, J)$  is projective.

Applying Theorem 2.9, we obtain that  $(M, J)$  admits a birational Lagrangian fibration for all  $J \in \text{Teich}_J$  if  $L$  is semi-ample for at least one point  $J \in \text{Teich}_\eta$ .

Consider now the action of the monodromy group  $\Gamma^I$  on  $H^2(M, \mathbb{Z})$ . As follows from Remark 1.16,  $\Gamma^I$  is an arithmetic subgroup in  $O(H^2((M, \mathbb{Z}), q))$ . Therefore,  $\Gamma^I$  contains an element  $\gamma$  such that  $\gamma(\eta) \neq \eta$ . It is easy to see that the divisors  $\text{Teich}_\eta$  and  $\text{Teich}_{\gamma(\eta)}$  intersect transversally. Their intersection corresponds to a manifold  $M$  with two birational Lagrangian fibrations  $A$  and  $B$ . Now let  $M'$  be a birationally equivalent hyperkähler manifold where  $A$  is holomorphic, we obtain the statement of Theorem 2.13. (As shown by Huybrechts [15], birationally equivalent hyperkähler manifolds belong to the same deformation class.)  $\square$

**COROLLARY 2.14.** *Let  $M$  be a simple hyperkähler manifold with a Lagrangian fibration. Assume  $b_2(M) \geq 7$ . Then the Kobayashi pseudometric vanishes for all ergodic complex structures on  $M$ .*

*Proof.* Consider a deformation  $(M, I')$  of  $M$  admitting two birational Lagrangian fibrations. Then the Kobayashi pseudometric of  $(M, I')$  vanishes by Theorem 2.12. For an ergodic complex structure  $I$ , we obtain

$$\text{diam}(I) \leq \inf_{I' \in \text{Teich}} \text{diam}(I') = 0$$

by upper semi-continuity. □

### 3. Vanishing of the infinitesimal pseudometric

In this section, we are interested in conditions that guarantee the vanishing of the infinitesimal Kobayashi pseudometric  $| \cdot |_M$  on a Zariski dense open subset of  $M$ . Recall that the SYZ conjecture predicts the existence of a Lagrangian fibration for every hyperkähler  $M$ ,  $\dim_{\mathbb{C}} M = 2n$ . Furthermore, if the base of the fibration is smooth (this is conjectured, see [17]), then the base is isomorphic to  $\mathbb{C}P^n$ , as shown by Hwang (see [14, 17]). If  $M$  is projective and admits an abelian fibration, then we have the following two results.

**THEOREM 3.1.** *Let  $M$  be a projective manifold with an equidimensional abelian fibration  $f : M \rightarrow B$  (holomorphic surjective with all fibers of the same dimension and general fibers isomorphic to abelian varieties), where  $B$  is a complex projective space of lower dimension. If  $f$  has no multiple fibers in codimension 1, then  $| \cdot |_M$  vanishes everywhere on  $M$ . In particular, if  $M$  is a projective hyperkähler manifold with a birational Lagrangian fibration over a non-singular base without multiple fibers in codimension 1, then  $| \cdot |_M$  vanishes everywhere.*

*Proof.* Let  $v \in T_x M$ . Then  $v$  can be regarded as the first-order part of some non-vertical  $k$ -jet  $\nu$ , which, in turn, push forward to a non-trivial jet prescription  $\mu$  at  $b = f(x) \in B$ . This jet prescription  $\mu$  is clearly satisfied by an algebraic holomorphic map  $h : \mathbb{C} \rightarrow B$ . Since this map can be chosen to avoid any subset of codimension 2 or more, we see by so doing that the pull back fibration  $M_h \rightarrow \mathbb{C}$  has no multiple fibers. Hence, all higher-order jet infinitesimal pseudometric vanishes on  $M_h$  by Theorem A.3. Since  $\nu$  is in the image of a  $k$ -jet on  $M_h$ , it also has zero  $k$ -jet infinitesimal pseudometric by the distance-decreasing property, and therefore  $|v|_M = 0$ . □

**THEOREM 3.2.** *Let  $M$  be a projective manifold. Let  $f : M \rightarrow B$  define an abelian fibration. Assume that there is a subvariety  $Z \subset X$  that dominates  $B$  and is birational to an abelian variety. Then  $| \cdot |_M$  vanishes everywhere above a Zariski dense open subset  $U$  in  $B$ . In particular, this holds for hyperkähler manifolds with  $b_2 \geq 5$  having two birational Lagrangian fibrations.*

*Proof.* By hypothesis and the resolution of singularity theorem,  $Z$  is the holomorphic image of a non-singular projective variety  $A$  obtained from an abelian variety by blowing up smooth centers. By construction, any vector in  $A$  is in the tangent space of an entire holomorphic curve. Let  $g : A \rightarrow B$  be the composition with the projection to  $B$  and  $\text{disc}(g)$  its discriminant locus. Let  $v \in TM$  be a non-zero vector above the complement  $U$  in  $B$  of  $\text{disc}(f) \cup \text{disc}(g)$ . If  $v$  is vertical, then it is a vector on the fiber  $A$  through  $p$ , which is an abelian variety and clearly  $|v|_M \leq |v|_A = 0$  in this case. If  $v$  is horizontal, then there is a vector  $v' \in TA$  by construction such that  $f_* v = g_* v'$ . Let  $h : \mathbb{C} \rightarrow A$  be such that  $h'(0) = v'$  and  $\pi : M_h \rightarrow \mathbb{C}$  be the pull back fibration via the base change by  $g \circ h$ . Then  $\pi$  has no multiple fibers and  $v$  lies in the image of  $TM_h$  by construction. Theorem A.2 from the Appendix now applies to show that  $| \cdot |_{M_h}$

vanishes, and so the distance-decreasing property yields  $|v|_M = 0$ . The last statement follows from the projectivity of  $M$  by the proof of Corollary 2.2.  $\square$

We also have the following result modulo the SYZ conjecture.

**THEOREM 3.3.** *Let  $M$  be a simple hyperkähler manifold with  $b_2 \geq 7$ . Assume that the SYZ conjecture is true for any deformation of  $M$ , and the Picard rank of  $M$  is maximal. Then  $M$  admits two birational Lagrangian fibrations.*

*Proof.* Consider a non-zero integral vector  $z \in \text{Pic}(M)$  such that  $q(z, z) = 0$ , where  $q$  denotes the Beauville–Bogomolov–Fujiki form. Since  $\text{rk Pic}(M) \geq 5$ , such a vector exists by Meyer’s Theorem (see [9, p. 75]). As shown in the proof of Theorem 2.13,  $z$  is associated with a birational Lagrangian fibration. Denote by  $\Gamma_1$  the group of automorphisms of the lattice  $\text{Pic}(M)$ . Since this group is arithmetic, it contains an element  $\gamma$  which does not preserve  $z$ . Then  $\gamma(z)$  is another vector associated with a birational Lagrangian fibration.  $\square$

The above theorems together imply the following corollary.

**COROLLARY 3.4.** *Let  $M$  be a simple hyperkähler manifold with  $b_2 \geq 7$ . Assume that the SYZ conjecture is true for any deformation of  $M$ . Then  $d_M$  is identically zero. If, further, the rank of  $\text{Pic}(M)$  is at least 5, then  $| \cdot |_M$  vanishes on a dense Zariski open subset of  $M$ .*

*Proof.* By the proof of Corollary 2.2,  $M$  is projective and therefore Theorem 3.2 applies.  $\square$

We remark again the expectation that the above assumption on the rank of the Picard group and on  $b_2(M)$  should always hold for hyperkähler manifolds of maximal Picard rank, and hence for non-ergodic hyperkähler manifolds.

### Appendix. On abelian fibrations

The following are some relevant basic results concerning abelian fibrations found in [23], which was cited and used in [7, 24, 25]. Recall that a fibration is a proper surjective map with connected fibers. All fibrations are assumed to be projective in this section, and abelian fibrations are those whose general fibers are abelian varieties.

**PROPOSITION A.1.** *Let  $e : P \rightarrow \mathbb{D}$  define an abelian fibration which, outside  $0 \in \mathbb{D}$ , is smooth with abelian varieties as fibers. Let  $n_0$  be the multiplicity of the central fiber  $P_0$ . Then there is a component of multiplicity  $n_0$  in  $P_0$ .*

*Proof.* We may reduce the problem to the case of  $n_0 = 1$  by the usual base change  $z \mapsto z^{n_0}$ , so that the resulting object (after normalization) is again such a fibration with an unramified cover to the original  $P$ . Let  $\{m_1, m_2, \dots, m_k\}$  be the set of multiplicities of the components of  $P_0$ . By assumption, there exists integers  $l_i$  such that  $l_1 m_1 + l_2 m_2 + \dots + l_k m_k = 1$ .

As fibrations are assumed to be projective in this paper, we may assume that  $f$  is algebraic. By restricting to  $\mathbb{D}_r = \{t : |t| < r\}$  for an  $r < 1$  if necessary, we can construct an algebraic multi-section  $s_i$  with multiplicity  $m_i$  above  $\mathbb{D}_r$  by simply taking a one-dimensional algebraic slice transversal to the  $i$ th component for each  $i$ . Above each point  $t$  outside 0,  $s_i$  consists of

$m_i$  points  $s_i^j(t)$ ,  $j = 1, 2, \dots, m_i$ . Then it is easy to verify that

$$\left( l_1 \sum_j s_1^j(t) \right) + \left( l_2 \sum_j s_2^j(t) \right) + \dots + \left( l_k \sum_j s_k^j(t) \right)$$

is independent of the choice of an origin in the abelian variety  $P_t$ . This gives a section  $s$  of the fibration outside 0, and we now show that  $s$  must be algebraic, giving a section of  $f$  and establishing our proposition.

This can be accomplished by looking at the base change via  $z \mapsto t = z^m$ , where  $m$  is the least common multiple of  $m_1, \dots, m_k$ . Then each  $s_i$  lifts to  $m_i$  sections which the cyclic Galois action permutes. Hence, the Galois action of  $Z_m$  acts transitively on the  $m$  sections constructed by replacing the  $i$ th term in the above expressing with each of the  $m_i$  sections, and so this set of sections descends to a section of the original fibration as desired.  $\square$

We remark that the above proposition is really a special case of a result of Lang and Tate found in [22].

This proposition allows us to do exactly the same analysis as in the case for elliptic fibrations done in [6] to obtain the following theorem. We refer the reader there or to [23] for the detail of the proof.

**THEOREM A.2.** *Let  $f : X \rightarrow C$  define an abelian fibration over a complex curve  $C$ . Then, for each  $s \in C$ , the multiplicity of the fiber  $X_s$  at  $s$  is the same as the minimum multiplicity  $m_s$  of the components of  $X_s$ . Let the  $\mathbb{Q}$ -divisor  $A = \sum_s (1 - 1/m_s)s$  be the resulting orbifold structure on  $C$ . Then the three conditions  $d_X = 0$  on  $X$ ,  $| \cdot |_X = 0$  on  $X$  and  $(C, A)$  is non-hyperbolic (that is,  $C$  is quasi-projective and  $e(C) - \deg A \geq 0$ ) are equivalent for such a fibration. In the case  $C$  is quasi-projective; these three conditions are equivalent to the absence of non-commutative free subgroups in  $\pi_1(X)$ , and to  $\pi_1(X)$  being solvable up to a finite extension.*

*Proof.* In the case  $(C, A)$  is uniformizable, we may pull back the fibration to the universal cover  $U$  of  $(C, A)$  with resulting fibration  $\tilde{f} : Y \rightarrow X$ . This is the case when  $C$  is not quasi-projective and otherwise when  $e(C) \leq \deg A$ , with equality if and only if  $U = \mathbb{C}$ , and when  $e(C) > \deg A$ , in which case either  $C = U = \mathbb{C}$  and  $A$  is supported at one point, or  $C = \mathbb{P}^1$  and  $A$  is supported at more than two points, see, for example, [12]. In all these cases,  $U$  is non-hyperbolic if and only if  $(C, A)$  is. By construction,  $Y$  has no multiple fibers over  $U$ , and is unramified over  $X$  (in codimension 1) so that all holomorphic curves in  $X$  lifts to  $Y$ . Hence, the Kobayashi pseudometrics and norms vanish on  $X$  if and only if it is so on  $Y$ , and so we only need to show the vanishing of  $| \cdot |_Y$  in this case since the fundamental group characterization in the quasi-projective case follows from the same characterization of the Galois group of the uniformization  $\tilde{f} : U \rightarrow C$ , and the exact sequence of fundamental groups of a fibration without multiple fibers. Note that in the case  $U = \mathbb{P}^1$ , to show that  $| \cdot |_Y$  vanish at a point above  $z \in U$  we may replace  $U$  by  $\mathbb{C} = U \setminus \{z\}$  since  $| \cdot |_Y \leq | \cdot |_{Y'}$ , where  $Y' := Y \setminus \tilde{f}^{-1}(z) \subset Y$ .

In the case  $(C, A)$  is not uniformizable, then  $C = \mathbb{P}^1$  and  $A$  is supported at one or two points, and the exact sequence of orbifold fundamental groups shows that  $\pi_1(X)$  is a quotient of  $\pi_1(X_s)$  for a general fiber  $X_s$ , hence abelian. Thus, it is suffice to show that for a point  $p \in X$ ,  $| \cdot |_X$  vanishes there in this case and, for this, it is sufficient to replace  $X$  by the complement of a fiber  $X_z$  different from the fiber  $X_w$  containing  $p$  and  $C$  by  $C \setminus \{z\}$ , where in the case  $w$  lies in the support of  $A$ , we choose  $z$  to be the other point in this support if one exists. Then  $(C, A)$  is uniformizable by  $\mathbb{C}$ .

Hence, it remains to show that  $|\cdot|_X$  vanishes at a point  $p$  for the case  $X$  has no multiple fibers and  $C = \mathbb{C}$ . In fact, given a finite jet prescription at  $p$ , we can find an entire holomorphic curve through  $p$  satisfying the jet prescription as follows. The jet prescription gives rise to a jet prescription at  $f(p) \in C$  which we assume, without loss of generality, to be the origin of  $C = \mathbb{C}$ . Let  $l$  be the first non-vanishing order of the latter jet and let  $\tilde{f}: Y \rightarrow \mathbb{C}$  be the pull back fibration by the base change  $z \mapsto z^l$ . Then the inverse function theorem allows us to translate the jet prescription at  $p$  to a section jet prescription on  $Y$  over  $0 \in \mathbb{C}$ . As there are no multiple fibers for  $\tilde{f}$ , Proposition A.1 yield the existence of local sections of  $\tilde{f}$  through any point of  $\mathbb{C}$ . The Cousin principles apply in this situation (that is, an analog of Weierstrass' theorem can be worked out, see [6, 23]) so that we can patch up a minimal covering family of such sections, including the one with the jet prescription, to give a global section of  $\tilde{f}$  with the jet prescription, and this gives the required entire holomorphic curve.  $\square$

Instead of restricting our attention to just the first-order jets for the infinitesimal pseudometric, one can generalize the definition of  $|\cdot|_X$  to jets of arbitrary finite order, see [23]. By their very definition, these infinitesimal pseudometrics dominates  $|\cdot|_X$  by truncating the jets to first order. The exact same proof as above yields the following generalization, see [23].

**THEOREM A.3.** *The above theorem holds if  $|\cdot|_X$  is replaced by its more general  $k$ th order jet version, for all integer  $k > 0$ .*

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### References

1. T. BAUER, F. CAMPANA, T. ECKL, S. KEBEKUS, T. PETERNELL, S. RAMS, T. SZEMBERG and L. WOTZLAW, 'A reduction map for nef line bundles', *Complex geometry (Göttingen, 2000)* (Springer, Berlin, 0000) 27–36.
2. A. BAYER and E. MACRI, 'MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations', Preprint, 2013, arXiv:1301.6968 [math.AG].
3. A. BEAUVILLE, 'Variétés Kähleriennes dont la première classe de Chern est nulle', *J. Diff. Geom.* 18 (1983) 755–782.
4. F. A. BOGOMOLOV, 'On the decomposition of Kähler manifolds with trivial canonical class', *Math. USSR-Sb.* 22 (1974) 580–583.
5. F. BOGOMOLOV, 'Hamiltonian Kähler manifolds', *Sov. Math. Dokl.* 19 (1978) 1462–1465.
6. G. BUZZARD and S. S.-Y. LU, 'Algebraic surfaces holomorphically dominable by  $\mathbb{C}^2$ ', *Invent. Math.* 139 (2000) 617–659.
7. F. CAMPANA, 'Orbifolds, special varieties and classification theory', *Ann. Inst. Fourier (Grenoble)* 54 (2004) 499–630.
8. F. CAMPANA, K. OGUSO and T. PETERNELL, 'Non-algebraic hyperkaehler manifolds', *J. Differential Geom.* 85 (2010) 397–424.
9. J. W. S. CASSELS, *Rational quadratic forms* (Academic Press, London, 1978).
10. F. CATANESE, 'A superficial working guide to deformations and moduli', Preprint, 2011, arXiv:1106.1368, *Advanced Lectures in Mathematics, Volume XXVI Handbook of Moduli, Volume III*, pp. 161–216 (International Press).
11. J.-P. DEMAILLY, 'Hyperbolic algebraic varieties and holomorphic differential equations', Talk given at the VIASM annual meeting, 25–26 August 2012, long expository version (135 pages), <http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/viasm2012expanded.pdf>.
12. H. M. FARKAS and I. KRA, *Riemann surfaces*, Graduate Texts in Mathematics 71 (Springer, New York, 1980).

13. A. FUJIKI, ‘On the de Rham cohomology group of a compact Kähler symplectic manifold’, *Adv. Stud. Pure Math.* 10 (1987) 105–165.
14. D. GREB and C. LEHN, ‘Base manifolds for Lagrangian fibrations on hyperkähler manifolds’, Preprint, 2013, arXiv:1303.3919v2.
15. D. HUYBRECHTS, ‘Compact hyperkähler manifolds: Basic results’, *Invent. Math.* 135 (1999) 63–113, alg-geom/9705025.
16. D. HUYBRECHTS, ‘Erratum to the paper: compact hyperkähler manifolds: basic results’, *Invent. math.* 152 (2003) 209–212, math.AG/0106014.
17. J.-M. HWANG, ‘Base manifolds for fibrations of projective irreducible symplectic manifolds’, *Invent. Math.* 174 (2008) 625–644.
18. L. KAMENOVA and M. VERBITSKY, ‘Families of Lagrangian fibrations on hyperkaehler manifolds’, Preprint, 2012, arXiv:1208.4626, 13 pages.
19. Y. KAWAMATA, ‘Pluricanonical systems on minimal algebraic varieties’, *Invent. Math.* 79 (1985) 567–588.
20. S. KOBAYASHI, ‘Intrinsic distances, measures and geometric function theory’, *Bull. AMS* 82 (1976) 357–416.
21. S. KOBAYASHI, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften 318 (Springer, Berlin, 1998).
22. S. LANG and J. TATE, ‘Principal homogeneous spaces over abelian varieties’, *Amer. J. Math.* 80 (1958) 659–684.
23. S. LU, ‘Multiply marked Riemann surfaces and the Kobayashi pseudometric on algebraic manifolds’, Preprint, 1999, arXiv:1308.1800.
24. S. LU, ‘The Kobayashi pseudometric on algebraic manifold and a canonical fibration’, Preprint, 2001, arXiv:math.AG/0206170 and MPIM online preprint.
25. S. S.-Y. LU, ‘Holomorphic curves on irregular varieties of general type starting from surfaces’, *Affine algebraic geometry*, CRM Proceedings & Lecture Notes 54 (American Mathematical Society, Providence, RI, 2011) 205–220.
26. E. MARKMAN, ‘Integral constraints on the monodromy group of the hyperkahler resolution of a symmetric product of a K3 surface’, *Int. J. Math.* 21 (2010) 169–223, arXiv:math/0601304.
27. E. MARKMAN, ‘Lagrangian fibrations of holomorphic-symplectic varieties of  $K3^{[n]}$ -type’, Preprint, 2013, arXiv:1301.6584 [math.AG].
28. D. MATSUSHITA, ‘On fibre space structures of a projective irreducible symplectic manifold’, alg-geom/9709033, math.AG/9903045, also in *Topology* 38 (1999) 79–83. Addendum, *Topology* 40 (2001) 431–432.
29. D. MATSUSHITA and D.-Q. ZHANG, ‘Zariski  $F$ -decomposition and Lagrangian fibration on hyperkähler manifolds’, Preprint, 2009, arXiv:0907.5311v4.
30. S. MORI and S. MUKAI, ‘The uniruledness of the moduli space of curves of genus 11’, *Lecture Notes in Math.* 1016 (1982) 334–353.
31. H. ROYDEN, ‘The extension of regular holomorphic maps’, *Proc. Amer. Math. Soc.* 43 (1974) 306–310.
32. J. SAWON, ‘Abelian fibred holomorphic symplectic manifolds’, *Turkish J. Math.* 27 (2003) 197–230, math.AG/0404362.
33. Y.-T. SIU, ‘Every Stein subvariety admits a Stein neighbourhood’, *Invent. Math.* 38 (1976) 89–100.
34. M. VERBITSKY, ‘Cohomology of compact hyperkähler manifolds and its applications’, *GAF* 6 (1996) 601–612.
35. M. VERBITSKY, ‘A global Torelli theorem for hyperkähler manifolds’, Preprint, 2009, arXiv: 0908.4121, 52 pages (to appear in *Duke Math. J.*).
36. M. VERBITSKY, ‘Hyperkähler SYZ conjecture and semipositive line bundles’, *GAF* 19 (2010) 1481–1493, arXiv:0811.0639, 21 pages.
37. M. VERBITSKY, ‘Ergodic complex structures on hyperkähler manifolds’, Preprint, 2013, arXiv:1306.1498, 22 pages.
38. C. VOISIN, ‘On some problems of Kobayashi and Lang; algebraic approaches’, *Cur. Dev. Math.* 2003 (2003) 53–125.
39. S. T. YAU, ‘On the Ricci-curvature of a complex Kähler manifold and the complex Monge-Ampère equation’, *Comm. Pure Appl. Math.* 31 (1978) 339–411.
40. M. G. ZAIDENBERG, ‘On hyperbolic embeddings of complements of divisors and the limiting behavior of the Kobayashi–Royden metric’, *Math. USSR-Sb.* 55 (1986) 55–70.

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