

ON THE PROBLEM OF OPTIMAL FAIR EXCHANGE

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Abstract. We consider the problem of optimal exchange which can be formulated as a kind of optimal transportation problem. The existence of an optimal solution and a duality theorem for the optimal exchange problem are proved in case of completely regular topological spaces. We show the connection between the problem of optimal exchange and the optimal transportation problem with density constraints. With the use of this connection we obtain a formula for the optimal value in the problem of optimal exchange.

Keywords: optimal transportation problem, Kantorovich problem, density constraints.

1. INTRODUCTION

Recently various new modifications of the Monge-Kantorovich optimal transportation problem have been widely studied (see [1], [3], [5], [6]). Recall that for Borel probability measures μ and ν on topological spaces X and Y respectively and a non-negative Borel measurable function h on $X \times Y$ the classical Kantorovich optimal transportation problem concerns the minimization of the integral

$$\int_{X \times Y} h(x, y) \sigma(dx dy)$$

over all measures σ from the set $\Pi(\mu, \nu)$ consisting of all Borel probability measures on $X \times Y$ with projections μ and ν on the factors, that is, $\sigma(A \times Y) = \mu(A)$ and $\sigma(X \times B) = \nu(B)$ for all Borel sets $A \subset X$ and $B \subset Y$. Denote by $Pr_X(\sigma)$ and $Pr_Y(\sigma)$ the projections of a measure σ on X and Y respectively. The measures μ and ν are called marginal distributions or marginals and h is called a cost function. Basic information about the Monge-Kantorovich problem can be found in [1], [5], [11], [12], [13]. In [7] and [12] economic applications of the optimal transportation problem are addressed.

In this work we consider the problem of optimal exchange which can be formulated as a kind of optimal transportation problem. Let X be the set of trading participants and let S be the set of goods. Let further $Y = X$ be a copy of X . We assume that a unit of the good $s \in S$ has a cost $c(s)$, where $c: S \rightarrow \mathbb{R}$ is a positive cost function.

Suppose we are given a discrete distribution π^+ on $X \times S$ and a discrete distribution π^- on $Y \times S$. For any $i \in X$, $s \in S$ the value $\pi_{i,s}^+$ is equal to the amount of the good s that the i -th participant is ready to share with other participants. Similarly

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$\pi_{i,s}^-$ is equal to the amount of the good s that the i -th participant wishes to receive. Denote by $\gamma_{i,j,s}$ the amount of the good s that the i -th participant sends to the j -th participant.

We obtain the following problem:

$$\begin{aligned} \sum_{i,j,s} c(s)\gamma_{i,j,s} &\rightarrow \max, \\ \sum_j \gamma_{i,j,s} &\leq \pi_{i,s}^+, \quad \sum_i \gamma_{i,j,s} \leq \pi_{j,s}^-, \\ \sum_{j,s} c(s)\gamma_{k,j,s} &= \sum_{i,s} c(s)\gamma_{i,k,s}. \end{aligned} \tag{1.1}$$

The last equality is the balance condition: for any participant k the total cost of the sent items of goods is equal to the total cost of the received items of goods.

Let us consider the continuous setting of the optimal exchange problem which is a natural generalization of the discrete problem (1.1). Let $X = Y$ and S be topological spaces. Let π^+ be a nonnegative Borel measure on $X \times S$ and let π^- be a nonnegative Borel measure on $Y \times S$. Assume also that $c: S \rightarrow \mathbb{R}$ is a positive Borel measurable function. The optimal exchange problem concerns the maximization of the integral functional

$$\int_{X \times Y \times S} c(s) d\gamma \rightarrow \max \tag{1.2}$$

over all nonnegative Borel measures γ on $X \times Y \times S$ satisfying the constraints

$$Pr_{X \times S}(\gamma) \leq \pi^+, \quad Pr_{Y \times S}(\gamma) \leq \pi^-, \tag{1.3}$$

$$\int_{Y \times S} c(s) d\gamma = \int_{X \times S} c(s) d\gamma. \tag{1.4}$$

Denote by

$$\Gamma(\pi^+, \pi^-, c) = \left\{ \gamma \in \mathcal{M}_+(X \times Y \times S) : Pr_{X \times S}(\gamma) \leq \pi^+, \quad Pr_{Y \times S}(\gamma) \leq \pi^-, \right. \\ \left. \int_{Y \times S} c(s) d\gamma = \int_{X \times S} c(s) d\gamma \right\}$$

the set of admissible measures $\gamma \in \mathcal{M}_+(X \times Y \times S)$ in the problem of optimal exchange.

Let us introduce the notation and terminology that will be used in this paper. The Borel σ -algebra of the topological space X is denoted by $\mathcal{B}(X)$. A nonnegative Borel measure μ on a topological space X is called Radon if for every Borel set B and every $\varepsilon > 0$ there exists a compact set $K \subset B$ with $\mu(B \setminus K) < \varepsilon$. The set of Radon nonnegative measures on X is denoted by $\mathcal{M}_+(X)$. A family $M \subset \mathcal{M}_+(X)$ is called uniformly tight if for every $\varepsilon > 0$ there exists a compact set K such that $\mu(X \setminus K) < \varepsilon$ for all $\mu \in M$.

The space $\mathcal{M}(X)$ of bounded signed Radon measures on X is endowed with the weak topology generated by the seminorms of the form

$$\mu \mapsto \left| \int_X f d\mu \right|,$$

where f is a bounded continuous function on X .

For a signed measure $\mu \in \mathcal{M}(X)$ denote by μ_+ and μ_- the positive and negative parts of the measure μ in the Hahn decomposition $\mu = \mu_+ - \mu_-$. For nonnegative measures $\mu, \nu \in \mathcal{M}_+(X)$ we denote $\mu \wedge \nu = \mu - (\mu - \nu)_+$.

For any measure $\sigma \in \mathcal{M}_+(X \times Y)$ with the projection μ on X conditional measures σ^x , $x \in X$, are nonnegative Radon measures on Y with the property

$$\int_{X \times Y} \varphi(x)\psi(y) \sigma(dx dy) = \int_X \int_Y \varphi(x)\psi(y) \sigma^x(dy) \mu(dx)$$

for all bounded Borel functions φ on X and ψ on Y , where it is also assumed that the function $x \mapsto \sigma^x(B)$ is μ -measurable for every Borel set $B \subset Y$. Conditional measures exist under rather broad assumptions, for example, in case of Souslin spaces. The fact that a measure σ with the projection μ on X has conditional measures is usually expressed as the equality

$$\sigma(dx dy) = \sigma^x(dy) \mu(dx).$$

We shall need below the so-called gluing lemma (see [4]). Let X_1, X_2, X_3 be completely regular topological spaces and let $\mu_{1,2}$ and $\mu_{2,3}$ be nonnegative Radon measures on $X_1 \times X_2$ and $X_2 \times X_3$ respectively, such that their projections on X_2 coincide. Then there exists a nonnegative Radon measure μ on $X_1 \times X_2 \times X_3$ such that its projection on $X_1 \times X_2$ is $\mu_{1,2}$ and its projection on $X_2 \times X_3$ is $\mu_{2,3}$.

If the measure $\mu_{2,3}$ has conditional measures

$$\mu_{2,3}(dx_2 dx_3) = \mu_{2,3}^{x_2}(dx_3) \mu_2(dx_2),$$

where μ_2 is the projection of $\mu_{2,3}$ on X_2 , then we can take the measure

$$\mu(dx_1 dx_2 dx_3) = \mu_{2,3}^{x_2}(dx_3) \mu_{1,2}(dx_1 dx_2)$$

as a gluing of measures $\mu_{1,2}$ and $\mu_{2,3}$.

In this paper we prove the existence of an optimal solution to the problem of optimal exchange (1.2)-(1.4) (see Section 2). Furthermore, in Section 2 the dual problem for the problem (1.2)-(1.4) is formulated and the duality formula is proved in case of completely regular topological spaces. In Section 3 we show the connection between the problem of optimal exchange and the optimal transportation problem with density constraints. With the use of this connection we propose a method of constructing the optimal solution in the optimal exchange problem. In Section 4 under the assumption that the supports of the measures π^+ and π^- are disjoint the optimal exchange problem is reduced to a Kantorovich optimal transportation problem with density constraints.

2. DUALITY IN THE OPTIMAL EXCHANGE PROBLEM

First we prove the existence of an optimal solution to the optimal exchange problem for Radon measures on completely regular topological spaces.

Theorem 2.1. *Let $X = Y$ and S be completely regular topological spaces, let $\pi^+ \in \mathcal{M}_+(X \times S)$, $\pi^- \in \mathcal{M}_+(Y \times S)$ be Radon nonnegative measures and let $c: S \rightarrow \mathbb{R}$*

be a positive Borel measurable cost function. Then the maximum in the problem (1.2)-(1.4) is attained, that is, there exists an optimal measure $\gamma \in \Gamma(\pi^+, \pi^-, c)$.

Proof. Note that the problem (1.2)-(1.4) with an arbitrary cost function $c(s)$ can be reduced to the problem with constant cost function $c(s) = 1$. Indeed, set $\tilde{\gamma} = c(s)\gamma$. Then the problem (1.2)-(1.4) is equivalent to the following problem of optimal exchange with unit cost function:

$$\begin{aligned} \int_{X \times Y \times S} d\tilde{\gamma} &\rightarrow \max, \\ Pr_{X \times S}(\tilde{\gamma}) &\leq \pi^+, \quad Pr_{Y \times S}(\tilde{\gamma}) \leq \pi^-, \\ \int_{Y \times S} d\tilde{\gamma} &= \int_{X \times S} d\tilde{\gamma}. \end{aligned}$$

Further without limitation of generality we assume that $c(s) = 1$.

Let us prove that the set $\Gamma(\pi^+, \pi^-, c)$ is compact in the weak topology. For this purpose we prove that the set $\Gamma(\pi^+, \pi^-, c)$ is uniformly tight. Since the measures π^+ and π^- are Radon, for any $\varepsilon > 0$ there exist compact sets $K_1 \subset X \times S$ and $K_2 \subset Y \times S$ such that $\pi^+((X \times S) \setminus K_1) < \varepsilon/2$ and $\pi^-((Y \times S) \setminus K_2) < \varepsilon/2$. Let

$$K = \{(x, y, s) \in X \times Y \times S : (x, s) \in K_1, (y, s) \in K_2\}.$$

Then K is a compact subset of $X \times Y \times S$ and for any measure $\gamma \in \Gamma(\pi^+, \pi^-, c)$ we have

$$\gamma((X \times Y \times S) \setminus K) \leq \pi^+((X \times S) \setminus K_1) + \pi^-((Y \times S) \setminus K_2) < \varepsilon.$$

Thus, the set $\Gamma(\pi^+, \pi^-, c)$ is uniformly tight. Furthermore, $\Gamma(\pi^+, \pi^-, c)$ is uniformly bounded in variation and closed in the weak topology. Therefore, by Prokhorov's theorem (see [2]) the set $\Gamma(\pi^+, \pi^-, c)$ is compact in the weak topology. The functional $\gamma \mapsto \int c(s)d\gamma$ is continuous in the weak topology and, therefore, attains a maximum on the compact set $\Gamma(\pi^+, \pi^-, c)$. □

Let us formulate the dual problem for the problem of optimal exchange (1.2)-(1.4): we aim to minimize the functional

$$\int_{X \times S} f(x, s)d\pi^+ + \int_{Y \times S} g(y, s)d\pi^- \rightarrow \min \quad (2.1)$$

under the constraints

$$f(x, s) + g(y, s) + c(s)h(x) - c(s)h(y) \geq c(s), \quad (2.2)$$

where

$$f: X \times S \rightarrow \mathbb{R}_+, \quad g: Y \times S \rightarrow \mathbb{R}_+, \quad h: X \rightarrow \mathbb{R} \quad (2.3)$$

are Borel measurable functions.

Proposition 2.2. *The maximum in the primal problem (1.2)-(1.4) is not greater than the infimum in the dual problem (2.1)-(2.3).*

Proof. Let $\gamma \in \Gamma(\pi^+, \pi^-, c)$ and Borel measurable functions $f: X \times S \rightarrow \mathbb{R}_+$, $g: Y \times S \rightarrow \mathbb{R}_+$, $h: X \rightarrow \mathbb{R}$ be such that $f(x, s) + g(y, s) + c(s)h(x) - c(s)h(y) \geq c(s)$ for all $x \in X$, $y \in Y$, $s \in S$. Then

$$\begin{aligned} \int_{X \times Y \times S} c(s) d\gamma &\leq \int_{X \times Y \times S} (f(x, s) + g(y, s) + c(s)h(x) - c(s)h(y)) d\gamma = \\ &= \int_{X \times S} f(x, s) dPr_{X \times S}(\gamma) + \int_{Y \times S} g(y, s) dPr_{Y \times S}(\gamma) + \\ &\quad + \int_X h(x) \int_{Y \times S} c(s) d\gamma - \int_Y h(y) \int_{X \times S} c(s) d\gamma \leq \\ &\leq \int_{X \times S} f(x, s) d\pi^+ + \int_{Y \times S} g(y, s) d\pi^- \end{aligned}$$

in virtue of (1.3) and (1.4). \square

We prove the equality of the optimal values in the primal and dual problem for Radon measures on completely regular topological spaces.

Theorem 2.3. *Let $X = Y$ and S be completely regular topological spaces, let $\pi^+ \in \mathcal{M}_+(X \times S)$, $\pi^- \in \mathcal{M}_+(Y \times S)$ be Radon nonnegative measures and let $c: S \rightarrow \mathbb{R}$ be a positive Borel measurable cost function. Then the maximum in the problem (1.2)-(1.4) is equal to the infimum in the dual problem (2.1)-(2.3).*

Proof. As in the proof of Theorem 2.1, without limitation of generality we may assume that $c(s) = 1$ for all $s \in S$.

First we prove Theorem 2.3 in the case where $X = Y$ and S are compact topological spaces. We shall use the Fenchel-Rockafellar duality theorem. Let $U = C_b(X \times Y \times S)$. By Riesz theorem we have $U^* = \mathcal{M}(X \times Y \times S)$. Let us define the functional Θ on $C_b(X \times Y \times S)$ in the following way. For any function $u \in C_b(X \times Y \times S)$ we set $\Theta(u) = 0$, if there exists a function $h \in C_b(X)$ such that

$$u(x, y, s) + h(x) - h(y) \geq 1 \quad \forall x \in X, y \in Y, s \in S, \quad (2.4)$$

and we set $\Theta(u) = +\infty$ otherwise. Next we define

$$\begin{aligned} \Xi(u) = \inf \left\{ \int_{X \times S} f(x, s) d\pi^+ + \int_{Y \times S} g(y, s) d\pi^- : f \in C_b(X \times S), g \in C_b(Y \times S), \right. \\ \left. f \geq 0, g \geq 0, u(x, y, s) = f(x, s) + g(y, s) \right\}. \end{aligned}$$

(we set $\Xi(u) = +\infty$ if such functions f and g do not exist). The functionals Θ and Ξ are convex on $C_b(X \times Y \times S)$. For the function u_0 which is equal to a constant greater than 1, we have $\Theta(u_0) = 0$, $\Xi(u_0) < +\infty$ and Θ is continuous at u_0 (since $\Theta(u) = 0$ for all functions $u \geq 1$). By Fenchel-Rockafellar theorem (see [13])

$$\inf_{u \in C_b(X \times Y \times S)} (\Theta(u) + \Xi(u)) = \max_{\gamma \in \mathcal{M}(X \times Y \times S)} (-\Theta^*(-\gamma) - \Xi^*(\gamma)).$$

Note that $\inf_{u \in U} (\Theta(u) + \Xi(u))$ is equal to the infimum in the dual problem (2.1)-(2.3). Let us show that the maximum on the right-hand side is equal to the maximum

in the primal problem (1.2)-(1.4). For this we prove that if the value $-\Theta^*(-\gamma) - \Xi^*(\gamma)$ is finite for a measure $\gamma \in \mathcal{M}(X \times Y \times S)$, then $\gamma \in \Gamma(\pi^+, \pi^-, c)$ and

$$-\Theta^*(-\gamma) - \Xi^*(\gamma) = \int_{X \times Y \times S} d\gamma.$$

First we consider the value $-\Theta^*(-\gamma)$. Let us prove that $-\Theta^*(-\gamma) = \int_{X \times Y \times S} d\gamma$, if the measure γ is nonnegative and $\int_{Y \times S} d\gamma = \int_{X \times S} d\gamma$, and $-\Theta^*(-\gamma) = -\infty$ otherwise.

We have

$$-\Theta^*(-\gamma) = -\sup_{u \in U} \left(-\int_{X \times Y \times S} u d\gamma - \Theta(u) \right) = \inf_{u \in U} \left(\int_{X \times Y \times S} u d\gamma + \Theta(u) \right).$$

If the measure γ is not nonnegative, then there exists a function $\varphi \in C_b(X \times Y \times S)$ such that $\varphi \geq 0$ and $\int_{X \times Y \times S} \varphi d\gamma < 0$. Let $u(x, y, s) = 1 + t\varphi(x, y, s)$, where $t > 0$. Then $\Theta(u) = 0$ and

$$\int_{X \times Y \times S} u d\gamma = \int_{X \times Y \times S} d\gamma + t \int_{X \times Y \times S} \varphi d\gamma \rightarrow -\infty, \quad t \rightarrow +\infty.$$

Therefore, $-\Theta^*(-\gamma) = -\infty$.

If the measure γ does not satisfy the condition that $\int_{Y \times S} d\gamma = \int_{X \times S} d\gamma$, then there exists a function $h \in C_b(X)$ such that

$$\int_{X \times Y \times S} h(x) d\gamma < \int_{X \times Y \times S} h(y) d\gamma.$$

Take

$$u(x, y, s) = 1 + t h(x) - t h(y),$$

where $t > 0$. Then $\Theta(u) = 0$ and

$$\begin{aligned} \int_{X \times Y \times S} u d\gamma &= \int_{X \times Y \times S} d\gamma + \\ &+ t \left(\int_{X \times Y \times S} h(x) d\gamma - \int_{X \times Y \times S} h(y) d\gamma \right) \rightarrow -\infty, \quad t \rightarrow +\infty. \end{aligned}$$

Therefore, $-\Theta^*(-\gamma) = -\infty$.

Suppose that the measure γ is nonnegative and satisfies the condition that $\int_{Y \times S} d\gamma = \int_{X \times S} d\gamma$. We show that

$$-\Theta^*(-\gamma) = \int_{X \times Y \times S} d\gamma. \quad (2.5)$$

Letting $u(x, y, s) = 1$, we obtain $-\Theta^*(-\gamma) \leq \int_{X \times Y \times S} d\gamma$. Let us prove that $-\Theta^*(-\gamma) \geq \int_{X \times Y \times S} d\gamma$. Indeed, if $\Theta(u) < +\infty$, then there exists a function

$h \in C_b(X)$ which satisfies the inequality (2.4) and hence

$$\begin{aligned} \int_{X \times Y \times S} u(x, y, s) d\gamma &\geq \int_{X \times Y \times S} d\gamma - \int_{X \times Y \times S} h(x) d\gamma + \int_{X \times Y \times S} h(y) d\gamma = \\ &= \int_{X \times Y \times S} d\gamma. \end{aligned}$$

Thus, $\inf_{u \in U} (\int_{X \times Y \times S} u d\gamma + \Theta(u)) \geq \int_{X \times Y \times S} d\gamma$. Therefore, the equality (2.5) holds true.

Now we consider the value $\Xi^*(\gamma)$. Let us prove that $\Xi^*(\gamma) = 0$, if $Pr_{X \times S}(\gamma) \leq \pi^+$ and $Pr_{X \times S}(\gamma) \leq \pi^-$, and $\Xi^*(\gamma) = +\infty$ otherwise.

We have

$$\begin{aligned} \Xi^*(\gamma) &= \sup_{u \in U} \left(\int_{X \times Y \times S} u d\gamma - \Xi(u) \right) = \\ &= \sup \left\{ \int_{X \times Y \times S} (f(x, s) + g(y, s)) d\gamma - \int_{X \times S} f(x, s) d\pi^+ - \int_{Y \times S} g(y, s) d\pi^- : \right. \\ &\quad \left. f \in C_b(X \times S), g \in C_b(Y \times S), f \geq 0, g \geq 0 \right\}. \end{aligned}$$

If the inequality $Pr_{X \times S}(\gamma) \leq \pi^+$ is not satisfied, then there exists a function $f \in C_b(X \times S)$, $f \geq 0$, such that

$$\int_{X \times Y \times S} f(x, s) d\gamma > \int_{X \times S} f(x, s) d\pi^+.$$

Then for the function $u(x, y, s) = t f(x, s)$, where $t > 0$, we obtain

$$\int_{X \times Y \times S} u d\gamma - \Xi(u) \geq t \left(\int_{X \times Y \times S} f(x, s) d\gamma - \int_{X \times S} f(x, s) d\pi^+ \right) \rightarrow +\infty, \quad t \rightarrow +\infty.$$

Thus, $\Xi^*(\gamma) = +\infty$. Similarly $\Xi^*(\gamma) = +\infty$, if the inequality $Pr_{Y \times S}(\gamma) \leq \pi^-$ is not satisfied. If $Pr_{X \times S}(\gamma) \leq \pi^+$ and $Pr_{Y \times S}(\gamma) \leq \pi^-$, then

$$\sup_{u \in U} \left(\int_{X \times Y \times S} u d\gamma - \Xi(u) \right) = 0.$$

Therefore, $-\Theta^*(-\gamma) - \Xi^*(\gamma) = -\infty$, if $\gamma \notin \Gamma(\pi^+, \pi^-, c)$, and

$$-\Theta^*(-\gamma) - \Xi^*(\gamma) = \int_{X \times Y \times S} d\gamma,$$

if $\gamma \in \Gamma(\pi^+, \pi^-, c)$.

Thus,

$$\max_{\gamma \in \mathcal{M}(X \times Y \times S)} (-\Theta^*(-\gamma) - \Xi^*(\gamma)) = \max_{\gamma \in \Gamma(\pi^+, \pi^-, c)} \int_{X \times Y \times S} d\gamma.$$

In the general case of completely regular topological spaces we consider the Stone-Cech compactifications $\beta X = \beta Y$ and βS of the spaces $X = Y$ and S . For compact spaces the duality formula is already proved. Note that every measure $\gamma \in \mathcal{M}_+(\beta X \times \beta Y \times \beta S)$ satisfying the conditions $Pr_{\beta X \times \beta S}(\gamma) \leq \pi^+$ and $Pr_{\beta Y \times \beta S}(\gamma) \leq \pi^-$, is concentrated on $X \times Y \times S$. Therefore, the maximum in the

primal problem over measures $\gamma \in \mathcal{M}_+(\beta X \times \beta Y \times \beta S)$ is equal to the maximum in the primal problem over measures $\gamma \in \mathcal{M}_+(X \times Y \times S)$. This implies the duality formula for the initial spaces $X = Y$ and S . \square

3. REDUCTION TO A TWO-DIMENSIONAL OPTIMIZATION PROBLEM WITH CONSTRAINTS

In this section we reduce the problem (1.2)-(1.4) to a two-dimensional optimization problem. Further without limitation of generality we assume that $c(s) = 1$ for all $s \in S$. Let $\gamma \in \Gamma(\pi^+, \pi^-, c)$. Denote by

$$\sigma^+ = Pr_{X \times S}(\gamma), \quad \sigma^- = Pr_{Y \times S}(\gamma)$$

the projections of the measure γ on $X \times S$ and $Y \times S$ respectively. By virtue of (1.3) we have

$$\sigma^+ \leq \pi^+, \quad \sigma^- \leq \pi^-.$$

The balance condition (1.4) implies that $Pr_X(\gamma) = Pr_Y(\gamma)$. Therefore, the measures σ^+ and σ^- satisfy the properties

$$Pr_X(\sigma^+) = Pr_Y(\sigma^-), \quad Pr_S(\sigma^+) = Pr_S(\sigma^-).$$

Consider the following optimization problem for nonnegative measures $\sigma^+ \in \mathcal{M}_+(X \times S)$, $\sigma^- \in \mathcal{M}_+(Y \times S)$: we aim to maximize the value

$$\sigma^+(X \times S) \rightarrow \max \tag{3.1}$$

under the condition that

$$\sigma^+ \leq \pi^+, \quad \sigma^- \leq \pi^-, \tag{3.2}$$

$$Pr_X(\sigma^+) = Pr_Y(\sigma^-), \quad Pr_S(\sigma^+) = Pr_S(\sigma^-). \tag{3.3}$$

Theorem 3.1. *Let $X = Y$ and S be completely regular topological spaces, $\pi^+ \in \mathcal{M}_+(X \times S)$ and $\pi^- \in \mathcal{M}_+(Y \times S)$ be Radon nonnegative measures. Then the maximum in the problem (1.2)-(1.4) is equal to the maximum in the problem (3.1)-(3.3). For every optimal solution (σ^+, σ^-) to the problem (3.1)-(3.3) there exists an optimal solution γ to the problem (1.2) such that $Pr_{X \times S}(\gamma) = \sigma^+$ and $Pr_{Y \times S}(\gamma) = \sigma^-$.*

Proof. Note that for given two measures $\sigma^+ \in \mathcal{M}_+(X \times S)$, $\sigma^- \in \mathcal{M}_+(Y \times S)$ satisfying the conditions (3.2) and (3.3), with the use of gluing lemma we can construct a measure $\gamma \in \mathcal{M}_+(X \times Y \times S)$ such that $Pr_{X \times S}(\gamma) = \sigma^+$ and $Pr_{Y \times S}(\gamma) = \sigma^-$. Then

$$Pr_X(\gamma) = Pr_X(\sigma^+) = Pr_Y(\sigma^-) = Pr_Y(\gamma)$$

and, therefore, $\gamma \in \Gamma(\pi^+, \pi^-, c)$. This implies that the maximum in the problem (1.2)-(1.4) is equal to the maximum in the problem (3.1)-(3.3), and moreover, a measure γ is optimal for the problem (1.2)-(1.4) if and only if measures σ^+ , σ^- form an optimal solution to the problem (3.1)-(3.3). \square

Thus, the initial (three-dimensional) optimization problem for a measure $\gamma \in \mathcal{M}_+(X \times Y \times S)$ can be reduced to (two-dimensional) optimization problem for measures σ^+ , σ^- .

Note that if measures σ^+ and σ^- satisfy the conditions (3.2) and (3.3), then $Pr_X(\sigma^+) \leq Pr_X(\pi^+) \wedge Pr_Y(\pi^-)$ and $Pr_S(\sigma^+) \leq Pr_S(\pi^+) \wedge Pr_S(\pi^-)$.

Denote

$$\hat{\mu} = Pr_X(\pi^+) \wedge Pr_Y(\pi^-), \quad \hat{\nu} = Pr_S(\pi^+) \wedge Pr_S(\pi^-).$$

Therefore, we obtain

Corollary 3.2. *The maximum in the problem (1.2)-(1.4) does not exceed $\min(\hat{\mu}(X), \hat{\nu}(S))$.*

Furthermore, Theorem 3.1 implies a necessary and sufficient condition under which it is possible to construct a measure γ satisfying the demand and supply for all trading participants.

Corollary 3.3. *The problem (1.2)-(1.4) has an optimal solution for which the inequalities (1.4) become equalities (that is, the demand and supply of every participant are completely satisfied) if and only if measures π^+ and π^- have the same projections:*

$$Pr_X(\pi^+) = Pr_Y(\pi^-), \quad Pr_S(\pi^+) = Pr_S(\pi^-).$$

Consider the following example where it is possible to find the optimal solution to the problem (1.2)-(1.4) in an explicit form.

Example 3.4. Let $X = Y = S = [0, 1]$. Let π^+ be Lebesgue measure on the domain

$$D = \{(x, s) \in [0, 1]^2 : x/2 \leq s \leq (x+1)/2\},$$

and let π^- be Lebesgue measure on the domain $[0, 1]^2 \setminus D$. Then

$$Pr_X(\pi^+) = Pr_Y(\pi^-) = 1/2\lambda,$$

$$Pr_S(\pi^+) = (1 - |2s - 1|)\lambda, \quad Pr_S(\pi^-) = |2s - 1|\lambda,$$

where λ is Lebesgue measure on the interval $[0, 1]$. We have

$$\hat{\mu} = 1/2\lambda, \quad \hat{\nu} = \min(1 - |2s - 1|, |2s - 1|)\lambda.$$

Then $\hat{\nu}(S) = 1/4$. It is easy to see that there exist measures $\sigma^+ \leq \pi^+$ and $\sigma^- \leq \pi^-$ with common projections $|x - 1/2|\lambda$ and $\hat{\nu}$ on the spaces X and S respectively: σ^+ is Lebesgue measure on the domain $D \setminus \tilde{D}$ and σ^- is Lebesgue measure on the domain $\tilde{D} \setminus D$, where \tilde{D} is the image of the domain D under the reflection in the line $x = 1/2$. We can construct the optimal measure γ on $X \times Y \times S$: the measure γ is concentrated on the set $\{y = 1 - x\}$ and has projections σ^+ and σ^- on $X \times S$ and $Y \times S$ respectively (we can define γ as the image of the measure σ^+ under the mapping $(x, s) \mapsto (x, 1 - x, s)$). That is, the participants x and $1 - x$ (where $x \in [0, 1/2]$) exchange goods with each other: the participant x gives the participant $1 - x$ goods s , where $x/2 \leq s \leq (1 - x)/2$, and receives from the participant $1 - x$ goods s , where $1 - x/2 \leq s \leq (x + 1)/2$.

Now we investigate the optimization problem (3.1)-(3.3). Denote

$$\mu = Pr_X(\sigma^+) = Pr_Y(\sigma^-), \quad \nu = Pr_S(\sigma^+) = Pr_S(\sigma^-).$$

In order to solve the problem (3.1)-(3.3) we need to find measures $\mu \in \mathcal{M}_+(X)$ and $\nu \in \mathcal{M}_+(S)$ with the maximum possible mass $\mu(X) = \nu(S)$, such that there exist measures $\sigma^+, \sigma^- \in \Pi(\mu, \nu)$ satisfying the conditions $\sigma^+ \leq \pi^+$ and $\sigma^- \leq \pi^-$.

The problem of optimization over the set of measures σ with given projections μ and ν and an additional constraint $\sigma \leq \pi$ has been considered in [8], [9], [10]. In [9] the following necessary and sufficient condition for the existence of a measure $\sigma \leq \pi$ with given projections μ and ν has been formulated.

Theorem 3.5 ([9]). *For given measures $\mu \in \mathcal{M}_+(X)$, $\nu \in \mathcal{M}_+(S)$ and $\pi \in \mathcal{M}_+(X \times S)$ there exists a measure $\sigma \in \mathcal{M}_+(X \times S)$ such that $Pr_X(\sigma) = \mu$, $Pr_S(\sigma) = \nu$ and $\sigma \leq \pi$ if and only if*

$$\int_X u(x)\mu(dx) + \int_S v(s)\nu(ds) \leq \int_{X \times S} [u(x) + v(s)]_+ \pi(dx ds)$$

for all functions $u \in C_b(X)$, $v \in C_b(S)$, where $[\cdot]_+ = \max(\cdot, 0)$.

Using Theorem 3.5, we can obtain the following formula for the optimal value in the problem (1.2)-(1.4).

Theorem 3.6. *Let X and S be completely regular topological spaces, let $\pi^+ \in \mathcal{M}_+(X \times S)$ and $\pi^- \in \mathcal{M}_+(X \times S)$ be Radon nonnegative measures. Then the maximum in the problem (1.2)-(1.4) is equal to*

$$\inf_{u \in C_b(X), v \in C_b(S)} \left\{ \int_{X \times S} [u(x) + v(s)]_+ \pi^+(dx ds) + \int_{X \times S} [1 - u(x) - v(s)]_+ \pi^-(dx ds) \right\}. \quad (3.4)$$

Proof. Let us define functionals Φ^+ and Φ^- on $C_b(X) \times C_b(S)$:

$$\Phi^+(u, v) = \int_{X \times S} [u(x) + v(s)]_+ \pi^+(dx ds),$$

$$\Phi^-(u, v) = \int_{X \times S} [1 - u(x) - v(s)]_+ \pi^-(dx ds)$$

for all $u \in C_b(X)$, $v \in C_b(S)$. Note that Φ^+ and Φ^- are continuous sublinear functionals on $C_b(X) \times C_b(S)$.

We first consider the case where X and S are compact topological spaces. Then by Riesz theorem every continuous linear functional L on $C_b(X) \times C_b(S)$ can be represented in the form

$$L(u, v) = \int_X u d\mu + \int_S v d\nu$$

for some (signed) measures $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(S)$. Moreover, the functional L is nonnegative (that is, $L(u, v) \geq 0$ for all pairs of nonnegative functions $u \in C_b(X)$ and $v \in C_b(S)$) if and only if measures μ and ν are nonnegative, i.e. $\mu \in \mathcal{M}_+(X)$, $\nu \in \mathcal{M}_+(S)$.

By virtue of Theorem 3.5 the problem of finding measures $\mu \in \mathcal{M}_+(X)$ and $\nu \in \mathcal{M}_+(S)$ with maximum possible mass $\mu(X) = \nu(S)$ for which there exist measures $\sigma^+, \sigma^- \in \Pi(\mu, \nu)$ such that $\sigma^+ \leq \pi^+$ and $\sigma^- \leq \pi^-$, is equivalent to finding a nonnegative continuous linear functional L on $C_b(X) \times C_b(S)$ satisfying the inequalities $L \leq \Phi^+$ and $L \leq \Phi^-$, for which the value $L(1, 1)$ is maximal.

Note that if L is a linear functional on $C_b(X) \times C_b(S)$ satisfying the conditions $L \leq \Phi^+$ and $L \leq \Phi^-$, then L is nonnegative and continuous. Indeed, for any nonnegative functions $u \in C_b(X)$ and $v \in C_b(S)$ we have $L(u, v) = -L(-u, -v) \geq -\Phi^+(-u, -v) = 0$. Furthermore, for any functions $u \in C_b(X)$ and $v \in C_b(S)$ it holds that $|L(u, v)| \leq \Phi^+(u, v) + \Phi^+(-u, -v)$, which implies the continuity of the linear functional L due to the continuity of Φ^+ .

Let

$$\Phi(u, v) = \inf \left\{ \Phi^+(u^+, v^+) + \Phi^-(u^-, v^-) : u = u^+ + u^-, v = v^+ + v^-, \right. \\ \left. u^+, u^- \in C_b(X), v^+, v^- \in C_b(S) \right\} \quad (3.5)$$

for all $u \in C_b(X)$, $v \in C_b(S)$.

Let us show that if a continuous linear functional L satisfies the conditions $L \leq \Phi^+$ and $L \leq \Phi^-$, then $L \leq \Phi$. Let $u \in C_b(X)$ and $v \in C_b(S)$. Then for any functions $u^+, u^- \in C_b(X)$, $v^+, v^- \in C_b(S)$ such that $u = u^+ + u^-$ and $v = v^+ + v^-$, we have

$$L(u, v) = L(u^+, v^+) + L(u^-, v^-) \leq \Phi^+(u^+, v^+) + \Phi^-(u^-, v^-).$$

Thus, $L(u, v) \leq \Phi(u, v)$.

We prove that the functional Φ is sublinear, that is,

$$\Phi(u_1 + u_2, v_1 + v_2) \leq \Phi(u_1, v_1) + \Phi(u_2, v_2)$$

for all $u_1, u_2 \in C_b(X)$, $v_1, v_2 \in C_b(S)$. Indeed, for any $u_i^+, u_i^- \in C_b(X)$ and $v_i^+, v_i^- \in C_b(S)$ such that $u_i = u_i^+ + u_i^-$, $v_i = v_i^+ + v_i^-$ for all $i \in \{1, 2\}$, we have

$$\Phi(u_1 + u_2, v_1 + v_2) \leq \Phi^+(u_1^+ + u_2^+, v_1^+ + v_2^+) + \Phi^-(u_1^- + u_2^-, v_1^- + v_2^-) \leq \\ \leq \Phi^+(u_1^+, v_1^+) + \Phi^+(u_2^+, v_2^+) + \Phi^-(u_1^-, v_1^-) + \Phi^-(u_2^-, v_2^-),$$

since the functionals Φ^+ and Φ^- are sublinear. Taking the infimum over all such functions u_i^+, u_i^- and v_i^+, v_i^- , $i \in \{1, 2\}$, we obtain that $\Phi(u_1 + u_2, v_1 + v_2) \leq \Phi(u_1, v_1) + \Phi(u_2, v_2)$.

By Hahn-Banach theorem for the sublinear functional Φ there exists a linear functional L on $C_b(X) \times C_b(S)$ such that $L \leq \Phi$ and $L(1, 1) = \Phi(1, 1)$. Then, as mentioned above, from the inequality $L \leq \Phi$ it follows that the linear functional L is continuous and nonnegative. Therefore, L can be represented in the form $L(u, v) = \int u d\mu + \int v d\nu$ for some nonnegative measures $\mu \in \mathcal{M}_+(X)$, $\nu \in \mathcal{M}_+(S)$. Then by Theorem 3.5 for measures μ and ν it is possible to find measures $\sigma^+ \leq \pi^+$ and $\sigma^- \leq \pi^-$ with projections μ and ν on X and S respectively. Using gluing lemma, we can construct a measure γ on $X \times Y \times S$ which gives the optimal solution to the problem (1.2)-(1.4). Therefore, the maximum in the problem (1.2)-(1.4) is equal to $\mu(X) = \nu(S) = L(1, 1)/2 = \Phi(1, 1)/2$ and by definition (3.5) this implies the equality (3.4).

The general case of completely regular topological spaces can be reduced to the case of compact topological spaces using the Stone-Cech compactifications βX and βS of the spaces X and S . □

In [9] another equivalent condition for the existence of a measure $\sigma \leq \pi$ with given projections μ and ν is also given. Denote

$$\Pi(\mu, \nu; \pi) = \{\sigma \in \Pi(\mu, \nu) : \sigma \leq \pi\}.$$

Theorem 3.7 ([9]). *Let $\mu \in \mathcal{M}_+(X)$, $\nu \in \mathcal{M}_+(S)$ and $\pi \in \mathcal{M}_+(X \times S)$. The set $\Pi(\mu, \nu; \pi)$ is non-empty if and only if*

$$\mu(A) + \nu(B) \leq \alpha + \pi(A \times B)$$

for all Borel sets $A \subset X$, $B \subset S$, where $\alpha = \mu(X) = \nu(S)$.

Let us denote

$$r(A, B) = \min(\pi^+(A \times B), \pi^-(A \times B))$$

for all Borel sets $A \subset X$, $B \subset S$. Then the problem (3.1)-(3.3) can be reduced to finding such measures $\mu \in \mathcal{M}_+(X)$ and $\nu \in \mathcal{M}_+(S)$ with maximum possible total mass $\mu(X) = \nu(S) = \alpha$ such that

$$\mu(A) + \nu(B) \leq \alpha + r(A, B) \tag{3.6}$$

for all Borel sets $A \subset X$, $B \subset S$.

The condition (3.6) may be rewritten in the following equivalent form:

$$\mu(A) \leq \nu(S \setminus B) + r(A, B), \quad \nu(B) \leq \mu(X \setminus A) + r(A, B). \tag{3.7}$$

Hence the optimal solution to the problem of optimal exchange can be constructed in the following way.

Step 1. We find measures $\mu \in \mathcal{M}_+(X)$ and $\nu \in \mathcal{M}_+(S)$ with maximum possible mass for which the condition (3.7) is satisfied.

Step 2. Using Theorem 3.7, we find measures $\sigma^+, \sigma^- \in \Pi(\mu, \nu)$ such that $\sigma^+ \leq \pi^+$ and $\sigma^- \leq \pi^-$.

Step 3. Applying gluing lemma for measures σ^+ and σ^- , we construct a measure γ which provides an optimal solution to the problem of optimal exchange.

4. REDUCTION TO A KANTOROVICH PROBLEM WITH DENSITY CONSTRAINTS

In this section we will assume that the measures π^+ and π^- are concentrated on disjoint sets A^+ and A^- , where $A^+, A^- \subset X \times S$ are Borel sets. This assumption means that for every trading participant the set of goods which he is ready to share with other participants and the set of goods which he wishes to receive do not intersect.

We show that in this case the problem of optimal exchange can be reduced to a Kantorovich problem with density constraints.

Let

$$\mu^+ = Pr_X \pi^+, \quad \nu^+ = Pr_S \pi^+, \quad \pi = \pi^+ + \pi^-.$$

We define the cost function $h: X \times S \rightarrow \mathbb{R}$ in the following way: $h(x, s) = 1$ for all $(x, s) \in A^+$ and $h(x, s) = 0$ otherwise.

Consider the following Kantorovich optimal transportation problem with density constraints: we aim to minimize the integral

$$\int_{X \times S} h(x, s) \tau(dx ds) \rightarrow \inf, \quad \tau \in \Pi(\mu^+, \nu^+; \pi), \quad (4.1)$$

over all measures τ from the set $\Pi(\mu^+, \nu^+; \pi) = \{\tau \in \Pi(\mu^+, \nu^+) : \tau \leq \pi\}$.

Denote by $K_h(\mu^+, \nu^+; \pi)$ the infimum in the Kantorovich problem (4.1).

Theorem 4.1. *The maximum in the problem (3.1)-(3.3) is equal to*

$$\pi^+(X \times S) - K_h(\mu^+, \nu^+; \pi).$$

The measures (σ^+, σ^-) constitute an optimal solution to the problem (3.1)-(3.3) if and only if the measure $\tau = \pi^+ - \sigma^+ + \sigma^-$ is an optimal solution to the problem (4.1).

Proof. We build a one-to-one correspondence between pairs of measures $\sigma^+, \sigma^- \in \mathcal{M}_+(X \times S)$ satisfying the conditions (3.2) and (3.3) and measures $\tau \in \Pi(\mu^+, \nu^+; \pi)$, such that

$$\int_{X \times S} h(x, s) \tau(dx ds) = \pi^+(X \times S) - \sigma^+(X \times S), \quad (4.2)$$

if a measure τ corresponds to the pair of measures (σ^+, σ^-) .

Let measures $\sigma^+, \sigma^- \in \mathcal{M}_+(X \times S)$ satisfy the conditions (3.2) and (3.3). Set

$$\tau = \pi^+ - \sigma^+ + \sigma^-.$$

Then $\tau \geq 0$, because $\sigma^+ \leq \pi^+$. Since measures σ^+ and σ^- have equal projections on X and S , we have $Pr_X \tau = Pr_X \pi^+ = \mu^+$ and $Pr_S \tau = Pr_S \pi^+ = \nu^+$, that is, $\tau \in \Pi(\mu^+, \nu^+)$. Moreover, $\tau \leq \pi = \pi^+ + \pi^-$, since $\sigma^- \leq \pi^-$. Therefore, $\tau \in \Pi(\mu^+, \nu^+; \pi)$.

Furthermore, we have

$$\int_{X \times S} h(x, s) \tau(dx ds) = \tau(A^+) = \pi^+(A^+) - \sigma^+(A^+)$$

since $\sigma^- \leq \pi^-$ and $\pi^-(A^+) = 0$. The last expression equals $\pi^+(X \times S) - \sigma^+(X \times S)$, because $\sigma^+ \leq \pi^+$ and the measure π^+ is concentrated on A^+ .

Conversely, let $\tau \in \Pi(\mu^+, \nu^+; \pi)$. We set

$$\sigma^+ = \pi^+ - \tau|_{A^+}, \quad \sigma^- = \tau|_{A^-}.$$

Since $\tau \leq \pi$, we have $\tau|_{A^+} \leq \pi|_{A^+} = \pi^+$ and $\tau|_{A^-} \leq \pi|_{A^-} = \pi^-$. Therefore, σ^+ and σ^- are nonnegative measures and $\sigma^+ \leq \pi^+$, $\sigma^- \leq \pi^-$.

Let us show that $Pr_X \sigma^+ = Pr_X \sigma^-$ and $Pr_S \sigma^+ = Pr_S \sigma^-$. Indeed, the difference of measures $\sigma^+ - \sigma^- = \pi^+ - \tau$ has zero projections on X and S , since $\tau \in \Pi(\mu^+, \nu^+)$.

Thus, the measures σ^+ and σ^- satisfy the conditions (3.2) and (3.3).

The constructed correspondences between pairs of measures (σ^+, σ^-) and measures τ are mutually inverse and satisfy the equality (4.2). Therefore, the maximum in the problem (3.1)-(3.3) is equal to

$$\pi^+(X \times S) - K_h(\mu^+, \nu^+; \pi),$$

and the optimality of measures (σ^+, σ^-) for the problem (3.1)-(3.3) is equivalent to the optimality of the measure $\tau = \pi^+ - \sigma^+ + \sigma^-$ for the problem (4.1). □

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