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TOPOLOGICAL RELATIONS ON WITTEN–KONTSEVICH AND HODGE POTENTIALS

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To the memory of our teacher
Vladimir Igorevich Arnold (1937–2010)

Abstract. Let $\overline{\mathcal{M}}_{g;n}$ denote the moduli space of genus g stable algebraic curves with n marked points. It carries the Mumford cohomology classes κ_i. A homology class in $H_*(\overline{\mathcal{M}}_{g;n})$ is said to be *κ-zero* if the integral of any monomial in the κ-classes vanishes on it. We show that any κ-zero class implies a partial differential equation for generating series for certain intersection indices on the moduli spaces. The genus homogeneous components of the Witten–Kontsevich potential, as well as of the more general Hodge potential, which include, in addition to ψ-classes, intersection indices for λ-classes, are special cases of these generating series, and the well-known partial differential equations for them are instances of our general construction.

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1. Introduction

Let $\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_{g;0}$ denote the moduli space of stable genus g complex algebraic curves, and let $\overline{\mathcal{M}}_{g;n}$ be the moduli space of stable genus g curves with n marked points. The spaces $\overline{\mathcal{M}}_{g;n}$ carry natural line bundles $\mathcal{L}_i$, $i = 1, \ldots, n$; the fiber of $\mathcal{L}_i$ at a point $(C; x_1, \ldots, x_n)$ is the cotangent line to C at the marked point x_i. As usual, we denote by $\psi_i \in H^2(\overline{\mathcal{M}}_{g;n})$ the first Chern class of $\mathcal{L}_i$, $\psi_i = c_1(\mathcal{L}_i)$. Witten [7] suggests notation

$$\langle \tau_{k_1} \ldots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{k_1} \ldots \psi_n^{k_n},$$

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where the genus g is determined from the dimension count

$$k_1 + \cdots + k_n = \dim \overline{\mathcal{M}}_{g;n} = 3g - 3 + n,$$

provided g is a nonnegative integer. He also introduces the generating function (the *Witten–Kontsevich potential*)

$$\begin{aligned} F(\hbar; t_0, t_1, t_2, \dots) &= F^0(t_0, t_1, \dots) + \hbar F^1(t_0, t_1, \dots) + \hbar^2 F^2(t_0, t_1, \dots) + \cdots \\ &= \sum_g \hbar^g \sum \langle \tau_{k_1} \dots \tau_{k_n} \rangle_g \frac{t_{k_1} \dots t_{k_n}}{|\mathrm{Aut}(k_1, \dots, k_n)|}, \\ &= \sum_g \hbar^g \sum \langle \tau_0^{l_0} \tau_1^{l_1} \dots \rangle_g \frac{t_0^{l_0} t_1^{l_1} \dots}{l_0!\, l_1! \dots}; \end{aligned}$$

here the internal sums are taken either over all unordered tuples $k_1, \dots, k_n$ of arbitrary length, or over all eventually zero sequences $l_0, l_1, l_2, \dots$ of nonnegative integers.

The Witten conjecture [7], now possessing a variety of different proofs, the first one due to Kontsevich, states that the function F is a solution to the KdV hierarchy of partial differential equations. In particular, it is a solution to the first KdV equation

$$\frac{\partial^2 F}{\partial t_0 \partial t_1} = \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 - \frac{\hbar}{12}\frac{\partial^4 F}{\partial t_0^4}. \tag{1}$$

In addition to the KdV equation, the Witten–Kontsevich potential satisfies the *string equation*

$$\frac{\partial F}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1} \frac{\partial F}{\partial t_i} + \frac{t_0^2}{2} \tag{2}$$

and the *dilaton equation*

$$\frac{\partial F}{\partial t_1} = \sum_{i=0}^{\infty} \frac{2i+1}{3} t_i \frac{\partial F}{\partial t_i} + \frac{\hbar}{24}.$$

Together, all these equations determine the Witten–Kontsevich potential uniquely (we refer the reader to the end of Sec. 4 for the proof of this well-known fact). It is convenient sometimes to rewrite the dilaton equation for the genus g component F^g of the Witten–Kontsevich potential in the form

$$\frac{\partial F^g}{\partial t_1} = \sum_{i=0}^{\infty} t_i \frac{\partial F^g}{\partial t_i} - \chi_g F^g + \frac{1}{24}\delta_{g,1}, \tag{3}$$

where we denote by $\chi_g = 2 - 2g$ the Euler characteristic of the curves used in the definition of the moduli space $\overline{\mathcal{M}}_g$; below, we make use of this form of the dilaton equation.

Respectively, the first terms of the KdV equation (1) acquire in the genus (that is, with respect to the parameter $\hbar$) expansion the form

$$\frac{\partial^2 F^0}{\partial t_0 \partial t_1} = \frac{1}{2}\left(\frac{\partial^2 F^0}{\partial t_0^2}\right)^2,$$
$$\frac{\partial^2 F^1}{\partial t_0 \partial t_1} - \frac{\partial^2 F^0}{\partial t_0^2}\frac{\partial^2 F^1}{\partial t_0^2} = \frac{1}{12}\frac{\partial^4 F^0}{\partial t_0^4},$$
$$\frac{\partial^2 F^2}{\partial t_0 \partial t_1} - \frac{\partial^2 F^0}{\partial t_0^2}\frac{\partial^2 F^2}{\partial t_0^2} = \frac{1}{2}\left(\frac{\partial^2 F^1}{\partial t_0^2}\right)^2 + \frac{1}{12}\frac{\partial^4 F^1}{\partial t_0^4}.$$

Here is a couple of examples of differential relations for the genus homogeneous parts F^g of the Witten potential obtained in a different way:

$$\frac{\partial F^1}{\partial t_1} - \frac{\partial^2 F^0}{\partial t_0^2}\frac{\partial F^1}{\partial t_0} = \frac{1}{24}\frac{\partial^3 F^0}{\partial t_0^3}, \tag{4}$$
$$\frac{\partial^3 F^0}{\partial t_0^3}\frac{\partial F^1}{\partial t_0} = \frac{1}{24}\frac{\partial^4 F^0}{\partial t_0^4}. \tag{5}$$

Examples 3.7 and 3.8 give a geometric derivation of these relations.

The relations coming from KdV also are of topological origin. They are as well related to the homology classes represented by linear combinations of boundary strata in the moduli spaces $\overline{\mathcal{M}}_g$. Below, we describe a systematic way to obtain such relations, show how each one of them can be expressed in this way and extend them to more general potentials.

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2. Calculus of Descendent Potentials

Descendent potentials incorporate behavior of intersection indices as the number of marked points increases and ψ-classes are added at the new points. It is a common knowledge that they satisfy certain generalizations of the string and the dilaton equation (see, for example, [6]). Below, we extend the name 'descendent potential' to generating functions for intersection indices of ψ-classes, in spite of the fact that these functions are not Gromov–Witten invariants of any actual variety.

In addition to genus homogeneous parts F^g of the Witten–Kontsevich potential, it is natural to consider their following generalizations. Pick a space $\mathcal{N} = \overline{\mathcal{M}}_{g;M}$, which is the moduli space of stable genus g curves with M marked points. We will treat these M marked points as being "frozen", add to them m "floating" points that we denote by $x_1, \ldots, x_m$, and consider the intersection indices of the ψ-classes attached to these latter points. Let $\mathcal{N}_m$ denote the space $\overline{\mathcal{M}}_{g;M+m}$, $m = 0, 1, 2, \ldots$, so that $\mathcal{N} = \mathcal{N}_0$. Introduce the generating function

$$f_{\mathcal{N}}(t_0, t_1, \ldots) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1,\ldots,k_m} t_{k_1} \ldots t_{k_m} \int_{\mathcal{N}_m} \psi_1^{k_1} \ldots \psi_m^{k_m},$$

where $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g;M+m})$ is the first Chern class of the line bundle of cotangent lines to the curve at the point x_i, $i = 1, \ldots, m$. Hence, for $g \geqslant 2$, F^g coincides with $f_{\mathcal{N}}$, where $\mathcal{N} = \overline{\mathcal{M}}_{g;0}$.

Knowing the function F^g allows one to determine immediately the functions $f_{\mathcal{N}}$ for all $\mathcal{N} = \overline{\mathcal{M}}_{g;M}$:

Proposition 2.1. *For $\mathcal{N} = \overline{\mathcal{M}}_{g;M}$, we have*

$$f_{\mathcal{N}} = \frac{\partial^M F^g}{\partial t_0^M}.$$

Proof. By definition, we have

$$\begin{aligned} f_{\mathcal{N}} &= \sum \langle \tau_0^{k_0+M} \tau_1^{k_1} \dots \rangle_g \frac{t_0^{k_0}}{k_0!} \frac{t_1^{k_1}}{k_1!} \dots \\ &= \frac{\partial^M}{\partial t_0^M} \sum \langle \tau_0^{k_0+M} \tau_1^{k_1} \dots \rangle_g \frac{t_0^{k_0+M}}{(k_0+M)!} \frac{t_1^{k_1}}{k_1!} \dots = \frac{\partial^M F^g}{\partial t_0^M}. \qquad \square \end{aligned}$$

This construction can be generalized. Namely, let $\beta \in H^*(\mathcal{N})$ be a cohomology class. Consider the generating function

$$f_{\mathcal{N};\beta}(t_0, t_1, \dots) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1, \dots, k_m} t_{k_1} \dots t_{k_m} \int_{\mathcal{N}_m} \pi_m^*(\beta) \psi_1^{k_1} \dots \psi_m^{k_m},$$

where $\pi_m \colon \mathcal{N}_m \to \mathcal{N} = \mathcal{N}_0$ is the morphism forgetting the m floating points. Such a generating function is called the *descendent potential* defined by the moduli space $\mathcal{N}$ and the cohomology class β.

The ψ-classes at the M frozen points can be added to the expression for the potential in the following way. Introduce notation $U = \frac{\partial^2 F^0}{\partial t_0^2}$, and consider the following sequence of vector fields in the variables t_i:

$$V_0 = \frac{\partial}{\partial t_0}, \quad V_1 = \frac{\partial}{\partial t_1} - U \frac{\partial}{\partial t_0}, \quad V_2 = \frac{\partial}{\partial t_2} - U \frac{\partial}{\partial t_1} + \frac{1}{2} U^2 \frac{\partial}{\partial t_0}, \quad \dots,$$

$$V_k = \sum_{i=0}^{k} (-1)^i \frac{U^i}{i!} \frac{\partial}{\partial t_{k-i}}, \quad \dots.$$

The derivatives of the potentials with respect to the vector fields V_k have the following geometric interpretation.

Proposition 2.2. *Let $\mathcal{N} = \mathcal{N}_0$, and let $\pi_0 : \mathcal{N} \to \mathcal{N}_{-1}$ be the morphism forgetting the last frozen marked point. For a cohomology class $\beta_{-1} \in H^*(\mathcal{N}_{-1})$, set $\beta_0 = \pi_0^* \beta_{-1}$ and $\beta = \beta_0 \Psi^k$, where Ψ is the ψ-class assigned to the forgotten frozen marked point. Then we have*

$$f_{\mathcal{N};\beta} = V_k f_{\mathcal{N}_{-1};\beta_{-1}}.$$

Proof. The proof is achieved by computing the class $\pi_m^*(\Psi^k) = (\pi_m^* \Psi)^k$ and its intersection indices with the monomials in the ψ-classes. The argument is based on the detailed study of the geometry of the moduli space $\mathcal{N}_m$.

For two sequences of nonnegative integers $r_0, r_1, \dots, r_s$ and $l_1, \dots, l_s$, we introduce a cohomology class $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)} \in H^*(\mathcal{N}_m)$ in the following way. This class is supported on a union of codimension s boundary strata. A typical singular curve in these boundary strata consists of $s+1$ irreducible components, namely, a "principal" one, of genus g, and of s additional rational components attached to one another in the form of a "chain" or a "tail" (that is, the corresponding modular graph is a chain). The tail of rational components is attached to the principal component at the last frozen marked point, and the marking of this point is moved to the opposite end component of the chain. Thus, all the rational components have two points of intersection with other components, except for the last one, which, in addition to the single point of intersection with the neighboring component, carries also a point marked with the marking of the last frozen point.

Now, taking the i th intersection point of the irreducible components (counted from the "principal" one), we assign the r_{i-1} th power of the ψ-class at this point to the branch which is closer to the "principal" component, and the l_i th power of the ψ-class at the far branch, so that the "frozen" marked point at the last irreducible component carries the r_s th power of the ψ-class. The cohomology class $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$ is the product of these powers of the ψ-classes, and its definition will be completed if we say that the floating marked points can be distributed among the irreducible components in an arbitrary way provided the stability conditions are satisfied. It belongs to the cohomology group $H^{2d}(\mathcal{N}_m)$, where $d = s+\sum l_i+\sum r_j$.

It is easy to see that the class $\pi_m^*\Psi^k = (\pi_m^*\Psi)^k$ can be represented as a linear combination of the classes of the form $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$. For example, for $k=1$, we have Witten's identity

$$\pi_m^*\Psi = \Delta_m^1 - \Delta_m^{0;(0,0)}. \tag{6}$$

For $k>1$, the assertion follows from the case $k=1$ by induction (see below).

The contribution of the summand $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$ to the potential $f_{\mathcal{N};\beta}$ is computed explicitly by the argument similar to that in the proof of Proposition 2.1:

$$\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k_1,\dots,k_m} t_{k_1}\dots t_{k_m}\int_{\mathcal{N}_m}\pi_m^*(\beta_0)\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}\psi_1^{k_1}\dots\psi_m^{k_m}$$
$$= \frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_{r_0}}\prod_{i=1}^{s}\frac{\partial^2F^0}{\partial t_{l_i}\partial t_{r_i}}.$$

The expression on the right can be simplified by means of the equations of the *dispersionless KdV hierarchy* that, for the genus zero potential F^0, yield

$$\frac{\partial^2F^0}{\partial t_l\partial t_r} = \frac{U^{k+l+1}}{k!\,l!\,(k+l+1)} \tag{7}$$

(recall that $U=\partial^2F^0/\partial t_0^2$).

We conclude, therefore, that, for the cohomology class $\beta = \pi_0^*(\Psi^k\beta_{-1})\in H^*(\mathcal{N})$, the potential $f_{\mathcal{N};\beta}$ is expressed as a linear combination of summands of the form $U^i\,\partial f_{\mathcal{N}_{-1};\beta_{n-1}}/\partial t_{k-i}$, $i=0,\dots,k$. It remains to compute the coefficients of this linear combination.

Let us compute the product of $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$ by the class (6) in a more explicit form. Obviously,

$$\Delta_m^1\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)} = \Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s+1)}.$$

The computation of the intersection with the divisor $\Delta_m^{0;(0,0)}$ is more involved. Consider an irreducible component of the stratum representing $\Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$ and one of irreducible components of the divisor $\Delta_m^{0;(0,0)}$. If the divisor does not contain the irreducible component in question, then the intersection is transversal; it can be taken into account by adding an extra double point to the curve. If the divisor contains the irreducible component in question, then multiplication by the divisor is multiplication by the first Chern class of its normal bundle, which is minus the sum of the ψ-classes assigned to the two branches of the curve at one of its singular points.

We conclude that the contribution of the i-th rational component to the potential associated with the product $\Delta_m^1 \Delta_m^{r_0;(l_1,r_1);\dots;(l_s,r_s)}$ is given by the product

$$\frac{\partial^2 F^0}{\partial t_{l_i+1}\partial t_{r_i}} + \frac{\partial^2 F^0}{\partial t_{l_i}\partial t_{r_i+1}} - \frac{\partial^2 F^0}{\partial t_{l_i}\partial t_0}\frac{\partial^2 F^0}{\partial t_0 \partial t_{r_i}}$$
$$= \frac{U^{l_i+r_i+2}}{(l_i+1)!\,(r_i+1)!\,(l_i+r_i+2)}((r_i+1)+(l_i+1)-(l_i+r_i+2)) = 0.$$

In other words, *only the principal component of the singular curve contributes nontrivially to the resulting potential.* Thus, making the induction assumption

$$f_{\mathcal{N};\Psi^k\beta_0} = \sum_{i=0}^{k}(-1)^i\frac{U^i}{i!}\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{t_{k-i}},$$

we conclude by induction that

$$\begin{aligned} f_{\mathcal{N};\Psi^{k+1}\beta_0} &= \sum_{i=0}^{k}(-1)^i\frac{U^i}{i!}\left(\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_{k+1-i}} - \frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_0}\frac{\partial^2 F^0}{\partial t_0\partial t_{k-i}}\right) \\ &= \sum_{i=0}^{k}(-1)^i\frac{U^i}{i!}\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_{k+1-i}} - \sum_{i=0}^{k}(-1)^i\frac{U^{k+1}}{i!\,(k+1-i)!}\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_0} \\ &= \sum_{i=0}^{k+1}(-1)^i\frac{U^i}{i!}\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_{k+1-i}} - \frac{U^{k+1}}{(k+1)!}\frac{\partial f_{\mathcal{N}_{-1};\beta_{-1}}}{\partial t_0}\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}. \end{aligned}$$

The first summand in the last expression is exactly the one we wish to obtain, while the sum in the second summand is $(1-1)^{k+1}=0$. This completes the proof of Proposition 2.2. □

The notion of descendent potential naturally extends to the case where the base space $\mathcal{N}$ is a product of moduli spaces,

$$\mathcal{N} = \overline{\mathcal{M}}_{g_1;n_1}\times\dots\times\overline{\mathcal{M}}_{g_r;n_r}.$$

In this case, $\mathcal{N}_m$ is the disjoint union of spaces of the form

$$\overline{\mathcal{M}}_{g_1;n_1+m_1}\times\dots\times\overline{\mathcal{M}}_{g_r;n_r+m_r}$$

with $m_1+\dots+m_r=m$ and the natural forgetting maps.

Propositions 2.3–2.5 below are clear.

Proposition 2.3. *If $\mathcal{N} = \mathcal{N}' \times \mathcal{N}''$ and $\beta = \beta' \times \beta'' \in H^*(\mathcal{N})$, for $\beta' \in H^*(\mathcal{N}')$, $\beta'' \in H^*(\mathcal{N}'')$, then*

$$f_{\mathcal{N};\beta} = f_{\mathcal{N}';\beta'} f_{\mathcal{N}'';\beta''}.$$

This property shows that the space of power series in the variables t_0, t_1, t_2, ... spanned by the descendent potentials is a subalgebra in the algebra of all power series.

Let us denote

$$F^g_{k_1,\dots,k_r} = V_{k_r} \dots V_{k_1} F^g.$$

By a *differential polynomial in* F we mean a polynomial combination of the functions $F^g_{k_1,\dots,k_r}$, for various k_i and g. Propositions 2.1–2.3 show that any differential polynomial can be treated as a descendent potential. Note that the vector fields V_k do not commute, which reflects distinction between the ψ-classes and their pull-backs under the forgetful map, so that, generally speaking $F^g_{k_1,k_2} \neq F^g_{k_2,k_1}$, for example.

The space $\overline{\mathcal{M}}_{g;M}$ is stratified and the strata are marked by combinatorics of their generic curves. The codimension of the stratum is the number of double points in such a curve. Each of the species of the spaces $\mathcal{N}$ that we considered above can be treated in one of the two ways. Namely, in addition to treating it as a separate moduli space, we can also consider a mapping of this space into another moduli space $\overline{\mathcal{M}}_{g;M}$ making it into a boundary stratum (or, more generally, a collection of boundary strata) in the latter space. In this way, we can associate a descendent potential to a collection of boundary strata in $\overline{\mathcal{M}}_{g;M}$. The next two propositions explain how the descendent potentials associated to the boundary strata can be recovered from those associated to the moduli spaces.

Proposition 2.4. *Let $\mathcal{N} = \overline{\mathcal{M}}_{g_1;n_1+1} \times \overline{\mathcal{M}}_{g_2;n_2+1}$, and let $j\colon \mathcal{N} \to \overline{\mathcal{M}}_{g_1+g_2;n_1+n_2}$ be the mapping taking a pair of curves to the singular curve obtained by identifying the last marked points in each of the two (whose range is thus a codimension one boundary stratum in $\overline{\mathcal{M}}_{g_1+g_2;n_1+n_2}$). Then, for a $\beta \in H^*(\mathcal{N})$, we have*

$$f_{\mathcal{N};\beta} = f_{\overline{\mathcal{M}}_{g_1+g_2;n_1+n_2};j_*(\beta)}.$$

Proposition 2.5. *Let $\mathcal{N} = \overline{\mathcal{M}}_{g;n+2}$, and let $j\colon \mathcal{N} \to \overline{\mathcal{M}}_{g+1;n}$ be the mapping taking a curve to the singular curve obtained by identifying the two last marked points (whose range is thus a codimension one boundary stratum in $\overline{\mathcal{M}}_{g+1;n}$). Then, for a $\beta \in H^*(\mathcal{N})$, we have*

$$f_{\mathcal{N};\beta} = f_{\overline{\mathcal{M}}_{g+1;n};j_*(\beta)}.$$

Consider a boundary stratum $\mathcal{N}$ in $\overline{\mathcal{M}}_g$. Let $\beta \in H^*(\mathcal{N})$ be a polynomial in the ψ-classes assigned to the branches of the curve at singular points. Pushing the class β forward to the ambient space, we can treat it as a cohomology class in $\overline{\mathcal{M}}_g$ supported on $\mathcal{N}$. Let us denote the resulting class by $\beta \frown [\mathcal{N}]$.

Corollary 2.6. *The generating function*

$$f_{\mathcal{N};\beta} = f_{\overline{\mathcal{M}}_g;\beta\frown[\mathcal{N}]} = \sum \int_{\overline{\mathcal{M}}_{g;n}} (\pi^*_{g;n}\beta \frown [\mathcal{N}])\, \psi_1^{k_1} \dots \psi_n^{k_n} \frac{t_{k_1} \dots t_{k_n}}{|\mathrm{Aut}(k_1, \dots, k_n)|},$$

where summation is carried over all integers n and all unordered tuples $(k_1, \dots, k_n)$, *is a differential polynomial in* F.

We call the stratum $\mathcal{N} \subset \overline{\mathcal{M}}_g$ the *base space* of the potential $F_{\mathcal{N};\beta}$.

Proof. More explicitly, let $\mathcal{N} = \mathcal{N}_0$ be a disjoint union of products of moduli spaces, and let $j\colon \mathcal{N} \to \overline{\mathcal{M}}_{g;M}$ be a mapping to the moduli space making $\mathcal{N}$ into a union of boundary strata. Then the class in the corollary has the form

$$\beta = j_*(\Psi_1^{k_1}\Psi_2^{k_2}\dots\Psi_r^{k_r}),$$

where the k_i-th power of the ψ-class denoted by $\Psi_i^{k_i}$ is assigned to the i-th frozen marked point, $i = 1, \dots, r$.

On the other hand, let $\mathcal{N}_0 \to \mathcal{N}_{-1} \to \dots \to \mathcal{N}_{-r}$ be a sequence of morphisms $\pi_i : \mathcal{N}_i \to \mathcal{N}_{i-1}$, $i = 0, -1, \dots, -r+1$, forgetting frozen marked points one-by-one. Denote by $\beta_0 \in H^*(\mathcal{N})$ the cohomology class

$$\beta_0 = \Psi_1^{k_1}\pi_0^*(\Psi_2^{k_2}\pi_{-1}^*(\dots\pi_{-r+2}^*(\Psi_r^{k_r})\dots))$$

and set $\widetilde{\beta} = j_*(\beta_0)$. Then the above propositions show that

$$f_{\mathcal{N};\widetilde{\beta}} = V_{k_1}\dots V_{k_r} f_{\mathcal{N}_{-r};1}$$

is a differential monomial.

In general, the classes β and $\widetilde{\beta}$ do not coincide, unless $k_1 = \dots = k_r = 0$, since $\Psi_i \neq \pi_{i-1}^*\Psi_i$. However, the difference between the classes β and $\widetilde{\beta}$ can be computed explicitly. It is clear that the correction term is expressed as a linear combination of similar classes supported on the strata of bigger codimension. Acting by induction, we express the potential $f_{\mathcal{N};\beta}$ not as a single differential monomial, but as a linear combination of several differential monomials, q.e.d. □

Let us associate to each stratum $\mathcal{N} \subset \overline{\mathcal{M}}_{g;M}$ the "virtual Euler characteristic"

$$\chi_{\mathcal{N}} = 2 - 2g - M.$$

It is easy to see that this number is independent of the particular space $\overline{\mathcal{M}}_{g;M}$ chosen as the ambient space of the stratum (that is, increasing of the genus g in the ambient space forces decreasing of M by twice the same value). Moreover, it behaves nicely with respect to the operations on these spaces considered in Propositions 2.1–2.5. For example,

$$\chi_{\mathcal{N}_m} = \chi_{\mathcal{N}} - m, \qquad \chi_{\mathcal{N}'\times\mathcal{N}''} = \chi_{\mathcal{N}'} + \chi_{\mathcal{N}''},$$

and so on. General descendent potentials $F_{\mathcal{N};\beta}$ satisfy the string and the dilaton equations similar to that for the genus homogeneous generating functions F^g:

Proposition 2.7. *Each descendent potential* $F_{\mathcal{N};\beta}$ *with* $-\chi_{\mathcal{N}} > 0$ *satisfies the string and the dilaton equations*

$$\frac{\partial f_{\mathcal{N};\beta}}{\partial t_0} = \sum_{i=0}^{\infty} t_{i+1}\frac{\partial f_{\mathcal{N};\beta}}{\partial t_i}, \qquad \frac{\partial f_{\mathcal{N};\beta}}{\partial t_1} = \sum_{i=0}^{\infty} t_i\frac{\partial f_{\mathcal{N};\beta}}{\partial t_i} - \chi_{\mathcal{N}} f_{\mathcal{N};\beta}.$$

The proof repeats word-for-word the one for the potential F, see [7].

3. Boundary Strata in Moduli Spaces

In this section we convert the calculus of descendent potentials developed in the previous one into a tool for producing partial differential equations for the genus homogeneous parts F^g of the Witten–Kontsevich potentials and their generalizations.

The moduli space $\overline{\mathcal{M}}_{g;n}$ is endowed with the *universal curve*, $u\colon \overline{\mathcal{C}}_{g;n} \to \overline{\mathcal{M}}_{g;n}$, which is a fiber bundle whose fiber over a point $C \in \overline{\mathcal{M}}_{g;n}$ is the quotient of the curve C modulo its automorphism group. The universal curve $\overline{\mathcal{C}}_{g;n}$ can be identified in a natural way with the moduli space $\overline{\mathcal{M}}_{g;n+1}$. The powers of the first Chern class of the relative cotangent bundle of u are projected to certain cohomology classes on $\overline{\mathcal{M}}_{g;n}$ called the *κ-classes*, $\kappa_i = u_*((c_1(T^\vee(u))^{i+1}) \in H^{2i}(\overline{\mathcal{M}}_{g;n})$, $i = 0, 1, \dots$.

Proposition 3.1. *The pushforward of any monomial in ψ-classes, under the projection $\pi_{n;0}\colon \overline{\mathcal{M}}_{g;n} \to \overline{\mathcal{M}}_g$, is a polynomial of κ-classes.*

This statement can be proved by induction using the following induction step.

Lemma 3.2. *Under the projection $\pi_n\colon \overline{\mathcal{M}}_{g;n+1} \to \overline{\mathcal{M}}_{g;n}$, the direct image of any monomial in ψ and κ-classes is a polynomial in ψ and κ-classes.*

The proof consists in an explicit algorithm for computing the action of $\pi_{n;0*}$ on the monomials, see, e.g., [3]. Denote by D_i the part of the boundary divisor in $\overline{\mathcal{M}}_{g;n+1}$ consisting of singular curves having a rational irreducible component with only two marked points x_i and x_{n+1} in it. Put

$$K = \psi_{n+1} = c_1(T^\vee(\pi_n)) + D_1 + \cdots + D_n \in H^2(\overline{\mathcal{M}}_{g;n+1}).$$

Then $\psi_i = \pi_n^*\psi_i + D_i$, $i = 1, 2, \dots, n$, and $\kappa_k = \pi_n^*\kappa_k + K^k$. Now express a given monomial in ψ- and κ-classes in $\overline{\mathcal{M}}_{g;n+1}$ in terms of the pullbacks of those classes in $\overline{\mathcal{M}}_{g;n}$. Taking into account the equations $D_i \cdot D_j = 0$, for $i \neq j$, $D_i \cdot K = 0$, $D_i^k = (\pi_n^*\psi_i)^{k-1} D_i$, we represent this monomial as a linear combination of classes of the form

$$\pi_n^* x \cdot D_i \text{ and } \pi_n^* x \cdot K^k,$$

where x is a monomial in ψ- and κ-classes in $\overline{\mathcal{M}}_{g;n}$. Now, by the projection formula, in order to compute the action of π_{n*}, it suffices to know that

$$\pi_{n*} D_i = 1, \quad \pi_{n*} K^k = \begin{cases} 0, & k = 0 \\ 2g - 2 + n = -\chi_{g;n}, & k = 1 \\ \kappa_{k-1}, & k > 1. \end{cases}$$

The induction step is justified.

Remark 3.3. The above algorithm provides an explicit formula for the pushforward of any monomial in ψ-classes entering it in powers 2 or greater:

$$\pi_{n;0*}\psi_1^{k_1+1} \dots \psi_n^{k_n+1} = \sum_{I_1 \sqcup \dots \sqcup I_s = \{1,\dots,n\}} \kappa_{|I_1|} \dots \kappa_{|I_S|},$$

where, for a subset $I = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, we put $|I| = k_{i_1} + \cdots + k_{i_m}$.

Now we introduce the main notion of the paper.

Definition 3.4. We say that a homology class in $H_*(\overline{\mathcal{M}}_{g;M})$ *is κ-zero* if the integral of any monomial of κ-classes over this homology class vanishes.

In terms of this definition, the previous lemma can be reformulated as follows.

Corollary 3.5. *If $\beta \in H^*(\overline{\mathcal{M}}_{g;M})$ is κ-zero, then the associated descendent potential $f_{\overline{\mathcal{M}}_{g;M};\beta}$ is identically equal to zero.*

Of course, all κ-zero homology classes form a vector space, the annihilator of the vector space of polynomials in the κ-classes. Combining this corollary with the assertions of the previous section, we obtain the following result.

Theorem 3.6. *Let a cohomology class $\beta \in H^*(\overline{\mathcal{M}}_{g;M})$ be represented as a linear combination of summands, each summand being either the fundamental class of a boundary stratum or the class of the stratum multiplied by a monomial in the ψ-classes associated to the branches at the singular points of the curves. If β is κ-zero, then equating to* 0 *the descendent potential $f_{\overline{\mathcal{M}}_{g;M};\beta}$, which is a differential polynomial in F, produces a partial differential equation satisfied by the functions F^h.*

Example 3.7. In our example in Sec. 1, Equation (4) recovers the equality $\Psi = \frac{1}{12}\delta$ in the (one-dimensional) space $H^2(\overline{\mathcal{M}}_{1;1}) = H_0(\overline{\mathcal{M}}_{1;1})$, where δ is the boundary divisor in $\overline{\mathcal{M}}_{1;1}$. The left hand side equal to $F^1_1 = V_1F^1$ represents the potential associated with the class Ψ, and the function $\frac{1}{2}F^0_{0,0,0}$ from the right hand side is the potential associated with the boundary divisor. It consists of genus 0 curves with three marked points two of which are identified, whence the third t_0-derivative of F^0. An extra factor $1/2$ leading to the coefficient $1/24$ is due to the fact that the natural mapping $\overline{\mathcal{M}}_{0;3} \to \overline{\mathcal{M}}_{1;1}$ identifying two of the three marked points is two-fold. Thus both sides of (4) represent descendent potentials of equal cohomology classes on $\overline{\mathcal{M}}_{1;1}$.

Example 3.8. Equation (5) relates descendent potentials associated to codimension 2, that is, of dim 1, boundary strata in the moduli space $\overline{\mathcal{M}}_2$. The boundary $\partial\overline{\mathcal{M}}_2$ contains two codimension 2 strata distinguished by whether a typical curve has one or two irreducible components. The left hand side corresponds to the stratum with a separating contracted cycle; a typical curve in this stratum has two irreducible components, namely, an elliptic curve intersecting a projective line whose two points are glued to one another, whence the third t_0-derivative of F^0 times the first t_0-derivative of F^1. The right hand side in Eq. (5) corresponds to the stratum without separating contracted cycle; a typical curve in this stratum is a projective line with four distinct points glued in pairs, whence the fourth derivative of F^0.

The cohomology classes represented by these strata are independent in $H^4(\overline{\mathcal{M}}_2) = H_2(\overline{\mathcal{M}}_2)$. However, there is a unique κ-monomial, namely, κ_1, in $H^2(\overline{\mathcal{M}}_2)$. Therefore, the class κ_1 vanishes when integrated over a linear combination of these strata. It follows that the descendent potentials of these strata are proportional.

The potentials in our considerations are associated to cohomology classes on $\overline{\mathcal{M}}_{g;M}$. In addition to the fundamental classes of the boundary strata and their Ψ-classes extensions, we will consider also monomials in the classes $\lambda_k = c_k(\Lambda) \in$

$H^{2k}(\overline{\mathcal{M}}_{g;M})$, the Chern classes of the rank g Hodge bundle over $\overline{\mathcal{M}}_{g;M}$. For such a λ-monomial β, we introduce notation

$$F^{g;\beta}=f_{\overline{\mathcal{M}}_g;\beta},\qquad F^{g;\beta}_{k_1,\dots,k_r}=V_{k_r}\dots V_{k_1}f_{\overline{\mathcal{M}}_g;\beta}.$$

Example 3.9. Consider the following functions:

$$F^0_{0_6},\quad F^1_0F^0_{0_5},\quad F^1_{0,0}F^0_{0_4},\quad F^1_{0,0,0}F^0_{0,0,0},\quad (F^1_0)^2F^0_{0_4},\quad F^1_0F^1_{0,0}F^0_{0,0,0},\quad (F^1_0)^3F^0_{0,0,0},$$
$$F^1_{0,0,0,1},\quad F^1_1F^2_{0,0,0},\quad F^1_{0,1}F^2_{0,0},\quad F^1_{0,0,1}F^2_0,\quad F^1_0F^2_{0,0,1},\quad F^1_{0,0}F^2_{0,1},\quad F^1_{0,0,0}F^2_1,$$
$$F^2_2F^1_0,\quad F^2_1F^1_1,\quad F^2_{0,2},\quad F^2_{1,1},\quad F^{2,\lambda_1^2}_{0,0},\quad F^{2,\lambda_1}_{0,1},\quad F^{3,\lambda_3},\quad F^{3,\lambda_1^3}.$$

This is a (far from being complete[1]) list of possible descendent potentials associated with certain cohomology classes in $H^6(\overline{\mathcal{M}}_3)=H_6(\overline{\mathcal{M}}_3)$. These include boundary strata of codimension 3 as well as some classes of the same degree but supported on the strata of smaller codimension. The cohomology group $H^6(\overline{\mathcal{M}}_3)$ has dimension 10. It is spanned by the classes of 9 three-dimensional boundary strata and by one extra class that can be chosen in many different ways, see [2]. The quotient over the subspace of κ-zero classes is even smaller, it only has dimension 3, since there are three monomials of degree 3 in κ-classes, namely, κ_1^3, $\kappa_1\kappa_2$, κ_3. It follows that the potentials of the above list span a three-dimensional space of functions. Checking the Taylor coefficients at monomials of small degree, we can recover the linear dependencies between these potentials. For example, every potential of this list can be represented as a linear combination of the first three ones:

$$\begin{aligned}
F^1_{0,0,0}F^0_{0,0,0}&=\frac{1}{24}F^0_{0_6}-F^1_0F^0_{0_5}-2F^1_{0,0}F^0_{0_4},\\
(F^1_0)^2F^0_{0_4}&=\frac{1}{24}F^1_0F^0_{0_5}-\frac{1}{24}F^1_{0,0}F^0_{0_4},\\
&\dots\dots\dots\dots\dots\dots\\
F^{3,\lambda_3}&=\frac{31}{967680}F^0_{0_6},\\
F^{3,\lambda_1^3}&=\frac{23}{96768}F^0_{0_6}-\frac{17}{30240}F^1_0F^0_{0_5}-\frac{1}{756}F^1_{0,0}F^0_{0_4}.
\end{aligned}$$

A part of these relations are obvious formal corollaries of similar relations arising from the smaller moduli spaces.

As an additional example, let us recall the λ_g-conjecture by Faber and Pandharipande [4]. It states that

$$\int_{\overline{\mathcal{M}}_{g;n}}\psi_1^{m_1}\dots\psi_n^{m_n}\lambda_g=\binom{2g+n-3}{m_1,\,\dots,\,m_n}b_g$$

provided that $m_1+\dots+m_n+g=3g+n-3=\dim\overline{\mathcal{M}}_{g;n}$. Here $\lambda_g=c_g(\Lambda)\in H^{2g}(\overline{\mathcal{M}}_{g;n})$ is the top Chern class of the Hodge bundle, and b_g is a constant depending only on the genus g,

$$b_g=\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!},$$

[1] E.g., we never use λ_2 because of Mumford's identity $\lambda_1^2=2\lambda_2$.

B_{2g} being the $2g$ th Bernoulli number. Similarly to the Witten conjecture, the λ_g-conjecture now also possesses a number of different proofs. From the point of view of the present paper, it is equivalent to the following statement.

Proposition 3.10. *We have*

$$F^{g;\lambda_g} = b_g \frac{\partial^{2g} F^0}{\partial t_0^{2g}}.$$

Equivalently, the cohomology class $\lambda_g \in H^{2g}(\overline{\mathcal{M}}_g)$ is the sum of the codimension g boundary class represented by a rational curve with g double points times b_g and a κ-zero class.

Indeed, the intersection indices of ψ-classes over the pull-backs of the above mentioned boundary class are exactly as predicted by the λ_g-conjecture.

This example served as a guiding one for the development of the present paper.

4. The Algebra of Descendent Potentials

The geometric approach described in the previous sections allows one to deduce partial differential equations for descendent potentials based on κ-zero cohomology classes. However, these equations, once written out, can be proved also in a variety of ways. In this section, we describe a well-known approach for proving equations for descendent potentials based on expressing them as polynomials in certain basic generating series.

The Itzykson–Zuber ansatz [5] is a presentation of the potential F^g as a polynomial in a collection of particular explicitly given series in the variables t_i. Namely, the power series $U = \frac{\partial^2 F^0}{\partial t_0^2}$ can be alternatively determined by the implicit functional equation

$$U = t_0 + t_1 U + t_2 \frac{U^2}{2!} + t_3 \frac{U^3}{3!} + \cdots .$$

Let us set

$$U_k = \frac{U^{(k)}}{(U')^k} = \frac{\partial^{k+2} F^0}{\partial t_0^{k+2}} \left(\frac{\partial^3 F^0}{\partial t_0^3} \right)^{-k}$$

(where the prime denotes the t_0-derivative) and

$$T_k = \frac{I_k}{1 - I_1}, \quad \text{where } I_k = \sum_{i=0}^{\infty} t_{i+k} \frac{U^i}{i!}.$$

The sequences U_k and T_k are defined for $k \geqslant 2$.

Taking the derivative of the implicit equation for U with respect to t_0, we obtain

$$U' = \frac{1}{1 - I_1}, \quad U'' = \frac{I_2}{(1 - I_1)^3}, \quad \ldots,$$

whence, in particular, $T_2 = U_2$.

Proposition 4.1. *The two sequences U_k and T_k can be taken into one another by a polynomial change.*

In addition to the equation $T_2 = U_2$ that we already know, we also have

$$T_3 = U_3 - 3U_2^3$$
$$T_4 = U_4 - 10U_2U_3 + 15U_2^3$$
$$T_5 = U_5 - 15U_4U_2 - 10U_3^2 + 105U_3U_2^2 - 105U_2^4,$$

and so on, and, conversely,

$$U_3 = T_3 + 3T_2^3$$
$$U_4 = T_4 + 10T_2T_3 + 15T_2^3$$
$$U_5 = T_5 + 15T_4T_2 - 10T_3^2 + 105T_3T_2^2 - 105T_2^4, \ldots$$

In order to prove Proposition 4.1, introduce the differential operator $D = \frac{1}{U'}\frac{\partial}{\partial t_0}$. The definitions immediately give

$$D(U_k) = U_{k+1} - kU_2U_k$$
$$D(T_k) = T_{k+1} + T_2T_k,$$

which, together with the fact that D satisfies the Leibniz rule, imply an inductive procedure to express one sequence of series polynomially in terms of the other one.

Theorem 4.2 (Itzykson–Zuber ansatz, [5]). *For any $g > 1$, the series $(U')^{2-2g}F^g$ can be represented as a polynomial in the series $T_1, T_2, T_3, \ldots$. Moreover, this polynomial is the restriction of the potential F^g to the codimension 2 subspace $t_0 = t_1 = 0$,*

$$(U')^{2-2g}F^g(t_0, t_1, t_2, \ldots) = F^g(0, 0, T_2, T_3, \ldots).$$

Thus,

$$\frac{F^2}{(U')^2} = \frac{7T_2^3}{1440} + \frac{29T_2T_3}{5760} + \frac{T_4}{1152}$$
$$\frac{F^3}{(U')^4} = \frac{245T_2^6}{20736} + \frac{193T_2^4T_3}{6912} + \frac{205T_2^2T_3^2}{13824} + \frac{583T_3^3}{580608} + \frac{53T_2^3T_4}{6912} + \frac{1121T_2T_3T_4}{241920}$$
$$+ \frac{607T_4^2}{2903040} + \frac{17T_2^2T_5}{11520} + \frac{503T_3T_5}{1451520} + \frac{77T_2T_6}{414720} + \frac{T_7}{82944},$$

and so on. Proposition 4.1 now guarantees existence of polynomial expressions of the same functions in terms of the series U_k:

$$\frac{F^2}{(U')^2} = \frac{U_2^3}{360} - \frac{7U_2U_3}{1920} + \frac{U_4}{1152}$$
$$\frac{F^3}{(U')^4} = -\frac{5U_2^6}{648} + \frac{59U_2^4U_3}{3024} - \frac{83U_2^2U_3^2}{7168} + \frac{59U_3^3}{64512} - \frac{83U_2^3U_4}{15120} + \frac{1273U_2U_3U_4}{322560}$$
$$- \frac{103U_4^2}{483840} + \frac{353U_2^2U_5}{322560} - \frac{53U_3U_5}{161280} - \frac{7U_2U_6}{46080} + \frac{U_7}{82944},$$

and so on. This form of the Itzykson–Zuber ansatz is the Dijkgraaf–Witten ansatz [1]. In view of the definition of the functions U_k, these equations can be regarded as partial differential equations on the components F^g of the Witten–Kontsevich potential equivalent to those considered in this paper.

The Itzykson–Zuber ansatz follows from the fact that the functions F^g satisfy the string and the dilaton equations. Since any descendent potential also satisfies these equations, the same is true for descendent potentials as well.

Theorem 4.3. *Any descendent potential $f_{\mathcal{N};\beta}$ of Section 2 with $-\chi_{\mathcal{N}} > 0$ rescaled by the factor $(U')^{\chi_{\mathcal{N}}}$ admits a polynomial expression in terms of the functions T_k (whence in terms of U_k):*

$$(U')^{\chi_{\mathcal{N}}} f_{\mathcal{N};\beta}(t_0, t_1, t_2, \dots) = f_{\mathcal{N};\beta}(0, 0, T_2, T_3, \dots).$$

Another consequence of the Itzykson–Zuber ansatz is the result mentioned in the introduction: *the KdV, string, and dilaton equations* (1), (2), (3) *together determine the Witten–Kontsevich potential uniquely.* Specialists know this fact for many years but we were able to find only a single recent reference [8] containing the proof. Here we reproduce the argument in [8] in a slightly simplified form. Let us make a coordinate change replacing the coordinates $t_0, t_1, t_2, \dots$ with $U, Z, t_2, t_3, \dots$, where $U = I_0 = \frac{\partial^2 F^0}{\partial t_0^2}$ and $Z = \frac{1}{1-I_1} = U' = \frac{\partial^3 F^0}{\partial t_0^3}$. The Itzykson–Zuber ansatz 4.2 (which is a consequence of the string and dilaton equations) implies that under this change the function F^g, $g \geqslant 2$, becomes a polynomial in Z, with the monomials Z^i of degrees satisfying $2g-1 \leqslant i \leqslant 5g-5$, whose coefficients are functions on the remaining variables. (In fact, these coefficients are also polynomials in $I_2, I_3, \dots$, but we do not make use of this fact). In the new coordinates, the KdV equation takes the form

$$Z^2 \frac{\partial}{\partial Z} \Phi^g = \frac{1}{2} \sum_{i=1}^{g-1} \partial_0 \Phi^i \, \partial_0 \Phi^{g-i} + \frac{1}{12} \partial_0^3 \Phi^{g-1}, \qquad \Phi^g = \partial_0 F^g,$$

where $\partial_0 = Z\frac{\partial}{\partial U} + b(U) Z^3 \frac{\partial}{\partial Z}$, $b(U) = I_2 = t_2 + U t_3 + t_4 \frac{U^2}{2!} + \dots$. It defines Φ^g inductively starting with $\Phi^1 = \frac{1}{24}(b(U)Z^2)$. Induction uses inversion of the operator $Z^2 \frac{\partial}{\partial Z}$. Finding the inverse is based on the requirement that the result is a polynomial in Z with monomials of prescribed degrees. Finally, inverting ∂_0 using the string equation we find F^g from Φ^g.

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